GENERALIZED SKEW DERIVATIONS ON SEMIPRIME RINGS

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Abstract. Let R be a semiprime ring with center $Z(R)$, Q its right Martindale quotient ring, C its extended centroid, F a generalized skew derivation of R, associated with a non-zero skew derivation d of R, and $n, m \geq 1$ fixed integers such that $F([x,y])^m = [x,y]^n$, for all $x, y \in R$. Then R contains a non-zero central ideal.
The case of centralizers (generalized skew derivations associated with a zero skew derivation) is also studied.

1. Introduction.

Let R be a prime ring with center $Z(R)$, extended centroid C and right Martindale quotient ring Q.
An additive mapping $d : R \to R$ is a *derivation* on R if $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$. Let $q \in R$ be a fixed element. A map $d : R \to R$ defined by $d(x) = [q, x] = qx - xq$, $x \in R$, is a derivation on R, which is called *inner derivation* defined by q. An additive map $G : R \to R$ is said to be a generalized derivation if there exists a derivation d of R such that, for all $x, y \in R$, $G(xy) = G(x)y + xd(y)$. Basic examples of generalized derivations are the usual derivations on R and left R-module mappings from R into itself. An important example is a map of the form $G(x) = ax + xb$, for some $a, b \in R$; such generalized derivations are called *inner*.
Generalized derivations have been primarily studied on operator algebras. Therefore any investigation from the algebraic point of view might be interesting (see for example [16] and [19]).
In [11] the following result is proved:

Theorem 1.1. *Let R be a prime ring, I a nonzero ideal of R and n a fixed positive integer. If R admits a generalized derivation F associated with a derivation d such that $F([x,y])^n = [x,y]$ for all $x, y \in I$. Then either R is commutative or $n = 1$, $d = 0$ and F is the identity map on R. Moreover in case R is a semiprime ring and $(F([x,y])^n = [x,y]$ for all $x, y \in R$, then either R is commutative or $n = 1$, $d(R) \subseteq Z(R)$, R contains a non-zero central ideal and $F(x) - x \in Z(R)$, for all $x \in R$.*

In [15], Huang and Davvaz consider a similar situation and prove the following:

Theorem 1.2. *Let R be a prime ring, F a generalized derivation of R, associated with a derivation d, U the right Utumi quotient ring of R and m, n fixed positive integers such that $F([x,y])^m = [x,y]^n$ for all $x, y \in R$. Then either R is commutative or $d = 0$. Moreover in case R is a semiprime ring then there exists a central*

2000 *Mathematics Subject Classification.* 16W25, 16W20, 16N60.
Key words and phrases. Generalized skew derivation, automorphism, (semi-)prime ring.

idempotent element $e \in U$ such that on the direct sum decomposition $eU \oplus (1-e)U$, d vanishes identically of eU and the ring $(1-e)U$ is commutative.

Here we continue this line of investigation and we examine what happens in case F is a generalized skew derivation of R such that $(F([x,y])^m = [x,y]^n$ for all $x, y \in R$. More specifically, let α be an automorphism of a ring R. An additive map $D : R \to R$ is called an *α-derivation* (or a *skew derivation*) on R if $D(xy) = D(x)y + \alpha(x)D(y)$ for all $x, y \in R$. In this case α is called an *associated automorphism* of D. Basic examples of α-derivations are the usual derivations and the map $\alpha - id$, where id denotes the identity map. Let $b \in Q$ be a fixed element. Then a map $D : R \to R$ defined by $D(x) = bx - \alpha(x)b$, $x \in R$, is an α-derivation on R and it is called an *inner α-derivation* (an *inner skew derivation*) defined by b. If a skew derivation D is not inner, then it is called *outer*.
An additive mapping $F : R \to R$ is called a *generalized α-derivation* (or a *generalized skew derivation*) on R if there exists an additive mapping d on R such that $F(xy) = F(x)y + \alpha(x)d(y)$ for all $x, y \in R$. A map d is uniquely determined by F and it is called an *associated additive map* of F. Moreover, it turns out that d is always an α-derivation (see [20, 21] for more details). A generalized α-derivation is said to be *regular* if the associated α-derivation d is not zero.
Let us also mention that an automorphism $\alpha : R \to R$ is *inner* if there exists an invertible $q \in Q$ such that $\alpha(x) = qxq^{-1}$ for all $x \in R$. If an automorphism $\alpha \in Aut(R)$ is not inner, then it is called *outer*.
In all that follows let Q be the right Martindale quotient ring of R, and $C = Z(Q)$ the center of Q. We refer the reader to [2] for the definitions and the related properties of these objects. Of course Q is a prime centrally closed C-algebra.
It is known that automorphisms, derivations and skew derivations of R can be extended both to Q and U. In [4] (Lemma 2), J.C. Chang extended the definition of a generalized skew derivation to the right Martindale quotient ring Q of R as follows: by a (right) generalized skew derivation we mean an additive mapping $F : Q \to Q$ such that $F(xy) = F(x)y + \alpha(x)d(y)$, for all $x, y \in Q$, where d is a skew derivation of R and α is an automorphism of R, moreover there exists $b \in Q$ such that $F(x) = bx + d(x)$, for all $x \in R$ (see also Theorem 3.1 and Corollary 3.2 in [6]).
In the first part of the paper we investigate generalized α-derivations associated with a non zero α-derivation. Finally we dedicate the last section to (left) centralizers, which is the case when the generalized α-derivation is not regular, that is F is an additive map on R such that $F(xy) = F(x)y$, for all $x, y \in R$.

The results we obtain are the following:

Theorem 1. *Let R be a prime ring with center $Z(R)$, Q its right Martindale quotient ring, C its extended centroid, F a regular generalized skew derivation of R, associated with a non-zero skew derivation d of R, and $n, m \geq 1$ fixed integers such that $F([x,y])^m = [x,y]^n$, for all $x, y \in R$. Then R is commutative.*

Theorem 2. *Let R be a prime ring with center $Z(R)$, Q its right Martindale quotient ring, C its extended centroid, F a left centralizer of R and $n, m \geq 1$ fixed integers such that $F([x,y])^m = [x,y]^n$, for all $x, y \in R$. Then*

- (a) *either R is commutative;*
- (b) *or there exists $\lambda \in C$ such that $F(x) = \lambda x$, for all $x \in R$, where $\lambda^m = 1$ and $u^m = u^n$ for all $u \in [R,R]$.*

Theorem 3. *Let R be a semiprime ring with center $Z(R)$, Q its right Martindale quotient ring, C its extended centroid, F a regular generalized skew derivation of R, associated with a non-zero skew derivation d of R, and $n, m \geq 1$ fixed integers such that $F([x, y])^m = [x, y]^n$, for all $x, y \in R$. Then R contains a non-zero central ideal.*

Theorem 4. *Let R be a semiprime ring with center $Z(R)$, Q its right Martindale quotient ring, C its extended centroid, F a left centralizer of R and $n, m \geq 1$ fixed integers such that $F([x, y])^m = [x, y]^n$, for all $x, y \in R$. Then*

(a) *either R contains a non-zero central ideal;*
(b) *or $F(x) = \lambda x$, for all $x \in R$, where $\lambda^m = 1$ and $u^m = u^n$ for all $u \in [R, R]$.*

2. The case of regular inner generalized skew derivations in prime rings.

In this section we consider the case when F is an inner generalized skew derivation induced by the elements $a, b \in R$ and $\alpha \in Aut(R)$, that is $F(x) = ax + \alpha(x)b$, for all $x \in R$. We recall that in this case, the skew derivation d of R, which is associated with F, is inner and has the following form: $d(x) = bx - \alpha(x)b$, for all $x \in R$. Of course we assume that $d \neq 0$.

In this sense, our aim will be to prove the following:

Proposition 1. *Let R be a prime ring, $n, m \geq 1$ fixed integers such that $(a[r_1, r_2] + \alpha([r_1, r_2])b)^m = [r_1, r_2]^n$, for all $r_1, r_2 \in R$, then R is commutative.*

Remark 2. We would like to point out that in case the automorphism α (associated with F and d) is the identity map on R, then d is merely an ordinary derivation and F is an ordinary generalized derivation of R. In this situation, by Theorem 1.2 we have that either R is commutative or $d(x) = bx - xb = 0$, for all $x \in R$. The last case is equivalent to $b \in Z(R)$ and drives us to a contradiction. Therefore in all that follows we consider the case when α is not the identity on R.

We begin with:

Lemma 3. *Let R be a dense subring of the ring of linear transformations of a vector space V over a division ring D and let R contain nonzero linear tranformations of finite rank. If $(a[r_1, r_2] + \alpha([r_1, r_2])b)^m = [r_1, r_2]^n$, for all $r_1, r_2 \in R$, then either R is commutative or R is a domain.*

Proof. Firstly we assume both R is not commutative and $dim_D V \geq 3$ and prove that a contradiction follows.
Since R is a primitive ring with non-zero socle, by [13] (p.79) there exists a semi-linear automorphism $T \in End(V)$ such that $\alpha(x) = TxT^{-1}$ for all $x \in R$, hence $(au + TuT^{-1}b)^m = u^n$, for all $u \in [R, R]$. Assume first that v and $T^{-1}bv$ are D-dependent for all $v \in V$. By Lemma 1 in [9], there exists $\lambda \in D$ such that $T^{-1}bv = v\lambda$, for all $v \in V$. In this case, for all $x \in R$,

$$d(x)v = (bx - TxT^{-1}b)v = bxv - TxT^{-1}bv = bxv - T(xv\lambda) =$$

$$bxv - T((xv)\lambda) = bxv - T(T^{-1}b)(xv) = bxv - bxv = 0.$$

This means that $d(x)V = (0)$, for all $x \in R$ and since V is faithful, it follows that $d(x) = 0$, for all $x \in R$, a contradiction.
Suppose now there exists $v \in V$ such that v and $T^{-1}bv$ are D-independent. Since

$dimV_D \geq 3$, then there exists $w \in V$ such that $v, T^{-1}bv, w$ are also D-independent. By the density of R, there exist $x, y \in R$ such that:

$$xv = v, \quad xT^{-1}bv = 0, \quad xw = T^{-1}v; \quad yv = v, \quad yT^{-1}bv = w.$$

These imply that $[x,y]v = 0$ and $(a[x,y] + T[x,y]T^{-1}b)^m v = (-1)^m v$, which leads to the contradiction $v = 0$.
Therefore $dim_D V \leq 2$. Once again we assume the conclusion of our Lemma does not hold in order to get a contradiction. That is, we suppose $dim_D V = 2$, which means that R is not a domain and it contains some non-trivial idempotent elements. Let $e^2 = e$ be an idempotent element of R. We recall that R satisfies

$$(a[x_1,x_2] + \alpha([x_1,x_2])b)^m = [x_1,x_2]^n. \tag{2.1}$$

In (2.1) we replace $[x_1,x_2]$ by $[\alpha^{-1}(1-e)x_1e, \alpha^{-1}(1-e)x_2e]$ and multiply on the right by $(1-e)$. Thus it follows that R satisfies

$$\Big([(1-e)\alpha(x_1)\alpha(e), (1-e)\alpha(x_2)\alpha(e)]b\Big)^m (1-e)$$

which is equivalent to

$$(1-e)\Big((\alpha(x_1)\alpha(e)(1-e)\alpha(x_2) - \alpha(x_2)\alpha(e)(1-e)\alpha(x_1))\alpha(e)b(1-e)\Big)^m. \tag{2.2}$$

By applying Remark 2.1 (1) in [24], we have that either $\alpha(e)(1-e) = 0$ or $\alpha(e)b(1-e) = 0$.
Suppose that there exists $e^2 = e$ such that $\alpha(e)(1-e) = 0$. In (2.1) replace $[x_1,x_2]$ by $[\alpha^{-1}(1-e)x_1e, x_2e]$ and multiply on the right by $(1-e)$. Therefore R satisfies

$$\Big((1-e)\alpha(x_1)\alpha(e)\alpha(x_2)\alpha(e)b\Big)^m (1-e)$$

that is R satisfies

$$\Big((\alpha(x_1)\alpha(e)(1-e)\alpha(x_2)\alpha(e)b(1-e)\Big)^{m+1}.$$

Hence by Levitzki's Lemma it follows $\alpha(e)R\alpha(e)b(1-e) = 0$, that is $\alpha(e)b(1-e) = 0$ in any case.
Analogously we may prove that $\alpha(1-e)be = 0$ for all $e^{=}e \in R$. This implies that $\alpha(e)b = \alpha(e)be = be$, for any idempotent element e of R. Let T be the additive subgroup generated by all idempotent elements in R, then it is easy to check that $\alpha(x)b = bx$, for all $x \in T$. Moreover, by [14] (page 18), it is well known that $[R,R] \subseteq T$, which implies $\alpha(x)b = bx$, for all $x \in [R,R]$. In this case, by [5] we get $F = 0$, and by our main assumption $[x_1,x_2]^n$ is satisfied by R. This last implies the contradiction R is commutative. $\square$

For the proof of our Proposition we premit the following:

Fact 4. Let R be a domain and $\alpha \in Aut(R)$ be an automorphism of R which is outer. In [17] Kharchenko proved that if $\Phi(x_i, \alpha(x_i))$ is a generalized polynomial identity for R, then R also satisfies the non-trivial generalized polynomial identity $\Phi(x_i, y_i)$, where x_i and y_i are distinct indeterminates.

2.1. **Proof of Proposition 1.** Suppose first that α is X-inner. Thus there exists an invertible element $q \in Q$ such that $\alpha(x) = qxq^{-1}$, for all $x \in R$. Thus $(au + quq^{-1}b)^m = u^n$, for all $u \in [R,R]$. Since R and Q satisfy the same generalized polynomial identities with coefficients in Q (see [7]), it follows that $(au+quq^{-1}b)^m = u^n$, for all $u \in [Q,Q]$. If $q^{-1}b \in C = Z(Q)$, then $d(x) = bx - bx = 0$, for all $x \in R$, a contradiction. So we may assume that $q^{-1}b \notin C$, and $(a[x_1,x_2] + q[x_1,x_2]q^{-1}b)^m - [x_1,x_2]^n$ is a non-trivial generalized polynomial identity for Q. By Martindale's theorem [22], Q is a primitive ring having non-zero socle with the field C as its associated division ring. Moreover Q is isomorphic to a dense subring of the ring of linear tranformations of a vector space V over C. By Lemma 3 $dim_C V = 1$, that is Q is commutative, as well as R.
Hence we may assume that α is X-outer. By Theorem 1 in [8], Q satisfies

$$(a[x_1,x_2] + \alpha([x_1,x_2])b)^m - [x_1,x_2]^n \tag{2.3}$$

moreover by Main Theorem in [8] Q is a GPI-ring. Thus Q is a primitive ring having non-zero socle and its associated division ring D is a finite-dimensional over C. By Lemma 3 $dim_D V = 1$, that is Q is a domain. In light of Fact 4 and by (2.3), Q satisfies

$$(a[x_1,x_2] + [y_1,y_2]b)^m - [x_1,x_2]^n$$

and in particular, for $x_1 = x_2 = 0$, $([r_1,r_2]b)^m = 0$ for all $r_1, r_2 \in Q$. By Lemma 1 in [5], either R is commutative, or $b = 0$ (which implies the contradiction $d = 0$).

3. The general case of regular generalized skew derivations in Prime Rings.

Here we can finally prove the main Theorem in case R is a prime ring and F is a regular generalized skew derivation of R. As remarked in the Introduction we can write $F(x) = bx + d(x)$ for all $x \in R$, $b \in Q_r$ and d is a non-zero skew derivation of R (see [4]). We also fix the following Fact which will be useful for our proof:

Fact 5. In [10] Chuang and Lee investigated polynomial identities with skew derivations. They proved that if $\Phi(x_i, D(x_i))$ is a generalized polynomial identity for R, where R is a prime ring and D in an outer skew derivation of R, then R also satisfies the generalized polynomial identity $\Phi(x_i, y_i)$, where x_i and y_i are distinct indeterminates. Furthermore, they proved [10, Theorem 1] that in the case $\Phi(x_i, D(x_i), \alpha(x_i))$ is a generalized polynomial identity for R, where R is a prime ring, D is an outer skew derivation of R and α is an outer automorphism of R, then R also satisfies the generalized polynomial identity $\Phi(x_i, y_i, z_i)$, where x_i, y_i, and z_i are distinct indeterminates.

Fact 6. By [10] (Theorem 1) we have the next result. If d is a non-zero skew-derivation of R and

$$\Phi\Big(x_1, \ldots, x_n, d(x_1), \ldots, d(x_n)\Big)$$

is a skew-differential identity of R, then one of the following statements holds:

(a) either d is inner ;
(b) or R satisfies the generalized polynomial identity

$$\Phi(x_1, \ldots, x_n, y_1, \ldots, y_n).$$

3.1. **Proof of Theorem 1.** By Theorem 2 in [10] R and Q satisfy the same generalized polynomial identities with a single skew derivation, then $F(u)^m = u^n$, for all $u \in [Q, Q]$. Suppose that d is X-inner, then there exist $b \in Q$ and $\alpha \in Aut(Q)$ such that $d(x) = bx - \alpha(x)b$, for all $x \in R$. In this case $F(x) = (a+b)x - \alpha(x)b$ and by Proposition 1 it follows that Q is commutative.
Assume finally that d is X-outer. Since Q satisfies

$$\big(a[x_1, x_2] + d([x_1, x_2])\big)^m - [x_1, x_2]^n \tag{3.1}$$

then Q satisfies

$$\big(a[x_1, x_2] + d(x_1)x_2 + \alpha(x_1)d(x_2) - d(x_2)x_1 - \alpha(x_2)d(x_1))\big)^m - [x_1, x_2]^n. \tag{3.2}$$

By Fact 6 and (3.2), Q satisfies

$$\big(a[x_1, x_2] + y_1x_2 + \alpha(x_1)y_2 - y_2x_1 - \alpha(x_2)y_1)\big)^m - [x_1, x_2]^n. \tag{3.3}$$

Moreover, in light of Fact 5, if α is an outer automorphism of Q, then by (3.3) it follows that Q satisfies

$$\big(a[x_1, x_2] + y_1x_2 + z_1y_2 - y_2x_1 - z_2y_1)\big)^m - [x_1, x_2]^n. \tag{3.4}$$

In particular, for $x_1 = x_2 = 0$, Q satisfies $(z_1y_2 - z_2y_1)^m$. Choose $z_1 = y_1$ and $z_2 = y_2$ so that $[y_1, y_2]^m$ is an identity for Q, which implies that Q is commutative.

Finally consider the case α is X-inner, then there exists an invertible element q of Q, such that $\alpha(x) = qxq^{-1}$, for all $x \in Q$. Starting from (3.3), Q satisfies the generalized identity

$$\big(a[x_1, x_2] + y_1x_2 + qx_1q^{-1}y_2 - y_2x_1 - qx_2q^{-1}y_1)\big)^m - [x_1, x_2]^n. \tag{3.5}$$

Since we assume α is not the identity map on Q, then $q \notin C$ and (3.5) is a non-trivial generalized polynomial identity for Q. By Martindale's theorem [22], Q is a primitive ring having non-zero socle with the field C as its associated division ring. Moreover Q is isomorphic to a dense subring of the ring of linear tranformations of a vector space V over C. Also in this case we prove that R must be commutative. Assume $dim_C V \geq 2$. Since $q \notin C$, there exists $v \in V$ such that $v, q^{-1}v$ are linearly C-independent. Moreover, by the density of Q, there exist $r_1, r_2, s_1, s_2 \in Q$, such that

$$r_1v = v, \quad r_2v = v, \quad s_1v = v, \quad s_2v = 0, \quad r_2q^{-1}v = 0.$$

Hence by (3.5) we get

$$0 = \left(\big(a[r_1, r_2] + s_1r_2 + qr_1q^{-1}s_2 - s_2r_1 - qr_2q^{-1}s_1)\big)^m - [x_1, x_2]^n\right)v = v$$

which is again a contradiction.

4. The Semiprime Case for regular generalized skew derivations.

In this second section we assume the following:

- Let R be a semiprime ring with center $Z(R)$, Q its right Martindale quotient ring, C its extended centroid, F a regular generalized skew derivation of R, associated with a non-zero skew derivation d of R, and $n, m \geq 1$ fixed integers such that $F([x, y])^m = [x, y]^n$, for all $x, y \in R$.

We will prove that R contains a non-zero central ideal.

Here we premit two results which will be useful in the sequel:

Fact 7. *Let R be prime ring and $\alpha \in Aut(R)$ be such that $xy - \alpha(y)x = 0$ for all $x, y \in I$, a non-zero ideal of R. Then R is commutative and α is the identity map on R.*

Proof. Let $x, y \in I$ and $r \in R$, then by our assumption $(xr)y = \alpha(y)xr = xyr$, that is $x[r, y] = 0$. Since $R[R, I] = (0)$ and by the primeness of R we easily conclude that R is commutative. Thus $xy = x\alpha(y)$ for all $x, y \in I$, which is $I(y - \alpha(y)) = (0)$. Once again by the primeness of R we have that α is the identity map on I. In particular, for all $r \in R$ and $x \in I$, it follows both $\alpha(xr) = xr$ and $\alpha(xr) = \alpha(x)\alpha(r) = x\alpha(r)$. Hence $x(\alpha(r) - r) = 0$, that is $I(\alpha(r) - r) = 0$ for any $r \in R$. Therefore we conclude that α is the identity on R. In light of this, we finally have $0 = xy - \alpha(y)x = [x, y]$, so that I is commutative, as well as R. □

Fact 8. *Let R be prime ring and $a, b \in R$ such that $([x, a]b)^m = 0$ for all $x \in R$. Then either $a \in Z(R)$ or $ab = ba = 0$.*

Proof. It is a consequence of Remark 2.1(1) in [24]. □

4.1. **The proof of Theorem 3.** Let P be a prime ideal of R, set $\overline{R} = R/P$ and write $\overline{x} = x + P \in \overline{R}$, for all $x \in R$. We start from

$$\overline{\Big(F([x, y])\Big)^m} = \overline{[x, y]^n}, \quad \forall x, y \in \overline{R}. \tag{4.1}$$

Let $x, y \in R$ and $p \in P$, then

$$\overline{\Big(F[xp, y])\Big)^m} = \overline{[px, y]^n}$$

that is

$$\overline{\Big(F(xpy - yxp)\Big)^m} = \overline{0}$$

which means

$$\overline{\Big(\alpha(x)d(p)y + \alpha(x)\alpha(p)d(y) - \alpha(y)\alpha(x)d(p)\Big)^m} = \overline{0} \tag{4.2}$$

We divide the proof into three cases:

CASE 1. $d(P) \subseteq P$.

In this case we may define $\overline{d} : \overline{R} \to \overline{R}$, such that $\overline{d}(\overline{x}) = \overline{d(x)}$, for all $\overline{x} \in \overline{R}$. From (4.2) we have

$$\overline{\Big(\alpha(x)\alpha(p)d(y)\Big)^m} = \overline{0}. \tag{4.3}$$

If $\alpha(P) \nsubseteq P$, then $\overline{\alpha(P)}$ is a non-zero ideal of $\overline{R}$. Hence, for $x \in P$ in (4.3) it follows $\overline{(\alpha(P)^2 d(R))^m} = \overline{0}$, which implies $d(R) \subseteq P$.
On the other hand, if $\alpha(P) \subseteq P$, then we define $\overline{\alpha} : \overline{R} \to \overline{R}$, such that $\overline{\alpha}(\overline{x}) = \overline{\alpha(x)}$, for all $\overline{x} \in \overline{R}$. Hence $\overline{d}$ is an α-derivation of $\overline{R}$ and $\overline{F}$ is a generalized α-derivation of $\overline{R}$. Therefore, by the prime case, either $\overline{d(R)} = \overline{0}$ or $\overline{R}$ is commutative, that is either $d(R) \subseteq P$ or $[R, R] \subseteq P$.

CASE 2. $d(P) \nsubseteq P$ *and* $\alpha(P) \subseteq P$.

Also in this case we may define $\overline{\alpha} : \overline{R} \to \overline{R}$, such that $\overline{\alpha}(\overline{x}) = \overline{\alpha(x)}$, for all $\overline{x} \in \overline{R}$. We also have that $\overline{d(P)}$ is a non-zero ideal of $\overline{R}$, moreover by (4.2) we get

$$\overline{\Big(\alpha(x)d(p)y - \alpha(y)\alpha(x)d(p)\Big)^m} = \overline{0} \tag{4.4}$$

that is

$$\overline{\Big(Xy - \alpha(y)X\Big)^m} = \overline{0}$$

for all $\overline{X} \in \overline{\alpha(R)d(P)}$, which is a non-zero ideal of $\overline{R}$. By [5] and using Fact 7, it follows $\overline{\alpha}$ is the identity map on $\overline{R}$ and $\overline{R}$ is commutative, that is $[R, R] \subseteq P$.

CASE 3. *$d(P) \nsubseteq P$ and $\alpha(P) \nsubseteq P$.*

In this case we remark that $\overline{d(P)}$ is a non zero left ideal of $\overline{P}$ and $\overline{\alpha(P)}$ is a non zero ideal of $\overline{R}$. Let $x \in P$, $r \in R$, then by (4.1) we have

$$\overline{\Big(F([\alpha^{-1}(r)x, x])\Big)^m} = \overline{[\alpha^{-1}(r)x, x]^n}$$

and by computations it follows

$$\overline{\Big([r, \alpha(x)]d(x)\Big)^m} = \overline{0}. \tag{4.5}$$

By Fact 8 we have that, for all $x \in P$, either $\overline{\alpha(x)} \in \overline{Z(R)}$ or $\overline{\alpha(x)d(x)} = \overline{d(x)\alpha(x)} = \overline{0}$.
Let $x \in P$ such that $\overline{0} \neq \overline{\alpha(x)} \notin \overline{Z(R)}$, so that $\overline{\alpha(x)d(x)} = \overline{d(x)\alpha(x)} = \overline{0}$. Thus, for all $r \in R$, $y \in P$, and by (4.1) we have

$$\overline{\Big(F([x, \alpha^{-1}(r)y])\Big)^m} = \overline{[x, \alpha^{-1}(r)y]^n}$$

that is

$$\overline{\Big(\alpha(x)rd(y) - r\alpha(y)d(x)\Big)^m} = \overline{0}. \tag{4.6}$$

Right multiplying by $\overline{\alpha(x)}$ it follows

$$\overline{\Big(\alpha(x)rd(y)\Big)^m \alpha(x)} = \overline{0}$$

which means

$$\overline{\Big(\alpha(x)Rd(P)\Big)^m} = \overline{0}.$$

Since $\overline{d(P)}$ is a non-zero left ideal of $\overline{R}$ and by Levitzki's Lemma, one has $\overline{\alpha(x)R} = \overline{0}$, that is $\overline{\alpha(x)} = \overline{0}$, a contradiction. Therefore $\overline{\alpha(x)} \in \overline{Z(R)}$, for all $x \in P$, so that $\overline{\alpha(P)} \in \overline{Z(R)}$. Since $\overline{\alpha(P)}$ is a non zero ideal of $\overline{R}$, we finally conclude that $\overline{R}$ is commutative, that is $[R, R] \subseteq P$.
In light of previous argument we have that, for any prime ideal P of R, either $d(R) \subseteq P$ or $[R, R] \subseteq P$, then both $d(R)[R, R] \subseteq \bigcap_i P_i = (0)$ and $[d(R), R] \subseteq \bigcap_i P_i = (0)$ (where P_i are all prime ideals of R). In particular, since $d(R) \neq 0$, the non-zero ideal generated by $d(R)$ is central in R, and we are done.

5. Analogous results for Centralizers.

In this final section we study similar previous conditions on prime and semiprime rings involving generalized α-derivations which are not regular, that is associated with a zero α-derivation. As remarked in the Introduction, in this case the generalized skew derivation F is called *(left) centralizer*, that is $F(xy) = F(x)y$, for all $x, y \in R$. Also here we firstly consider the case of prime rings and prove Theorem 2. Then we generalize the result to the semiprime case and prove Theorem 4.

We premit the following:

Lemma 9. *Let R be a semiprime ring and F a left centralizer of R such that $[F(x), x] = 0$, for all $x \in R$. Then either there exists $\lambda \in C$ such that $F(x) = \lambda x$ for all $x \in R$, or R contains a non-zero central ideal.*

Proof. By [3], there exist $\lambda \in C$ and $\vartheta : R \to C$ such that $F(x) = \lambda x + \vartheta(x)$, for all $x \in R$.
Assume there exists $x_0 \in R$ such that $\vartheta(x_0) \neq 0$. Thus $F(x_0) = \lambda x_0 + \vartheta(x_0)$ and right mulitplying by any $y \in R$ we have

$$F(x_0 y) = F(x_0)y = \lambda x_0 y + \vartheta(x_0)y. \tag{5.1}$$

On the other hand, for any $y \in R$ there exists $\vartheta(x_0 y) \in C$ such that

$$F(x_0 y) = \lambda x_0 y + \vartheta(x_0 y) \tag{5.2}$$

and comparing (5.1) with (5.2) it follows $\vartheta(x_0)y = \vartheta(x_0 y) \in C$, for all $y \in R$. Hence, by (5.1) we also get $(F(x_0) - \lambda x_0)y \in C$, for any $y \in R$. We notice that, in light of $\vartheta(x_0) \neq 0$, one has $(F(x_0) - \lambda x_0)R \neq (0)$, therefore there exists $y_0 \in R$ such that $0 \neq (F(x_0) - \lambda x_0)y_0 \in C$. In particular $R(F(x_0) - \lambda x_0)y_0)R \subseteq C$, that is the ideal generated by $(F(x_0) - \lambda x_0)y_0$ is central. □

5.1. **The proof of Theorem 2.** Since F is a (left) centralizer, there exists $b \in Q$ such that $F(x) = bx$, for all $x \in R$ and R satisfies the generalized polynomial identitiy $(b[x_1, x_2])^m - [x_1, x_2]^n$. By a theorem due to Beidar (Theorem 2 in [1]) this generalized polynomial identity is also satisfied by Q. In case C is infinite, we have $(b[r_1, r_2])^m - [r_1, r_2]^n = 0$ for all $r_1, r_2 \in Q \bigotimes_C \overline{C}$, where $\overline{C}$ is the algebraic closure of C. Since both Q and $Q \bigotimes_C \overline{C}$ are centrally closed ([12], Theorems 2.5 and 3.5], we may replace R by Q or $Q \bigotimes_C \overline{C}$ according as C is finite or infinite. Thus we may assume that R is centrally closed over C which is either finite or algebraically closed. By Martindale's theorem [22], R is a primitive ring having a non-zero socle H, with C as the associated division ring. Moreover eHe is a simple central algebra finite dimensional over C, for any minimal idempotent element $e \in RC$. We may assume H non-commutative, otherwise also R must be commutative. Notice that H satisfies $(b[x_1, x_2])^m - [x_1, x_2]^n)$ (see for example proof of Theorem 1 in [18]).
Since H is a simple ring then one of the following holds: either H does not contain any non-trivial idempotent element or H is generated by its idempotents.
In this last case, assume $e^2 = e \in H$. Choose any $r \in H$ and let $x_1 = er(1 - e)$ and $x_2 = 1 - e$. Hence $(ber(1 - e))^m = 0$, that is $((1 - e)ber)^{m+1} = 0$ implying $((1 - e)beR)^{m+1} = (0)$. By Levitzki's Lemma and primeness of R, it follows $(1 - e)be = 0$. This implies that, for any idempotent element e of rank 1, $(1 - e)be = 0$. Hence $[b, e] = 0$, for any idempotent of rank 1, and $[b, H] = 0$, since H is generated by these idempotent elements. This argument gives the contradiction that $b \in C$.

Therefore H cannot contain any non-trivial idempotent elements, then H is a finite dimensional division algebra over C and $b \in H = RC = Q$. If C is finite then H is a finite division ring, that is H is a commutative field and so R is commutative too.
If C is infinite then $H \otimes_C K \cong M_r(K)$, the ring of $r \times r$ matrices over K, where K is a splitting field of H. In this case, a Vandermonde determinant argument shows that $(b[x_1, x_2])^m - [x_1, x_2]^n$ is still an identity in $M_r(K)$. As above one can see that if $r \geq 2$ then b commutes with any idempotent element in $M_r(K)$. In any case we have the contradiction.
By the previous argument we may assume that $b \in C$. Thus H satisfies $b^m[x_1, x_2]^m - [x_1, x_2]^n$, which is a polynomial identity with coefficients in C. Hence H is a P.I.-ring, then there exists a field K such that H and $M_r(K)$ satisfies the same polynomial identities, in particular $M_r(K)$ satisfies $b^m[x_1, x_2]^m - [x_1, x_2]^n$. Fix i, j and denote e_{ij} the matrix unit with 1 in (i, j)-entry and zero elsewhere.
Let $[x_1, x_2] = e_{ii} - e_{jj}$. Notice that, if $r = 2$ and $char(K) = 2$, then $e_{ii} - e_{jj} = I_2$ is the identity matrix in $M_2(K)$ and it follows easily that $b^m = I_2$. Therefore we consider the case either $r \geq 3$ or $r = 2$ and $char(K) \neq 2$. In any case, $e_{ii} - e_{jj} \neq I_r$, the identity matrix in $M_r(K)$, and by computation we have $b^m(e_{ii} + (-1)^m e_{jj}) = e_{ii} + (-1)^n e_{jj}$. Right multiplying by e_{ii} we get $b^m e_{ii} = e_{ii}$. Repeating this process for any $i = 1, \dots, r$, we conclude that:

- any (i, j)-entry of the matrix b^m is zero, for $i \neq j$;
- any (i, i)-entry of the matrix b^m is 1

that is b^m is the identity matrix in $M_r(K)$. This mean $b^m x = x$, for all $x \in M_r(K)$, and as above remarked we also have $b^m x = x$, for all $x \in H$, as well as in Q and R. The proof is now complete.

We finally extend the previous Theorem to the semiprime case:

5.2. **The proof of Theorem 4.** Let now R be a semiprime ring and P be a prime ideal of R. As above, set $\overline{R} = R/P$ and write $\overline{x} = x + P \in \overline{R}$, for all $x \in R$. We start from

$$\overline{\left(F([x, y])\right)^m} = \overline{[x, y]^n}, \quad \forall x, y \in \overline{R}. \tag{5.3}$$

In particular for $y \in P$ and $x \in R$, we get $\overline{\left(F(y)x\right)^m} = \overline{0}$, that is $\overline{\left(F(P)R\right)^m} = \overline{0}$ and, by Levitzki's Lemma, $\overline{F(P)} = \overline{0}$, so that $\overline{F(P)} \subseteq P$. In this case we may define $\overline{F} : \overline{R} \to \overline{R}$, such that $\overline{F}(\overline{x}) = \overline{F(x)}$, for all $\overline{x} \in \overline{R}$. By Theorem 2, for all $\overline{R}$ we have:

(a) either $\overline{R}$ is commutative;
(b) or there exists $\mu \in C$ such that $\overline{F(x)} = \mu\overline{x}$, for all $x \in R$, and $u^m = u^n$ for all $u \in \overline{[R, R]}$.

In any case, for any prime ideal P and for all $x, y \in R$, $[x, y]^m - [x, y]^n \subseteq \bigcap_i P_i = (0)$, then $u^m = u^n$, for all $u \in [R, R]$.
Moreover, for all $x \in R$, $[F(x), x]R[R, R] \subseteq \bigcap_i P_i = (0)$, then $[F(x), x]R[F(x), x] = (0)$ and by the semiprimeness of R it follows $[F(x), x] = 0$ for all $x \in R$.
Thus by Lemma 9, either R contains a non-zero central ideal or there exist $\lambda \in C$

such that $F(x) = \lambda x$, for all $x \in R$. In the last case:

$$\overline{\left(\lambda([x,y])\right)^m} = \overline{[x,y]^n}, \quad \forall x, y \in \overline{R}.$$

Applying again Theorem 2 for any prime ring $\overline{R}$, we have:

(a) either $\overline{R}$ is commutative;

(b) or $\overline{\lambda^m x} = \overline{x^n}$, for all $\overline{x} \in \overline{R}$.

Thus, for any prime ideal P and for all $x \in R$, one has $(\lambda^m x - x)[R, R] \subseteq \bigcap_i P_i = (0)$. In case $(\lambda^m x - x) = 0$, for all $x \in R$, then λ^m is the identity element of R. On the other hand, in case there exists $x_0 \in R$ such that $(\lambda^m x_0 - x_0) \neq 0$, then by [23] (see Lemma 1.3), we conclude that $0 \neq (\lambda^m x_0 - x_0) \in Z(R)$, and R contains a non zero central ideal (it is the ideal generated by $\lambda^m x_0 - x_0$).

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