MINIMAL SUPERALGEBRAS GENERATING MINIMAL SUPERVARIETIES

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Abstract. It has been showed that in characteristic zero the generators of the minimal supervarieties of finite basic rank belong to the class of *minimal superalgebras* introduced by Giambruno and Zaicev in 2003. In the present paper the complete list of the minimal supervarieties generated by minimal superalgebras whose maximal semisimple homogeneous subalgebra is sum of three graded simple algebras is provided. As a consequence, we negatively answer the question of whether any minimal superalgebra generates a minimal supervariety.

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1. Introduction

Let F be a field of characteristic zero. An actual quantitative measure of the polynomial identities satisfied by an associative F-algebra A is given by the sequence of its codimensions $\{c_n(A)\}_{n\geq 1}$, whose n-th term is the dimension of the space of multilinear polynomials in n variables in the corresponding relatively free algebra of countable rank. It was introduced by Regev in the seminal paper [11], where it was proved that when A satisfies a non-zero polynomial identity (in the sequel we shall refer to these algebras as PI algebras) $\{c_n(A)\}_{n\geq 1}$ is exponentially bounded. Later a fundamental contribution of Giambruno and Zaicev ([6] and [7]) has showed that

$$\exp(A) := \lim_{m\to+\infty} \sqrt[m]{c_m(A)}$$

exists and is a non-negative integer, which is called the *exponent* of A.

This provides an integral scale allowing to measure the growth of any variety and in a natural manner has addressed the research towards a classification of varieties according to the asymptotic behaviour of their codimensions. Along this direction, among varieties of some fixed exponent a prominent role is played by the *minimal* ones, namely those varieties of exponent d such that every proper subvariety has exponent strictly less than d. In [8] it has been proved that a variety of exponential growth is minimal

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if, and only if, it is generated by the Grassmann envelope of a so called *minimal superalgebra*.

More in general, superalgebras are a key ingredient in the structure theory of PI algebras, as shown by Kemer in the solution of the Specht Problem ([10]). From his work also the relevance of their graded polynomial identities appears clear and this has deeply motivated their study. The point of view we are going to explore here consists into asking information about the set of graded identities of a F-algebra A endowed with a $\mathbb{Z}_2$-grading, which we denote by $T_{\mathbb{Z}_2}(A)$. From an algebraic point of view, it is a $T_{\mathbb{Z}_2}$-ideal of the free F-superalgebra $F\langle Y \cup Z\rangle$, namely a two-sided ideal of $F\langle Y \cup Z\rangle$ invariant under every graded endomorphism, which is completely determined by multilinear polynomials it contains (as we are working in characteristic zero). In particular, extending into this setting the approach of Regev, we are interested to the graded codimensions $\{c_n^{\mathbb{Z}_2}(A)\}_{n\geq 1}$ of A, whose n-th term is defined as the dimension of the space of multilinear $\mathbb{Z}_2$-graded polynomials in n variables in the corresponding relatively free $\mathbb{Z}_2$-graded algebra of countable rank.

In [5] it was proved that this sequence is exponentially bounded if, and only if, A is a PI algebra. Under the extra assumption that A is also finitely generated, in [1] the authors stated that

$$\exp_{\mathbb{Z}_2}(A) := \lim_{m\to+\infty} \sqrt[m]{c_m^{\mathbb{Z}_2}(A)}$$

exists and is a non-negative integer, which is called the $\mathbb{Z}_2$-*graded exponent* or *superexponent* of A.

By virtue of this result, as in the ordinary case, it becomes natural and interesting to investigate minimal varieties of PI associative superalgebras (or supervarieties) of *finite basic rank* (that is, generated by a finitely generated superalgebra satisfying an ordinary polynomial identity) of fixed graded exponent. The starting point for the problem we are going to focus in the present paper on is the following statement in which minimal superalgebras come again into the picture.

Theorem 1.1 (Proposition 3.2 of [4])**.** *Let $\mathcal{V}^{sup}$ be a supervariety of finite basic rank. If $\mathcal{V}^{sup}$ is minimal of superexponent $d \geq 2$, then $\mathcal{V}^{sup}$ is generated by a suitable minimal superalgebra.*

According to it, the complete characterization of minimal supervarieties of finite basic rank of exponential growth is reduced to decide whether any minimal superalgebra generates a minimal supervariety. This problem is still open and its possible solution seems to be more involved than that of the ungraded case. In more detail, a minimal superalgebra A is finite-dimensional and defined on an algebraically closed field. Hence, by the generalization of the Wedderburn-Malcev Theorem we can write $A = A_{ss} + J(A)$, where A_{ss} is a maximal semisimple subalgebra of A homogeneous in the $\mathbb{Z}_2$-grading and $J(A)$ is its Jacobson radical (which is homogeneous as well). Also A_{ss} can be written as the direct sum of graded simple algebras which can be of two types: either simple or non-simple as algebras. It has been proved that in the case in which the sequence of the graded simple components of A_{ss} has in some sense a regular distribution, the supervariety generated by A is minimal (Theorems 4.7 and 5.4 of [4] and 3.6 of [3]).

In spite of this positive result, in the present article we provide a family of minimal superalgebras not generating minimal supervarieties. This is done by characterizing all minimal supervarieties generated by minimal superalgebras whose maximal semisimple homogeneous subalgebra has three graded simple summands.

2. Preliminaries and Announcement of the Main Results

Throughout the rest of the paper, unless otherwise stated, F is a field of characteristic zero and all the algebras are assumed to be associative and to have the same ground field F. For any pair of positive integers s and t the symbol $M_{s\times t}$ means the space of all rectangular matrices with s rows and t columns over F and set $M_s := M_{s\times s}$; whereas, if $m_1, \ldots, m_n$ is a sequence of positive integers, let $UT(m_1, \ldots, m_n)$ be the upper block triangular matrix algebra of size $m_1, \ldots, m_n$. Finally, if $F\langle X\rangle$ is the free associative algebra on a countable set $X := \{x_1, x_2, \ldots\}$ over F, for any positive integer q the *Standard polynomial* in q variables $\mathrm{St}_q(x_1, \ldots, x_q)$ is the element of $F\langle X\rangle$ defined as

$$\sum_{\sigma\in S_q} \mathrm{sgn}(\sigma)x_{\sigma(1)}x_{\sigma(2)}\cdots x_{\sigma(q)}.$$

An algebra A is a $\mathbb{Z}_2$*-graded algebra* or a *superalgebra* if it has a vector space decomposition $A = A^{(0)} \oplus A^{(1)}$ such that $A^{(i)}A^{(j)} \subseteq A^{(i+j)}$. The elements of $A^{(0)}$ are called *homogeneous of degree* 0 and those of $A^{(1)}$ *homogeneous of degree* 1. An element w of A is *homogeneous* if it is homogeneous of degree 0 or 1 (and denote its degree by $|w|$), whereas a subalgebra or an ideal $V \subseteq A$ is *homogeneous* if $V = (V\cap A^{(0)})\oplus(V\cap A^{(1)})$. The superalgebra A is called *simple* (or $\mathbb{Z}_2$*-simple*) if the multiplication is non-trivial and it has no non-trivial homogeneous ideals. In this case, we shall also refer to A as a *graded simple* algebra.

Let $F\langle Y \cup Z\rangle$ be the free associative algebra on the disjoint countable sets of variables $Y := \{y_1, y_2, \ldots\}$ and $Z := \{z_1, z_2, \ldots\}$. It has a natural superalgebra structure if we require that the variables from Y have degree 0 and those from Z have degree 1. The superalgebra $F\langle Y\cup Z\rangle$ is said to be the *free superalgebra* over F. An element $f(y_1, \ldots, y_m, z_1, \ldots, z_n)$ of $F\langle Y \cup Z\rangle$ is a $\mathbb{Z}_2$*-graded polynomial identity* for a superalgebra $A = A^{(0)} \oplus A^{(1)}$ if $f(a_1, \ldots, a_m, b_1, \ldots, b_n) = 0_A$ for every $a_1, \ldots, a_m \in A^{(0)}$ and $b_1, \ldots, b_n \in A^{(1)}$. Given a $T_{\mathbb{Z}_2}$-ideal I of $F\langle Y \cup Z\rangle$, the *variety of superalgebras* or *supervariety* $\mathcal{V}^{sup}$ associated to I is the class of all F-superalgebras whose $T_{\mathbb{Z}_2}$-ideals of graded polynomial identities contain I. The $T_{\mathbb{Z}_2}$-ideal I is denoted by $T_{\mathbb{Z}_2}(\mathcal{V}^{sup})$. The supervariety $\mathcal{V}^{sup}$ is generated by the superalgebra A if $T_{\mathbb{Z}_2}(\mathcal{V}^{sup}) = T_{\mathbb{Z}_2}(A)$, and in this case we write $\mathcal{V}^{sup} = \mathrm{supvar}(A)$. Furthermore, set $\exp_{\mathbb{Z}_2}(\mathcal{V}^{sup}) := \exp_{\mathbb{Z}_2}(A) = \lim_{m\to+\infty} \sqrt[m]{c_m^{\mathbb{Z}_2}(A)}$, the *superexponent* of the supervariety $\mathcal{V}^{sup}$ (we recall that the *m-th* $\mathbb{Z}_2$*-graded codimension* $c_m^{\mathbb{Z}_2}(A)$ of A is the dimension of the vector space $\frac{P_m^{sup}}{P_m^{sup}\cap T_{\mathbb{Z}_2}(A)}$, where P_m^{sup} is the space of multilinear polynomials of degree m of $F\langle Y \cup Z\rangle$ in the variables $y_1, \ldots, y_m, z_1, \ldots, z_m$).

Assume that A is a finite-dimensional superalgebra and let $A = A_{ss} + J(A)$ be its Wedderburn-Malcev decomposition. Furthermore the maximal semisimple homogeneous subalgebra A_{ss} of A can be written as the direct sum of graded simple algebras whose structure is well known, at least when the ground field is algebraically closed. In fact, they are one of the following types:

(a) $M_{k,l} := \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, where $k \geq l \geq 0$, $k \neq 0$, $A \in M_k$, $D \in M_l$, $B \in M_{k\times l}$ and $C \in M_{l\times k}$, endowed with the grading $M_{k,l}^{(0)} := \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}$ and $M_{k,l}^{(1)} := \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}$;

(b) $M_m(F \oplus tF)$, where $t^2 = 1_F$, with grading (M_m, tM_m).

Giambruno and Zaicev in [8] introduced the definition of *minimal* superalgebra.

Definition 2.1. *Let F be an algebraically closed field. A superalgebra A is called* minimal *if it is finite-dimensional and $A = A_{ss} + J(A)$ where*

(i) *$A_{ss} = A_1 \oplus \cdots \oplus A_n$ with $A_1, \ldots, A_n$ graded simple algebras;*

(ii) *there exist homogeneous elements $w_{12}, \ldots, w_{n-1,n} \in J(A)$ and minimal homogeneous idempotents $e_1 \in A_1, \ldots, e_n \in A_n$ such that*

$$e_i w_{i,i+1} = w_{i,i+1} e_{i+1} = w_{i,i+1} \qquad 1 \leq i \leq n-1$$

and

$$w_{12}w_{23}\cdots w_{n-1,n} \neq 0_A;$$

(iii) *$w_{12}, \ldots, w_{n-1,n}$ generate $J(A)$ as a two-sided ideal of A.*

In Lemma 3.5 of [8] it was shown that the minimal superalgebra $A = A_{ss} + J(A)$ has the following vector space decomposition

$$A = \bigoplus_{1\leq i\leq j\leq n} A_{ij}, \tag{1}$$

where $A_{11} := A_1, \ldots, A_{nn} := A_n$ and, for all $i < j$,

$$A_{ij} := A_i w_{i,i+1} A_{i+1} \cdots A_{j-1} w_{j-1,j} A_j.$$

Moreover $J(A) = \oplus_{i<j} A_{ij}$ and $A_{ij}A_{kl} = \delta_{jk} A_{il}$, where δ_{jk} is the Kronecker delta. Finally, as stressed in Chapter 8 of [9], the order of the components $A_1, \ldots, A_n$ of A_{ss} is important. For this reason, in the sequel we shall tacitly agree that if $A_{ss} = A_1 \oplus \cdots \oplus A_n$, then $A_1 J(A) A_2 J(A) \cdots A_n \neq 0_A$. According to the main result of [1], $\exp_{\mathbb{Z}_2}(A) = \dim_F(A_{ss})$.

The aim of the paper is to contribute to the classification of minimal supervarieties of fixed graded exponent. We recall the definition.

Definition 2.2. *A variety $\mathcal{V}^{sup}$ of PI associative superalgebras is said to be* minimal of superexponent d *if $\exp_{\mathbb{Z}_2}(\mathcal{V}^{sup}) = d$ and $\exp_{\mathbb{Z}_2}(\mathcal{U}^{sup}) < d$ for every proper subvariety $\mathcal{U}^{sup}$ of $\mathcal{V}^{sup}$.*

As observed in the Introduction, in the case of finite basic rank the question which remains still open is to characterize those minimal superalgebras generating minimal supervarieties. Along this direction the main contribution can be summarized in the following

Theorem 2.3 (3.6 of [3])**.** *Let $A = A_{ss} + J(A)$ be a minimal superalgebra. If $A_{ss} = A_1 \oplus \cdots \oplus A_n$ and there exists $1 \leq h \leq n$ such that $A_1, \ldots, A_h$ are non-simple graded simple and $A_{h+1}, \ldots, A_n$ are simple graded simple algebras (or conversely), then the supervariety generated by A is minimal of superexponent $\dim_F(A_1 \oplus \cdots \oplus A_n)$.*

According to it, the possible smallest number of graded simple summands of the maximal semisimple homogeneous subalgebra of a minimal superalgebra A such that $\operatorname{supvar}(A)$ is not minimal is $n = 3$ (for the sake of completeness, we recall that the cases $n = 1$ and $n = 2$ were originally settled in Corollary 3.5 and Theorem 5.4 of [4], respectively). For this reason it becomes interesting to investigate what happens when $A_{ss} = A_1 \oplus A_2 \oplus A_3$. Obviously, by virtue of Theorem 2.3, for our aims nothing is to do when either A_2 and at least one between A_1 and A_3 are both simple graded simple or they are both non-simple graded simple. Hence the unique situations we remain to deal with are when:

- A_1 and A_3 are non-simple graded simple and A_2 is simple graded simple;
- A_1 and A_3 are simple graded simple and A_2 is non-simple graded simple.

We shall show that in all these cases A generates a minimal supervariety unless in the latter one. Indeed, the main result we prove, which is the core of the present paper and the starting point for further developments, shows that in such an event if the subspace A_{13} appearing in the decomposition (1), which is a non-zero (A_1, A_3)-bimodule, is not irreducible, then $\operatorname{supvar}(A)$ is not minimal except for one case, namely

Theorem 2.4. *Let $A = A_{ss} + J(A)$ be a minimal superalgebra such that $A_{ss} = A_1 \oplus A_2 \oplus A_3$ with*

$$A_1 = M_{k,l}, \qquad A_2 = M_m(F \oplus tF) \quad \text{and} \quad A_3 = M_{r,s}.$$

If A_{13} is not irreducible as a (A_1, A_3)-bimodule, then A generates a minimal supervariety (of superexponent $\dim_F(A_1 \oplus A_2 \oplus A_3)$) if, and only if, either $k = l$ or $r = s$.

3. The case in which A_1 and A_3 are non-simple graded simple

Assume throughout this Section that $A_1 = M_m(F \oplus tF)$, $A_2 = M_{k,l}$ and $A_3 = M_r(F \oplus sF)$ (where $t^2 = s^2 = 1_F$). We aim to show that any minimal superalgebra whose maximal semisimple homogeneous subalgebra coincides with $A_1 \oplus A_2 \oplus A_3$ generates a minimal supervariety. To this end we need to investigate in more details the structure of such a superalgebra: this is done via the language of actions of automorphisms. In fact, it is well known that any superalgebra A can be viewed as an algebra with action of an automorphism ϕ of A of order at most 2. Indeed, the homomorphism ϕ of $A = A^{(0)} \oplus A^{(1)}$ defined by $\phi(a_0) := a_0$ and $\phi(a_1) := -a_1$ for any $a_0 \in A^{(0)}$ and $a_1 \in A^{(1)}$ is an automorphism of A of order at most 2. Conversely, if A is an algebra with an automorphism ϕ of order at most 2, then, setting $A^{(0)} := \{a|\, a \in A,\ \phi(a) = a\}$ and $A^{(1)} := \{a|\, a \in A,\ \phi(a) = -a\}$, A is a superalgebra with grading $(A^{(0)}, A^{(1)})$.

Let $A = A_{ss} + J(A)$ be a minimal superalgebra such that $A_{ss} = A_1 \oplus A_2 \oplus A_3$. By regarding A as a ϕ-algebra, for $i \in \{1, 3\}$ we can write A_i as $A_i = I_i \oplus \phi(I_i)$, where I_i is a minimal two-sided ideal of A_i, and the corresponding homogeneous idempotents (of degree zero) e_i appearing in the Definition 2.1 as $e_i = \rho_i + \phi(\rho_i)$ with ρ_i a non-homogeneous minimal idempotent of I_i. For simplicity, set $\bar\rho_i := \phi(\rho_i)$ and $\bar I_i := \phi(I_i)$.

Let us consider the element $w_{13} := w_{12}w_{23}$ and the subspace A_{13} of the decomposition (1). As for the homogeneous radical elements $w_{j,j+1}$ defining A the equality

$$e_j w_{j,j+1} e_{j+1} = e_j w_{j,j+1} = w_{j,j+1} e_{j+1} = w_{j,j+1}$$

is satisfied, one has that

$$\begin{aligned} w_{13} &= (\rho_1 + \bar\rho_1) w_{12} w_{23} (\rho_3 + \bar\rho_3) \\ &= \rho_1 w_{12} w_{23} \rho_3 + \bar\rho_1 w_{12} w_{23} \bar\rho_3 + \bar\rho_1 w_{12} w_{23} \rho_3 + \rho_1 w_{12} w_{23} \bar\rho_3 \end{aligned}$$

and

$$A_{13} = A_1 w_{12} A_2 w_{23} A_3 = A_1 w_{12} e_2 A_2 e_2 w_{23} A_3 = A_1 w_{12} w_{23} A_3.$$

Thus

$$\begin{aligned} A_{13} =& I_1 \rho_1 w_{12} w_{23} \rho_3 I_3 \oplus \bar I_1 \bar\rho_1 w_{12} w_{23} \bar\rho_3 \bar I_3 \oplus \\ & I_1 \rho_1 w_{12} w_{23} \bar\rho_3 \bar I_3 \oplus \bar I_1 \bar\rho_1 w_{12} w_{23} \rho_3 I_3. \end{aligned}$$

As by the definition of minimal superalgebra $w_{13} \neq 0_A$, we deduce that at least one of the homogeneous summands $\rho_1 w_{12} w_{23} \rho_3 + \bar\rho_1 w_{12} w_{23} \bar\rho_3$ and $\bar\rho_1 w_{12} w_{23} \rho_3 + \rho_1 w_{12} w_{23} \bar\rho_3$ of w_{13} is non-zero. Suppose that just one of those is non-zero. In such an event we shall say also in the sequel that A_{13} is direct sum of two terms. In particular, if $\rho_1 w_{12} w_{23} \rho_3 + \bar\rho_1 w_{12} w_{23} \bar\rho_3 \neq 0_A$, then

$$A_{13} = I_1 \rho_1 w_{12} w_{23} \rho_3 I_3 \oplus \bar I_1 \bar\rho_1 w_{12} w_{23} \bar\rho_3 \bar I_3,$$

otherwise

$$A_{13} = I_1 \rho_1 w_{12} w_{23} \bar\rho_3 \bar I_3 \oplus \bar I_1 \bar\rho_1 w_{12} w_{23} \rho_3 I_3.$$

Set $\mathcal{I}_3 := I_3$ and $\epsilon_3 := \rho_3$ if the first case occurs and $\mathcal{I}_3 := \bar I_3$ and $\epsilon_3 := \bar\rho_3$ otherwise; moreover, put $\mathcal{I}_1 := I_1$ and $\epsilon_1 := \rho_1$ (and, as before, $\bar\epsilon_i := \phi(\epsilon_i)$ and $\bar{\mathcal{I}}_i := \phi(\mathcal{I}_i)$ for $i \in \{1, 3\}$), in any event we can write

$$A_{13} = \mathcal{I}_1 \epsilon_1 w_{12} w_{23} \epsilon_3 \mathcal{I}_3 \oplus \bar{\mathcal{I}}_1 \bar\epsilon_1 w_{12} w_{23} \bar\epsilon_3 \bar{\mathcal{I}}_3.$$

Furthermore let us define

$$v_{12} := \begin{cases} \epsilon_1 w_{12} + \bar\epsilon_1 w_{12} & \text{if } |w_{12}| = 0; \\ \epsilon_1 w_{12} - \bar\epsilon_1 w_{12} & \text{otherwise} \end{cases}$$

and

$$v_{23} := \begin{cases} w_{23}\epsilon_3 + w_{23}\bar{\epsilon}_3 & \text{if } |w_{23}| = 0; \\ w_{23}\epsilon_3 - w_{23}\bar{\epsilon}_3 & \text{otherwise.} \end{cases}$$

It is straightforward to check that the subalgebra of A generated by A_1, A_2, A_3 and the homogeneous elements v_{12} and v_{23} is a minimal superalgebra coinciding with A. Hence we can always assume that the radical elements generating $J(A)$ are of degree 0.

Claim. *There exists one isomorphism-type for a minimal superalgebra $A = (A_1 \oplus A_2 \oplus A_3) + J(A)$ such that A_{13} is direct sum of two non-zero terms.*

In fact, take another minimal superalgebra $B = B_{ss} + J(B)$ such that $B_{ss} = B_1 \oplus B_2 \oplus B_3$ with $B_j = A_j$ for every $1 \leq j \leq 3$ and B_{13} is the direct sum of two summands. Let us call z_{12} and z_{23} the homogeneous radical elements defining B (which we can suppose of degree zero) and let $f_j \in B_j$ be the minimal idempotents appearing in the Definition 2.1. Using the same above arguments one has that

$$B_{13} = \mathcal{J}_1\nu_1 z_{12}z_{23}\nu_3\mathcal{J}_3 \oplus \bar{\mathcal{J}}_1\bar{\nu}_1 z_{12}z_{23}\bar{\nu}_3\bar{\mathcal{J}}_3,$$

where, for $i \in \{1,3\}$, $B_i = \mathcal{J}_i \oplus \bar{\mathcal{J}}_i$, with $\mathcal{J}_i$ a minimal two-sided ideal of B_i, and ν_i is the non-homogeneous minimal idempotent of $\mathcal{J}_i$ such that $f_i = \nu_i + \bar{\nu}_i$ (here we are regarding B as an algebra with action of an automorphism of order 2, we call ϕ_B to distinguish it from that of A, and set $\bar{\mathcal{J}}_i := \phi_B(\mathcal{J}_i)$ and $\bar{\nu}_i := \phi_B(\nu_i)$).

For $1 \leq j \leq 3$, let us consider the superalgebras isomorphisms

$$\Psi_{jj} : A_j \longrightarrow B_j$$

such that $\Psi_{jj}(\epsilon_j) = \nu_j$ (and hence $\Psi_{jj}(\bar{\epsilon}_j) = \bar{\nu}_j$) if $j \neq 2$ and $\Psi_{22}(e_2) = f_2$. Since $\mathcal{I}_1\epsilon_1 \otimes e_2A_2$ is irreducible as a $(\mathcal{I}_1, A_2)$-bimodule, the map

$$\eta : \mathcal{I}_1\epsilon_1 \otimes e_2A_2 \longrightarrow \mathcal{I}_1\epsilon_1 v_{12}e_2A_2, \qquad a_1\epsilon_1 \otimes e_2a_2 \longmapsto a_1\epsilon_1 v_{12}e_2a_2$$

is a bimodules isomorphism. In analogous manner we define an isomorphism from $\mathcal{J}_1\nu_1 \otimes f_2B_2$ into $\mathcal{J}_1\nu_1 z_{12}f_2B_2$. On the other hand the action of the maps Ψ_{11} and Ψ_{22} on $\mathcal{I}_1e_1$ and e_2A_2 respectively induces an isomorphism from $\mathcal{I}_1\epsilon_1 \otimes e_2A_2$ into $\mathcal{J}_1\nu_1 \otimes f_2B_2$. The final outcome of these deduction is that there exists a vector spaces isomorphism

$$\psi_{12} : \mathcal{I}_1\epsilon_1 v_{12}e_2A_2 \longrightarrow \mathcal{J}_1\nu_1 z_{12}f_2B_2, \quad a_1\epsilon_1 v_{12}e_2a_2 \longmapsto \Psi_{11}(a_1)\nu_1 z_{12}f_2\Psi_{22}(b_2).$$

Now, as

$$A_{12} = \mathcal{I}_1\epsilon_1 v_{12}e_2A_2 \oplus \bar{\mathcal{I}}_1\bar{\epsilon}_1 v_{12}e_2A_2$$

and

$$B_{12} = \mathcal{J}_1\nu_1 z_{12}f_2B_2 \oplus \bar{\mathcal{J}}_1\bar{\nu}_1 z_{12}f_2B_2,$$

the map

$$\Psi_{12} : A_{12} \longrightarrow B_{12}, \qquad h + k \longmapsto \psi_{12}(h) + \overline{\psi_{12}(\bar{k})}$$

(where, obviously, $h \in \mathcal{I}_1\epsilon_1 v_{12}e_2A_2$, $k \in \bar{\mathcal{I}}_1\bar{\epsilon}_1 v_{12}e_2A_2$ and $\overline{\psi_{12}(\bar{k})} := \phi_B(\psi_{12}(\phi(k))))$ is a vector spaces isomorphism preserving the $\mathbb{Z}_2$-gradings.

The same argument yields that the map

$$\psi_{23} : A_2e_2v_{23}\epsilon_3\mathcal{I}_3 \longrightarrow B_2f_2z_{23}\nu_3\mathcal{J}_3, \quad a_2e_2v_{23}\epsilon_3a_3 \longmapsto \Psi_{22}(a_2)f_2z_{23}\nu_3\Psi_{33}(a_3)$$

induces a vector spaces isomorphism preserving the $\mathbb{Z}_2$-gradings, let us call Ψ_{23}, from $A_{23} = A_2 e_2 v_{23} \epsilon_3 \mathcal{I}_3 \oplus A_2 e_2 v_{23} \bar{\epsilon}_3 \bar{\mathcal{I}}_3$ into $B_{23} = B_2 f_2 z_{23} \nu_3 \mathcal{J}_3 \oplus B_2 f_2 z_{23} \bar{\nu}_3 \bar{\mathcal{J}}_3$.

Finally, the same conclusion holds for

$$\Psi_{13} : A_{13} \longrightarrow B_{13},$$

$$a_1 \epsilon_1 v_{12} v_{23} \epsilon_3 a_3 + a_1' \bar{\epsilon}_1 v_{12} v_{23} \bar{\epsilon}_3 a_3' \longmapsto \Psi_{11}(a_1) \nu_1 z_{12} z_{23} \nu_3 \Psi_{33}(a_3) + \Psi_{11}(a_1') \bar{\nu}_1 z_{12} z_{23} \bar{\nu}_3 \Psi_{33}(a_3').$$

But $A = \oplus_{1 \le i \le j \le 3} A_{ij}$ and $B = \oplus_{1 \le i \le j \le 3} B_{ij}$, hence, gluing the maps Ψ_{ij}, we have actually constructed a vector spaces isomorphism from A into B preserving the $\mathbb{Z}_2$-gradings, which is easily seen to be a superalgebras isomorphism.

If we drop the assumption on the decomposition of A_{13} we are able to show that non-isomorphic minimal superalgebras with the same semisimple part satisfy the same $\mathbb{Z}_2$-graded polynomial identities.

Theorem 3.1. *Let $A_1 = M_m(F \oplus tF)$, $A_2 = M_{k,l}$ and $A_3 = M_r(F \oplus sF)$ (where $t^2 = s^2 = 1_F$). Any minimal superalgebra whose maximal semisimple homogeneous subalgebra coincides with $A_1 \oplus A_2 \oplus A_3$ has the same $T_{\mathbb{Z}_2}$-ideal of graded polynomial identities.*

Proof. Let $A = A_{ss} + J(A)$ be a minimal superalgebra such that $A_{ss} = A_1 \oplus A_2 \oplus A_3$ and A_{13} is sum of four distinct terms, namely (using the same above notations)

$$\begin{aligned} A_{13} = & I_1 \rho_1 w_{12} w_{23} \rho_3 I_3 \oplus \bar{I}_1 \bar{\rho}_1 w_{12} w_{23} \bar{\rho}_3 \bar{I}_3 \oplus \\ & I_1 \rho_1 w_{12} w_{23} \bar{\rho}_3 \bar{I}_3 \oplus \bar{I}_1 \bar{\rho}_1 w_{12} w_{23} \rho_3 I_3. \end{aligned}$$

Set $H := I_1 \rho_1 w_{12} w_{23} \rho_3 I_3 \oplus \bar{I}_1 \bar{\rho}_1 w_{12} w_{23} \bar{\rho}_3 \bar{I}_3$, which is a two-sided homogeneous ideal of A, let us consider the superalgebra $A' := A/H$. We observe that its maximal semisimple subalgebra A'_{ss} coincides with A_{ss} and, as $H \subseteq J(A)$, its Jacobson radical $J(A')$ is equal to $J(A)/H$. As a consequence, the homogeneous elements $w_{12} + H$ and $w_{23} + H$ of A' generate $J(A')$. Furthermore

$$(w_{12} + H) \cdot (w_{23} + H) = w_{12} w_{23} + H \neq 0_{A'}$$

otherwise also $I_1 \rho_1 w_{12} w_{23} \bar{\rho}_3 \bar{I}_3 \oplus \bar{I}_1 \bar{\rho}_1 w_{12} w_{23} \rho_3 I_3$ should be in H, which contradicts the original assumption on A_{13}. Therefore we conclude that A' is a minimal superalgebra such that $A'_{13} = A_{13}/H$ is direct sum of two summands.

Now, take the homogeneous two-sided ideal $K := I_1 \rho_1 w_{12} w_{23} \bar{\rho}_3 \bar{I}_3 \oplus \bar{I}_1 \bar{\rho}_1 w_{12} w_{23} \rho_3 I_3$ of A. Proceeding in the same way, we obtain that $A'' := A/K$ is a minimal superalgebra such that $A''_{ss} = A_{ss}$ and A''_{13} is the direct sum of two non-zero terms. Thus, according to the claim, A' is isomorphic to A''.

Looking at the identities satisfied by these superalgebras, it is easily seen that

$$T_{\mathbb{Z}_2}(A) \subseteq T_{\mathbb{Z}_2}(A') = T_{\mathbb{Z}_2}(A''). \tag{2}$$

On the other hand, let $f \in F\langle Y \cup Z\rangle$ be a graded polynomial identity for A'. Since $T_{\mathbb{Z}_2}(A') = T_{\mathbb{Z}_2}(A'')$, for any graded evaluation $\mu : F\langle Y \cup Z\rangle \longrightarrow A$ one has that

$$\mu(f) \in H \cap K = 0_A.$$

Therefore f is a graded polynomial identity for A. Hence $T_{\mathbb{Z}_2}(A') \subseteq T_{\mathbb{Z}_2}(A)$ and, by virtue of (2), the equality holds. $\square$

Easy consequence of the above result is the following

Theorem 3.2. *Let $A = A_{ss} + J(A)$ be a minimal superalgebra such that $A_{ss} = A_1 \oplus A_2 \oplus A_3$ with*

$$A_1 = M_m(F \oplus tF), \qquad A_2 = M_{k,l} \quad \text{and} \quad M_r(F \oplus sF).$$

Then A generates a minimal supervariety of superexponent $\dim_F(A_1 \oplus A_2 \oplus A_3)$.

Proof. Set $\mathcal{V}^{sup} := \mathrm{supvar}(A)$ and let us consider a subvariety $\mathcal{U}^{sup} \subseteq \mathcal{V}^{sup}$ such that $\exp_{\mathbb{Z}_2}(\mathcal{V}^{sup}) = \exp_{\mathbb{Z}_2}(\mathcal{U}^{sup})$. Since $\mathcal{V}^{sup}$ satisfies some Capelli identities, $\mathcal{U}^{sup}$ has finite basic rank (see Theorem 11.4.3 of [9]). Hence, by a result of Kemer, $\mathcal{U}^{sup}$ is generated by a finite-dimensional superalgebra $\tilde{B}$. According to Lemma 8.1.4 of [9], there exists a minimal superalgebra B such that $T_{\mathbb{Z}_2}(\tilde{B}) \subseteq T_{\mathbb{Z}_2}(B)$ and $\exp_{\mathbb{Z}_2}(\tilde{B}) = \exp_{\mathbb{Z}_2}(B)$. Therefore $T_{\mathbb{Z}_2}(A) \subseteq T_{\mathbb{Z}_2}(B)$ and $\exp_{\mathbb{Z}_2}(A) = \exp_{\mathbb{Z}_2}(B)$ as well. Furthermore from Lemma 3.3 of [4] we know that $B_{ss} = A_1 \oplus A_2 \oplus A_3$.

At this point, Theorem 3.1 yields that $T_{\mathbb{Z}_2}(A) = T_{\mathbb{Z}_2}(B)$, and this concludes the proof. $\square$

4. The case in which A_1 and A_3 are simple graded simple

Throughout this Section let $A_1 = M_{k,l}$, $A_2 = M_m(F \oplus tF)$ and $A_3 = M_{r,s}$ and consider a minimal superalgebra A such that $A_{ss} = A_1 \oplus A_2 \oplus A_3$ (for the elements defining A we use the notations of Definition 2.1). As before, regarding A as a ϕ-algebra, write $A_2 = I_2 \oplus \phi(I_2)$, where I_2 is a minimal two-sided ideal of A_2, and its corresponding homogeneous idempotents (of degree zero) e_2 as $\rho_2 + \phi(\rho_2)$ with ρ_2 a non-homogeneous minimal idempotent of I_2. For simplicity, set $\bar{\rho}_2 := \phi(\rho_2)$ and $\bar{I}_2 := \phi(I_2)$. Using the usual arguments, one has that

$$A_{13} = A_1 w_{12} \rho_2 w_{23} A_3 + A_1 w_{12} \bar{\rho}_2 w_{23} A_3$$

is a (A_1, A_3)-bimodule such that each of its summands is an irreducible (A_1, A_3)-bimodule.

We make a preliminary observation.

Remark. *If the elements $w_{12}\rho_2 w_{23}$ and $w_{12}\bar{\rho}_2 w_{23}$ are linearly dependent, then they coincide.*

Proof. Assume that there exist $\alpha, \beta \in F \setminus \{0_F\}$ such that

$$\alpha w_{12} \rho_2 w_{23} + \beta w_{12} \bar{\rho}_2 w_{23} = 0_A.$$

Consequently

$$(-1)^{|w_{12}|+|w_{23}|}(\alpha w_{12} \bar{\rho}_2 w_{23} + \beta w_{12} \rho_2 w_{23}) = 0_A$$

as well. The combination of the above equalities yields

$$\begin{cases} \alpha w_{12}\rho_2 w_{23} + \beta w_{12}\bar{\rho}_2 w_{23} = 0_A; \\ \beta w_{12}\rho_2 w_{23} + \alpha w_{12}\bar{\rho}_2 w_{23} = 0_A. \end{cases}$$

Now, if $\alpha^2 - \beta^2 \neq 0_F$ then $w_{12}\rho_2 w_{23} = w_{12}\bar{\rho}_2 w_{23} = 0_A$, and hence

$$w_{12}w_{23} = w_{12}e_2 w_{23} = w_{12}(\rho_2 + \bar{\rho}_2)w_{23} = 0_A,$$

which is not allowed since, according to Definition 2.1, that element is non-zero. Thus suppose that $\alpha^2 = \beta^2$. If $\alpha = \beta$ one has again that $w_{12}w_{23} = 0_A$, which is not allowed. Therefore it must be $\alpha = -\beta$, and this implies that $w_{12}\rho_2 w_{23} = w_{12}\bar{\rho}_2 w_{23}$. □

Assume now that A_{13} is irreducible as a (A_1, A_3)-bimodule. Then

$$A_{13} = A_1 w_{12}\rho_2 w_{23} A_3 = A_1 w_{12}\bar{\rho}_2 w_{23} A_3.$$

This means that there exist an integer k and, for every $1 \leq i \leq k$, elements $a_i \in A_1$ and $b_i \in A_3$ such that $w_{12}\bar{\rho}_2 w_{23} = \sum_{i=1}^{k} a_i w_{12}\rho_2 w_{23} b_i$. It follows that

$$w_{12}\bar{\rho}_2 w_{23} = e_1 w_{12}\bar{\rho}_2 w_{23} e_3 = \sum_{i=1}^{k} e_1 a_i w_{12}\rho_2 w_{23} b_i e_3 = \sum_{i=1}^{k} e_1 a_i e_1 w_{12}\rho_2 w_{23} e_3 b_i e_3$$
$$= \sum_{i=1}^{k} \alpha_i e_1 w_{12}\rho_2 w_{23} \beta_i e_3 = \gamma w_{12}\rho_2 w_{23},$$

since $e_1 a_i e_1 = \alpha_i e_1$ and $e_3 b_i e_3 = \beta_i e_3$ for suitable $\alpha_i, \beta_i \in F$ and $\gamma := \sum_{i=1}^{k} \alpha_i \beta_i$ is in $F \setminus \{0_F\}$. By the above remark, we conclude that

$$w_{12}\rho_2 w_{23} = w_{12}\bar{\rho}_2 w_{23} \tag{3}$$

and it is a homogeneous element of degree $|w_{12}| + |w_{23}|$.

Before of proceeding, we construct two examples of minimal superalgebras belonging to the class we are considering. To this end, we recall that a $\mathbb{Z}_2$-grading on the complete matrix algebra M_n is called *elementary* if there exists a n-tuple $(g_1, \ldots, g_n) \in \mathbb{Z}_2^n$ such that the matrix units E_{ij} of M_n are homogeneous and $E_{ij} \in M_n^{(\tau)}$ if, and only if, $\tau = g_j - g_i$. In an equivalent manner, we can say that it is defined a map $|\,| : \{1, \ldots, n\} \longrightarrow \mathbb{Z}_2$ inducing a grading on M_n by setting the degree of E_{ij} equal to $|j| - |i|$. Obviously the algebra of upper block triangular matrices also admits elementary gradings. In fact, the embedding of such an algebra into a full matrix algebra with an elementary grading makes it a homogeneous subalgebra.

Now, let us consider the subalgebra of $UT(k + l, 2m, r + s)$ consisting of matrices of the form

$$\begin{pmatrix} C & J_1 & J_2 & J_3 \\ 0 & D & E & J_4 \\ 0 & E & D & J_5 \\ 0 & 0 & 0 & H \end{pmatrix},$$

where $C \in M_{k+l}$, $D, E \in M_m$, $H \in M_{r+s}$, $J_1, J_2 \in M_{(k+l)\times m}$, $J_3 \in M_{(k+l)\times(r+s)}$, $J_4, J_5 \in M_{m\times(r+s)}$. We endowe it with two gradings induced by the $(k+l+2m+r+s)$-tuples $(\underbrace{0,\dots,0}_{k\text{ times}},\underbrace{1,\dots,1}_{l\text{ times}},\underbrace{0,\dots,0}_{m\text{ times}},\underbrace{1,\dots,1}_{m\text{ times}},\underbrace{0,\dots,0}_{r\text{ times}},\underbrace{1,\dots,1}_{s\text{ times}})$ and $(\underbrace{0,\dots,0}_{k\text{ times}},\underbrace{1,\dots,1}_{l\text{ times}},\underbrace{0,\dots,0}_{m\text{ times}},\underbrace{1,\dots,1}_{m\text{ times}},\underbrace{1,\dots,1}_{r\text{ times}},\underbrace{0,\dots,0}_{s\text{ times}})$. Let us denote these superalgebras by $(\hat{A},|\,|_{\hat{A}})$ and $(\hat{B},|\,|_{\hat{B}})$ (and their matrix units by $E_{ij}^{(\hat{A})}$ and $E_{ij}^{(\hat{B})}$) respectively. It is easily seen that the maximal semisimple homogeneous subalgebra of $\hat{A}$ is equal to $\hat{A}_1\oplus\hat{A}_2\oplus\hat{A}_3$ where

$$\hat{A}_1:=\langle E_{ij}^{(\hat{A})}\,|\,1\le i,j\le k+l\rangle\cong M_{k,l},$$

$$\hat{A}_2:=\langle E_{ij}^{(\hat{A})}+E_{i+m,j+m}^{(\hat{A})},\,E_{pq}^{(\hat{A})}+E_{p+m,q-m}^{(\hat{A})}\,|\,k+l+1\le i,j,p\le k+l+m,\\ k+l+m+1\le q\le k+l+2m\rangle\cong M_m(F\oplus tF),$$

$$\hat{A}_3:=\langle E_{ij}^{(\hat{A})}\,|\,k+l+2m+1\le i,j\le k+l+2m+r+s\rangle\cong M_{r,s}$$

and its Jacobson radical is generated as a two-sided ideal by the homogeneous elements of degree zero $w_{12}^{(\hat{A})}:=E_{1,k+l+1}^{(\hat{A})}$ and $w_{23}^{(\hat{A})}:=E_{k+l+1,k+l+2m+1}^{(\hat{A})}$. Finally, since for $w_{12}^{(\hat{A})}$ and $w_{23}^{(\hat{A})}$ and the homogeneous (minimal) idempotents $e_1^{(\hat{A})}:=E_{11}^{(\hat{A})}\in\hat{A}_1$, $e_2^{(\hat{A})}:=E_{k+l+1,k+l+1}^{(\hat{A})}+E_{k+l+m+1,k+l+m+1}^{(\hat{A})}\in\hat{A}_2$ and $e_3^{(\hat{A})}:=E_{k+l+2m+1,k+l+2m+1}^{(\hat{A})}\in\hat{A}_3$ the relations appearing in Definition 2.1 are satisfied, we have that $\hat{A}$ is a minimal superalgebra. Moreover the subspace $\hat{A}_{13}$ is irreducible as a $(\hat{A}_1,\hat{A}_3)$-bimodule.

The same conclusion holds for the superalgebra $\hat{B}$, which has semisimple part $\hat{B}_{ss}=\hat{B}_1\oplus\hat{B}_2\oplus\hat{B}_3$ coinciding with that of $\hat{A}$ (for the elements defining $\hat{B}$ it is sufficient to replace the supscpript $(\hat{A})$ with $(\hat{B})$ and observe that, in this case, $w_{23}^{(\hat{B})}$ is homogeneous of degree 1).

Lemma 4.1. *If $k>l$ and $r>s$, for the minimal superalgebras $(\hat{A},|\,|_{\hat{A}})$ and $(\hat{B},|\,|_{\hat{B}})$ one has that $T_{\mathbb{Z}_2}(\hat{A})\not\subseteq T_{\mathbb{Z}_2}(\hat{B})$ and $T_{\mathbb{Z}_2}(\hat{A})\not\supseteq T_{\mathbb{Z}_2}(\hat{B})$. Consequently, $\hat{A}$ and $\hat{B}$ are not isomorphic as graded algebras.*

Proof. As a first step, we prove that $T_{\mathbb{Z}_2}(\hat{A})\not\subseteq T_{\mathbb{Z}_2}(\hat{B})$. To this end, let us consider the element of $F\langle Y\cup Z\rangle$

(4)

$$f:=\mathrm{St}_{2(m+k)-1}(y_1,\dots,y_{2(m+k)-1})z_1\,\mathrm{St}_{2(m+r)-1}(y_{2(m+k)},\dots,y_{2(2m+k+r-1)})$$

and observe that any non-zero graded evaluation of the Standard polynomials $\mathrm{St}_{2(m+k)-1}(y_1,\dots,y_{2(m+k)-1})$ and $\mathrm{St}_{2(m+r)-1}(y_{2(m+k)+1},\dots,y_{2(2m+k+r)-1})$ in $\hat{A}$ is in $J(\hat{A})\oplus\hat{A}_3$ and $\hat{A}_1\oplus J(\hat{A})$, respectively. Therefore any non-zero graded evaluation of f in $\hat{A}$ is in $J(\hat{A})^2$. In particular, it has to be a linear combination of the matrix units $E_{ij}^{(\hat{A})}$ with either $1\le i\le k$ and $k+l+2m+r+1\le j\le k+l+2m+r+s$ or $k+1\le i\le k+l$ and $k+l+2m+1\le j\le k+l+2m+r$. Now, take the polynomial

$$g:=\mathrm{St}_{2l}(\hat{y}_1,\dots,\hat{y}_{2l})f\,\mathrm{St}_{2s}(\hat{y}_{2l+1},\dots,\hat{y}_{2(s+l)}), \tag{5}$$

where $\hat{y}_1, \ldots, \hat{y}_{2(s+l)}$ are pairwise different variables of degree zero of $F\langle Y \cup Z\rangle$ not involved in f. Let $\mu : F\langle Y \cup Z\rangle \longrightarrow \hat{A}$ be a non-zero graded evaluation of g in $\hat{A}$. Since g is multilinear, for our aims we can assume that such an evaluation is made at a homogeneous basis of $\hat{A}$ including the matrix units $E_{ij}^{(\hat{A})}$ of $\hat{A}_1$ and $\hat{A}_3$. According to the above discussion, $\mu(\mathrm{St}_{2l}(\hat{y}_1, \ldots, \hat{y}_{2l}))$ must be in $\hat{A}_1$ and $\mu(\mathrm{St}_{2s}(\hat{y}_{2l+1}, \ldots, \hat{y}_{2(s+l)}))$ must be in $\hat{A}_3$. Taking in account the homogeneous degree of these factors and the original assumption that $k > l$ and $r > s$, the Amitsur-Levitzki Theorem yields that $\mu(\mathrm{St}_{2l}(\hat{y}_1, \ldots, \hat{y}_{2l}))$ is linear combination of the matrices $E_{\alpha\beta}^{(\hat{A})}$ and $\mu(\mathrm{St}_{2s}(\hat{y}_{2l+1}, \ldots, \hat{y}_{2(s+l)}))$ of the matrices $E_{pq}^{(\hat{A})}$, where $1 \leq \alpha, \beta \leq k$ and $k + l + 2m + 1 \leq p, q \leq k + l + 2m + r$. This fact combined with the previous observations on the graded evaluations of the polynomial f allows to conclude that g is an element of $T_{\mathbb{Z}_2}(\hat{A})$.

Finally, as $(\hat{B}_1 \oplus \hat{B}_{12} \oplus \hat{B}_2)^{(0)}$ contains a subalgebra isomorphic to $UT(k, m)$, for every $1 \leq i \leq k$ and $k+l+1 \leq j \leq k+l+m$ there exists a graded evaluation of $\mathrm{St}_{2(m+k)-1}(y_1, \ldots, y_{2(m+k)-1})$ in $\hat{B}$ equal to $E_{ij}^{(\hat{B})}$. Analogously, for every $k+l+m+1 \leq p \leq k+l+2m$ and $k+l+2m+1 \leq q \leq k+l+2m+r$ there is an evaluation of $\mathrm{St}_{2(m+r)-1}(y_{2(m+k)}, \ldots, y_{2(2m+k+r-1)})$ equal to $E_{pq}^{(\hat{B})}$. Thus, fixed integers i, j, p, q as above with the extra assumption that $i > 1$ if $k > 1$ and $q < k+l+2m+r$ if $r > 1$, since that in any case we can find an evaluation of $\mathrm{St}_{2l}(\hat{y}_1, \ldots, \hat{y}_{2l})$ equal to $E_{1i}^{(\hat{B})}$ and one of $\mathrm{St}_{2s}(\hat{y}_{2l+1}, \ldots, \hat{y}_{2(s+l)})$ equal to $E_{q,k+l+2m+r}^{(\hat{B})}$, evaluating the variable z_1 at $E_{jp}^{(\hat{B})} + E_{j+m,p-m}^{(\hat{B})}$ we have found a graded evaluation of the polynomial g in $\hat{B}$ equal to $E_{1,k+l+2m+r}^{(\hat{B})}$. Therefore g is not a graded polynomial identity for $\hat{B}$, and the desired conclusion holds.

On the other hand, the same arguments used above allow to conclude that the polynomial

$$\Gamma := \mathrm{St}_{2l}(\hat{y}_1, \ldots, \hat{y}_{2l})\delta\, \mathrm{St}_{2s}(\hat{y}_{2l+1}, \ldots, \hat{y}_{2(s+l)}),$$

where

$$\delta := \mathrm{St}_{2(m+k)-1}(y_1, \ldots, y_{2(m+k)-1})y_{2(m+k)}\, \mathrm{St}_{2(m+r)-1}(y_{2(m+k)+1}, \ldots, y_{2(2m+k+r)-1})$$

and $\hat{y}_1, \ldots, \hat{y}_{2(s+l)}$ are pairwise different elements of degree zero of $F\langle Y \cup Z\rangle$ not involved in δ, is in $T_{\mathbb{Z}_2}(\hat{B}) \setminus T_{\mathbb{Z}_2}(\hat{A})$, and this completes the proof. $\square$

We prove now that the graded algebras $\hat{A}$ and $\hat{B}$ are, up to isomorphisms, the unique elements of the class of minimal superalgebras we are dealing with (we continue to use the notations introduced at the beginning of the Section).

Lemma 4.2. *For a minimal superalgebra $A = (A_1 \oplus A_2 \oplus A_3) + J(A)$ such that $k > l$, $r > s$ and A_{13} is irreducible as a (A_1, A_3)-bimodule there exist two isomorphism-types (according to $|w_{12}| + |w_{23}| \pmod 2$).*

Proof. Let us consider the elements

$$u_{12} := w_{12}\rho_2 - w_{12}\bar{\rho}_2 \qquad \text{and} \qquad u_{23} := \rho_2 w_{23} - \bar{\rho}_2 w_{23}$$

of A. When $|w_{12}| = |w_{23}| = 1$, both of them are of degree 0 and, from the fact that $u_{12}u_{23} = w_{12}w_{23} \neq 0_A$, it is easily seen that the subalgebra of A generated by A_1, A_2 and A_3 and u_{12} and u_{23} is a minimal superalgebra coinciding with A. In the same manner, if $|w_{12}| = 1$ and $|w_{23}| = 0$, u_{12} has degree 0, whereas u_{23} has degree 1. In this case if we replace the elements w_{12} and w_{23} with u_{12} and u_{23} respectively, we also obtain the superalgebra A. Therefore we conclude that it is always possible to assume that $|w_{12}| = 0$, and hence we are left with two possibilities (according to $|w_{23}|$).

At this point, take a minimal superalgebra B with maximal semisimple homogeneous subalgebra $B_{ss} = B_1 \oplus B_2 \oplus B_3$ coinciding with A_{ss} and homogeneous radical elements z_{12} (which, as w_{12}, we can assume of degree zero) and z_{23} such that $|z_{23}| = |w_{23}|$ and B_{13} is irreducible as a (B_1, B_3)-bimodule. We aim to show that A and B are isomorphic as graded algebras. Now, for every $1 \leq j \leq 3$, call f_j the minimal idempotents (of degree zero) of B_j and write f_2 as $f_2 = \nu_2 + \bar{\nu}_2$, where ν_2 is the the non-homogeneous minimal idempotent of the minimal two-sided ideal $\mathcal{J}_2$ of B_2 such that $B_2 = \mathcal{J}_2 \oplus \bar{\mathcal{J}}_2$ (we are regarding B as an algebra with action of an automorphism ϕ_B of order 2 and setting $\bar{\mathcal{J}}_2 := \phi_B(J_2)$ and $\bar{\nu}_2 := \phi_B(\nu_2)$). Let us consider the superalgebras isomorphisms

$$\Psi_{jj} : A_j \longrightarrow B_j$$

such that $\Psi_{jj}(e_j) = f_j$ if $j \neq 2$ and $\Psi_{22}(\rho_2) = \nu_2$ (and hence $\Psi_{22}(\bar{\rho}_2) = \bar{\nu}_2$). Applying the same arguments of the previous Section, for every $1 \leq i < j \leq 3$ one constructs a vector space isomorphism Ψ_{ij} from the subspace A_{ij} of A into the subspace B_{ij} of B, which clearly preserves the $\mathbb{Z}_2$-grading when $(i, j) \neq (1, 3)$. For what concerns the latter case, for the map

$$\Psi_{13} : A_1 w_{12}\rho_2 w_{23} A_3 \longrightarrow B_1 z_{12}\nu_2 z_{23} B_3,\ a_1 w_{12}\rho_2 w_{23} a_3 \longmapsto \Psi_{11}(a_1) z_{12}\nu_2 z_{23}\Psi_{33}(a_3)$$

invoking (3) one has that

$$\begin{aligned}\Psi_{13}(\phi(w_{12}\rho_2 w_{23})) &= \Psi_{13}((-1)^{|w_{23}|} w_{12}\bar{\rho}_2 w_{23}) = \Psi_{13}((-1)^{|w_{23}|} w_{12}\rho_2 w_{23})\\ &= (-1)^{|z_{23}|} z_{12}\nu_2 z_{23} = (-1)^{|z_{23}|} z_{12}\bar{\nu}_2 z_{23}\\ &= \phi_B(z_{12}\nu_2 z_{23}) = \phi_B(\Psi_{13}(w_{12}\rho_2 w_{23})),\end{aligned}$$

from which it follows that the $\mathbb{Z}_2$-grading is still preserved.

Since $A = \oplus_{1\leq i\leq j\leq 3} A_{ij}$ and $B = \oplus_{1\leq i\leq j\leq 3} B_{ij}$, these maps induce a vector space isomorphism from A into B, which is easily verified (the details are left to the reader) to actually be a superalgebras isomorphism.

Therefore we are left with at most two isomorphism-types for the superalgebras we are considering. From the fact that the previously constructed minimal non-isomorphic superalgebras, $\hat{A}$ and $\hat{B}$, satisfy all the assumptions of the Lemma, the desired conclusion follows. $\square$

We are now in a position to state the first main result of this Section.

Theorem 4.3. *Let $A = A_{ss} + J(A)$ be a minimal superalgebra such that $A_{ss} = A_1 \oplus A_2 \oplus A_3$ with*

$$A_1 = M_{k,l}, \qquad A_2 = M_m(F \oplus tF) \quad \textit{and} \quad A_3 = M_{r,s}.$$

If A_{13} is irreducible as a (A_1, A_3)-bimodule, then A generates a minimal supervariety of superexponent $\dim_F(A_1 \oplus A_2 \oplus A_3)$.

Proof. Using verbatim the same arguments of the proof of Theorem 3.2, we reduce to considering a minimal superalgebra $B = B_{ss} + J(B)$ such that $B_{ss} = B_1 \oplus B_2 \oplus B_3$ with $B_i = A_i$ and homogeneous minimal idempotents $f_i \in B_i$ for every $1 \leq i \leq 3$, its Jacobson radical is generated by homogeneous elements z_{12} and z_{23} with $z_{12}z_{23} \neq 0_B$ and $T_{\mathbb{Z}_2}(A) \subseteq T_{\mathbb{Z}_2}(B)$. It is sufficient to show that $T_{\mathbb{Z}_2}(A) = T_{\mathbb{Z}_2}(B)$.

To this aim, we preliminarly observe that we can assume that B_{13} is irreducible as well. In fact, suppose that this is not the case. Hence, writing as usual f_2 as $\nu_2 + \bar{\nu}_2$, one has that

$$B_{13} = B_1 z_{12} \nu_2 z_{23} B_3 \oplus B_1 z_{12} \bar{\nu}_2 z_{23} B_3.$$

Let I be the ideal of B generated by $z_{12}\nu_2 z_{23} - z_{12}\bar{\nu}_2 z_{23}$, which is obviously homogeneous. Now, for the superalgebra $B' := B/I$ it is easily seen that its maximal semisimple homogeneous subalgebra coincides with B_{ss} and, since $I \subseteq B_{13}$, its Jacobson radical is equal to $J(B)/I$. Furthermore $(z_{12} + I) \cdot (z_{23} + I) \neq 0_{B'}$, since $z_{12}z_{23}$ is not in I. Therefore B' is a minimal superalgebra such that B'_{13} is irreducible and

$$T_{\mathbb{Z}_2}(B) \subseteq T_{\mathbb{Z}_2}(B').$$

As $T_{\mathbb{Z}_2}(A) \subseteq T_{\mathbb{Z}_2}(B)$, for our aims it is sufficient to replace the superalgebra B with B'.

If $k > l$ and $r > s$ Lemma 4.1 and 4.2 yield that A and B are isomorphic either to $\hat{A}$ or to $\hat{B}$. In particular, from Lemma 4.1 it follows that the containment $T_{\mathbb{Z}_2}(A) \subseteq T_{\mathbb{Z}_2}(B)$ implies that A is isomorphic to B as a graded algebra and, consequently, $T_{\mathbb{Z}_2}(A) = T_{\mathbb{Z}_2}(B)$.

Finally, assume that $k = l$ (analogous arguments can be used when $r = s$, and for this reason we avoid to discuss it). We aim to show that all the superalgebras satisfying these hypothesis have the same $T_{\mathbb{Z}_2}$-ideal of graded polynomial identities, namely $T_{\mathbb{Z}_2}(A_1) \cdot T_{\mathbb{Z}_2}(A_2) \cdot T_{\mathbb{Z}_2}(A_3)$ (we could actually strenghten the conclusion of Lemma 4.2 and prove that in this case there exists only one isomorphism-type of minimal superalgebra with the required properties, but we have preferred to adopt this other startegy since it will be useful in the sequel). We notice that, as the first part of the proof of Lemma 4.2 does not depend on the assumption on the pairs of integers (k, l) and (r, s), A and B are isomorphic either to $\hat{A}$ or to $\hat{B}$. In any case, both of these superalgebras can be written as

$$\begin{pmatrix} V & U \\ 0 & W \end{pmatrix},$$

where $V = M_{k,l}$, $U = M_{(k+l)\times(2m+r+s)}$ and $W \subseteq M_{2m+r+s}$ is the subalgebra of $\hat{A}$ ($\hat{B}$, respectively) generated by $\hat{A}_2, \hat{A}_3$ and $w_{23}^{(\hat{A})}$ ($\hat{B}_2, \hat{B}_3$ and $z_{23}^{(\hat{B})}$, respectively). Since $k = l$, from Proposition 5.3 of [2] we deduce that V is $\mathbb{Z}_2$-regular and Theorem 4.5 of [2] yields that the ideal of graded polynomial identities satisfied by this algebra is equal to $T_{\mathbb{Z}_2}(V) \cdot T_{\mathbb{Z}_2}(W) = T_{\mathbb{Z}_2}(A_1) \cdot T_{\mathbb{Z}_2}(W)$. But, according to the discussion of Section 2 of [3], in any event W is a minimal superalgebra with maximal semisimple homogeneous subalgebra coinciding with $A_2 \oplus A_3$. At this stage, from Theorem 5.3 of [4] one has that $T_{\mathbb{Z}_2}(W) = T_{\mathbb{Z}_2}(A_2) \cdot T_{\mathbb{Z}_2}(A_3)$, and this concludes the proof. $\square$

It remains to analyze what happens when the (A_1, A_3)-bimodule A_{13} is not irreducible, namely to prove the main result of this paper which is claimed in Theorem 2.4.

Proof of Theorem 2.4. Using the previously presented arguments, replacing the element w_{12} with $u_{12} := w_{12}\rho_2 - w_{12}\bar{\rho}_2$ if $|w_{12}| = 1$ and w_{23} with $u_{23} := \rho_2 w_{23} - \bar{\rho}_2 w_{23}$ again if $|w_{23}| = 1$, we can always assume that the radical elements of A appearing in Definition 2.1 have degree zero (we notice that, since A_{13} is not irreducible, we also have $u_{12}w_{23} \neq 0_A$ and $w_{12}u_{23} \neq 0_A$). Furthermore, the same lines of reasoning applied in the previous situations allow to conclude that there exists one isomorphism type for the minimal superalgebra A when A_{13} is not irreducible as a (A_1, A_3)-bimodule (the easy details are left to the reader).

Now, suppose first that either $k = l$ or $r = s$. We aim to show that $\mathrm{supvar}(A)$ is minimal. To this purpose, as in the proof of Theorem 3.2 take a minimal superalgebra $B = B_{ss} + J(B)$ such that $T_{\mathbb{Z}_2}(A) \subseteq T_{\mathbb{Z}_2}(B)$ and $B_{ss} = A_1 \oplus A_2 \oplus A_3$. We have to prove that A and B satisfy the same graded polynomial identities. Now, if B_{13} is not irreducible, by the above remark A is isomorphic to B as a graded algebra, and we are done. In the remaining case, in the proof of Theorem 4.3 we have established that

$$T_{\mathbb{Z}_2}(B) = T_{\mathbb{Z}_2}(A_1) \cdot T_{\mathbb{Z}_2}(A_2) \cdot T_{\mathbb{Z}_2}(A_3).$$

As the second term of the above equality is contained in $T_{\mathbb{Z}_2}(A)$, the desired conclusion holds.

Conversely, assume that $k > l$ and $r > s$. The final target is to construct a minimal superalgebra A' such that $T_{\mathbb{Z}_2}(A) \subsetneq T_{\mathbb{Z}_2}(A')$ and $\exp_{\mathbb{Z}_2}(A) = \exp_{\mathbb{Z}_2}(A')$. To this end, let I be the ideal of A generated by the element $w_{12}\rho_2 w_{23} - w_{12}\bar{\rho}_2 w_{23}$, which is clearly homogeneous, and set $A' := A/I$. Obviously,

$$T_{\mathbb{Z}_2}(A) \subseteq T_{\mathbb{Z}_2}(A').$$

As seen in the proof of Theorem 4.3 (in that case for the algebra B), A' is a minimal superalgebra with maximal semisimple homogeneous subalgebra equal to $A_1 \oplus A_2 \oplus A_3$. Furthermore, if ϕ' is the action induced by ϕ on A', one has that

$$\phi'(w_{12}\rho_2 w_{23} + I) = w_{12}\bar{\rho}_2 w_{23} + I = w_{12}\rho_2 w_{23} + I$$

(we have supposed that $|w_{12}| = |w_{23}| = 0$). This means that A'_{13} is irreducible. Therefore, A' is isomorphic to the superalgebra $\hat{A}$.

On the other hand, let us consider the subalgebra of $UT(2(k+l), 2m, 2(r+s))$ consisting of matrices of the form

$$\begin{pmatrix} K & 0 & I_1 & I_2 & I_3 & I_4 \\ 0 & K & I_2 & I_1 & I_4 & I_3 \\ 0 & 0 & L & P & I_5 & I_6 \\ 0 & 0 & P & L & I_6 & I_5 \\ 0 & 0 & 0 & 0 & Q & 0 \\ 0 & 0 & 0 & 0 & 0 & Q \end{pmatrix},$$

where $K \in M_{k+l},\ L, P \in M_m,\ Q \in M_{r+s},\ I_1, I_2 \in M_{(k+l)\times m},\ I_3, I_4 \in M_{(k+l)\times(r+s)},\ I_5, I_6 \in M_{m\times(r+s)}$. We endowe it with the grading induced by the $2(k+l+m+r+s)$-tuple

$$(\underbrace{0,\dots,0}_{k\,\text{times}},\underbrace{1,\dots,1}_{l\,\text{times}},\underbrace{1,\dots,1}_{k\,\text{times}},\underbrace{0,\dots,0}_{l\,\text{times}}\underbrace{0,\dots,0}_{m\,\text{times}},\underbrace{1,\dots,1}_{m\,\text{times}},\underbrace{0,\dots,0}_{r\,\text{times}},\underbrace{1,\dots,1}_{s\,\text{times}},\underbrace{1,\dots,1}_{r\,\text{times}},\underbrace{0,\dots,0}_{s\,\text{times}}),$$

which we denote by $\check{A}$. If $E_{ij}^{(\check{A})}$ are its matrix units, it is easily seen that the maximal semisimple subalgebra of $\check{A}$ is equal to $\check{A}_1 \oplus \check{A}_2 \oplus \check{A}_3$ where

$$\check{A}_1 := \langle E_{ij}^{(\check{A})} + E_{i+k+l,j+k+l}^{(\check{A})} \,|\, 1 \leq i, j \leq k+l \rangle \cong M_{k,l},$$

$$\check{A}_2 := \langle E_{ij}^{(\check{A})} + E_{i+m,j+m}^{(\check{A})},\, E_{pq}^{(\check{A})} + E_{p+m,q-m}^{(\check{A})} \,|\, 2(k+l)+1 \leq i, j, p \leq 2(k+l)+m,$$
$$2(k+l)+m+1 \leq q \leq 2(k+l+m)\rangle \cong M_m(F \oplus tF),$$

$$\check{A}_3 := \langle E_{ij}^{(\check{A})} + E_{i+r+s,j+r+s}^{(\check{A})} \,|\, 2(k+l+m)+1 \leq i, j \leq 2(k+l+m)+r+s \rangle \cong M_{r,s}$$

and its Jacobson radical is generated as a two-sided ideal by the homogeneous elements of degree zero $w_{12}^{(\check{A})} := E_{1,2(k+l)+1}^{(\check{A})} + E_{k+l+1,2(k+l)+m+1}^{(\check{A})}$ and $w_{23}^{(\check{A})} := E_{2(k+l)+1,2(k+l+m)+1}^{(\check{A})} + E_{2(k+l)+m+1,2(k+l+m)+r+s+1}^{(\check{A})}$. Finally, since for $w_{12}^{(\check{A})}$ and $w_{23}^{(\check{A})}$ and the homogeneous (minimal) idempotents $e_1^{(\check{A})} := E_{11}^{(\check{A})} + E_{k+l+1,k+l+1}^{(\check{A})} \in \check{A}_1$, $e_2^{(\check{A})} := E_{2(k+l)+1,2(k+l)+1}^{(\check{A})} + E_{2(k+l)+m+1,2(k+l)+m+1}^{(\check{A})} \in \check{A}_2$ and $e_3^{(\check{A})} := E_{2(k+l+m)+1,2(k+l+m)+1}^{(\check{A})} + E_{2(k+l+m)+r+s+1,2(k+l+m)+r+s+1}^{(\check{A})} \in \check{A}_3$ the relations appearing in Definition 2.1 are satisfied, we have that $\check{A}$ is a minimal superalgebra. Furthermore $\check{A}_{13}$ is not irreducible as $(\check{A}_1, \check{A}_3)$-bimodule. Hence from the uniqueness, up to isomorphism, of the superalgebra A we conclude that A is isomorphic to $\check{A}$.

At this stage, take the polynomials f and g defined in (4) and (5), respectively. We have shown there that $g \in T_{\mathbb{Z}_2}(\hat{A}) = T_{\mathbb{Z}_2}(A')$. We claim that it is not a graded polynomial identity for the superalgebra $\check{A}$, and hence for A. In fact, for every $1 \leq i \leq k$ and $2(k+l)+1 \leq j \leq 2(k+l)+m$ there exists a graded evaluation of $\mathrm{St}_{2(m+k)-1}(y_1, \dots, y_{2(m+k)-1})$ in $\check{A}$ equal to $E_{ij}^{(\check{A})} + E_{i+k+l,j+m}^{(\check{A})}$. Analogously, for every $2(k+l)+1 \leq p \leq 2(k+l)+m$ and $2(k+l+m)+1 \leq q \leq 2(k+l+m)+r$ there is an evaluation of $\mathrm{St}_{2(m+r)-1}(y_{2(m+k)}, \dots, y_{2(2m+k+r-1)})$ equal to $E_{pq}^{(\check{A})} + E_{p+m,q+r+s}^{(\check{A})}$. Thus, fixed integers i, j, p, q as above with the extra assumption that $i > 1$ if

$k > 1$ and $q < 2(k+l+m)+r$ if $r > 1$, evaluating the variable z_1 at $E^{(\check{A})}_{j+m,p} + E^{(\check{A})}_{j,p+m}$, we have found a graded evaluation of the polynomial f in $\check{A}$ equal to $E^{(\check{A})}_{i,q+r+s} + E^{(\check{A})}_{i+k+l,q}$. Since we can find in any event an evaluation of $\mathrm{St}_{2l}(\hat{y}_1, \ldots, \hat{y}_{2l})$ equal to $E^{(\check{A})}_{1i} + E^{(\check{A})}_{k+l+1,k+l+i}$ and one of $\mathrm{St}_{2s}(\hat{y}_{2l+1}, \ldots, \hat{y}_{2(s+l)})$ equal to $E^{(\check{A})}_{q,2(k+l+m)+r} + E^{(\check{A})}_{q+r+s,2(k+l+m+r)+s}$, the claim is confirmed. Therefore g is in $T_{\mathbb{Z}_2}(A') \setminus T_{\mathbb{Z}_2}(A)$, and this completes the proof. □

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