A numerical method for the solution of exterior Neumann problems for the Laplace equation in domains with corners

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Abstract

In this paper we propose a new boundary integral method for the numerical solution of Neumann problems for the Laplace equation, posed in exterior planar domains with piecewise smooth boundaries. Using the single layer representation of the potential, the differential problem is reformulated as a classical boundary integral equation. The use of a smoothing transformation and the introduction of a modified Gauss-Legendre quadrature formula for the approximation of the singular integrals, which turns out to be convergent, lead us to apply a Nyström type method for the numerical solution of the integral equation. We solve some test problems and present the numerical results in order to show the efficiency of the proposed procedure.

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1. Introduction

In this work we are concerned with the exterior Neumann problem for the Laplace equation in two-dimensional domains with piecewise smooth boundaries. Let $\Omega \subset \mathbb{R}^2$ be an open bounded simply connected domain, with a piecewise smooth Lipschitz boundary Γ. We shall assume that the boundary curve Γ contains n corner points $P_1, \dots, P_n$ and is otherwise smooth. We consider the problem

$$\begin{aligned} &\Delta u = 0, && \text{in } \mathbb{R}^2 \setminus \bar{\Omega}, \\ &\frac{\partial u}{\partial \mathbf{n}} = f, && \text{on } \Gamma, \\ &|u(x)| = o(1), && \text{as } |x| \to \infty, \end{aligned} \tag{1}$$

where $\mathbf{n}$ denotes the outward unit normal vector at Γ. We shall assume that the given Neumann data f is a sufficiently smooth function which satisfies

$$\int_\Gamma f ds = 0. \tag{2}$$

It is well known that the solution of the exterior problem (1)-(2) exists and is unique (see, for instance, [21, p. 73],[20, p. 351],[5, p. 152]). A boundary integral equation (BIE) formulation of the exterior Neumann problem (1) is obtained by using the single layer representation of the potential u, i.e.

$$u(x) = -\int_\Gamma \phi(y) \log |x - y| dS(y), \quad x \in \mathbb{R}^2 \setminus \bar{\Omega}, \tag{3}$$

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where $|x-y|$ is the Euclidean distance between x and y, $dS(y)$ is the element of arc length and ϕ is the single-layer density function. The single layer potential (3) is a solution of (1) provided that the density ϕ is a solution of the integral equation

$$-\pi\phi(x) - \int_{\Gamma} \frac{\partial}{\partial \mathbf{n}(x)} \log|x-y|\phi(y)dS(y) = f(x), \quad x \in \Gamma, \tag{4}$$

and in addition satisfies

$$\int_{\Gamma} \phi(y)dS(y) = 0. \tag{5}$$

If the boundary Γ of the domain Ω is twice continuously differentiable, then the integral operator

$$\mathcal{K}\phi(x) = \int_{\Gamma} \frac{\partial}{\partial \mathbf{n}(x)} \log|x-y|\phi(y)dS(y) \tag{6}$$

is a compact operator on the space $C(\Gamma)$ of continuous functions on the curve Γ, equipped with the uniform norm. Then, by applying the Fredholm theory one can deduce that $(-\pi - \mathcal{K})$ is bounded and invertible as an operator on $C(\Gamma)$. Moreover, the standard numerical analysis applies. This case has been widely studied in the literature. In this paper we consider domains with piecewise smooth boundaries having corner points, that means points on Γ at which the limits of the normal derivative as one approaches from the clockwise and the counter-clockwise directions exist but are not equal. The case of nonsmooth boundaries is different from the smooth boundary one: both the kernel of the integral operator $\mathcal{K}$ and the solution of equation (4) are singular. Consequently, the operator $(-\pi-\mathcal{K})$ is not bounded with respect to the uniform norm and the standard analysis does not apply. Nevertheless, as shown in [19, 29], $(-\pi - \mathcal{K})$ is bounded and invertible as an operator on

$$L_0^2(\Gamma) = \left\{ \psi \in L^2(\Gamma) \ : \ \int_{\Gamma} \psi dS = 0 \right\},$$

where, as usual, $L^2(\Gamma)$ denotes the space of square integrable functions on Γ.

An extensive literature on efficient boundary integral methods for the numerical solution of elliptic problems in domains with corners has been developed (see, for instance, [6, 12, 18, 22, 26, 27] and, more recently, [1, 2, 3, 4, 5, 13, 14, 16, 17], and the references therein).

Some of these methods are based on piecewise polynomial approximations on graded meshes ([6, 12, 18, 22]). Appropriate mesh refinements close to the corner points allow to achieve arbitrarily high order of convergence. Neverthless, such procedures could produce ill-conditioned linear systems as the local degree increases. Numerical methods using global approximations have been recently proposed, too (see [1, 2, 3, 4, 5, 26, 28]). Since the solutions of integral equations on piecewise smooth curves could be unbounded at the corner points of the boundary, smoothing transformations are often introduced to improve the behavior of the unknown functions and, consequently, the convergence rate of the approximate solutions (see, for instance, [26, 28]). In order to get high accuracy in the numerical solution of Neumann problems for the Laplace equation in domains with corners, analytical subtraction of singularities and special treatment of nearly non-integrable integrands are carried out in the implementation of the Nyström method described in [5]. Suitable discretization techniques, along with compression and preconditioning schemes for the linear systems, have been recently proposed in [2] for a Nyström discretization of the boundary integral equation (4). In [16, 17] a scheme dubbed “recursive inverse preconditioning” was developed in order to overcome the negative effects of the ill-conditioning of matrices arising in the application of Nyström methods to singular integral equations on non-smooth domains.

Moreover, suitable modifications of the classical methods near the corners are often introduced in order to prove the stability of the numerical procedure (see, for instance, [18, 22, 28, 13, 14]). In many cases the modification is performed by applying some cutoff technique. In [13, 14] a different approach is adopted to avoid instability: the employed quadrature rules are a little bit modified around each corner without resorting to any cutoff. In particular, [14] is focused on the same problem of interest in this paper and proposes a modified Nyström method for the numerical solution of a boundary integral equation of the second kind which is different from (4). The unknown of such equation is the harmonic solution u on the boundary, while the right-hand side does not coincide with the Neumann data f (as in (4)), but it is given by an integral which involves f and needs to be approximated, when its analytical expression is not known.

In this paper a "modified" Nyström method is proposed for the numerical solution of equation (4). First, a decomposition of the boundary and a proper smoothing changes of variables are introduced. In such a way (4) is converted into a system of integral equations of the second kind whose solutions are sufficiently smooth and the right-hand sides are directly provided by the Neumann data on the boundary.
Then, the Nyström discretization of the system is obtained using a slight modification of the classical Gauss-Legendre quadrature formula, which we prove to be convergent. The theoretical investigation of stability and convergence for the proposed procedure is left for future work. Anyway, here, they are amply demonstrated through a variety of numerical tests.
The method is not computationally too expensive. The evaluation of the matrix entries, for the linear system arising from the discretization, does not require the computation of integrals as it occurs when collocation or Galerkin methods are applied. Moreover, compared with the procedure described in [14], it turns out to be cheaper. In fact, due to the different decomposition of the boundary we reduce to solve linear systems of smaller dimension. Furthermore, none additional computational cost is needed in order to compute their right-hand sides.
Finally, the numerical evidence shows that the proposed method produces well conditioned linear systems.

The contents of the paper are as follows. Section 2 provides preliminary definitions and notation. In Section 3 we show how to reduce the BIE (4) to a system of integral equations on the interval $[0,1]$. In Section 4 the numerical procedure is described. Section 5 contains the proofs of the theoretical results, and, finally, in Section 6 some numerical tests showing the efficiency of the method are presented.

2. Preliminaries

In this section we introduce some function spaces in which we are going to study our problem.
Let $L^2 \equiv L^2([0,1])$ be the space of all square integrable functions F on $[0,1]$, equipped with the norm

$$\|F\|_2 = \left(\int_0^1 |F(t)|^2 dt \right)^{\frac{1}{2}}.$$

For more regular functions we consider the following Sobolev type subspaces of $L^2([0,1])$

$$W_r^2 = \left\{ F \in L^2 : \; F^{(r-1)} \in AC(0,1), \; \|F^{(r)} \varphi^r\|_2 < +\infty \right\},$$

where r is a positive integer, $\varphi(t) = \sqrt{t(1-t)}$ and $AC(0,1)$ denotes the collection of all functions which are absolutely continuous on every closed subset of $(0,1)$. We equip the space W_r^2 with the norm

$$\|F\|_{W_r^2} = \|F\|_2 + \|F^{(r)}\varphi^r\|_2.$$

For $r = 0$ we set $W_r^2 \equiv L^2$. Moreover, we introduce the space

$$\mathcal{X} = \left\{ \bar{F} = (F_1, \dots, F_n) \; : \; F_i \in L^2, \; i = 1, \dots, n \right\}, \tag{7}$$

endowed with the norm

$$\|\bar{F}\| = \left(\sum_{i=1}^n \|F_i\|_2^2 \right)^{\frac{1}{2}}, \quad \bar{F} = (F_1, \dots, F_n) \in \mathcal{X} \tag{8}$$

and the subspace of $\mathcal{X}$

$$\mathcal{X}^r = \left\{ \bar{F} = (F_1, \dots, F_n) \; : \; F_i \in W_r^2, \; i = 1, \dots, n \right\}, \tag{9}$$

equipped with the norm

$$\|\bar{F}\|_r = \left(\sum_{i=1}^{n} \|F_i\|^2_{W^2_r} \right)^{\frac{1}{2}}, \quad \bar{F} = (F_1, \ldots, F_n) \in \mathcal{X}^r. \tag{10}$$

Throughout the paper $\mathcal{C}$ denotes a positive constant which may have different values in different formulas. We will write $\mathcal{C}(a, b, \ldots)$ to say that $\mathcal{C}$ depends on the parameters $a, b, \ldots$ and $\mathcal{C} \neq \mathcal{C}(a, b, \ldots)$ to say that $\mathcal{C}$ is independent of the parameters $a, b, \ldots$.

3. A decomposition of the BIE

Let us describe how the boundary integral equation (4) can be converted into a system of n integral equations over the interval $[0, 1]$. Begin by subdividing the curve Γ into sections $\Gamma_1, \Gamma_2, \ldots, \Gamma_n$ such that the arc Γ_i connects the consecutive corner points P_{i-1} and P_i ($P_0 \equiv P_n$) of the boundary. In the sequel we will suppose for simplicity that Γ_i, $i = 1, \ldots, n$, are smooth arcs. Moreover, we denote by β_i the interior angle that Γ forms at the corner point P_i. In our analysis we shall make the stronger assumption that each arc is straight in some neighborhood of the corners. However, using perturbation arguments it should be possible to derive the same result without this restriction (see [12, 28] and the references therein).
For each arc Γ_i, $i = 1, \ldots, n$, we consider a parametric representation

$$\bar{\sigma}_i(\bar{t}) = (\bar{\xi}_i(\bar{t}), \bar{\eta}_i(\bar{t})) \in \Gamma_i, \quad 0 \leq \bar{t} \leq 1 \tag{11}$$

which traverses Γ_i in a counter-clockwise direction, such that $\bar{\sigma}_i \in C^2([0, 1])$, $|\bar{\sigma}_i'(\bar{t})| \neq 0$ for each $\bar{t} \in [0, 1]$, $\bar{\sigma}_i(0) = P_{i-1}$, $\bar{\sigma}_i(1) = P_i$ and

$$\begin{array}{ll} \bar{\sigma}_i(\bar{t}) - \bar{\sigma}_i(0) = c_{i-1} e^{\mathrm{i}\beta_{i-1}} \bar{t}, & \bar{t} \in [0, \varepsilon] \\ \bar{\sigma}_i(\bar{t}) - \bar{\sigma}_i(1) = c_i (1 - \bar{t}), & \bar{t} \in [1 - \varepsilon, 1] \end{array} \tag{12}$$

for some sufficiently small $0 < \varepsilon < 1/2$, with c_{i-1} and c_i complex constants (points in $\mathbb{R}^2$ are here identified with complex numbers as usual), $\mathrm{i}^2 = -1$.

We can reformulate the equation (4) as the following system of boundary integral equations

$$-\pi \phi_i(x) - \sum_{j=1}^{n} \int_{\Gamma_j} \frac{\partial}{\partial \mathbf{n}(x)} \log |x - y| \phi_j(y) dS(y) = f_i(x), \quad x \in \Gamma_i, \quad i = 1, \ldots, n \tag{13}$$

where we have used the notation ϕ_i and f_i to denote the restriction of ϕ and f to the section Γ_i of the boundary, respectively. We also require that the solutions ϕ_i, $i = 1, \ldots, n$, of (13) satisfy the additional condition (see (5))

$$\sum_{i=1}^{n} \int_{\Gamma_i} \phi_i(y) dS(y) = 0. \tag{14}$$

In order to transform the BIE system (13) into a 1D system of integral equations defined on the interval $[0, 1]$, we use the parametric representations introduced in (11). Moreover, taking into account that the behavior of the solution ϕ near the corners P_i is given by

$$\phi(x) = c(\theta) \rho^{s_i} + \text{smoother terms}, \quad s_i = \min \left\{ \frac{\pi}{\beta_i}, \frac{\pi}{2\pi - \beta_i} \right\} - 1, \quad x \in \Gamma, \tag{15}$$

with (ρ, θ) the polar coordinates centered at P_i (see, for instance, [5, 7] and the references therein), we propose a smoothing transformation which can improve such behavior. We consider a smoothing change of variable $\bar{t} = \gamma(t)$ such that

$$\gamma(t) = \left\{ \begin{array}{ll} t^q, & t \in [0, \epsilon] \\ 1 - (1 - t)^q, & t \in [1 - \epsilon, 1] \end{array} \right. , \tag{16}$$

for some small $\epsilon > 0$ and some integer parameter $q \geq 2$. This type of transformations have already been applied in different works for dealing with endpoint singularities (see, for instance, [9, 10, 12, 22, 24, 25]). In this paper, in the practical computations, we will take

$$\gamma(t) = \frac{t^q}{t^q + (1-t)^q}, \quad 0 \leq t \leq 1. \tag{17}$$

The regularity of the unknown function depends on the value of the parameter q involved in (16): the larger the value of q is, the smoother the solution will be. Setting

$$\sigma_i(t) = \bar{\sigma}_i(\gamma(t)), \quad \sigma_i(t) =: (\xi_i(t), \eta_i(t)), \quad 0 \leq t \leq 1, \tag{18}$$

substituting $x = \sigma_i(t)$ and $y = \sigma_i(s)$ in (13), and multiplying both sides of the i-th equation in (13) by $|\sigma_i'(t)|$, the system (13) is converted into the following system of integral equations on $[0,1]$

$$-\pi\tilde{\phi}_i(t) - \sum_{j=1}^{n} \int_0^1 K_{ij}(t,s)\tilde{\phi}_j(s)ds = \tilde{f}_i(t), \quad i = 1, \ldots, n \tag{19}$$

with $\tilde{\phi}_i(t) = \phi_i(\sigma_i(t))|\sigma_i'(t)|$, $\tilde{f}_i(t) = f_i(\sigma_i(t))|\sigma_i'(t)|$ and the kernels $K_{ij}(t,s)$ given by

$$K_{ij}(t,s) = \begin{cases} \dfrac{\eta_i'(t)[\xi_j(s) - \xi_i(t)] - \xi_i'(t)[\eta_j(s) - \eta_i(t)]}{[\xi_i(t) - \xi_j(s)]^2 + [\eta_i(t) - \eta_j(s)]^2}, & i \neq j \text{ or } t \neq s \\ \dfrac{1}{2}\dfrac{\eta_i'(t)\xi_i''(t) - \xi_i'(t)\eta_i''(t)}{[\xi_i'(t)]^2 + [\eta_i'(t)]^2}, & i = j \text{ and } t = s \end{cases}.$$

Moreover, with this notation, the condition (14) can be reformulated as

$$\sum_{i=1}^{n} \int_0^1 \tilde{\phi}_i(t)dt = 0. \tag{20}$$

After the introduction of the smoothing change of variable (16), the behavior of the solutions $\tilde{\phi}_i$, $i = 1, \ldots, n$, of the system (19) near the endpoints of the interval $[0,1]$ is given by

$$\tilde{\phi}_i(t) = \begin{cases} \mathcal{C}t^{q(1+s_i)-1} + \text{smoother terms} & t \in [0, \epsilon] \\ \mathcal{C}(1-t)^{q(1+s_i)-1} + \text{smoother terms} & t \in [1-\epsilon, 1] \end{cases}, \tag{21}$$

with s_i given in (15). Defining the matrices of operators

$$\bar{\mathcal{I}} = \begin{pmatrix} I & 0 & \cdots & 0 \\ 0 & I & \cdots & 0 \\ \vdots & \cdots & \ddots & \vdots \\ 0 & \cdots & \cdots & I \end{pmatrix}, \qquad \bar{\mathcal{K}} = (\mathcal{K}_{ij})_{i,j=1,\ldots,n} \tag{22}$$

with I the identity operator and the operators $\mathcal{K}_{ij}$ given by

$$(\mathcal{K}_{ij}F)(t) = \int_0^1 K_{ij}(t,s)F(s)ds, \tag{23}$$

system (19) can be written in a more compact form as

$$\left(-\pi\bar{\mathcal{I}} - \bar{\mathcal{K}}\right)\bar{\phi}(t) = \bar{f}(t), \quad t \in [0,1] \tag{24}$$

where we set $\bar{\phi} = (\tilde{\phi}_1, \ldots, \tilde{\phi}_n)^T$ and $\bar{f} = (\tilde{f}_1, \ldots, \tilde{f}_n)^T$.

Now, we consider the linear subspace of the space $\mathcal{X}$ in (7) defined as follows

$$\mathcal{X}_0 = \left\{ \bar{F} \in \mathcal{X} \ : \ \sum_{i=1}^{n} \int_0^1 F_i(t)dt = 0 \right\} \tag{25}$$

and the bijective map $\mathcal{P} : L_0^2(\Gamma) \to \mathcal{X}_0$ such that $\mathcal{P}\psi = (\tilde{\psi}_1, \dots, \tilde{\psi}_n),\ \tilde{\psi}_i = \psi(\sigma_i(t))|\sigma_i'(t)|,\ t \in [0,1]$.
Our aim becomes to look for a solution of equation (24) belonging to $\mathcal{X}_0$, for a fixed right-hand side $\bar{f} \in \mathcal{X}_0$. We note that this solution exists and is unique in $\mathcal{X}_0$. In fact, the operator $(-\pi - \mathcal{K}) : L_0^2(\Gamma) \to L_0^2(\Gamma)$ is bijective (see, for instance, [5] and the references therein) and $\left(-\pi\bar{\mathcal{I}} - \bar{\mathcal{K}}\right) = \mathcal{P}(-\pi - \mathcal{K})\mathcal{P}^{-1}$. Consequently, the operator $\left(-\pi\bar{\mathcal{I}} - \bar{\mathcal{K}}\right)^{-1} : \mathcal{X}_0 \to \mathcal{X}_0$ exists and is bounded.

The main difficulty in carrying out the analysis of system (19) (which in a compact form is (24)) arises only when t is near 0 and s is near 1 or vice versa, i.e, the values of the parameter corresponding to the corners of the boundary Γ by means of the parameterization (18). More precisely, in such a neighborhood any kernel $K_{ij}(t,s)$ with $|i-j| = 1$ behaves like a Mellin convolution.
In order to show that each integral operator $\mathcal{K}_{ij}$ with $|i-j| = 1$ can be split into the sum of Mellin operators near the corners and a compact operator $\mathcal{E}_{ij}$ on L^2 (see [12, 28]), we introduce smooth cut-off functions χ_0 and χ_1 on the interval $[0,1]$ such that

$$\begin{array}{lll} \chi_0(t) = 1, & t \in [0, \varepsilon/2], & \operatorname{supp}(\chi_0) \subset [0, \varepsilon] \\ \chi_1(t) = 1, & t \in [1-\varepsilon/2, 1], & \operatorname{supp}(\chi_1) \subset [1-\varepsilon, 1] \end{array} \tag{26}$$

for some $0 < \varepsilon < 1/2$, and $0 \le \chi_0(t), \chi_1(t) \le 1$ for $t \in [0,1]$.

We also define the following integral operators

$$\left(\mathcal{K}_0^\beta F\right)(t) = \int_0^1 K_0^\beta(t,s)F(s)ds \tag{27}$$

with

$$K_0^\beta(t,s) = \frac{qt^{q-1}(1-s)^q \sin\beta}{t^{2q} - 2t^q(1-s)^q \cos\beta + (1-s)^{2q}} \tag{28}$$

and

$$\left(\mathcal{K}_1^\beta F\right)(t) = \int_0^1 K_1^\beta(t,s)F(s)ds \tag{29}$$

with

$$K_1^\beta(t,s) = \frac{q(1-t)^{q-1}s^q \sin\beta}{(1-t)^{2q} - 2(1-t)^q s^q \cos\beta + s^{2q}}. \tag{30}$$

We observe that these operators are Mellin convolution ones, since their kernels can be rewritten in the form

$$K_0^\beta(t,s) = \frac{1}{1-s} k^\beta\left(\frac{t}{1-s}\right), \quad K_1^\beta(t,s) = \frac{1}{s} k^\beta\left(\frac{1-t}{s}\right)$$

where

$$k^\beta(\tau) = \frac{q\tau^{q-1}\sin\beta}{\tau^{2q} - 2\tau^q\cos\beta + 1}. \tag{31}$$

The integral operators $\mathcal{K}_{ij}$ defined by (23), when $|i-j| = 1$, can be represented as in the following lemma (see [11], [12, Lemma 5.1]).

Lemma 3.1. *If $n = 1$, the integral operator $\mathcal{K}_{11}$ satisfies*

1. $\mathcal{K}_{11} = \chi_0\mathcal{K}_0^{\beta_1}\chi_1 + \chi_1\mathcal{K}_1^{\beta_1}\chi_0 + \mathcal{E}_{11}$,

with χ_k, $k = 0,1$, the cut-off functions defined in (26) *and $\mathcal{E}_{11}$ a compact operator on L^2.*
If $n \ge 2$, for each couple of indices (i,j) with $i,j \in \{1,\dots,n\}$ and $|i-j| = 1$, the following equalities hold

2. $\mathcal{K}_{ij} = \chi_0 \mathcal{K}_0^{\beta_{i-1}} \chi_1 + \mathcal{E}_{ij}, \quad j = i-1$
3. $\mathcal{K}_{ij} = \chi_1 \mathcal{K}_1^{\beta_i} \chi_0 + \mathcal{E}_{ij}, \quad j = i+1,$

where χ_k*,* $k = 0, 1$*, are given by* (26) *and* $\mathcal{E}_{ij}$ *denotes a compact operator on* L^2*.*

As a consequence of the previous lemma we can split the matrix $\bar{\mathcal{K}}$ in (22) as

$$\bar{\mathcal{K}} = \bar{\mathcal{M}} + \bar{\mathcal{S}} \tag{32}$$

where $\bar{\mathcal{M}}$ is the bidiagonal matrix defined as follows

$$\bar{\mathcal{M}} = \begin{pmatrix} 0 & \mathcal{M}_{12} & 0 & \cdots & \cdots & 0 \\ \mathcal{M}_{21} & 0 & \mathcal{M}_{23} & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & \mathcal{M}_{n-1,n-2} & 0 & \mathcal{M}_{n-1,n} \\ 0 & \cdots & \cdots & 0 & \mathcal{M}_{n,n-1} & 0 \end{pmatrix} \tag{33}$$

with

$$\mathcal{M}_{ij} = \begin{cases} \chi_0 \mathcal{K}_0^{\beta_{i-1}} \chi_1, & j = i-1 \\ \chi_1 \mathcal{K}_1^{\beta_i} \chi_0, & j = i+1 \end{cases} \tag{34}$$

and

$$\bar{\mathcal{S}} = \begin{pmatrix} \mathcal{K}_{11} & \mathcal{E}_{12} & \mathcal{K}_{13} & \cdots & \cdots & \mathcal{K}_{1n} \\ \mathcal{E}_{21} & \mathcal{K}_{22} & \mathcal{E}_{23} & \cdots & \cdots & \mathcal{K}_{2n} \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \mathcal{K}_{n-1,1} & \mathcal{K}_{n-1,2} & \vdots & \mathcal{E}_{n-1,n-2} & \mathcal{K}_{n-1,n-1} & \mathcal{E}_{n-1,n} \\ \mathcal{K}_{n1} & \mathcal{K}_{n2} & \cdots & \mathcal{K}_{n-1,n-2} & \mathcal{E}_{n,n-1} & \mathcal{K}_{nn} \end{pmatrix}. \tag{35}$$

In the special case $n = 1$ the relation (32) holds true with

$$\mathcal{M}_{11} = \chi_0 \mathcal{K}_0^{\beta_1} \chi_1 + \chi_1 \mathcal{K}_1^{\beta_1} \chi_0, \quad \bar{\mathcal{M}} = \mathcal{M}_{11}, \quad \bar{\mathcal{S}} = \mathcal{E}_{11}. \tag{36}$$

Taking into account the splitting of the matrix $\bar{\mathcal{K}}$ given by (32), the system (24) can be also rewritten as follows

$$\left(-\pi\bar{\mathcal{I}} - \bar{\mathcal{M}} - \bar{\mathcal{S}}\right) \bar{\phi} = \bar{f}. \tag{37}$$

Now, considering the normed space $(\mathcal{X}, \|\cdot\|)$ defined in (7)-(8), we are able to prove the following theorem.

Theorem 3.2. *Let us assume that* $\mathrm{Ker}(-\pi\bar{\mathcal{I}} - \bar{\mathcal{M}} - \bar{\mathcal{S}}) = \{0\}$ *in* $\mathcal{X}$*. Then the system* (37) *has a unique solution in* $\mathcal{X}$ *for each given right hand side* $\bar{f} \in \mathcal{X}$*. In particular, if* $\bar{f} \in \mathcal{X}_0$*,*

4. The numerical method

In this section we propose a "modified" Nyström type method for the approximation of the solution $\bar{\phi} \in \mathcal{X}_0$ of the system of integral equations (37), provided that the right-hand side $\bar{f} \in \mathcal{X}_0$.
For any fixed $m \in \mathbb{N}$, we denote by $\lambda_{m,k}$ and $s_{m,k}$, $k = 1, \dots, m$, the coefficients and the nodes of the Gauss quadrature formula with respect to the Legendre weight. Then, setting

$$(\mathcal{E}_{ij} F)(t) = \int_0^1 E_{ij}(t,s) F(s) ds,$$

for $|i-j|=1$, and recalling (23), we define the following discrete operators, approximating $\mathcal{K}_{ij}$ and $\mathcal{E}_{ij}$,

$$(\mathcal{K}_{ij}^m F)(t) = \sum_{k=1}^{m} \lambda_{m,k} K_{ij}(t, s_{m,k}) F(s_{m,k}), \tag{38}$$

$$(\mathcal{E}_{ij}^m F)(t) = \sum_{k=1}^{m} \lambda_{m,k} E_{ij}(t, s_{m,k}) F(s_{m,k}). \tag{39}$$

Moreover, in order to introduce suitable approximations of the Mellin type integral operators $\mathcal{M}_{ij}$, defined in (34) and (36), we follow an idea recently proposed in [8]. We set

$$t_m = \frac{c}{m^{2-2\delta}}, \qquad \bar{t}_m = \frac{c}{m^{2-2\mu\delta}}, \tag{40}$$

where c is a fixed positive constant, δ is an arbitrarily small positive quantity and μ is a parameter chosen in the interval $\left(1, \frac{1}{\delta}\right)$. Then, for $N \ll m$, we define the following two sets of equispaced points

$$z_{N,i}^0 = \bar{t}_m + \frac{(i-1)\bar{t}_m}{N}, \qquad z_{N,i}^1 = (1-\bar{t}_m) + \frac{(i-1)\bar{t}_m}{N}, \qquad i = 1, \dots, N \tag{41}$$

and denote by $L_N^0(F,t)$ and by $L_N^1(F,t)$ the Lagrange polynomials interpolating the function F at the knots $z_{N,i}^0$, $i=1,\dots,N$, and $z_{N,i}^1$, $i=1,\dots,N$, respectively, i.e., $L_N^h(F,t) = \sum_{i=1}^N l_{N,i}^h(t) F(z_{N,i}^h)$, $h=0,1$, with $l_{N,i}^h$, $i=1,\dots,N$, the fundamental Lagrange polynomials.
With this notation, as first step, we introduce the operator $\mathcal{K}_0^{\beta,m}$ approximating the operator $\mathcal{K}_0^{\beta}$ in (27)

$$\left(\mathcal{K}_0^{\beta,m} F\right)(t) = \sum_{k=1}^{m} \lambda_{m,k} K_0^{\beta}(t, s_{m,k}) F(s_{m,k})$$

and its "modified" version $\tilde{\mathcal{K}}_0^{\beta,m}$ defined as follows

$$\left(\tilde{\mathcal{K}}_0^{\beta,m} F\right)(t) = \begin{cases} L_N^0(\mathcal{K}_0^{\beta,m} F, t), & 0 \le t < t_m \\ (\mathcal{K}_0^{\beta,m} F)(t), & t_m \le t \le 1 \end{cases}.$$

In analogous way, setting

$$\left(\mathcal{K}_1^{\beta,m} F\right)(t) = \sum_{k=1}^{m} \lambda_{m,k} K_1^{\beta}(t, s_{m,k}) F(s_{m,k}),$$

we consider, as approximation of the Mellin integral operator $\mathcal{K}_1^{\beta}$ in (29), the following one

$$\left(\tilde{\mathcal{K}}_1^{\beta,m} F\right)(t) = \begin{cases} (\mathcal{K}_1^{\beta,m} F)(t), & 0 \le t \le 1 - t_m \\ L_N^1(\mathcal{K}_1^{\beta,m} F, t), & 1 - t_m < t \le 1 \end{cases}.$$

Finally, when $n > 1$, for $i, j = 1, \dots, n$, with $|i-j|=1$, we define the operator $\mathcal{M}_{ij}^m$ as

$$\mathcal{M}_{ij}^m = \begin{cases} \chi_0 \tilde{\mathcal{K}}_0^{\beta_{i-1},m} \chi_1, & j = i-1 \\ \chi_1 \tilde{\mathcal{K}}_1^{\beta_i,m} \chi_0, & j = i+1 \end{cases}, \tag{42}$$

and, for $n=1$, let

$$\mathcal{M}_{11}^m = \chi_0 \tilde{\mathcal{K}}_0^{\beta_1,m} \chi_1 + \chi_1 \tilde{\mathcal{K}}_1^{\beta_1,m} \chi_0. \tag{43}$$

Setting, for $n > 1$

$$\bar{\mathcal{M}}_m = \begin{pmatrix} 0 & \mathcal{M}_{12}^m & 0 & \cdots & \cdots & 0 \\ \mathcal{M}_{21}^m & 0 & \mathcal{M}_{23}^m & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & \mathcal{M}_{n-1,n-2}^m & 0 & \mathcal{M}_{n-1,n}^m \\ 0 & \cdots & \cdots & 0 & \mathcal{M}_{n,n-1}^m & 0 \end{pmatrix}, \tag{44}$$

$$\bar{\mathcal{S}}_m = \begin{pmatrix} \mathcal{K}_{11}^m & \mathcal{E}_{12}^m & \mathcal{K}_{13}^m & \cdots & \cdots & \mathcal{K}_{1n}^m \\ \mathcal{E}_{21}^m & \mathcal{K}_{22}^m & \mathcal{E}_{23}^m & \cdots & \cdots & \mathcal{K}_{2n}^m \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \mathcal{K}_{n-1,1}^m & \mathcal{K}_{n-1,2}^m & \vdots & \mathcal{E}_{n-1,n-2}^m & \mathcal{K}_{n-1,n-1}^m & \mathcal{E}_{n-1,n}^m \\ \mathcal{K}_{n1}^m & \mathcal{K}_{n2}^m & \cdots & \mathcal{K}_{n-1,n-2}^m & \mathcal{E}_{n,n-1}^m & \mathcal{K}_{nn}^m \end{pmatrix}, \tag{45}$$

and, in the case $n = 1$,

$$\bar{\mathcal{M}}_m = \mathcal{M}_{11}^m, \quad \bar{\mathcal{S}}_m = \mathcal{E}_{11}^m, \tag{46}$$

we are able to prove the following properties satisfied by the operator sequences $\{\bar{\mathcal{M}}_m\}_m$ and $\{\bar{\mathcal{S}}_m\}_m$.

Theorem 4.1. *Let $\bar{\mathcal{M}}$ and $\bar{\mathcal{M}}_m$ be defined as in (33), (36) and (44), (46), respectively. Then, for any integer $r < 2q + 1$, the operators $\bar{\mathcal{M}}_m : \mathcal{X}^r \to \mathcal{X}$ are linear maps such that*

$$\limsup_{m\to\infty} \|\bar{\mathcal{M}}_m\|_{\mathcal{X}^r \to \mathcal{X}} < \pi \tag{47}$$

and

$$\lim_{m\to\infty} \|(\bar{\mathcal{M}}_m - \bar{\mathcal{M}})\bar{F}\| = 0, \quad \forall\ \bar{F} \in \mathcal{X}^r. \tag{48}$$

Theorem 4.2. *Let $\bar{\mathcal{S}}$ and $\bar{\mathcal{S}}_m$ be defined as in (35) and (45), respectively. Then, provided that the curve $\Gamma \setminus \{P_1, \ldots, P_n\}$ is of class C^{r+2}, for some $r \in \mathbb{N}$, the operators $\bar{\mathcal{S}}_m : \mathcal{X}^r \to \mathcal{X}$ are linear maps such that*

$$\lim_{m\to\infty} \|(\bar{\mathcal{S}}_m - \bar{\mathcal{S}})\bar{F}\| = 0, \quad \forall \bar{F} \in \mathcal{X}^r \tag{49}$$

and $\left\{\bar{\mathcal{S}}_m\right\}_m$ is collectively compact.

Let us remark that, recalling the behavior of the solutions $\tilde{\phi}_i$, $i = 1, \ldots, n$, of system (19) (or (24)) near the endpoints of the interval $[0, 1]$, it can be proved that, for any arbitrarily large $r \in \mathbb{N}$, if the functions f_i in (13) are sufficiently smooth and the exponent q in the smoothing transformation (16) satisfies

$$q > \frac{(r+1)/2}{1+s_i}, \quad s_i = \min\left\{\frac{\pi}{\beta_i}, \frac{\pi}{2\pi - \beta_i}\right\} - 1, \text{ for } i = 1, \ldots, n \tag{50}$$

(β_i is the interior angle at the corner point P_i of the boundary Γ), then the solutions $\tilde{\phi}_i$, $i = 1, \ldots, n$, of (24) belong to the Sobolev-type space W_r^2 (see (21)).

Such properties encourage us to apply a modified Nyström method in which the matrices of integral operators $\bar{\mathcal{M}}$ and $\bar{\mathcal{S}}$ are discretized by the matrices $\bar{\mathcal{M}}_m$ and $\bar{\mathcal{S}}_m$, respectively. Therefore, assuming that $\bar{f} = (\tilde{f}_1, \ldots, \tilde{f}_n)^T \in \mathcal{X}_0$ and the functions $\tilde{f}_i$ are sufficiently smooth, the method we propose consists in solving, instead of the system (37), the approximating system

$$\left(-\pi\bar{\mathcal{I}} - \bar{\mathcal{M}}_m - \bar{\mathcal{S}}_m\right)\bar{\phi}_m = \bar{f}, \tag{51}$$

whose unknown is the array of functions denoted by $\bar{\phi}_m = (\tilde{\phi}_{m,1}, \ldots, \tilde{\phi}_{m,n})^T$. Moreover, since the solution $\bar{\phi}_m$ of (51) has to approximate the unique solution $\bar{\phi} \in \mathcal{X}_0$ of system (37), we also require that it satisfies the following condition

$$\sum_{i=1}^{n} \sum_{k=1}^{m} \lambda_{m,k} \tilde{\phi}_{m,i}(s_{m,k}) = 0 \tag{52}$$

which represents the discrete analog of condition (20).

In order to compute the solution $\bar{\phi}_m = (\tilde{\phi}_{m,1}, \ldots, \tilde{\phi}_{m,n})^T$ of the finite dimensional problem (51)-(52), which the proposed modified Nyström method leads to, we collocate each equation of the system (51) at the Legendre quadrature knots $s_{m,k}$, $k = 1, \ldots, m$. Then we get the linear system

$$\left(-\pi \bar{\mathcal{I}} - \bar{\mathcal{M}}_m - \bar{\mathcal{S}}_m\right) \bar{\phi}_m(s_{m,k}) = \bar{f}(s_{m,k}), \quad k = 1, \ldots, m, \tag{53}$$

of nm equations in the nm unknowns $\tilde{\phi}_{m,i}(s_{m,k})$, $i = 1, \ldots, n$, $k = 1, \ldots, m$. Adding to the nm equations in (53) the linear equation (52), we get an overdetermined system of $(nm + 1)$ equations in nm unknowns. Hence, we have to omit one equation, in order to obtain a square system. Numerical evidence shows that a good choice is to omit the first or the last equation in (53).

We note that for each solution $\bar{\phi}_m \in \mathcal{X}$ of (51)-(52) there is a solution of the linear system (53)-(52) (with an omitted equation in (53)) obtained by evaluating $\bar{\phi}_m$ at the quadrature points $\{s_{m,k}\}_{k=1,m}$. Vice versa, if $\mathbf{a} = (a_{1,1}, \ldots, a_{n,1}, \ldots, a_{1,m}, \ldots, a_{n,m}) \in \mathbb{R}^{nm}$ is a solution of (53)-(52), then there is a unique solution $\bar{\phi}_m$ of (51)-(52) that agrees with $\mathbf{a}$ at the node points, i.e. such that $a_{i,k} = \tilde{\phi}_{m,i}(s_{m,k})$, $i = 1, \ldots, n$, $k = 1, \ldots, m$. Moreover, in order to improve the conditioning of the linear system, inspired by a preconditioning technique described in [23], we propose to solve instead of (53)-(52) the following equivalent system

$$\begin{cases} \sqrt{\lambda_{m,k}} \left(-\pi \bar{\mathcal{I}} - \bar{\mathcal{M}}_m - \bar{\mathcal{S}}_m\right) \bar{\phi}_m(s_{m,k}) = \sqrt{\lambda_{m,k}} \bar{f}(s_{m,k}), \ k = 1, \ldots, m \\ \sum_{i=1}^{n} \sum_{k=1}^{m} \lambda_{m,k} \tilde{\phi}_{m,i}(s_{m,k}) = 0 \end{cases} \tag{54}$$

for the new unknowns $\tilde{a}_{i,k} = \sqrt{\lambda_{m,k}} \tilde{\phi}_{m,i}(s_{m,k})$, $i = 1, \ldots, n$, $k = 1, \ldots, m$.

The above system is obtained by multiplying from the left both the matrix of the coefficients and the vector of the right-hand sides by the nonsingular block diagonal matrix

$$D = \begin{pmatrix} D_1 & & & \\ & D_2 & & \\ & & \ddots & \\ & & & D_m \end{pmatrix} \in \mathbb{R}^{nm \times nm}, \qquad D_k = \begin{pmatrix} \sqrt{\lambda_{m,k}} & & & \\ & \sqrt{\lambda_{m,k}} & & \\ & & \ddots & \\ & & & \sqrt{\lambda_{m,k}} \end{pmatrix} \in \mathbb{R}^{n \times n}.$$

Once given a solution $(\tilde{a}_{1,1}, \ldots, \tilde{a}_{n,1}, \ldots, \tilde{a}_{1,m}, \ldots, \tilde{a}_{n,m}) \in \mathbb{R}^{nm}$ to (54), using the values $\tilde{\phi}_{m,i}(s_{m,k}) = \frac{\tilde{a}_{i,k}}{\sqrt{\lambda_{m,k}}}$ (see the definition (44)-(45) of the matrices of operators $\bar{\mathcal{M}}_m$ and $\bar{\mathcal{S}}_m$), the approximating solution $\bar{\phi}_m$ can be computed by means of the following formula

$$\bar{\phi}_m(t) = -\frac{1}{\pi} \left[\bar{f}(t) + \left(\bar{\mathcal{M}}_m + \bar{\mathcal{S}}_m\right) \bar{\phi}_m(t)\right]. \tag{55}$$

The last step of our procedure consists in the approximate computation of the solution $u(x)$ of the boundary value problem (1), for all $x = (x_1, x_2) \in \mathbb{R}^2 \setminus \bar{\Omega}$.

We note that, using the single layer potential representation of $u(x)$ given by (3) and taking into account the decomposition of the boundary $\Gamma = \Gamma_1 \cup \ldots \cup \Gamma_n$ and the parametrization (18) for each arc Γ_i, $i = 1, \ldots, n$, one can write

$$u(x_1, x_2) = -\sum_{i=1}^{n} \int_0^1 \tilde{\phi}_i(t) \log |(x_1, x_2) - (\xi_i(t), \eta_i(t))| dt.$$

We propose to approximate $u(x_1,x_2)$ by means of the function

$$u_m(x_1,x_2) = -\sum_{i=1}^{n}\sum_{k=1}^{m} \lambda_{m,k}\tilde{\phi}_{m,i}(s_{m,k}) \log|(x_1,x_2) - (\xi_i(s_{m,k}),\eta_i(s_{m,k}))|. \tag{56}$$

We observe that the quantities $\sqrt{\lambda_{m,k}}\tilde{\phi}_{m,i}(s_{m,k})$ involved in (56) are just the solution of the system (54).

The theoretical analysis of the stability and convergence is left for future work. Anyway, they are amply demonstrated through the results obtained for a variety of numerical examples and showed in Section 6.

5. Proofs

Lemma 5.1. *For any integer $0 \le r < 2q+1$, the kernels $K_h^\beta(t,s)$, $h = 0,1$, defined by* (28) *and* (30) *satisfy the following conditions*

$$\left\|K_0^\beta(t,\cdot)\right\|_{W_r^2} \le \mathcal{C}t^{-\frac{r}{2}-\frac{1}{2}}, \qquad \left\|K_1^\beta(t,\cdot)\right\|_{W_r^2} \le \mathcal{C}(1-t)^{-\frac{r}{2}-\frac{1}{2}}$$

with the constant $\mathcal{C} = \mathcal{C}(q,r)$.

Proof. It is sufficient to prove, for the prototype kernel $K^\beta(t,s) = \dfrac{qt^{q-1}s^q\sin\beta}{t^{2q}-2t^qs^q\cos\beta+s^{2q}}$, that

$$\left\|K^\beta(t,\cdot)\right\|_{W_r^2} \le \mathcal{C}t^{-\frac{r}{2}-\frac{1}{2}}, \quad \mathcal{C} = \mathcal{C}(q,r). \tag{57}$$

We begin by noting that $\|K^\beta(t,\cdot)\|_2 \le \mathcal{C}t^{-\frac{1}{2}}$, since

$$\begin{aligned} \|K^\beta(t,\cdot)\|_2 &= qt^{q-1}|\sin\beta|t^{-q+\frac{1}{2}}\left(\int_0^{\frac{1}{t}} \frac{y^{2q}}{(1-2y^q\cos\beta+y^{2q})^2}dy\right)^{\frac{1}{2}} \\ &\le qt^{-\frac{1}{2}}\left(\int_0^{\infty} \frac{y^{2q}}{(1-2y^q\cos\beta+y^{2q})^2}dy\right)^{\frac{1}{2}}. \end{aligned} \tag{58}$$

Next, by representing the function $K^\beta(t,s)$ as

$$K^\beta(t,s) = \frac{1}{2\mathrm{i}}qt^{q-1}\left(\frac{1}{t^q-e^{\mathrm{i}\beta}s^q} - \frac{1}{t^q-e^{-\mathrm{i}\beta}s^q}\right)$$

($\mathrm{i}^2=-1$), for the r-th partial derivative one can easily obtain

$$\frac{\partial^r K^\beta(t,s)}{\partial s^r} = \frac{1}{2\mathrm{i}}qt^{q-1}\sum_{k=1}^{r} c_k(q)s^{kq-r}\left[\frac{k!(e^{\mathrm{i}\beta})^k}{(t^q-e^{\mathrm{i}\beta}s^q)^{k+1}} - \frac{k!(e^{-\mathrm{i}\beta})^k}{(t^q-e^{-\mathrm{i}\beta}s^q)^{k+1}}\right],$$

for suitable constants $c_k(q)$, $k=1,\dots,r$, depending on the parameter q.
Then, recalling that $\varphi(s)=\sqrt{s(1-s)}$, one can deduce

$$\begin{aligned} &\left\|\frac{\partial^r K^\beta(t,\cdot)}{\partial s^r}\varphi^r\right\|_2 \le \frac{1}{|2\mathrm{i}|}qt^{q-1}\sum_{k=1}^{r}|c_k(q)|k! \times \\ &\left(\int_0^1 \left|s^{\frac{r}{2}}(1-s)^{\frac{r}{2}}s^{kq-r}\frac{\sum_{j=0}^{k+1}\binom{k+1}{j}(s^q)^j(t^q)^{k+1-j}\left(e^{\mathrm{i}\beta(k-j)}-e^{-\mathrm{i}\beta(k-j)}\right)}{(t^{2q}-2t^qs^q\cos\beta+s^{2q})^{k+1}}\right|^2 ds\right)^{\frac{1}{2}} \\ &\qquad \le qt^{q-1}\sum_{k=1}^{r}|c_k(q)|k!\left(\int_0^1 s^{2kq-r}\frac{(s^q+t^q)^{2k+2}}{(t^{2q}-2t^qs^q\cos\beta+s^{2q})^{2k+2}}ds\right)^{\frac{1}{2}}, \end{aligned}$$

from which, setting $s = ty$, it follows

$$\begin{aligned}\left\| \frac{\partial^r K^\beta(t,\cdot)}{\partial s^r}\varphi^r \right\|_2 &\le q t^{q-1} \sum_{k=1}^{r} |c_k(q)| k! t^{\frac{1}{2}-\frac{r}{2}-q} \left(\int_0^{\frac{1}{t}} \frac{y^{2kq-r}(y^q+1)^{2k+2}}{(1-2y^q\cos\beta + y^{2q})^{2k+2}} dy \right)^{\frac{1}{2}} \\ &\le q t^{-\frac{r}{2}-\frac{1}{2}} \sum_{k=1}^{r} |c_k(q)| k! \left(\int_0^{\infty} \frac{y^{2kq-r}(y^q+1)^{2k+2}}{(1-2y^q\cos\beta + y^{2q})^{2k+2}} dy \right)^{\frac{1}{2}} \le \mathcal{C}(r,q) t^{-\frac{r}{2}-\frac{1}{2}}.\end{aligned}$$

The proof of (57) is complete. □

Lemma 5.2. *For any positive integer parameter q, the function $k^\beta(\tau)$ in (31) satisfies the following inequalities*

$$\int_0^\infty \frac{\left|k^\beta(\tau)\right|}{\tau^{\frac{1}{2}}} d\tau < \pi. \tag{59}$$

Proof. Setting $z = \sqrt{\tau}$, we can write (see, for instance, [15, formula 3.242])

$$\begin{aligned}\int_0^\infty \frac{\left|k^\beta(\tau)\right|}{\tau^{\frac{1}{2}}} d\tau &= 2q|\sin\beta| \int_0^\infty \frac{z^{2(q-1)}}{1-2z^{2q}\cos\beta + z^{4q}} dz = 2q|\sin\beta| \int_0^\infty \frac{z^{2(q-1)}}{1+2z^{2q}\cos(\pi-\beta) + z^{4q}} dz \\ &= \pi|\sin\beta| \sin\left(\frac{\pi-\beta}{2q}\right) \csc(\pi-\beta) \csc\left(\frac{2q-1}{2q}\pi\right) = \pi \frac{|\sin\beta|}{\sin\beta} \frac{\sin\left(\frac{\pi-\beta}{2q}\right)}{\sin\left(\frac{\pi}{2q}\right)}.\end{aligned}$$

Now, we can deduce (59) by observing that in the case when $0 < \beta < \pi$ one has

$$\int_0^\infty \frac{\left|k^\beta(\tau)\right|}{\tau^{\frac{1}{2}}} d\tau = \frac{\sin\left(\frac{\pi-\beta}{2q}\right)}{\sin\left(\frac{\pi}{2q}\right)} \pi < \pi,$$

while for $\pi < \beta < 2\pi$

$$\int_0^\infty \frac{\left|k^\beta(\tau)\right|}{\tau^{\frac{1}{2}}} d\tau = -\frac{\sin\left(\frac{\pi-\beta}{2q}\right)}{\sin\left(\frac{\pi}{2q}\right)} \pi = \frac{\sin\left(\frac{\beta-\pi}{2q}\right)}{\sin\left(\frac{\pi}{2q}\right)} < \pi.$$

□

Lemma 5.3. *For any $F, G \in L^2$, the following inequalities hold*

$$\|\mathcal{M}_{i,i-1}F\|_2 < \pi \|\chi_1 F\|_2, \quad i = 2, \dots, n, \tag{60}$$

$$\|\mathcal{M}_{i,i+1}F\|_2 < \pi \|\chi_0 F\|_2, \quad i = 1, \dots, n-1, \tag{61}$$

$$\|\mathcal{M}_{i,i-1}F + \mathcal{M}_{i,i+1}G\|_2 < \pi \left(\|\chi_1 F\|_2^2 + \|\chi_0 G\|_2^2 \right)^{\frac{1}{2}}, \quad i = 2, \dots, n-1. \tag{62}$$

Proof. In order to prove the inequality (60), we observe that for any function $F \in L^2([0,1])$, taking into account (59), one has

$$\begin{aligned}\|\mathcal{M}_{i,i-1}F\|_2 &= \left(\int_0^1 \left| \left(\chi_0 \mathcal{K}_0^{\beta_{i-1}} \chi_1 F\right)(t) \right|^2 dt \right)^{\frac{1}{2}} \le \left(\int_0^\varepsilon \left| \left(\mathcal{K}_0^{\beta_{i-1}} \chi_1 F\right)(t) \right|^2 dt \right)^{\frac{1}{2}} \\ &\le \left(\int_0^\infty \frac{\left|k^{\beta_{i-1}}(\tau)\right|}{\tau^{\frac{1}{2}}} d\tau \right) \|\chi_1 F\|_2 < \pi \|\chi_1 F\|_2.\end{aligned}$$

The proof of (61) is analogous. Now let us prove (62). By using (59), for any functions $F, G \in L^2$ we have

$$
\begin{aligned}
\|\mathcal{M}_{i,i-1}F + \mathcal{M}_{i,i+1}G\|_2 &= \left(\int_0^1 \left|\left(\chi_0 \mathcal{K}_0^{\beta_{i-1}} \chi_1 F + \chi_1 \mathcal{K}_1^{\beta_i} \chi_0 G\right)(t)\right|^2 dt\right)^{\frac{1}{2}} \\
\leq & \left(\int_0^{\varepsilon} \left|\left(\mathcal{K}_0^{\beta_{i-1}} \chi_1 F\right)(t)\right|^2 dt + \int_{1-\varepsilon}^1 \left|\left(\mathcal{K}_1^{\beta_i} \chi_0 G\right)(t)\right|^2\right)^{\frac{1}{2}} \\
\leq & \left(\int_0^1 \left|\left(\mathcal{K}_0^{\beta_{i-1}} \chi_1 F\right)(t)\right|^2 dt + \int_0^1 \left|\left(\mathcal{K}_1^{\beta_i} \chi_0 G\right)(t)\right|^2\right)^{\frac{1}{2}} \\
\leq & \left(\left(\int_0^{\infty} \frac{\left|k^{\beta_{i-1}}(\tau)\right|}{\tau^{\frac{1}{2}}} d\tau\right)^2 \|\chi_1 F\|_2^2 + \left(\int_0^{\infty} \frac{\left|k^{\beta_i}(\tau)\right|}{\tau^{\frac{1}{2}}} d\tau\right)^2 \|\chi_0 G\|_2^2\right)^{\frac{1}{2}} < \pi \left(\|\chi_1 F\|_2^2 + \|\chi_0 G\|_2^2\right)^{\frac{1}{2}}.
\end{aligned}
$$

□

Remark 5.4. *By proceeding exactly as in the proof of* (62)*, one can show that the operator* $\mathcal{M}_{11}$ *defined in* (36)*, for any* $F \in L^2$*, satisfies*

$$
\|\mathcal{M}_{1,1}F\|_2 < \pi \left(\|\chi_0 F\|_2^2 + \|\chi_1 F\|_2^2\right)^{\frac{1}{2}}. \tag{63}
$$

Proof of Theorem 3.2. We first observe that the norm of the operator $-\pi\bar{\mathcal{I}}$ as a map acting from $\mathcal{X}$ to $\mathcal{X}$ is given by

$$
\|-\pi\bar{\mathcal{I}}\| = \pi. \tag{64}
$$

Now, we are going to show that the operator $\bar{\mathcal{M}} : \mathcal{X} \to \mathcal{X}$ is a bounded linear operator and satisfies

$$
\|\bar{\mathcal{M}}\| < \pi. \tag{65}
$$

The proof of linearity of the operator $\bar{\mathcal{M}}$ is trivial. Let us prove the inequality (65). In the case $n = 1$, for $F \in L^2$, from (63) we deduce that

$$
\|\bar{\mathcal{M}}F\|_2 < \pi\|F\|_2. \tag{66}
$$

When $n > 1$, for any array $\bar{F} = (F_1, \ldots, F_n) \in \mathcal{X}$, one has

$$
\|\bar{\mathcal{M}}\bar{F}\| = \left(\|\mathcal{M}_{12}F_2\|_2^2 + \sum_{i=2}^{n-1} \|\mathcal{M}_{i,i-1}F_{i-1} + \mathcal{M}_{i,i+1}F_{i+1}\|_2^2 + \|\mathcal{M}_{n,n-1}F_{n-1}\|_2^2\right)^{\frac{1}{2}}
$$

from which, in virtue of (60)-(62), it follows that

$$
\begin{aligned}
\|\bar{\mathcal{M}}\bar{F}\| &< \pi \left(\|\chi_0 F_2\|_2^2 + \sum_{i=2}^{n-1} \left(\|\chi_1 F_{i-1}\|_2^2 + \|\chi_0 F_{i+1}\|_2^2\right) + \|\chi_1 F_{n-1}\|_2^2\right)^{\frac{1}{2}} \\
&= \pi \left(\|\chi_1 F_1\|_2^2 + \sum_{i=2}^{n-1} \left(\|\chi_0 F_i\|_2^2 + \|\chi_1 F_i\|_2^2\right) + \|\chi_0 F_n\|_2^2\right)^{\frac{1}{2}} \leq \pi \left(\sum_{i=1}^{n} \|F_i\|_2^2\right)^{\frac{1}{2}} = \pi\|\bar{F}\|.
\end{aligned} \tag{67}
$$

The estimates (66) and (67) imply the inequality (65). Taking into account (64) and (65) and applying the geometric series theorem, we can deduce that the operator $(-\pi\bar{\mathcal{I}} - \bar{\mathcal{M}})^{-1}$ exists and is a bounded operator on $\mathcal{X}$ into $\mathcal{X}$ with $\left\|(-\pi\bar{\mathcal{I}} - \bar{\mathcal{M}})^{-1}\right\| \leq \dfrac{1}{\pi - \|\bar{\mathcal{M}}\|}$. Then we can reformulate the equation (37) as the following equivalent one

$$
\bar{\phi} - \left(-\pi\bar{\mathcal{I}} - \bar{\mathcal{M}}\right)^{-1} \bar{\mathcal{S}}\bar{\phi} = \left(-\pi\bar{\mathcal{I}} - \bar{\mathcal{M}}\right)^{-1} \bar{f}. \tag{68}
$$

Now, we observe that the operator $\bar{\mathcal{S}}$ as a map from $\mathcal{X}$ into $\mathcal{X}$ is compact since it is a matrix of compact operators. Hence the operator $\left(-\pi\bar{\mathcal{I}}-\bar{\mathcal{M}}\right)^{-1}\bar{\mathcal{S}}$ is a compact operator and, consequently, the Fredholm alternative can be applied to the equation (68). Then, system (37) is unisolvent in $\mathcal{X}$ for each right-hand side $\bar{f}\in\mathcal{X}$ if and only if the homogeneous problem admits only the trivial solution.
In particular, if $\bar{f}\in\mathcal{X}_0$ then the vector $\bar{\phi}=\left(-\pi\bar{\mathcal{I}}-\bar{\mathcal{K}}\right)^{-1}\bar{f}$ also belongs to the subspace $\mathcal{X}_0$. In fact, in virtue of the invertibility of the operator $\left(-\pi\bar{\mathcal{I}}-\bar{\mathcal{K}}\right)$ in $\mathcal{X}_0$, there exists an array $\bar{\psi}\in\mathcal{X}_0$ such that $\bar{\psi}=\left(-\pi\bar{\mathcal{I}}-\bar{\mathcal{K}}\right)^{-1}\bar{f}$. Then, by the assumption, it follows that $\bar{\phi}=\bar{\psi}$. □

Proof of Theorem 4.1. Since, for $\bar{F}=(F_1,\dots,F_n)\in\mathcal{X}$, one has

$$\|\bar{\mathcal{M}}_m\bar{F}\|=\left(\|\mathcal{M}_{12}^m F_2\|_2^2+\sum_{i=2}^{n-1}\|\mathcal{M}_{i,i-1}^m F_{i-1}+\mathcal{M}_{i,i+1}^m F_{i+1}\|_2^2+\|\mathcal{M}_{n,n-1}^m F_{n-1}\|_2^2\right)^{\frac{1}{2}},$$

we begin by estimating the norms $\left\|\mathcal{M}_{i,i-1}^m F\right\|_2$, $\left\|\mathcal{M}_{i,i+1}^m F\right\|_2$, $\left\|\mathcal{M}_{i,i-1}^m F+\mathcal{M}_{i,i+1}^m G\right\|_2$ under the assumption $F,G\in W_r^2$. We can proceed as in the proof of Theorem 3.3 in [8] and obtain

$$\begin{aligned}\left\|\mathcal{M}_{i,i-1}^m F\right\|_2 &= \left(\int_0^1\left|\left(\chi_0\tilde{\mathcal{K}}_0^{\beta_{i-1},m}\chi_1 F\right)(t)\right|^2dt\right)^{\frac{1}{2}}\le\left(\int_0^1\left|\left(\tilde{\mathcal{K}}_0^{\beta_{i-1},m}\chi_1 F\right)(t)\right|^2dt\right)^{\frac{1}{2}}\\ &\le \|\chi_1F\|_2\int_0^\infty\frac{\left|k^{\beta_{i-1}}(\tau)\right|}{\tau^{\frac{1}{2}}}d\tau+\frac{\mathcal{C}}{m^r}t_m^{-\frac{r}{2}}\|\chi_1F\|_{W_r^2}+\frac{\mathcal{C}}{m^{(\mu-1)\delta}}\|\chi_1F\|_2\\ &\le \|\chi_1F\|_2\int_0^\infty\frac{\left|k^{\beta_{i-1}}(\tau)\right|}{\tau^{\frac{1}{2}}}d\tau+\left(\frac{\mathcal{C}}{m^{r\delta}}+\frac{\mathcal{C}}{m^{(\mu-1)\delta}}\right)\|\chi_1F\|_{W_r^2},\end{aligned}$$

where $\mathcal{C}=\mathcal{C}(r,N)$. Now, setting $\gamma=\max_{j=1,\dots,n}\int_0^\infty\frac{\left|k^{\beta_j}(\tau)\right|}{\tau^{\frac{1}{2}}}d\tau$ and $\rho=\min\{r,\mu-1\}$, we have

$$\left\|\mathcal{M}_{i,i-1}^m F\right\|_2\le\gamma\|\chi_1F\|_2+\frac{\mathcal{C}}{m^{\rho\delta}}\|\chi_1F\|_{W_r^2}$$

In the same manner we can see that

$$\left\|\mathcal{M}_{i,i+1}^m F\right\|_2\le\gamma\|\chi_0F\|_2+\frac{\mathcal{C}}{m^{\rho\delta}}\|\chi_0F\|_{W_r^2}.$$

From the previous relations we can also deduce that

$$\begin{aligned}\left\|\mathcal{M}_{i,i-1}^m F+\mathcal{M}_{i,i+1}^m G\right\|_2^2 &\le \int_0^\varepsilon\left|\left(\tilde{\mathcal{K}}_0^{\beta_{i-1},m}\chi_1F\right)(t)\right|^2dt+\int_{1-\varepsilon}^1\left|\left(\tilde{\mathcal{K}}_1^{\beta_i,m}\chi_0G\right)(t)\right|^2dt\\ &\le \left(\gamma\|\chi_1F\|_2+\frac{\mathcal{C}}{m^{\rho\delta}}\|\chi_1F\|_{W_r^2}\right)^2+\left(\gamma\|\chi_0F\|_2+\frac{\mathcal{C}}{m^{\rho\delta}}\|\chi_0F\|_{W_r^2}\right)^2\end{aligned}$$

with $\mathcal{C}=\mathcal{C}(r,N)$. Then, a trivial verification shows that

$$\begin{aligned}\|\bar{\mathcal{M}}_m\bar{F}\| &\le \gamma\left(\sum_{i=1}^n\|F_i\|_2^2\right)^{\frac{1}{2}}+\frac{\mathcal{C}}{m^{\rho\delta/2}}\left(\sum_{i=1}^n\|F_i\|_{W_r^2}^2\right)^{\frac{1}{2}}\\ &\le \left(\gamma+\frac{\mathcal{C}}{m^{\rho\delta/2}}\right)\|\bar{F}\|_r\end{aligned}\tag{69}$$

and, consequently,

$$\|\bar{\mathcal{M}}_m\|_{\mathcal{X}^r\to\mathcal{X}}\le\gamma+\frac{\mathcal{C}}{m^{\rho\delta/2}},\tag{70}$$

from which we deduce

$$\limsup_{m\to\infty} \|\bar{\mathcal{M}}_m\|_{\mathcal{X}^r\to\mathcal{X}} \le \gamma.$$

Finally, taking into account (59), (47) follows. Now let us prove (48). For $\bar{F} = (F_1, \ldots, F_n) \in \mathcal{X}^r$ one has

$$\|(\bar{\mathcal{M}}_m - \bar{\mathcal{M}})\bar{F}\| = \Big(\|(\mathcal{M}_{12}^m - \mathcal{M}_{12})\, F_2\|_2^2 \tag{71}$$

$$+ \sum_{i=2}^{n-1} \left\| \left(\mathcal{M}_{i,i-1}^m - \mathcal{M}_{i,i-1}\right) F_{i-1} + \left(\mathcal{M}_{i,i+1}^m - \mathcal{M}_{i,i+1}\right) F_{i+1} \right\|_2^2 + \left\| \left(\mathcal{M}_{n,n-1}^m - \mathcal{M}_{n,n-1}\right) F_{n-1} \right\|_2^2 \Big)^{\frac{1}{2}}.$$

Moreover, the following inequalities hold true

$$\left\| \left(\mathcal{M}_{i,i-1}^m - \mathcal{M}_{i,i-1}\right) F_{i-1} \right\|_2 \le \left\| \left(\tilde{\mathcal{K}}_0^{\beta_{i-1},m} - \mathcal{K}_0^{\beta_{i-1}} \right) \chi_1 F_{i-1} \right\|_2 \tag{72}$$

$$\left\| \left(\mathcal{M}_{i,i+1}^m - \mathcal{M}_{i,i+1}\right) F_{i+1} \right\|_2 \le \left\| \left(\tilde{\mathcal{K}}_1^{\beta_i,m} - \mathcal{K}_1^{\beta_i} \right) \chi_0 F_{i+1} \right\|_2 \tag{73}$$

$$\left\| \left(\mathcal{M}_{i,i-1}^m - \mathcal{M}_{i,i-1}\right) F_{i-1} + \left(\mathcal{M}_{i,i+1}^m - \mathcal{M}_{i,i+1}\right) F_{i+1} \right\|_2^2 \le$$

$$\left\| \left(\tilde{\mathcal{K}}_0^{\beta_{i-1},m} - \mathcal{K}_0^{\beta_{i-1}} \right) \chi_1 F_{i-1} \right\|_2^2 + \left\| \left(\tilde{\mathcal{K}}_1^{\beta_i,m} - \mathcal{K}_1^{\beta_i} \right) \chi_0 F_{i+1} \right\|_2^2. \tag{74}$$

Then, taking into account (59) and Lemma 5.1, we can apply Theorem 3.3 in [8] and deduce the thesis. □

Proof of Theorem 4.2. First, note that, for all $\bar{F} = (F_1, \ldots, F_n) \in \mathcal{X}^r$, if

$$\|\mathcal{K}_{ij}^m F_j\| \le \mathcal{C}\|F_j\|_{W_r^2}, \quad \mathcal{C} \ne \mathcal{C}(m, F_j), \quad |i-j| \ne 1 \tag{75}$$

$$\|\mathcal{E}_{ij}^m F_j\| \le \mathcal{C}\|F_j\|_{W_r^2}, \quad \mathcal{C} \ne \mathcal{C}(m, F_j), \quad |i-j| = 1, \tag{76}$$

then one has

$$\|\bar{\mathcal{S}}_m F\| \le \mathcal{C}\|\bar{F}\|_r, \quad \mathcal{C} \ne \mathcal{C}(m, \bar{F}). \tag{77}$$

Moreover, if

$$\lim_{m\to\infty} \left\| \left(\mathcal{K}_{ij}^m - \mathcal{K}_{ij}\right) F_j \right\| = 0, \quad |i-j| \ne 1 \tag{78}$$

$$\lim_{m\to\infty} \left\| \left(\mathcal{E}_{ij}^m - \mathcal{E}_{ij}\right) F_j \right\| = 0, \quad |i-j| = 1, \tag{79}$$

it immediately follows that $\lim_{m\to\infty} \left\| \left(\bar{\mathcal{S}}_m - \bar{\mathcal{S}}\right) \bar{F} \right\| = 0$, i.e. (49). Finally, the set $\left\{\bar{\mathcal{S}}_m\right\}_m$ is collectively compact if the sets $\left\{\mathcal{K}_{ij}^m\right\}_m$, for any $|i-j| \ne 1$, and $\left\{\mathcal{E}_{ij}^m\right\}_m$, for any $|i-j| = 1$, are also collectively compact. Since, under the assumptions, the kernels of the operators $\mathcal{K}_{ij}$ and $\mathcal{E}_{ij}$, satisfy the conditions

$$\sup_{0\le s\le 1} \|K_{ij}(\cdot, s)\|_{W_r^2} < +\infty, \quad \sup_{0\le t\le 1} \|K_{ij}(t, \cdot)\|_{W_r^2} < +\infty, \quad |i-j| \ne 1 \tag{80}$$

$$\sup_{0\le s\le 1} \|E_{ij}(\cdot, s)\|_{W_r^2} < +\infty, \quad \sup_{0\le t\le 1} \|E_{ij}(t, \cdot)\|_{W_r^2} < +\infty, \quad |i-j| = 1, \tag{81}$$

in virtue of [8, theorems 3.1-3.2], (75)-(76) and (78)-(79) are satisfied. Furthermore, the collective compactness of the sequences of operators $\left\{\mathcal{K}_{ij}^m\right\}_m$ and $\left\{\mathcal{E}_{ij}^m\right\}_m$ are guaranteed (see, also, [8, Remark 3.1]) and the proof is complete. □

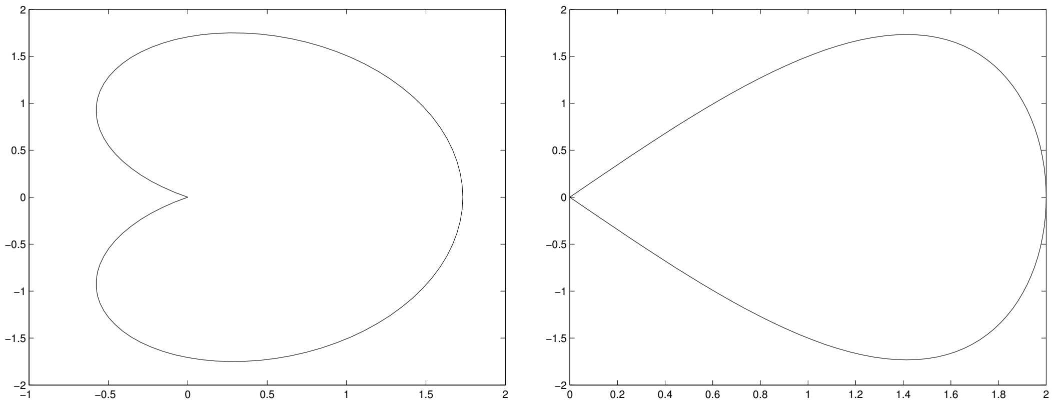

Figure 1: Γ in Example 6.1 with $\beta = \frac{5}{3}\pi$ (left) and in Example 6.2 with $\beta = \frac{2}{3}\pi$ (right)

6. Numerical examples

In this section we show the numerical results obtained by applying the method described in Section 4 to some examples of the exterior Neumann problem in planar domains with corners. In order to give the boundary condition f, we choose a test harmonic function u and we compute the absolute errors

$$e_m(x_1, x_2) = |u(x_1, x_2) - u_m(x_1, x_2)| \tag{82}$$

at some points $(x_1, x_2) \in \mathbb{R}^2 \backslash \overline{\Omega}$. The approximate function u_m is given by (56), where the involved quantities $\tilde{\phi}_{m,i}(s_{m,k})$ are provided by solving the preconditioned linear system (54). Moreover, the corresponding estimated orders of convergence

$$EOC = \frac{\log\left(e_m(x_1, x_2)/e_{2m}(x_1, x_2)\right)}{\log 2}$$

are presented. For each test, we also give the errors in the computation of the single layer density. Since the exact solution $\bar{\phi} = (\tilde{\phi}_1, \ldots, \tilde{\phi}_n)^T$ of system (24) is unknown, we have computed the approximation $\bar{\phi}_{\bar{m}} = (\tilde{\phi}_{\bar{m},1}, \ldots, \tilde{\phi}_{\bar{m},n})^T$ for $\bar{m} = 512$ or $\bar{m} = 1024$ and used it as exact solution. Then, we report the errors

$$err_m = \max_{i=1,\ldots,n} err_{m,i}, \qquad err_{m,i} = \max_{j=1,\ldots,100} \left|\tilde{\phi}_{\bar{m},i}(s_j) - \tilde{\phi}_{m,i}(s_j)\right|,$$

where $s_1, \ldots, s_{100}$ are equispaced points in the interval $(0, 1)$, and the related estimated orders of convergence

$$eoc = \frac{\log\left(err_m/err_{2m}\right)}{\log 2}.$$

We also show the values of the condition numbers in the spectral norm both for the matrix associated with the linear system (52), (53) (CN) and for the matrix of the preconditioned system (54) (PCN). In all the examples we specify the value of the smoothing parameter q in (17). We remark that the values of the parameters c, δ, μ and N involved in (40)-(41) have been chosen according to the criteria proposed in [8, Subsection 4.1] taking $\sigma = \varrho$, with $\varrho = q(1+s) - 1$ and $s = \min\left\{\frac{\pi}{\beta}, \frac{\pi}{2\pi-\beta}\right\} - 1$ (β denotes the interior angle at the corner).

Example 6.1. *We consider a family of "heart-shaped" domains bounded by the curves*

$$\sigma(\bar{t}) = \begin{pmatrix} \cos(1+\frac{\beta}{\pi})\pi\bar{t} - \sin(1+\frac{\beta}{\pi})\pi\bar{t} \\ \sin(1+\frac{\beta}{\pi})\pi\bar{t} + \cos(1+\frac{\beta}{\pi})\pi\bar{t} \end{pmatrix} \begin{pmatrix} \tan\frac{\beta}{2} \\ 1 \end{pmatrix} - \begin{pmatrix} \tan\frac{\beta}{2} \\ \cos\pi\bar{t} \end{pmatrix}, \quad \bar{t} \in [0, 1],$$

where $\beta \in (\pi, 2\pi)$ is the interior angle of the single outward-pointing corner $P_1 = (0,0)$ (see Figure 1 on the left). For this test we choose boundary data corresponding to the exact solution

$$u(x_1, x_2) = \log|(x_1, x_2) - (0.5, 0)| - \log|(x_1, x_2) - (0.2, 0)|.$$

Tables 1-3 report the results obtained applying our method in the case $\beta = \frac{5}{3}\pi$ and for the two different choices of the smoothing parameter $q = 2$ and $q = 3$. Moreover, in Table 4 we compare the errors $e_m(x_1, x_2)$ obtained by the proposed method with the errors $\bar{e}_m(x_1, x_2)$ presented in [14] (m is the dimension of the linear system). Taking, also, into account that in the latter case some additional computational cost has to be paid in order to compute the right-hand side of the linear system, we can deduce that the performance of the new procedure is far better.

Table 1: Example 6.1 $c = a \cdot 10^{2\varrho}$, $\delta = 10^{-2}$, $\mu = 2\varrho + 1$, $N = 3$

	$q=2, \quad a=0.9$		$q=3, \quad a=0.3$	
m	err_m	eoc	err_m	eoc
16	1.32e-01		6.39e-01	
		4.44		3.56
32	6.05e-03		5.41e-02	
		4.95		8.04
64	1.95e-04		2.05e-04	
		4.81		7.95
128	6.95e-06		8.28e-07	
		4.84		7.26
256	2.41e-07		5.38e-09	

Table 2: Example 6.1 $c = a \cdot 10^{2\varrho}$, $\delta = 10^{-2}$, $\mu = 2\varrho + 1$, $N = 3$

	$q=2, \quad a=0.9$				$q=3, \quad a=0.3$			
m	$e_m(-0.1,0)$	EOC	$e_m(3,3)$	EOC	$e_m(-0.1,0)$	EOC	$e_m(3,3)$	EOC
16	1.46e-01		8.31e-03		4.64e-01		3.22e-02	
		5.81		6.18		3.55		4.03
32	2.60e-03		1.14e-04		3.95e-02		1.96e-03	
		12.83		12.29		8.06		8.00
64	3.55e-07		2.27e-08		1.47e-04		7.65e-06	
		18.31		23.95		16.48		16.61
128	1.09e-12		1.39e-15		1.60e-09		7.60e-11	
		8.26		1.31		18.23		17.71
256	3.55e-15		5.62e-16		5.21e-15		3.53e-16	

Figure 2 contains the plot of the absolute error on the linear segment defined by the equation $x_2 = \frac{x_1}{3}$, with $x_1 \in [-200, 0)$. We can observe that the more m is large, the better u_m approximates the exact solution u and, consequently, satisfies the infinity condition. In Figure 3 we show a plot of the absolute errors on the boundary condition $\left|f(x) - \frac{\partial u_m(x)}{\partial \mathbf{n}}\right|$, with $x = \bar{\sigma}(\gamma(t)) \in \Gamma$, $t \in [0,1]$, and in Table 5 we report the condition numbers. We note that the proposed preconditioning tecnique yields linear systems whose condition numbers do not grow up so fast when m increases.

Example 6.2. *We consider the "teardrop domain" Ω bounded by the curve Γ parameterized by*

$$\bar{\sigma}(\bar{t}) = \left(2\sin\pi\bar{t}, -\tan\frac{\beta}{2}\sin 2\pi\bar{t}\right), \quad \bar{t} \in [0,1], \tag{83}$$

Table 3: Example 6.1 $c = a \cdot 10^{2\varrho}$, $\delta = 10^{-2}$, $\mu = 2\varrho + 1$, $N = 3$

	$q=2,\quad a=0.9$				$q=3,\quad a=0.3$			
m	$e_m(-40,-50)$	EOC	$e_m(100,-100)$	EOC	$e_m(-40,-50)$	EOC	$e_m(100,-100)$	EOC
16	6.58e-04		3.35e-04		2.01e-03		1.02e-03	
		5.82		5.82		3.40		3.40
32	1.16e-05		5.93e-06		1.90e-04		9.70e-05	
		12.29		12.29		8.00		8.00
64	2.32e-09		1.18e-09		7.40e-07		3.77e-07	
		24.53		20.57		16.44		16.44
128	9.54e-17		7.57e-16		8.28e-12		4.22e-12	
		-2.39		0.53		15.98		12.92
256	5.01e-16		5.24e-16		1.27e-16		5.41e-16	

Table 4: Example 6.1 Absolute errors $\bar{e}_m(x_1, x_2)$ in [14] and $e_m(x_1, x_2)$ defined by (82)

	$(-0.1, 0)$		$(3, 3)$		$(-40, -50)$		$(100, -100)$	
m	$\bar{e}_m$	e_m	$\bar{e}_m$	e_m	$\bar{e}_m$	e_m	$\bar{e}_m$	e_m
51	6.62e-03	1.53e-05	2.35e-05	7.04e-07	4.52e-05	7.53e-08	5.9e-05	3.83e-08
99	6.95e-03	1.18e-09	1.89e-04	1.97e-12	1.12e-05	1.95e-13	5.3e-06	9.87e-14
195	6.78e-04	1.66e-15	1.81e-05	3.46e-16	1.05e-06	2.85e-16	5.3e-07	8.51e-16

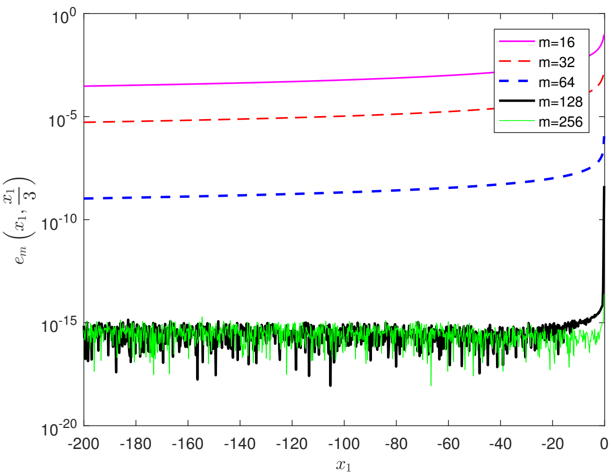

Figure 2: Example 6.1 Absolute errors $e_m(x_1, x_2)$ with $x_1 \in [-200, 0)$ e $x_2 = \dfrac{x_1}{3}$

where $\beta \in (0, \pi)$ is the interior angle of the single outward-pointing corner $P_1 = (0, 0)$ (see Figure 1 on the right) and we choose the boundary data f as the normal derivative of the following function

$$u(x_1, x_2) = \arctan\left(\frac{x_2 - 0.2}{x_1 - 0.8}\right) - \arctan\left(\frac{x_2}{x_1 - 0.8}\right).$$

The numerical results showed in tables 6-10 and in figures 4-5 are obtained for $\beta = \frac{2}{3}\pi$. One can repeat the same comments made in Example 6.1.

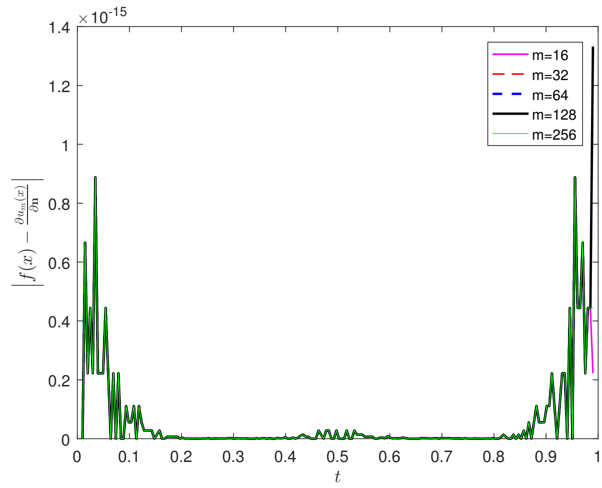

Figure 3: Example 6.1 Absolute errors $\left|f(x) - \frac{\partial u_m(x)}{\partial \mathbf{n}}\right|$ with $x = \bar{\sigma}(\gamma(t)) \in \Gamma$

Table 5: Example 6.1 Condition numbers

	$q = 2$		$q = 3$	
m	CN	PCN	CN	PCN
16	4.56e+03	2.65e+02	1.84e+07	4.48e+02
32	1.46e+04	5.17e+02	5.31e+04	9.17e+02
64	5.29e+04	1.02e+03	1.72e+05	1.81e+03
128	2.03e+05	2.03e+03	6.56e+05	3.61e+03
256	8.03e+05	4.06e+03	2.59e+06	7.22e+03

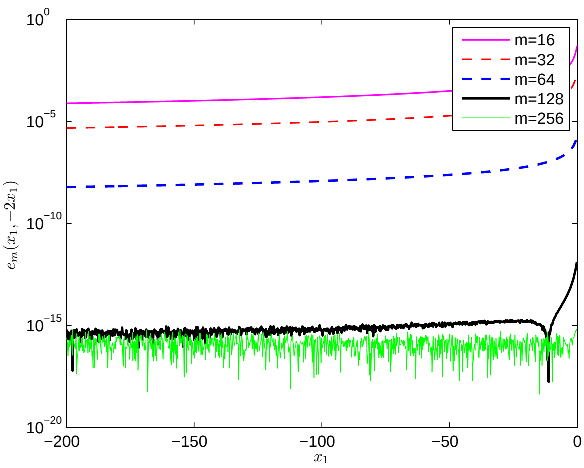

Figure 4: Example 6.2 Absolute errors $e_m(x_1, x_2)$ with $x_1 \in [-200, 0)$ e $x_2 = -2x_1$

We also solved the exterior Neumann problem when the angle β of the teardrop domain Ω is close to 0

Table 6: Example 6.2 $c = a \cdot 10^{2\varrho}$, $\delta = 10^{-1}$, $\mu = 2\varrho + 1$, $N = 3$

	$q = 2,\quad a = 0.3$		$q = 3,\quad a = 0.01$	
m	err_m	eoc	err_m	eoc
16	1.13e-01		3.65e-01	
		4.30		2.36
32	5.73e-03		7.10e-02	
		10.27		6.21
64	4.61e-06		9.54e-04	
		18.54		14.38
128	1.20e-11		4.47e-08	
		0.17		13.83
256	1.07e-11		3.05e-12	

Table 7: Example 6.2 $c = a \cdot 10^{2\varrho}$, $\delta = 10^{-1}$, $\mu = 2\varrho + 1$, $N = 3$

	$q = 2,\quad a = 0.3$				$q = 3,\quad a = 0.01$			
m	$e_m(-0.01, 0)$	EOC	$e_m(3, 3)$	EOC	$e_m(-0.01, 0)$	EOC	$e_m(3, 3)$	EOC
16	1.42e-01		9.39e-03		3.70e-01		2.59e-02	
		4.57		5.09		2.88		3.04
32	5.96e-03		2.74e-04		5.02e-02		3.14e-03	
		10.34		10.43		6.25		7.48
64	4.58e-06		1.97e-07		6.55e-04		1.75e-05	
		20.39		18.20		15.49		11.57
128	3.33e-12		6.56e-13		1.41e-08		5.73e-09	
		9.56		11.40		21.96		27.62
256	4.38e-15		2.42e-16		3.46e-15		2.77e-17	

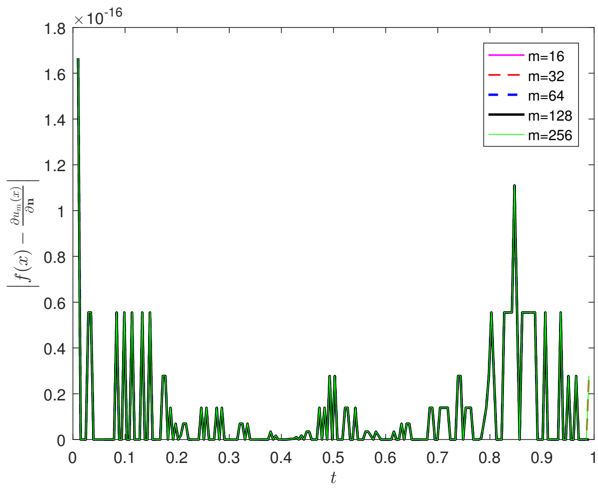

Figure 5: Example 6.2 Absolute errors $\left| f(x) - \frac{\partial u_m(x)}{\partial \mathbf{n}} \right|$ with $x = \bar{\sigma}(\gamma(t)) \in \Gamma$

Table 8: Example 6.2 $c = a \cdot 10^{2\varrho}$, $\delta = 10^{-1}$, $\mu = 2\varrho + 1$, $N = 3$

	$q = 2, \quad a = 0.3$				$q = 3, \quad a = 0.01$			
m	$e_m(-40,-50)$	EOC	$e_m(100,-100)$	EOC	$e_m(-40,-50)$	EOC	$e_m(100,-100)$	EOC
16	7.26e-04		3.79e-04		1.72e-03		8.13e-04	
		4.63		4.18		2.67		2.33
32	2.92e-05		2.09e-05		2.68e-04		1.61e-04	
		11.52		9.86		6.56		6.01
64	9.93e-09		2.24e-08		2.84e-06		2.49e-06	
		18.16		22.41		14.37		14.02
128	3.37e-14		4.00e-15		1.34e-10		1.49e-10	
		10.72		6.58		22.11		20.20
256	1.99e-17		4.16e-17		2.99e-17		1.23e-16	

Table 9: Example 6.2 Absolute errors $\bar{e}_m(x_1, x_2)$ in [14] and $e_m(x_1, x_2)$ defined by (82)

	$(-0.01, 0)$		$(3, 3)$		$(-40, -50)$		$(100, -100)$	
m	$\bar{e}_m$	e_m	$\bar{e}_m$	e_m	$\bar{e}_m$	e_m	$\bar{e}_m$	e_m
51	4.15e-03	9.80e-05	6.39e-04	6.01e-07	1.47e-03	3.64e-07	1.80e-03	4.04e-07
99	4.80e-05	4.34e-10	8.81e-06	3.57e-10	4.34e-06	1.31e-11	5.57e-06	3.73e-12
195	3.70e-06	1.63e-15	2.54e-07	2.08e-17	1.98e-08	7.50e-17	8.05e-09	1.81e-16

and π. For this tests, we have chosen the boundary data f equal to the normal derivative of the potential

$$u(x_1, x_2) = \arctan\left(\frac{x_2 - 0.001}{x_1 - 1.0}\right) - \arctan\left(\frac{x_2 + 0.001}{x_1 - 1.2}\right).$$

Note that the points $(1.0, 0.001)$ and $(1.2, -0.001)$ are contained in the interior domain Ω for all values of the angle β considered in Table 11. Here E_m denotes the largest absolute error observed while approximating the solution u by u_m at a collection of 600 points sampled randomly in the box with corners $(3, 4)$ and $(6, 0)$. When the interior angle at the corner becomes very close to 0 and π, a greater computational effort is required in order to achieve high accuracy in the approximation of the potential in the exterior domain.

Example 6.3. *In this example the family of "boomerang-shaped" domains with the boundary Γ given by the following parametric equation*

$$\Gamma : \bar{\sigma}(\bar{t}) = \left(2 \sin 3\pi\bar{t}, -\tan\frac{\beta}{2}\sin 2\pi\bar{t}\right), \quad \bar{t} \in [0, 1]$$

Table 10: Example 6.2 Condition numbers

	$q = 2$		$q = 3$	
m	CN	PCN	CN	PCN
16	7.42e+02	9.19e+01	7.37e+02	9.20e+01
32	2.84e+03	1.78e+02	2.93e+03	1.86e+02
64	1.10e+04	3.50e+02	1.15e+04	3.69e+02
128	4.32e+04	6.86e+02	4.55e+04	7.32e+02
256	1.71e+05	1.36e+03	1.81e+05	1.46e+03

Table 11: Example 6.2 Numerical results for degenerate domains

β	m	E_m
$\pi/10$	512	1.79e-15
$\pi/20$	1024	1.54e-14
$\pi/50$	2048	6.16e-14
$\pi/100$	4096	1.68e-13
$\pi-\pi/10$	1024	2.99e-15
$\pi-\pi/20$	2048	1.23e-14
$\pi-\pi/50$	4096	4.34e-15
$\pi-\pi/100$	8192	2.17e-13

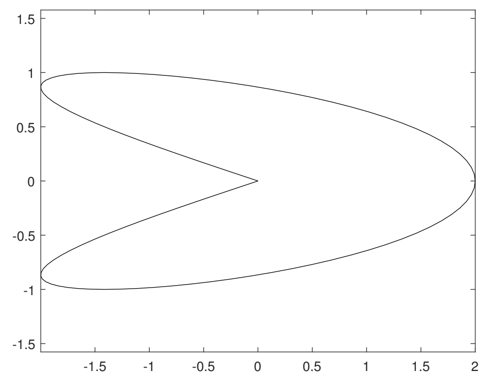

Figure 6: Γ in Example 6.3 with $\beta = \frac{3}{2}\pi$

is considered. The curve Γ has a single corner point at $P_1 = (0,0)$ where forms the interior angle $\beta \in (0,\pi)$ (see Figure 6 which depicts Γ in the case $\beta = \frac{3}{2}\pi$). For this numerical test, we choose $\beta = \frac{3}{2}\pi$ and the boundary data corresponding to the exact solution

$$u(x_1,x_2) = \log|(x_1,x_2) - (1.0,0.0)| - \log|(x_1,x_2) - (1.1,0.1)|. \tag{84}$$

In Table 13 we report the maximum absolute error, denoted by $E_m(3)$, measured at 300 points on the circle of radius 3 centered at the origin, and the condition numbers of the linear system (54).

Looking at the errors, these numerical results seem to be competitive if compared to the results presented in [2, Table 3]. Neverthless, the condition numbers of the linear system matrix obtained for the same test in [2], when inner product preserving Nyström discretizations is used, are smaller than those we get by applying our method.

Example 6.4. *Let us consider the family of boomerang-shaped domains introduced in the previous test. In this experiment we consider $\beta = \frac{4}{3}\pi$ and we take as Neumann data the function*

$$f(\sigma(t)) = (1-3t^2)/|\sigma'(t)|, \quad \sigma(t) = \bar{\sigma}(\gamma(t)), \quad t \in [0,1].$$

It, obviously, satisfies condition (2), but the corresponding exact solution u is unknown. Therefore, we will use our computed value $u_{2048}(x_1,x_2)$ as the exact value. In Table 14 the largest absolute errors E_m measured at 100 points on the linear segment $x_2 = -\frac{1}{10}x_1$ with $x_1 \in [-200,0)$ and the corresponding numerical order of convergence $EOC = \dfrac{\log(E_m/E_{2m})}{\log 2}$ are presented.

Table 12: Example 6.3 $c = a \cdot 10^{2\varrho}$, $\delta = 10^{-1}$, $\mu = 2\varrho + 1$, $N = 2$

	$q = 2, \quad a = 0.4$		$q = 3, \quad a = 0.1$	
m	err_m	eoc	err_m	eoc
64	1.48e+00		3.56e+00	
		5.51		2.67
128	3.26e-02		5.58e-01	
		12.24		8.43
256	6.73e-06		1.60e-03	
		9.37		17.86
512	1.01e-08		6.75e-09	

Table 13: Example 6.3 $c = a \cdot 10^{2\varrho}$, $\delta = 10^{-2}$, $\mu = 2\varrho + 1$, $N = 2$

	$q = 2, \quad a = 0.4$		$q = 3, \quad a = 0.1$	
m	$E_m(3)$	PCN	$E_m(3)$	PCN
16	3.98e-01	1.76e+02	2.05e-01	5.65e+02
32	9.73e-02	1.92e+03	8.45e-02	8.26e+02
64	2.28e-02	2.90e+03	4.32e-02	4.00e+03
128	5.22e-04	5.57e+03	6.57e-03	7.87e+03
256	1.08e-07	1.10e+04	1.61e-05	1.56e+04
512	6.21e-15	2.21e+04	1.13e-10	3.12e+04
1024	2.90e-15	4.43e+04	5.09e-15	6.78e+04

Example 6.5. *In this example we have employed a domain Ω whose boundary is formed by four circular arcs, centered at ± 1 and ± 3, respectively, and each of radius* 3.64 *(see Figure 7 on the left). We assume as potential over the above domain the harmonic function*

$$u(x_1, x_2) = \frac{x_1}{(x_1^2 + x_2^2)^2}.$$

The results obtained by applying the proposed method to the problem (1)*, with $f = \dfrac{\partial u}{\partial \mathbf{n}}$, are reported in tables 15 and 16. Here $E_m(\rho)$ is the maximum absolute error at* 200 *points on a circle of radius ρ centered at the origin. We can note that, this time, for sufficiently large m, we get more accurate results by choosing as smoothing paramater in* (17) *$q = 3$ rather than $q = 2$. Moreover, the linear systems we solved are in both cases well conditioned.*

Example 6.6. *In this last test we solve an exterior Neumann problem on the "inkblot" domain Ω depicted in Figure 7 on the right. The contour Γ of the domain has 8 corner points and is parameterized by the polar equation*

$$r(\theta) = 4 + 2|\cos(4\theta)| \sin(4\theta), \quad 0 \le \theta \le 2\pi.$$

The boundary data f is assumed to be the normal derivative of the potential

$$u(x_1, x_2) = \frac{x_1^2 - x_2^2}{(x_1^2 + x_2^2)^2}.$$

By looking at the numerical results shown in tables 17 and 18, one can draw conclusions analogous to the ones reported for the previous example.

Table 14: Example 6.4 Maximum absolute errors on the linear segment $x_2 = -\frac{1}{10}x_1$ with $x_1 \in [-200, 0)$ and EOC

	$q=2$		$q=3$	
m	E_m	EOC	E_m	EOC
16	1.50e-02		9.17e-02	
		2.11		2.41
32	3.49e-03		1.71e-02	
		7.92		4.63
64	1.44e-05		6.90e-04	
		15.33		10.19
128	3.48e-10		5.88e-07	
		17.10		20.49
256	2.47e-15		3.98e-13	
		0.57		6.53
512	1.65e-15		4.28e-15	

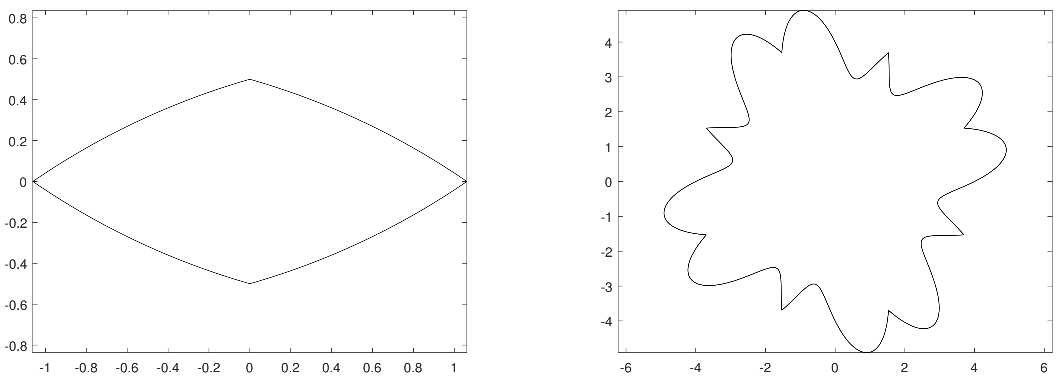

Figure 7: Γ in Example 6.5 (left) and in Example 6.6 (right)

7. Conclusions

A Nyström type method based on global approximation is proposed for the numerical solution of exterior Neumann problems for the Laplace equation in domains with corners.
The differential problem is reformulated as a system of boundary integral equations by using the single layer representation of the potential and a suitable decomposition of the piecewise smooth contour of the planar domain.
Since the solutions of the system could be unbounded at the corner points of the boundary, a proper smoothing transformation is introduced to improve the behavior of the unknown functions.
The Nyström discretization of the system is obtained by applying the classical Gauss-Legendre quadrature formula suitably modified near the corner points in order to assure the convergence of the rule.
Moreover, the well conditioning of the linear systems arising from the discretization of the integral equations is achieved by means of an appropriate preconditioning technique.
Finally, a formula for the computation of the approximating single layer potential is provided.
Several numerical examples confirm the applicability of the proposed method and illustrate its performance.
The stability and the convergence are showed by numerical evidence. A forthcoming work could be addressed to their theoretical proof.

Table 15: Example 6.5 $c = 10$ if $q = 2$ and $c = 0.1$ if $q = 3$, $\delta = 10^{-3}$, $\mu = 1.5$, $N = 3$

	$q=2$		$q=3$	
m	err_m	eoc	err_m	eoc
16	5.70e-05		1.21e-03	
		4.06		11.32
32	3.40e-06		4.73e-07	
		3.95		11.29
64	2.18e-07		1.89e-10	
		3.98		1.60
128	1.38e-08		6.19e-11	
		4.07		0.04
256	8.22e-10		6.01e-11	

Table 16: Example 6.5 $c = 10$ if $q = 2$ and $c = 0.1$ if $q = 3$, $\delta = 10^{-3}$, $\mu = 1.5$, $N = 3$

m	$E_m(1.1)$		$E_m(2)$		$E_m(6)$		$E_m(20)$		PCN	
	$q=2$	$q=3$	$q=2$	$q=3$	$q=2$	$q=3$	$q=2$	$q=3$	$q=2$	$q=3$
16	6.19e-05	2.39e-03	4.95e-05	1.15e-03	7.06e-05	4.18e-04	1.06e-04	1.27e-04	7.87	15.17
32	3.21e-06	3.99e-07	2.99e-06	1.64e-07	4.60e-06	6.34e-08	7.01e-06	2.78e-08	8.14	14.37
64	2.06e-07	8.15e-11	1.92e-07	7.59e-11	2.96e-07	1.16e-10	4.50e-07	1.77e-10	8.25	14.18
128	1.31e-08	1.30e-12	1.22e-08	1.21e-12	1.87e-08	1.86e-12	2.85e-08	2.83e-12	8.30	14.14
256	8.26e-10	1.95e-14	7.70e-10	1.88e-14	1.18e-09	3.09e-14	1.80e-09	4.63e-14	8.31	15.05
512	5.18e-11	5.55e-15	4.83e-11	4.32e-15	7.42e-11	5.45e-15	1.12e-10	4.80e-15	8.31	12.36

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Table 17: Example 6.6 $c = 5$, $\delta = 10^{-1}$, $\mu = 1.2$, $N = 3$

	$q = 2$		$q = 3$	
m	err_m	eoc	err_m	eoc
16	6.26e-03		3.99e-02	
		4.63		4.39
32	2.51e-04		1.89e-03	
		5.93		6.87
64	4.12e-06		1.61e-05	
		0.52		3.69
128	2.86e-06		1.24e-06	
		1.95		3.68
256	7.36e-07		9.67e-08	

Table 18: Example 6.6 $c = 5$, $\delta = 10^{-1}$, $\mu = 1.2$, $N = 3$

m	$E_m(6)$		$E_m(15)$		$E_m(30)$		$E_m(50)$		PCN	
	$q = 2$	$q = 3$	$q = 2$	$q = 3$	$q = 2$	$q = 3$	$q = 2$	$q = 3$	$q = 2$	$q = 3$
16	7.62e-04	1.51e-03	1.18e-04	2.49e-04	5.23e-05	6.26e-05	4.07e-05	2.27e-05	22.52	52.57
32	5.32e-05	2.70e-05	8.35e-06	2.86e-06	3.59e-06	7.12e-07	2.75e-06	2.59e-07	11.13	14.37
64	9.04e-07	2.25e-06	2.12e-07	2.91e-07	1.50e-07	7.26e-08	1.48e-07	2.61e-08	9.40	12.51
128	6.51e-07	1.76e-07	9.04e-08	2.28e-08	2.87e-08	5.69e-09	1.63e-08	2.05e-09	9.42	12.01
256	2.01e-07	1.49e-08	2.65e-08	1.92e-09	6.99e-09	4.81e-10	2.89e-09	1.74e-10	9.47	11.74
512	4.71e-08	1.28e-09	6.13e-09	1.66e-10	1.55e-09	4.20e-11	5.82e-10	1.57e-11	9.51	11.59

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