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Differential consequences of balance laws in extended irreversible thermodynamics of rigid heat conductors

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We consider a system of balance laws arising in extended irreversible thermodynamics of rigid heat conductors, together with its differential consequences, namely the higher-order system obtained by taking into account the time and space derivatives of the original system. We point out some mathematical properties of the differential consequences, with particular attention to the problem of the propagation of thermal perturbations with finite speed. We prove that, under an opportune choice of the initial conditions, a solution of the Cauchy problem for the system of differential consequences is also a solution of the Cauchy problem for the original system. We investigate the thermodynamic compatibility of the system at hand by applying a generalized Coleman–Noll procedure. On the example of a generalized Guyer–Krumhansl heat-transport model, we show that it is possible to get a hyperbolic system of evolution equations even when the state space is non-local.

1. Introduction

In extended irreversible thermodynamics (EIT) of rigid heat conductors [1–3], it is very frequent to consider the following system of balance laws

$$e_{,t} + q_{,x} = \sigma, \quad (1.1)$$

$$q_{,t} + \Phi_{,x} = r, \quad (1.2)$$

and
$$\Phi_{,t} + \Psi_{,x} = p, \quad (1.3)$$

where e , q and Φ are, respectively, the internal energy per unit volume, the heat flux and the flux of heat flux, σ , r and p are the productions of e , q and Φ , respectively, and Ψ is the flux of Φ . The system above represents the one-dimensional version of the 13-fields system of EIT [1,3,4], which, in turn, is directly amenable to Grad's 13-moments system [5]. The closure of equations (1.1)–(1.3) is achieved by assigning suitable constitutive equations for the flux Ψ and for the production terms σ , r and p . More details on the derivation of the system (1.1)–(1.3) and on its properties are given in the appendix.

Beside the system above, it is possible to consider the higher-order differential equations obtained by differentiating the previous system with respect to space and/or time. A classical example is represented by the Maxwell–Cattaneo–Vernotte system [6], namely

$$e_{,t} + q_{,x} = 0, \quad (1.4)$$

and

$$\tau q_{,t} + q = -kT_{,x}, \quad (1.5)$$

where τ is a relaxation time, k the thermal conductivity and T the temperature. Indeed, under the constitutive hypothesis $e = c_v T$, where c_v denotes the volumetric heat capacity, a governing equation for the temperature is obtained by deriving equation (1.4) with respect to time and equation (1.5) with respect to space. In this way, one obtains

$$c_v T_{,tt} + q_{,xt} = 0, \quad (1.6)$$

and

$$\tau q_{,tx} + q_{,x} = -kT_{,xx}. \quad (1.7)$$

The coupling of the previous equations with the balance of energy (1.4) yields the classical telegraph equation [6]

$$\tau T_{,tt} + T_{,t} - \frac{k}{c_v} T_{,xx} = 0. \quad (1.8)$$

We observe that in the system (1.4)–(1.5) only first-order derivatives appear, while in the system (1.6)–(1.7) the second-order time derivative of the temperature, and the second-order mixed derivative of the heat flux are present. At very short time scale, the second-order time derivative of temperature takes high values and is predominant with respect to the first-order time derivative. On the other hand, at very small length scale, the second-order mixed derivative of the heat flux becomes predominant with respect to the first-order time derivative. Such an observation suggests that the introduction of the time and space derivatives of the usual variables as new variables is important in the description of fast and short-wave phenomena, in which these derivatives may be considerably higher than the average value of the standard variables. Therefore, focusing the interest on the evolution of such derivatives as especially relevant variables of equations is very appealing from the physical perspective, and much more general than heat transport. For instance, in low-amplitude perturbations of temperature, T does not change very much, but the time and space derivatives of T may be considerably high and may be the central physical quantities of interest.

We note also that in deriving equation (1.8), the system (1.6)–(1.7) has been coupled with the local energy balance (1.4) (first law of thermodynamics), in order to be sure that the solutions of equation (1.8) satisfy such a fundamental physical law.

It is worth observing that the system (1.4)–(1.5) requires initial conditions for T and q , while the temperature equation (1.8) needs initial conditions for T and its time derivative. Such a second condition is, in general, assigned on physical grounds, and the obtained solution of (1.8) corresponds to the solution of system (1.4)–(1.5) which satisfies such an additional initial

condition. In this paper, we are aiming to discuss the possibility of substituting to the system (1.1)–(1.3) a new system constituted by its differential consequences. It reads

$$e_{,tt} + q_{,xt} = \sigma_{,t}, \quad (1.9)$$

$$q_{,tt} + \Phi_{,xt} = r_{,t}, \quad (1.10)$$

and
$$\Phi_{,tt} + \Psi_{,xt} = p_{,t}. \quad (1.11)$$

Then, the coupling of equations (1.9)–(1.11) and (1.1)–(1.3), yields

$$e_{,tt} - \Phi_{,xx} + r_{,x} = \sigma_{,t}, \quad (1.12)$$

$$q_{,tt} - \Psi_{,xx} + p_{,x} = r_{,t}, \quad (1.13)$$

and
$$\Phi_{,tt} + \Psi_{,xt} = p_{,t}. \quad (1.14)$$

In what follows we will focus our attention on the system (1.12)–(1.14). One of the possible advantages following by the replacement of the system (1.1)–(1.3) with the system (1.12)–(1.14) is that this last may contain some higher-order derivatives which are negligible, and hence its mathematical analysis is simplified with respect to that of the original system. Such a statement will be clarified in §3 on the example of the classical Guyer–Krumhansl [7] heat-transport equation.

As far as the physical interpretation of the solutions of the system (1.12)–(1.14) is concerned, it is worth observing that if one proves that such a system is compatible with the second law of thermodynamics [3], then its solutions, in principle, may represent real processes. On the other hand, there could be some solutions of system (1.12)–(1.14) which obey the second law of thermodynamics but do not occur really in nature, perhaps because they do not meet some other fundamental physical properties such as, Galilean invariance or causality [3]. However, some of these processes could not be described by the system (1.1)–(1.3). In fact, a solution of (1.1)–(1.3) is a solution of (1.12)–(1.14) too. This can be easily seen by direct substitution of the time and space derivatives of such solution into the system (1.12)–(1.14). On the other hand, a solution of (1.12)–(1.14) does not necessarily satisfy the system (1.1)–(1.3). This can be seen by observing that a solution of the Cauchy problem for the system (1.1)–(1.3) depends on a single integration constant, while a solution of the Cauchy problem for the system (1.12)–(1.14) depends on two integration constants. Thus, the system (1.12)–(1.14) admits a more general set of solutions, which contains the solutions of (1.1)–(1.3) as a subset.

However, it is possible to prove the following

Theorem 1.1. *Let $\mathbf{u} = (e, q, \Phi)$ and $\mathbf{v} = (e, q, \Phi)$ denote the solutions of the systems (1.1)–(1.3) and (1.12)–(1.14), respectively. Moreover, let*

$$\mathbf{u}(x, 0) = \mathbf{u}_0(x), \quad (1.15)$$

an initial condition for the system (1.1)–(1.3), and

$$\mathbf{v}(x, 0) = \mathbf{u}_0(x), \quad \mathbf{v}_{,t}(x, 0) = \mathbf{V}_0(x), \quad (1.16)$$

an initial condition for the system (1.12)–(1.14), where $\mathbf{V}_0(x)$ is the value assumed by $\mathbf{u}_{,t}(x, t)$ at $t = 0$. Then, the solution $\mathbf{v}(x, t)$ of the Cauchy problem (1.12)–(1.14) with the initial condition (1.16) is also solution of the system (1.1)–(1.3) with initial condition (1.15).

Proof. Let's suppose that both systems (1.1)–(1.3) and (1.12)–(1.14) have the necessary regularity for the validity of the Cauchy–Kovaleskaya theorem [8]. Then, the solution $\mathbf{u}(x, t)$ of (1.1)–(1.3) satisfying the initial condition (1.15) is unique. By time derivation of $\mathbf{u}(x, t)$ we can determine the initial condition (1.16) for (1.12)–(1.14) by evaluating such a derivative at $t = 0$. Then, $\mathbf{u}(x, t)$ is also a solution of the Cauchy problem for the system (1.12)–(1.14) with initial condition (1.16). By the Cauchy–Kovaleskaya theorem, it follows that such a solution is unique. This proves that any solution of a given Cauchy problem for the system (1.12)–(1.14) with initial condition (1.16), is also a solution of the system (1.1)–(1.3) with initial condition (1.15). ■

Example 1.2. In order to illustrate the previous result by a concrete example, let's consider the following differential equation

$$u_{,t} = at, \quad (1.17)$$

where a is a constant. Such a problem admits the general solution

$$u(t) = a\frac{t^2}{2} + C, \quad (1.18)$$

where the integration constant C can be determined by a suitable initial condition of the type

$$u(0) = u_0. \quad (1.19)$$

Let's consider now the following differential consequence of equation (1.17)

$$u_{,tt} = a, \quad (1.20)$$

whose general integral is

$$u(t) = a\frac{t^2}{2} + C_1t + C_2. \quad (1.21)$$

Thus, under the following initial conditions

$$u(0) = u_0, \quad u_{,t}(0) = 0, \quad (1.22)$$

the solution (1.18) of equation (1.17), and the solution (1.21) of its differential consequence (1.20), coincide. If, instead, we consider the initial conditions

$$u(0) = u_0, \quad u_{,t}(0) = v_0, \quad v_0 \neq 0. \quad (1.23)$$

Equation (1.20) and the initial conditions (1.23) lead to the solution

$$u(t) = a\frac{t^2}{2} + v_0t + u_0, \quad (1.24)$$

which does not satisfy equation (1.17) with the initial condition (1.19).

Remark 1.3. Let's suppose that u and u_0 are displacements, a is an acceleration, v_0 is a velocity and t is the time. Moreover, let's suppose that the order of magnitude of a is 10^{-2} m s^{-2} , the order of magnitude of v_0 is 10^{-8} m s^{-1} and the order of magnitude of u_0 is 10^{-14} m . Then, at the length scale of the order of m and at the time scale of the order of s , the first term in equations (1.18) and (1.24) is preponderant with respect to the other ones, which can be neglected, and then the solutions of (1.17) and (1.20) coincide. In such a case, the initial conditions (1.23) reduce to

$$u(0) = 0, \quad u_{,t}(0) = 0, \quad (1.25)$$

which are satisfied also by the solution of (1.17) with initial condition $u(0) = 0$. Let's choose now a smaller scale for space and time with respect to the previous case, for instance, 10^{-14} m and 10^{-6} s . In such a case, neither u_0 nor v_0 are negligible. Moreover, all the terms in equations (1.18) and (1.24) have the same order of magnitude, namely 10^{-14} m , and none of them is negligible. Thus, the solutions of (1.17) and (1.20) are different.

The considerations above justify the hypothesis that the solutions of the system (1.12)–(1.14) which are not solutions of the system (1.1)–(1.3) could describe fast and short-wavelength phenomena, which occur at very short time and space scale.

In a previous paper [9], the mathematical properties of a particular system of differential consequences arising in weakly non-local thermodynamics have been studied. Following the way paved in [9], in this paper, we study the properties of the system (1.12)–(1.14) connected with the speeds of propagation of its solutions.

We prove that this equation can be hyperbolic, leading so to finite speeds of propagation of thermal disturbances, even if the constitutive equations are non-local, namely they depend on the gradients of the unknown fields.

Indeed, the propagation of thermomechanical perturbations with finite speed is a hot topic in continuum thermodynamics, which in the past decades led to a new analysis of the classical tenets of rational thermodynamics (RT) [3], and to the foundation of two new thermodynamic theories [2,3], namely, extended irreversible thermodynamics [1], and rational extended thermodynamics (RET) [10]. The fundamentals of those theories are illustrated in detail in the appendix.

Heat pulses have been detected in several systems in different situations. Thermal effects coupled with mechanical effects are present in second sound propagation in dielectric crystals at low temperature [11–13]. For instance, Jackson & Walker [12] detected heat waves in sodium fluoride (NaF) in the range from 12 K to 15 K, and observed:

- a fastest peak, travelling with the speed of a longitudinal sound wave, independent of temperature;
- a next peak, also independent of temperature, travelling with the speed of a transversal sound wave; and
- a third peak, that occurs only at 13 K and 14.5 K, which can be interpreted as a temperature wave.

Similar experimental results have been obtained by Narayanamurti & Dynes [13] in Bismuth (Bi) in the range from 3 K to 5 K.

Recently, heat pulse experiments have been carried out in further experiments. For instance, in [14] two experimental setups are used to evidence effects beyond Fourier law. In [15], the non-Fourier heat conduction experiment in heterogeneous materials at room temperature is carried out. The results seem to prove that the Guyer–Krumhansl equation can be the appropriate extension of Fourier law for room temperature phenomena in modelling of heterogeneous materials.

Heat pulses have also been detected at nanoscale [16], since the very short length allows the thermal disturbances to reach any point of the system before than the heat flux relaxes to its equilibrium value. Currently, high-frequency thermal waves received a new impetus with the analysis of thermal transport in graphene, which has an exceptionally large heat conductivity and a long relaxation time of heat flux [17].

A further problem which is worth to be investigated is the form of the constitutive equations of the entropy and of the entropy flux. To achieve that task, we exploit the second law of thermodynamics, proving that the entropy is local and depends on the dissipative fluxes too, while the entropy flux is non-local.

The paper has the following layout.

In §2, we exploit the second law of thermodynamics in order to show that the system (1.12)–(1.14) is physically admissible and that its solutions can represent, in principle, real processes. [3]. To this end, we apply a generalized Coleman–Noll procedure [18,19], which is based on the substitution in the entropy inequality of the differential consequences of the basic balance laws. In this way, suitable constitutive equations for the entropy and for the entropy flux are obtained.

In §3, we consider the system formed by the local balances of energy and heat flux, with non-local constitutive equations. We show that such a system is hyperbolic, i.e. it allows the propagation of thermal perturbations with finite speed [10].

In §4, a discussion of the results obtained in §§2 and 3 is carried out. Possible future developments of the theory are examined as well.

Finally, in the appendix, the mathematical structure of two recent thermodynamic theories, i.e. EIT and RET, is illustrated in detail. Their connection with Grad's approximation of the Boltzmann equation is pointed out.

2. Compatibility with second law of thermodynamics

Nowadays, in the theory of heat conduction, there is a current interest for mesoscopic modelization, based on generalized heat-transport equations [4,20–23] which is simpler than the much more complex and detailed microscopic approach. Mesoscopic modelization applies different schemes and procedures, which derive by different thermodynamic theories [2,3] such as, for example, EIT, a modern thermodynamic theory in which both the dissipative fluxes and their gradients are regarded as non-equilibrium state variables [1]. Within such a framework, the system (1.1)–(1.3) is closed by assigning suitable constitutive equations for the last flux Ψ and for the productions σ , r and p . According to this point of view, the previous constitutive quantities are defined on the following state space

$$Z = \{e, e_{,x}, q, q_{,x}, \Phi, \Phi_{,x}\}. \quad (2.1)$$

Assumption 2.1. We calculate the system of differential consequences on Z under the hypothesis that the products of the partial derivatives of the unknown fields, as well as the terms in the third spatial derivatives of the unknown fields, are negligible.

Due to the assumption above, on Z the system of differential consequences (1.12)–(1.14) reads

$$e_{,tt} = \Phi_{,xx} - r_{,x} - \frac{\partial \sigma}{\partial e} q_{,x} + \frac{\partial \sigma}{\partial e} \sigma + \frac{\partial \sigma}{\partial e_{,x}} (-q_{,xx} + \sigma_{,x}) + \frac{\partial \sigma}{\partial q} (r - \Phi_{,x}) + \frac{\partial \sigma}{\partial q_{,x}} (r_{,x} - \Phi_{,xx}) + \frac{\partial \sigma}{\partial \Phi} (p - \Psi_{,x}) + \frac{\partial \sigma}{\partial \Phi_{,x}} \left(p_{,x} - \frac{\partial \Psi}{\partial e} e_{,xx} - \frac{\partial \Psi}{\partial q} q_{,xx} - \frac{\partial \Psi}{\partial \Phi} \Phi_{,xx} \right), \quad (2.2)$$

$$q_{,tt} = \frac{\partial \Psi}{\partial e} e_{,xx} + \frac{\partial \Psi}{\partial q} q_{,xx} + \frac{\partial \Psi}{\partial \Phi} \Phi_{,xx} - p_{,x} - \frac{\partial r}{\partial e} q_{,x} + \frac{\partial r}{\partial e} \sigma + \frac{\partial r}{\partial e_{,x}} (-q_{,xx} + \sigma_{,x}) + \frac{\partial r}{\partial q} (r - \Phi_{,x}) + \frac{\partial r}{\partial q_{,x}} (r_{,x} - \Phi_{,xx}) + \frac{\partial r}{\partial \Phi} (p - \Psi_{,x}) + \frac{\partial r}{\partial \Phi_{,x}} \left(p_{,x} - \frac{\partial \Psi}{\partial e} e_{,xx} - \frac{\partial \Psi}{\partial q} q_{,xx} - \frac{\partial \Psi}{\partial \Phi} \Phi_{,xx} \right), \quad (2.3)$$

$$\text{and } \Phi_{,tt} = -\frac{\partial \Psi}{\partial e} (-q_{,xx} + \sigma_{,x}) - \frac{\partial \Psi}{\partial q} (-\Phi_{,xx} + r_{,x}) + \frac{\partial \Psi}{\partial \Phi} \left(\frac{\partial \Psi}{\partial e} e_{,xx} + \frac{\partial \Psi}{\partial q} q_{,xx} + \frac{\partial \Psi}{\partial \Phi} \Phi_{,xx} + p_{,x} \right) - \frac{\partial p}{\partial e} (q_{,x} + \sigma) + \frac{\partial p}{\partial e_{,x}} (-q_{,xx} + \sigma_{,x}) + \frac{\partial p}{\partial q} (r - \Phi_{,x}) + \frac{\partial p}{\partial q_{,x}} (r_{,x} - \Phi_{,xx}) + \frac{\partial p}{\partial \Phi} (p - \Psi_{,x}) + \frac{\partial p}{\partial \Phi_{,x}} \left(p_{,x} - \frac{\partial \Psi}{\partial e} e_{,xx} - \frac{\partial \Psi}{\partial q} q_{,xx} - \frac{\partial \Psi}{\partial \Phi} \Phi_{,xx} \right), \quad (2.4)$$

where, for the sake of brevity, we have not explicitated the first spatial derivatives of σ , r , p , and Ψ .

Theorem 2.2. The vector $\mathbf{u} = (e, q, \Phi)$ is solution of the system (2.2)–(2.4) if, and only if, it is a solution of the system (1.1)–(1.3), too.

Proof. Firstly, we observe that the system (2.2)–(2.4) is nothing but that the system (1.12)–(1.14) on the state space Z . Thus, since a solution of (1.1)–(1.3) is also a solution of (1.12)–(1.14), it also satisfies (2.2)–(2.4). In order to prove the inverse, we observe that the system (2.2)–(2.4) has been obtained as a consequence of (1.12)–(1.14) by:

- writing in explicit form the time and space derivatives of the constitutive quantities on the state space; and
- substituting in the obtained expressions the equations (1.1)–(1.3).

This is enough to ensure that any solution of (2.2)–(2.4) is also solution of (1.1)–(1.3). ■

Example 2.3. Let's consider again the differential equation of example 1.2, namely,

$$u_{,t} = at. \quad (2.5)$$

As said above, such a problem admits the general solution

$$u(t) = a \frac{t^2}{2} + C. \quad (2.6)$$

Let's consider now the following differential consequence of equation (2.5)

$$u_{,tt} = a. \quad (2.7)$$

By substituting equation (2.5) into equation (2.7) we get

$$u_{,tt} = \frac{1}{t} u_{,t}. \quad (2.8)$$

It is immediate to verify that the general solution of equation (2.8) is

$$u(t) = C_1 \frac{t^2}{2} + C_2. \quad (2.9)$$

By this equation, it follows

$$u_{,t} = C_1 t. \quad (2.10)$$

By recalling now that we have obtained the differential equation (2.8) under the hypothesis

$$u_{,t} = at, \quad (2.11)$$

the comparison of equations (2.10) and (2.11) yields $C_1 = a$. Thus, the solution (2.9) takes the form (2.6) of the general solution of the original equation (2.5).

Remark 2.4. It is particularly important to observe that, as a consequence of theorem 2.2, we get that the solutions of (2.2)–(2.4) also obey the first law of thermodynamics, namely, equation (1.1).

Let's explore now the thermodynamic admissibility of the thermal processes ruled by equations (1.12)–(1.14). The second law of thermodynamics requires that only those thermodynamic transformations leading to a non-negative entropy production are physically admissible. Locally, such a production reads

$$\Sigma^{(s)} = s_{,t} + J_{,x}, \quad (2.12)$$

where s is the entropy per unit volume and J is the local entropy flux. On Z , second law of thermodynamics reads

$$\begin{aligned} \Sigma^{(s)} = & \frac{\partial s}{\partial e} e_{,t} + \frac{\partial s}{\partial e_{,x}} e_{,xt} + \frac{\partial s}{\partial q} q_{,t} + \frac{\partial s}{\partial q_{,x}} q_{,xt} + \frac{\partial s}{\partial \Phi} \Phi_{,t} + \frac{\partial s}{\partial \Phi_{,x}} \Phi_{,xt} + \frac{\partial J}{\partial e} e_{,x} \\ & + \frac{\partial J}{\partial e_{,x}} e_{,xx} + \frac{\partial J}{\partial q} q_{,x} + \frac{\partial J}{\partial \Phi} \Phi_{,x} + \frac{\partial J}{\partial q_{,x}} q_{,xx} + \frac{\partial J}{\partial \Phi_{,x}} \Phi_{,xx} \geq 0. \end{aligned} \quad (2.13)$$

The consequences of the inequality (2.13) can be pointed out by applying a generalized Coleman–Noll procedure [18,19], which is based on the substitution in (2.13) of the system of differential consequences (1.12)–(1.14). In this way, the following proposition can be proved.

Theorem 2.5. *The inequality (2.13) is satisfied in any thermodynamic process if, and only if, the following thermodynamic restrictions hold*

$$\frac{\partial s}{\partial e_x} = \frac{\partial s}{\partial q_x} = \frac{\partial s}{\partial \Phi_x} = 0, \quad (2.14)$$

$$\frac{\partial J}{\partial e_x} = \frac{\partial s}{\partial \Phi} \frac{\partial \Psi}{\partial e_x}, \quad (2.15)$$

$$\frac{\partial J}{\partial q_x} = \frac{\partial s}{\partial \Phi} \frac{\partial \Psi}{\partial q_x}, \quad (2.16)$$

$$\frac{\partial J}{\partial \Phi_x} = \frac{\partial s}{\partial \Phi} \frac{\partial \Psi}{\partial \Phi_x}, \quad (2.17)$$

and
$$F(e, e_x, q, q_x, \Phi, \Phi_x) \geq 0. \quad (2.18)$$

where F is a suitable function defined on Z .

Proof. Preliminarily, let's observe that by substitution of equations (1.12)–(1.14) into the inequality (2.13), one gets

$$\begin{aligned} & \frac{\partial s}{\partial e}(-q_x + \sigma) + \frac{\partial s}{\partial e_x} \left[\left(\frac{\partial \Psi}{\partial e} \right)^{-1} (-\Phi_{tt} + p_{,t}) - \left(\frac{\partial \Psi}{\partial e} \right)^{-1} \frac{\partial \Psi}{\partial q} q_{,xt} - \left(\frac{\partial \Psi}{\partial e} \right)^{-1} \frac{\partial \Psi}{\partial \Phi} \Phi_{,xt} \right] \\ & + \frac{\partial s}{\partial q}(r - \Phi_x) + \frac{\partial s}{\partial q_x}(e_{,tt} + \sigma_{,t}) + \frac{\partial s}{\partial \Phi}(p - \Psi_x) + \frac{\partial s}{\partial \Phi_x}(r_{,t} - q_{,tt}) + \frac{\partial J}{\partial e} e_x \\ & + \frac{\partial J}{\partial e_x} e_{,xx} + \frac{\partial J}{\partial q} q_x + \frac{\partial J}{\partial \Phi} \Phi_x + \frac{\partial J}{\partial q_x} q_{,xx} + \frac{\partial J}{\partial \Phi_x} \Phi_{,xx} \geq 0. \end{aligned} \quad (2.19)$$

The above inequality contains the three terms $(\partial s/\partial q_x)e_{,tt}$, $(\partial s/\partial \Phi_x)q_{,tt}$ and $(\partial s/\partial e_x)\Phi_{,tt}$ in which the higher-order time derivatives $e_{,tt}$, $q_{,tt}$ and $\Phi_{,tt}$ are independent of the elements of the state space and can assume arbitrary values, while their coefficients $\partial s/\partial q_x$, $\partial s/\partial \Phi_x$ and $\partial s/\partial e_x$ are defined on the state space. Then, these coefficients must vanish; otherwise, the three terms above could assume arbitrary negative values and the inequality could be easily violated [18]. This leads to the restriction (2.14).

As a consequence of (2.14), the inequality (2.19) reduces to

$$\begin{aligned} & \frac{\partial s}{\partial e}(-q_x + \sigma) + \frac{\partial s}{\partial q}(r - \Phi_x) + \frac{\partial s}{\partial \Phi}(p - \Psi_x) + \frac{\partial J}{\partial e} e_x \\ & + \frac{\partial J}{\partial e_x} e_{,xx} + \frac{\partial J}{\partial q} q_x + \frac{\partial J}{\partial \Phi} \Phi_x + \frac{\partial J}{\partial q_x} q_{,xx} + \frac{\partial J}{\partial \Phi_x} \Phi_{,xx} \geq 0. \end{aligned} \quad (2.20)$$

Finally, once the space derivative Ψ_x has been developed and substituted in (2.20), one obtains

$$\begin{aligned} & \left(\frac{\partial J}{\partial e_x} - \frac{\partial s}{\partial \Phi} \frac{\partial \Psi}{\partial e_x} \right) e_{,xx} + \left(\frac{\partial J}{\partial q_x} - \frac{\partial s}{\partial \Phi} \frac{\partial \Psi}{\partial q_x} \right) q_{,xx} \\ & + \left(\frac{\partial J}{\partial \Phi_x} - \frac{\partial s}{\partial \Phi} \frac{\partial \Psi}{\partial \Phi_x} \right) \Phi_{,xx} + F(e, e_x, q, q_x, \Phi, \Phi_x) \geq 0, \end{aligned} \quad (2.21)$$

where, for the sake of abbreviation, we have denoted by F the sum of all the terms in (2.20) which are defined on Z . The same considerations used to obtain (2.14) can be applied to the first three terms in the inequality (2.21), which are given by the product of some coefficients defined on the state space and the higher-order derivatives $e_{,xx}$, $q_{,xx}$ and $\Phi_{,xx}$, which do not belong to Z . Also in this case, these coefficients must vanish, otherwise the inequality could be violated, and this yields the thermodynamic restrictions (2.15)–(2.17), together with the reduced entropy inequality (2.18). ■

The thermodynamic restrictions (2.15)–(2.17) allow, in principle, to determine the constitutive equation for the entropy flux J , once the form of the functions s and Ψ is known. To achieve that

task, let's postulate the following constitutive equations

$$s = s_1(e) - \frac{1}{2}s_2(e)q^2 - \frac{1}{2}s_3(e)\Phi^2, \quad (2.22)$$

and

$$\Psi = \Psi_0(e, q, \Phi) + \Psi_1(e, q, \Phi)e_{,x} + \Psi_2(e, q, \Phi)q_{,x} + \Psi_3(e, q, \Phi)\Phi_{,x}. \quad (2.23)$$

Equation (2.22) is typical of EIT, [1,3,4,24], in which the entropy does not depend only on the equilibrium variable e but by the dissipative fluxes too. Its form is in accordance with the theory of representation of isotropic scalar functions of vectors and tensors. The functions $s_2(e)$ and $s_3(e)$ are assumed positive definite, in such a way that the principle of maximum entropy at the equilibrium [1,3,4] is always fulfilled.

Corollary 2.6. *Under the constitutive hypotheses (2.22) and (2.23), the entropy flux admits the representation*

$$J = \left(\frac{\partial s_1}{\partial e} - \frac{1}{2} \frac{\partial s_2}{\partial e} q^2 - \frac{1}{2} \frac{\partial s_3}{\partial e} \Phi^2 \right) q - s_2 q \Phi - s_3 \Phi (\Psi_1 e_{,x} + \Psi_2 q_{,x} + \Psi_3 \Phi_{,x}). \quad (2.24)$$

Proof. Due to (2.22) and (2.23), the thermodynamic restrictions (2.15)–(2.17) become

$$\frac{\partial J}{\partial e_{,x}} = -s_3 \Phi \Psi_1, \quad (2.25)$$

$$\frac{\partial J}{\partial q_{,x}} = -s_3 \Phi \Psi_2, \quad (2.26)$$

and

$$\frac{\partial J}{\partial \Phi_{,x}} = -s_3 \Phi \Psi_3. \quad (2.27)$$

Thus, by direct integration of the previous expressions, we get

$$J = -s_3 \Phi \Psi_1 e_{,x} + J_1(e, q, \Phi, q_{,x}, \Phi_{,x}), \quad (2.28)$$

$$J = -s_3 \Phi \Psi_2 q_{,x} + J_2(e, q, \Phi, e_{,x}, \Phi_{,x}), \quad (2.29)$$

and

$$J = -s_3 \Phi \Psi_3 \Phi_{,x} + J_3(e, q, \Phi, e_{,x}, q_{,x}). \quad (2.30)$$

Then, we observe that nothing prevents choosing

$$J_1 = -s_3 \Phi \Psi_2 q_{,x} - s_3 \Phi \Psi_3 \Phi_{,x} + J_0(e, q, \Phi), \quad (2.31)$$

$$J_2 = -s_3 \Phi \Psi_1 e_{,x} - s_3 \Phi \Psi_3 \Phi_{,x} + J_0(e, q, \Phi), \quad (2.32)$$

and

$$J_3 = -s_3 \Phi \Psi_1 e_{,x} - s_3 \Phi \Psi_2 q_{,x} + J_0(e, q, \Phi). \quad (2.33)$$

Thus, we get

$$J = -s_3 \Phi \Psi_1 e_{,x} - s_3 \Phi \Psi_2 q_{,x} - s_3 \Phi \Psi_3 \Phi_{,x} + J_0(e, q, \Phi). \quad (2.34)$$

In order to determine $J_0(e, q, \Phi)$, we apply a classical result proved by Verhás [25], according to which, in the presence of the higher-order fluxes q and Φ in the constitutive equation for the entropy, the local entropy flux can be represented as follows

$$J_0 = \frac{\partial s}{\partial e} q + \frac{\partial s}{\partial q} \Phi. \quad (2.35)$$

By taking into account equation (2.22), we obtain

$$J_0 = \left(\frac{\partial s_1}{\partial e} - \frac{1}{2} \frac{\partial s_2}{\partial e} q^2 - \frac{1}{2} \frac{\partial s_3}{\partial e} \Phi^2 \right) q - s_2 q \Phi, \quad (2.36)$$

and, as a consequence, the constitutive equation (2.24) is proved. ■

Remark 2.7. We observe that equation (2.24) generalizes the classical constitutive equation $J = (\partial s / \partial e) q$ postulated in rational thermodynamics [18].

Remark 2.8. The quantity $\partial s/\partial e$ is usually identified as the reciprocal of the absolute temperature. Thus, the corresponding parenthesis in the first term of the right-hand side of equation (2.36) could be interpreted as a generalization of the local-equilibrium temperature. This is an interesting physical topic since the appropriate definition of non-equilibrium temperature is still an open problem in continuum non-equilibrium thermodynamics. For an extensive discussion, let's refer the reader to [26].

Remark 2.9. Indeed, according to information theory, the entropy flux has the form of the sum of several fluxes, each of them multiplied by the thermodynamic conjugate variable of the quantity transported by the flux (such conjugate variable is given by the derivative of the entropy with respect to the variable) [27–29]. Note that this is satisfied by the particular expression in equation (2.35). The second term in equation (2.24) has the form of the flux of q (i.e. Φ) times the thermodynamic conjugate of q (i.e. $\partial s/\partial q = -s_2q$), which has indeed the mentioned form. The last term in equation (2.24), namely the term involving s_3 , is peculiar of the present model, as in information theory the gradients of the fields are not usually included. However, such term in equation (2.24) has also the form required in information theory, provided the term Ψ_0 in equation (2.23) is zero. Indeed, the term $s_3\Phi$ is the conjugate to Φ according to the constitutive equation (2.22) for the entropy, whereas the quantity multiplying it is the flux of Φ (i.e. Ψ), according to equation (2.23), if the first term in equation (2.23) is zero.

3. Non-local constitutive equations and hyperbolic systems of balance laws

In this section, we consider the subsystem of equations (1.1)–(1.3) constituted by the local balances of energy and heat flux in the absence of the production term σ

$$e_{,t} + q_{,x} = 0, \quad (3.1)$$

and

$$q_{,t} + \Phi_{,x} = r. \quad (3.2)$$

This system plays a central role in EIT of rigid heat conductors, since it allows to obtain the most important heat equations [9], such as the Maxwell–Cattaneo–Vernotte equation (1.5) [6], and the Guyer–Krumhansl equation [7]

$$\tau q_{,t} + q = -kT_{,x} + 3l^2q_{,xx}, \quad (3.3)$$

where l is the mean free path of the heat carriers at microscopic level. In the case of the Maxwell–Cattaneo–Vernotte equation, the system above requires local constitutive equations for Φ and r , while for the Guyer–Krumhansl equation the constitutive equation for Φ needs to be non-local. The non-local nature of the constitutive equation is due to the presence of both the spatial derivative of q (which yields the term in $q_{,xx}$) and of the mean free path of the phonons, which for the dielectric crystals analysed by Guyer and Krumhansl are the sole responsible of the heat transport. In fact, for miniaturized systems (for instance nanosystems), the mean free path of the heat carriers has the same order of magnitude of the length of the system. Hence, the heat carriers can reach any point of the conductor, experiencing so non-local interactions with the other carriers and with the crystal lattice. At the macroscopic scale, instead, the mean free path is negligible with respect to the physical dimension of the system, and local interactions are recovered.

Our aim is to prove that the system above allows the propagation of thermal perturbations with finite speed, even in the presence of non-local constitutive equation for Φ . The system of differential consequences of equations (3.1)–(3.2) can be put in the form

$$e_{,tt} + r_{,x} - \Phi_{,xx} = 0, \quad (3.4)$$

and

$$q_{,tt} + \Phi_{,xt} = r_{,t}, \quad (3.5)$$

while the constitutive equations for Φ and r are taken as follows

$$\Phi = \frac{nk}{\tau c_v} e - \frac{3l^2}{\tau} q_{,x} + \frac{3l}{\tau} q, \quad r = -\frac{q}{\tau}, \quad (3.6)$$

where n denotes a real control parameter whose role will be clarified later.

Remark 3.1. For three-dimensional systems, the constitutive equations (3.6) can be generalized as follows

$$\Phi_{ij} = \frac{nk}{\tau c_v} e \delta_{ij} - \frac{l^2}{\tau} \left(\frac{\partial q_i}{\partial x_j} + 2 \frac{\partial q_j}{\partial x_i} \right) + \frac{3}{\tau} l_i q_j, \quad r_i = -\frac{q_i}{\tau}, \quad (3.7)$$

where δ_{ij} denotes the identity matrix and l_i are the components of the vectorial mean free path of the phonons. The mean free path has a vectorial nature whenever the phonons' scattering depends on the direction. For instance, l has a vectorial character in:

- long nanowires, in which the mean free path in the direction parallel to the cross section can be different than that in the direction orthogonal to the cross section;
- anisotropic systems, for which the mean free path depends on the direction followed by the phonons;
- graded systems, i.e. alloys of different materials, with the composition depending on the position, in which l could have the direction set by the composition gradient.

By the constitutive equation for Φ , we obtain the second-order derivatives

$$\Phi_{,xx} = \frac{nk}{\tau c_v} e_{,xx} - \frac{3l^2}{\tau} q_{,xxx} + \frac{3l}{\tau} q_{,xx}, \quad (3.8)$$

and

$$\Phi_{,xt} = \frac{nk}{\tau c_v} e_{,xt} - \frac{3l^2}{\tau} q_{,xxt} + \frac{3l}{\tau} q_{,xt}. \quad (3.9)$$

To go further, we observe that according to the hypothesis 2.1 of §1, the terms in l^2 entering the derivatives above can be considered to be negligible with respect to the other terms. Also in this case, the non-locality is still conserved by the presence in equations (3.8) and (3.9) of the terms $(3l/\tau)q_{,xx}$ and $(3l/\tau)q_{,xt}$.

Remark 3.2. The approximation made above is in accordance with the Guyer–Krumhansl derivation of their celebrated heat-transport equation [7], which has been obtained by solving the linearized Boltzmann transport equation for phonons up to the second-order approximation in the spatial derivatives of the heat flux. Finally, we observe that, under the same approximation, the system constituted by equations (3.1) and (3.3) is parabolic.

Remark 3.3. It is important to note that, in our approximation we are not neglecting the term $(3l^2/\tau)q_{,x}$ in the flux of heat flux but the terms $(3l^2/\tau)q_{,xxx}$ and $(3l^2/\tau)q_{,xxt}$ in the second derivatives of the flux of heat flux, since our model is developed under the hypothesis that the third derivatives of the unknown fields are small (see assumption 2.1). Under the approximation above, in order to conserve the non-locality in the evolution equation for the heat flux, the term $(3l/\tau)q$ plays a central role. On the other hand, in the classical approach, in which the differential consequences of (3.1)–(3.2) are not taken into consideration, the term $(3l^2/\tau)q_{,x}$ is essential to obtain the Guyer–Krumhansl equation.

Thus, the coupling of equations (3.2) and (3.5) yields

$$\tau^2 q_{,tt} + \tau q_{,t} + \tau \Phi_{,x} = \tau r - \tau^2 \Phi_{,xt} + \tau^2 r_{,t}. \quad (3.10)$$

Then, if in equation (3.10), the sum $-\tau^2 \Phi_{,xt} + \tau^2 r_{,t}$ is negligible with respect to the other terms in the equation, the following hyperbolic generalization of the Guyer–Krumhansl model

$$\tau^2 q_{,tt} + \tau q_{,t} + q = -kT_{,x} + 3l^2 q_{,xx}, \quad (3.11)$$

can be recovered under the constitutive assumptions

$$r = -q/\tau, \quad \Phi = \frac{k}{\tau c_v} e - \frac{3l^2}{\tau} q_{,x}. \quad (3.12)$$

This proves that several different physical situations can occur in which the parabolic Guyer–Krumhansl equation can be generalized into hyperbolic schemes.

By virtue of the the hypothesis 2.1 of §1, the third-order derivatives in equations (3.8)–(3.9) can be neglected, and the system (3.4)–(3.5) can be written as follows

$$e_{,tt} - \frac{nk}{\tau c_v} e_{,xx} - \frac{3l}{\tau} q_{,xx} = \frac{q_{,x}}{\tau}, \quad (3.13)$$

and

$$q_{,tt} + \frac{nk}{\tau c_v} e_{,xt} + \frac{3l}{\tau} q_{,xt} = -\frac{q_{,t}}{\tau}. \quad (3.14)$$

Finally with the positions

$$u = e_{,t}, \quad v = e_{,x}, \quad z = q_{,t}, \quad w = q_{,x}, \quad (3.15)$$

equations (3.13)–(3.14) yield the following first-order, quasi-linear system of partial differential equations

$$u_{,t} - \frac{nk}{\tau c_v} v_{,x} - \frac{3l}{\tau} w_{,x} = \frac{w}{\tau}, \quad (3.16)$$

$$z_{,t} + \frac{nk}{\tau c_v} u_{,x} = -\left(\frac{1}{\tau} + \frac{3l}{\tau}\right)z, \quad (3.17)$$

$$v_{,t} - u_{,x} = 0, \quad (3.18)$$

and

$$w_{,t} - z_{,x} = 0. \quad (3.19)$$

In a single spatial dimension x , the systems of governing equations of continuum physics often may be also put in the first-order quasi-linear form

$$A_0(\mathbf{U})\mathbf{U}_{,t} + A_1(\mathbf{U})\mathbf{U}_{,x} = \mathbf{f}(\mathbf{U}), \quad (3.20)$$

with the unknown N -column vector $\mathbf{U}(x, t) = (U_1, U_2, \dots, U_N)^T$, where A_0 and A_i are real $N \times N$ matrices and \mathbf{f} is an N -column vector too. The wave speeds and the amplitudes of the acceleration waves are given, respectively, by the eigenvalues λ and the eigenvectors \mathbf{r} of the following eigenvalue problem

$$(A_1 - \lambda A_0)\mathbf{r} = \mathbf{0}. \quad (3.21)$$

Definition 3.4. The system (3.20) is said hyperbolic in the t -direction if $\det A_0 \neq 0$, and the problem (3.21) has only real eigenvalues (characteristic speeds).

Theorem 3.5. Under the constitutive assumptions (3.6), the system (3.16)–(3.19) is hyperbolic provided the following relation is true

$$l \leq \frac{2}{9} \sqrt{n} \sqrt{\frac{k}{3\tau c_v}}. \quad (3.22)$$

Moreover, the speeds of propagation of thermal disturbances are given by

$$\lambda_{2,3,4} = -2\sqrt{\frac{3nk}{\tau c_v}} \sin \left[\frac{1}{3} \arcsin \left(-\frac{9}{2} l \sqrt{\frac{3\tau c_v}{nk}} \right) + 2m \frac{\pi}{3} \right], \quad m = -1, 0, 1 \quad (3.23)$$

Proof. By straightforward calculations, it follows that the system (3.16)–(3.19) can be rearranged in the form (3.20), with $\mathbf{U} = (u, z, v, w)^T$, $\mathbf{f} = (w/\tau, -(1/\tau + 3l/\tau)z, 0, 0)^T$ and

$$A_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad (3.24)$$

$$A_1 = \begin{bmatrix} 0 & 0 & -\frac{nk}{\tau c_v} & -\frac{3l}{\tau} \\ \frac{nk}{\tau c_v} & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}. \quad (3.25)$$

Then, the speeds of propagation of thermal perturbations are the solutions of the following characteristic equation

$$\det \begin{bmatrix} -\lambda & 0 & -\frac{nk}{\tau c_v} & -\frac{3l}{\tau} \\ \frac{nk}{\tau c_v n} & -\lambda & 0 & 0 \\ -1 & 0 & -\lambda & 0 \\ 0 & -1 & 0 & -\lambda \end{bmatrix} = 0, \quad (3.26)$$

namely,

$$\lambda \left(\lambda^3 - \frac{nk}{\tau c_v} \lambda + \frac{3lnk}{c_v \tau} \right) = 0. \quad (3.27)$$

Beside the solution $\lambda_1 = 0$, equation (3.27) yields the roots of the cubic equation

$$\lambda^3 - \frac{nk}{\tau c_v} \lambda + \frac{3lnk}{c_v \tau} = 0, \quad (3.28)$$

which admits three real solutions if, and only if,

$$\frac{9l^2}{4} - \frac{nk}{27\tau c_v} \leq 0. \quad (3.29)$$

Finally, from the general theory of cubic equations, it follows that those real solutions admit the representation (3.23). This proves our proposition. \blacksquare

The inequality (3.29) provides an upper bound for the mean free path of the heat carriers which guarantees that the system is hyperbolic. For l exceeding such a limit, the diffusive heat transport takes over.

The previous expression proves that the mean free path of the heat carriers, which is related to the non-local nature of the constitutive equations, contributes to the determination of the characteristic speeds. Finally, if the mean free path is negligible, $l \simeq 0$, the characteristic speeds reduce to

$$\lambda_{2,3,4} = -2\sqrt{\frac{3nk}{\tau c_v}} \sin\left(2m\frac{\pi}{3}\right), \quad m = -1, 0, 1. \quad (3.30)$$

For $m = 0$, we get the speed $\lambda_3 = 0$, while for $m = \mp 1$, we get the speeds $\lambda_{2,4} = \pm 3\sqrt{2}\sqrt{nk/\tau c_v}$. On the other hand, for vanishing non-local effects, ($l \simeq 0$), the characteristic speeds of second sound propagation, namely $\lambda_{2,4} = \pm\sqrt{k/\tau c_v}$, should be recovered. To obtain these speeds, it is sufficient to choose the control parameter as $n = 1/18$ in equation (3.30).

4. Discussion

In this paper, we have considered a system of balance laws arising in EIT of rigid heat conductors together with its differential consequences, namely the higher-order system obtained by taking into account the time and space derivatives of the original system. We have pointed out some mathematical properties of the differential consequences. Particular emphasis has been given to the problem of the propagation of thermal perturbations with finite speed, which is a hot topic in modern continuum thermodynamics [2]. We have proved that:

- An opportune choice of the initial conditions ensures that a solution of the Cauchy problem for the system of differential consequences is also a solution of the Cauchy problem for the original system.
- The system of differential consequences can be rewritten in a particular form which is equivalent to the original one.
- The system of differential consequences is compatible with the second law of thermodynamics, and hence, it may correspond, in principle, to thermodynamic processes which are physically admissible.
- The constitutive equation for the entropy flux is non-local and generalizes the classical constitutive equation $J = (\partial s / \partial e) q$ postulated in rational thermodynamics [18].
- Non-locality of the constitutive equations and finite speed of propagation of thermal disturbances are compatible, since it is possible to get a hyperbolic system of balance laws even when the state space is non-local.

In our next researches, we aim at extending the previous results to more general systems of balance laws following by Grad's hierarchic system approximating the Boltzmann equation [5].

Moreover, we would like to investigate if the generalized flux of heat flux considered in §3 allows for a more accurate description of heat transport in nanosystems, where non-local effects play a central role.

A further topic for a possible future work could be comparing different physical situations in which the parabolic Guyer–Krumhansl equation can be generalized into hyperbolic schemes.

As far as the constitutive equation (2.24) is concerned, it would be interesting to investigate if the non-local terms appearing therein could show their effects in some situations which are interesting in the applications, for instance, in the description of heat transport in low-dimensional nanosystems.

Data accessibility. This work does not contain any experimental data.

Authors' contributions. P.R. and V.A.C. determined the system of balance laws and its differential consequences; formulated second law of thermodynamics for the system at hand; carried out the thermodynamic analysis and determined the restrictions placed by the entropy principle; proved that the system of differential consequences is hyperbolic and calculated the speeds of propagation of thermal disturbances; chosen the list of references; delineated the general structure of the article.

Competing interests. We declare we have no competing interests.

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Appendix A. Extended thermodynamics of rigid heat conductors

Notation A.1. In this section, we deal with three-dimensional systems. Hence, we will use the following notation: $\mathbf{x} = \{x_i\}$, $i = 1, 2, 3$, will denote the vector position of the particles of the system and $\mathbf{v} = \{v_i\}$, $i = 1, 2, 3$, their velocity. The symbol $f(\mathbf{x}, \mathbf{v}, t)$ will denote a scalar-valued function defined in $[R^3 \times R^3 \times [0, \infty[$, which depends on position \mathbf{x} , velocity \mathbf{v} and time t . Moreover, the symbols $\partial_t f$ and $\partial_i f$ will denote the partial derivative of function f with respect to t and x_i , respectively.

In non-equilibrium thermodynamics, modelling non-equilibrium phenomena with steep gradients and rates represents a challenging task, which can be tackled by means of two complementary approaches: the continuum approach and the kinetic approach.

The continuum approach consists in describing the system by means of macroscopic equations obtained on the basis of conservation laws together with appropriate constitutive equations. In such a case, the applicability of the classical macroscopic theory is restricted to processes characterized by small Knudsen numbers (for instance, dense gases). On the other hand, the approach based on kinetic theory, which postulates that the state of the system can be described by a velocity distribution function, whose evolution is governed by the Boltzmann equation, is applicable to processes characterized by a large Knudsen number (for instance, rarefied gases).

The extended theories of thermodynamics are phenomenological theories aiming at filling the gap between these two limit cases. To this end, these theories apply the mesoscopic approach [4], in which the macroscopic systems of governing equations are postulated in the form dictated by kinetic theory [1,10]. Currently, the name extended thermodynamics characterizes two different thermodynamic theories, namely rational extended thermodynamics (RET) [10], and extended irreversible thermodynamics (EIT) [1]. The mathematical structure of extended theories is based on a hierarchy of balance laws that have the same form of the moments of the distribution function [5] in Boltzmann equation, truncated at some arbitrary order N . Although these theories take into account the kinetic approach, they also use additional physical laws such as, for instance, the entropy principle and the principle of material frame indifference [3]. The difference between RET and EIT lies on a different interpretation and use of those universal laws. We refer the reader to [2,3] for more details.

We recall that the kinetic theory of monatomic gases is based on the assumption that the state of a gas can be described by the distribution function $f(\mathbf{x}, \mathbf{v}, t)$, which obeys the Boltzmann equation

$$\partial_t f + v_i \partial_i f = Q(f), \quad (\text{A } 1)$$

where the collision integral $Q(f)$ represents the rate of change of the distribution function due to the particle collisions.

The solution of equation (A 1) is, in general, an extremely complicated task. However, since one is usually interested in the evolution of macroscopic observable quantities, a way to avoid the direct solution of the Boltzmann equation is to define macroscopic quantities by averaging f over the velocity space, by constructing the following moments of the distribution function

$$F = \int_{\mathbf{R}^3} m f d\mathbf{v}, \quad F_{i \dots l} = \int_{\mathbf{R}^3} m v_i \dots v_l f d\mathbf{v} \quad i \dots l \in \{1, 2, 3\}, \quad (\text{A } 2)$$

where m represents the mass of the particles, and to determine appropriate governing equations for these new quantities. In 1949, Grad [5] proved that the moments of the distribution function satisfy an infinite hierarchy of balance laws in which the flux in one equation at the step N becomes the density in the next one, namely, at the step $N + 1$. Thus, we have

$$\partial_t F + \partial_i F_i = 0, \quad (\text{A } 3)$$

↙

$$\partial_t F_i + \partial_j F_{ij} = 0, \quad (\text{A } 4)$$

↙

$$\partial_t F_{ij} + \partial_k F_{ijk} = P_{ij}, \quad (\text{A } 5)$$

↙

$$\partial_t F_{ijk} + \partial_l F_{ijkl} = P_{ijk}, \quad (\text{A } 6)$$

$$\vdots$$

$$\partial_t F_{ij\dots m} + \partial_n F_{ij\dots mn} = P_{ij\dots m}, \quad (\text{A } 7)$$

$$\vdots$$

where

$$P_{i\dots l} = \int_{\mathbf{R}^3} Q(f) m v_i \cdots v_l \mathbf{d}\mathbf{v}, \quad (\text{A } 8)$$

with $P_{ll} = 0$.

When the infinite hierarchy above is stopped at the step N , namely at the unknown density of tensorial rank $N - 1$, and the unknown fields are $\{F, F_i, F_{ij}, \dots, F_{ij\dots m}\}$, the problem of closure of the system arises, because the last flux and the production terms cannot be determined as solutions of the above system of partial differential equations. The basic idea of extended theories is that the evolution of any continuum system is governed by a phenomenological system of equations whose mathematical structure is that of equations (A 3)–(A 7). To close the system, it is necessary to assign the last flux and the production terms by suitable constitutive functions. In RET [10], it is assumed that the constitutive quantities may depend on the unknown fields $(F, F_i, F_{ij}, \dots, F_{ij\dots m})$ only. In EIT [1], instead, the constitutive equations may depend on the gradients of the unknown fields, too [2]. In both theories, the constitutive equations must be assigned in such a way that any solution $(F, F_i, F_{ij}, \dots, F_{ij\dots m})$ of the system (A 3)–(A 7) satisfies second law of thermodynamics [3,18].

Since one might wonder what is the role of higher-order fluxes is, we observe that the kinetic theory points out that the relaxation times of the higher-order fluxes are not always shorter than the collision time. Thus, the only use of the lower-order fluxes as independent variables is not always satisfactory to describe high-frequency processes, because when the frequency becomes comparable to the inverse of the relaxation time of the first-order flux, all the higher-order fluxes also behave like independent variables and must be incorporated in the formalism.

Indeed, in several processes involving rigid heat conductors, beside the local balance of energy (first law of thermodynamics), it is sufficient to consider additional balance equations only for the first-order and second-order fluxes, which can be identified with the heat flux \mathbf{q} , and the flux of heat flux Φ [4,20]. Thus, the following hierarchic system arises

$$\partial_t e + \partial_i q_i = \sigma, \quad (\text{A } 9)$$

$$\swarrow$$

$$\partial_t q_i + \partial_j \Phi_{ij} = r_i, \quad (\text{A } 10)$$

$$\swarrow$$

$$\partial_t \Phi_{ij} + \partial_k \Psi_{ijk} = p_{ij}, \quad (\text{A } 11)$$

where e is the internal energy per unit volume, q_i are the components of the heat flux, Φ_{ij} are the components of the flux of heat flux (second-order tensor), Ψ_{ijk} are the components of the flux of Φ_{ij} (third-order tensor), σ is the local production of energy, r_i are the components of the local production of heat flux and p_{ij} are the components of the local production of the flux of heat flux.

It is worth noticing that the system above is not given by the first three equations of the hierarchic system (A 3)–(A 7), but it is a phenomenological system which is postulated on macroscopic grounds with the same hierarchic structure of (A 3)–(A 7), accordingly with kinetic theory. As we said above, the theory is macroscopic but the form of the field equations is postulated on microscopic grounds (mesoscopic approach) [4]. In such a framework, let's recall that the heat flux vector \mathbf{q} represents the amount of heat transferred per unit time and per unit area and its dimension is $\text{J s}^{-2} \text{m}^{-2}$. The second-order tensor Φ , i.e. the flux of heat flux, is

introduced to discuss several problems in heat-transport theory, for instance, for describing heat pulse experiments at room temperature [21], heat conduction in nanosystems [30] or second-order non-local effects [31]. Its physical dimension is $\text{J s}^{-2} \text{m}^{-1}$, and its divergence represents the energy rate (i.e. the energy per second) per unit area and per second. Analogously, the divergence of the flux Ψ is the rate of energy rate per unit area and per squared seconds. The physical dimension of Ψ is J s^{-3} .

The system of differential consequences of equations (A 9)–(A 11) is

$$\partial_{tt}^2 e + \partial_{ii}^2 q_i = \partial_t \sigma, \quad (\text{A } 12)$$

$$\partial_{tt}^2 q_i + \partial_{ij}^2 \Phi_{ij} = \partial_t r_i, \quad (\text{A } 13)$$

and

$$\partial_{tt}^2 \Phi_{ij} + \partial_{ik}^2 \Psi_{ijk} = \partial_t p_{ij}. \quad (\text{A } 14)$$

It can be put in the form

$$\partial_{tt}^2 e = -\partial_{ij}^2 \Phi_{ij} + \partial_t r_i + \partial_t \sigma, \quad (\text{A } 15)$$

$$\partial_{tt}^2 q_i = -\partial_{jk}^2 \Psi_{ijk} + \partial_j p_{ij} + \partial_t r_i, \quad (\text{A } 16)$$

and

$$\partial_{tt}^2 \Phi_{ij} = -\partial_{ik}^2 \Psi_{ijk} + \partial_t p_{ij}. \quad (\text{A } 17)$$

In the equations above, the partial derivatives with respect to space and time of the constitutive quantities σ , r_i , p_{ij} and Ψ_{ijk} appear. In order to calculate explicitly these derivatives, it is necessary to establish the state space, namely the set of functions on which the constitutive quantities depend. As we said in §1, the model considered here is developed within the frame of EIT and the state space is supposed to contain also the first gradients of the unknown fields, namely

$$Z = \{e, \partial_k e, q_i, \partial_k q_i, \Phi_{ij}, \partial_k \Phi_{ij}\}. \quad (\text{A } 18)$$

Finally, under the validity of the assumption 2.1, once the account is taken of the composition of Z , the application of the chain rule leads to

$$\begin{aligned} \partial_{tt}^2 e &= \partial_{ij}^2 \Phi_{ij} - \partial_t r_i - \frac{\partial \sigma}{\partial e} \partial_i q_i + \frac{\partial \sigma}{\partial e} \sigma + \frac{\partial \sigma}{\partial (\partial_k e)} (-\partial_{ki}^2 q_i + \partial_k \sigma) + \frac{\partial \sigma}{\partial q_i} (r_i - \partial_j \Phi_{ij}) \\ &\quad + \frac{\partial \sigma}{\partial (\partial_k q_i)} (\partial_k r_i - \partial_{kj}^2 \Phi_{ij}) + \frac{\partial \sigma}{\partial \Phi_{ij}} (p_{ij} - \partial_k \Psi_{ijk}) + \frac{\partial \sigma}{\partial (\partial_k \Phi_{ij})} (\partial_k p_{ij} - \partial_{kl}^2 \Psi_{ijl}), \end{aligned} \quad (\text{A } 19)$$

$$\begin{aligned} \partial_{tt}^2 q_i &= \partial_{jk}^2 \Psi_{ijk} - \partial_j p_{ij} - \frac{\partial r_i}{\partial e} \partial_l q_l + \frac{\partial r_i}{\partial e} \sigma + \frac{\partial r_i}{\partial (\partial_k e)} (-\partial_{kl}^2 q_l + \partial_k \sigma) + \frac{\partial r_i}{\partial q_l} (r_l - \partial_j \Phi_{lj}) \\ &\quad + \frac{\partial r_i}{\partial (\partial_k q_l)} (\partial_k r_l - \partial_{kj}^2 \Phi_{lj}) + \frac{\partial r_i}{\partial \Phi_{ij}} (p_{lj} - \partial_k \Psi_{ljk}) + \frac{\partial r_i}{\partial (\partial_k \Phi_{lj})} (\partial_k p_{lj} - \partial_{km}^2 \Psi_{ljm}), \end{aligned} \quad (\text{A } 20)$$

and

$$\begin{aligned} \partial_{tt}^2 \Phi_{ij} &= -\frac{\partial \Psi_{ijk}}{\partial e} (-\partial_{kl}^2 q_l + \partial_k \sigma) - \frac{\partial \Psi_{ijk}}{\partial q_m} (-\partial_{kl}^2 \Phi_{ml} + \partial_k r_m) \\ &\quad - \frac{\partial \Psi_{ijk}}{\partial \Phi_{mn}} (-\partial_{kl}^2 \Psi_{mnl} + \partial_k p_{mn}) + \frac{\partial p_{ij}}{\partial e} (\partial_k q_k + \sigma) \\ &\quad + \frac{\partial p_{ij}}{\partial (\partial_l e)} (-\partial_{lk}^2 q_k + \partial_l \sigma) + \frac{\partial p_{ij}}{\partial q_l} (r_l - \partial_m \Phi_{lm}) + \frac{\partial p_{ij}}{\partial (\partial_k q_l)} (\partial_k r_l - \partial_{km}^2 \Phi_{lm}) \\ &\quad + \frac{\partial p_{ij}}{\partial \Phi_{lm}} (p_{lm} - \partial_k \Psi_{lmk}) + \frac{\partial p_{ij}}{\partial (\partial_n \Phi_{lm})} (\partial_n p_{lm} - \partial_{nk}^2 \Psi_{lmk}), \end{aligned} \quad (\text{A } 21)$$

where, for the sake of brevity, we did not develop the spatial derivatives of the constitutive quantities.

Also in this case, in deriving equation (A 19), we have used the following identity

$$\frac{\partial \sigma}{\partial e} \partial_t e = -\frac{\partial \sigma}{\partial e} \partial_i q_i + \frac{\partial \sigma}{\partial e} \sigma, \quad (\text{A } 22)$$

namely, we have used equation (A 9), which represents the first law of thermodynamics. Thus, in the derivation of the system of differential consequences, such fundamental physical law has been taken into account, so that the solutions of (A 19)–(A 21) also satisfy the first law of thermodynamics.

The partial derivatives of the constitutive quantities with respect to the basic fields, as, for instance, $\partial\Psi_{ijk}\partial\Phi_{mm}$ (a fifth-order tensor), should be regarded as mathematical coefficients entering the partial derivatives of the constitutive functions, without a particular physical meaning. Their specific form depends on the form of the constitutive equations for σ , r_i , p_{ij} and Ψ_{ijk} . These equations must be assigned in such a way that the local entropy production $\Sigma^{(s)} = s_{,t} + J_{i,i}$, where s is the entropy per unit volume and J_i are the components of the entropy flux, is non-negative for arbitrary thermodynamic processes, namely $\Sigma^{(s)} \geq 0$ for any solution of the system (A 9)–(A 11). Such an inequality, on the state space Z reads

$$\begin{aligned} \frac{\partial s}{\partial e} e_{,t} + \frac{\partial s}{\partial(\partial_i e)} \partial_{ii}^2 e + \frac{\partial s}{\partial q_i} \partial_t q_i + \frac{\partial s}{\partial(\partial_k q_i)} \partial_{ik}^2 q_i + \frac{\partial s}{\partial\Phi_{ij}} \partial_t \Phi_{ij} + \frac{\partial s}{\partial(\partial_k \Phi_{ij})} \partial_{ik}^2 \Phi_{ij} \\ + \frac{\partial J_i}{\partial e} \partial_t e + \frac{\partial J_i}{\partial(\partial_k e)} \partial_{ik}^2 e + \frac{\partial J_i}{\partial q_k} \partial_t q_k + \frac{\partial J_i}{\partial(\partial_l q_k)} \partial_{il}^2 q_k + \frac{\partial J_i}{\partial\Phi_{jk}} \partial_t \Phi_{jk} + \frac{\partial J_i}{\partial(\partial_l \Phi_{jk})} \partial_{il}^2 \Phi_{jk} \geq 0. \end{aligned} \quad (\text{A } 23)$$

The exploitation of the inequality above is carried out in a different way in EIT and in RET [3]. In fact, in EIT, the inequality (A 23) is analysed by the classical Coleman–Noll [18] or Liu [32] procedures, in order to obtain restrictions on the constitutive equations, (see, for instance, [24]). In RET, instead, the inequality (A 23) is used to determine the so-called main field, namely, a particular form of the unknown fields which renders the system (A 3)–(A 7) symmetric. It is possible to prove that in such a case well-posedness of the Cauchy problem for (A 3)–(A 7), and finite speeds of propagation of its solutions are guaranteed. We refer the reader to [3] for an extensive discussion of this topic. In this paper, we apply the EIT approach and hence, in §2, we derive the consequences of the inequality (A 23) by a generalized Coleman–Noll procedure [19].

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