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Nyström methods for bivariate Fredholm integral equations on unbounded domains

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ABSTRACT

In this paper we propose a numerical procedure in order to approximate the solution of two-dimensional Fredholm integral equations on unbounded domains like strips, half-planes or the whole real plane. We consider global methods of Nyström types, which are based on the zeros of suitable orthogonal polynomials. One of the main interesting aspects of our procedures regards the "quality" of the involved functions, since we can successfully manage functions which are singular on the finite boundaries and can have an exponential growth on the infinite boundaries of the domains. Moreover the errors of the methods are comparable with the error of best polynomial approximation in the weighted spaces of functions that we go to treat. The convergence and the stability of the methods and the well conditioning of the final linear systems are proved and some numerical tests, which confirm the theoretical estimates, are given.

1. Introduction

This paper deals with the numerical approximation of the solution of bivariate Fredholm integral equations of the second kind

$$f(x,y)-\mu\int_S k(x,y,s,t)f(s,t)\mathbf{w}(s,t)\,ds\,dt=g(x,y),\quad (x,y)\in S, \tag{1.1}$$

defined on the set $S=(a,b)\times(c,d)$, where the ranges $(a,\,b)$ and $(c,\,d)$ can be bounded intervals or unbounded ones, the function $\mathbf{w}(x,\,y)$ is the product of two weights functions $w_1(x), w_2(y)$ defined on $(a,\,b)$, $(c,\,d)$ respectively and μ is a real parameter. k and g are given functions defined on $S\times S$ and S respectively, while f is the unknown.

We will discuss two cases in details: (1) S is the first quadrant in the coordinate plane, i.e. $S=(0,+\infty)\times(0,+\infty)$ and w_1, w_2 are two Generalized Laguerre weights; (2) S is defined as the horizontal strip of the plane between the axes $y=\pm1$, i.e. $S=(-\infty,+\infty)\backslash\{0\}\times(-1,1)$ and w_1, w_2 are a Generalized Freud type weight and a Jacobi weight, respectively. However, the numerical schemes we are going to propose could be successfully applied in order to handle, mutatis mutandis, other different unbounded domains like the entire plane, the half-plane, other kinds of strips etc.

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Fredholm integral equations, both in the univariate and in the bivariate cases, appear in several problems arising in many different areas, like computer graphics, engineering, mathematical physics etc. Moreover some differential schemes with boundary conditions can be reduced to Fredholm integral equations.

While the numerical solution of the Fredholm integral equation in the univariate case was extensively studied and discussed (see for instance [1,2]), the bivariate problem was investigated only recently. The interested reader can consult the bibliography in [3] which gives an overview on the numerical methods appearing in the literature for solving the problem in the case of finite domains of the plane (see also [4,5]).

To our knowledge, the problem of unbounded domains has not been treated so far. On the other hand, the methods based on the reduction of the problem to finite ranges, usually require non linear changes of variables, which can make poor the smoothness of the known functions, and consequently of the solution. For instance by means of the transformation $\psi(x) = \frac{x}{1-x}$ the interval $[0, +\infty)$ is mapped into [0, 1], but a regular function like $\log(1+y)$ on $[0, +\infty)$, by the change $y = \frac{x}{1-x}$ is transformed into a weakly singular one on [0, 1]. In addition, the new known functions of the equation could contain not suitable or non standard weight functions, for which the main tools of the approximation theory are unknown.

Following an approach already used by the authors in [4] for $S = [-1, 1]^2$, in the present paper we propose a global method in order to approximate the solution of equations of the type (1.1) by means of a suitable Nyst röm method. The latter is essentially based on a tensor product of univariate Gaussian rules, like Gauss–Laguerre rules in the first case and Gauss–Freud, Gauss–Jacobi rules in the second one. In both of the cases we are able to treat equations whose solutions belong to some spaces of weighted continuous functions which can be singular along the finite boundaries of S. To be more precise, in the first case the solution can have singularities of algebraic type on one or both the positive semi-axes, and an exponential growth at infinity at most. In the second case, the solution can be singular along the straight lines of equations $y = \pm 1$, algebraically diverging on the internal line $\{(x, y) : x = 0,\ y \in (-1, 1)\}$ and exponentially growing for $x \to \pm\infty$.

The unboundedness of the domains add difficulties to the numerical approach that need suitable truncation techniques (see for instance [6]) in order to avoid a slow convergence.

We prove that under suitable assumptions on the known functions and the weights, the proposed procedures are stable and convergent and that the involved linear systems are well conditioned. We underline that, under the same conditions the rate of convergence of the methods is comparable with the error of best polynomial approximation in the functional space where the solution lives.

Moreover, since we use “truncated” Gaussian rules, the dimension of the corresponding linear systems is smaller than the classical ones and therefore the computational costs can be significantly reduced.

We remark that, due to the global nature of the approximation, our method requires that the kernel function k and the right–hand side g of the equation have to be analytically known and therefore the method cannot be used if these functions are only point wisely given.

We organized the outline of the paper as follows. In Section 2 we introduce the notations, recall some auxiliary results and give an estimate of the bivariate best polynomial approximation error in weighted uniform norm, in terms of the best polynomial approximation w.r.t. the single variables. Therefore, for a better readability of the paper, we decided to separate the description of the numerical method and the statements of the main results from the proofs. Hence in Sections 3 and 4 we show the Nyström method in the case S is the first quadrant of the plane, by stating the convergence and stability results of the method, by giving computational details for the implementation and some numerical tests which confirm the theoretical estimates. Similarly Sections 5 and 6 are devoted to an analogous study in the case of the strip $S = \mathbb{R} - \{0\} \times (-1, 1)$. All the proofs of the main results stated in Sections 3 and 5 can be found in Section 7. Finally Section 8 contains the conclusions.

2. Notations and basic polynomial approximation results

Along all the paper the constant $\mathcal{C}$ will be used several times, having different meaning in different formulas. Moreover from now on we will write $\mathcal{C} \neq \mathcal{C}(a, b, \ldots)$ in order to say that $\mathcal{C}$ is a positive constant independent of the parameters $a, b, \ldots,$ and $\mathcal{C} = \mathcal{C}(a, b, \ldots)$ to say that $\mathcal{C}$ depends on $a, b, \ldots$. Moreover, if $A, B \geq 0$ are quantities depending on some parameters, we write $A \sim B$, if there exists a positive constant $\mathcal{C}$ independent of the parameters of A and B, such that

$$\frac{B}{\mathcal{C}} \leq A \leq \mathcal{C}B.$$

The notations f_x and f_y will be used when the function $f(x, y)$ is considered as a function of the only variable y or x, respectively. Moreover, $k_{(s,t)}$ (respectively $k_{(x,y)}$) will mean that the function of four variables $k(x, y, s, t)$ is considered as a function of only (x, y) (respectively (s, t)). Finally $\mathbf{P} \equiv (x, y)$ will denote a generic point of the domain S.

Denoting by $\mathbb{P}_m$ the space of the univariate polynomials of degree m, let $\mathbb{P}_{m,m}$ be the space of all algebraic polynomials of two variables of degree at most m in each variable. Setting

$$\begin{aligned} \omega_1(z) &= (1-z)^{\rho}(1+z)^{\sigma}, \quad z \in I_1 := [-1, 1], \quad \rho, \sigma \geq 0, \\ \omega_2(z) &= e^{-\frac{z^{\delta_1}}{2}} z^{\zeta}(1+z)^{\nu}, \quad z \in I_2 := [0, +\infty), \quad \zeta, \nu \geq 0,\ \delta_1 > \frac{1}{2}, \\ \omega_3(z) &= e^{-\frac{|z|^{\delta_2}}{2}} |z|^{\xi}(1+|z|)^{\eta}, \quad z \in I_3 := \mathbb{R}, \quad \xi, \eta \geq 0,\ \delta_2 > 1. \end{aligned} \tag{2.1}$$

let us introduce the following space of functions

$$C_{\omega_1}(I_1) = \{h \in C(-1,1) : \lim_{x\to\pm 1} h(x)\omega_1(x) = 0\}, \tag{2.2}$$

$$C_{\omega_2}(I_2) = \{h \in C(0,\infty) : \lim_{x\to 0^+} h(x)\omega_2(x) = 0 = \lim_{x\to+\infty} h(x)\omega_2(x)\} \tag{2.3}$$

$$C_{\omega_3}(I_3) = \{h \in C(\mathbb{R}-\{0\}) : \lim_{x\to 0^\pm} h(x)\omega_3(x) = 0 = \lim_{x\to\pm\infty} h(x)\omega_3(x)\}. \tag{2.4}$$

Then, for any function $h \in C_{\omega_j}(I_j),\ j = 1,2,3,$ let

$$E_m(h)_{\omega_j} := \inf_{p\in\mathbb{P}_m} \|(h-p)\omega_j\|_\infty = \inf_{p\in\mathbb{P}_m} \max_{x\in I_j} |(h(x)-p(x))\omega_j(x)|$$

denote the error of best polynomial approximation of h in weighted uniform norm.

Let

$$\hat{a}_m = \hat{a}_m(\omega_2) \sim \left[\frac{2^{2\delta_1-1}\Gamma^2(\delta_1)}{\Gamma(2\delta_1)}\right]^{\frac{1}{\delta_1}} \left(1+\frac{2\zeta+1}{8m}\right)^{\frac{1}{\delta_1}} m^{\frac{1}{\delta_1}} = \mathcal{C} m^{\frac{1}{\delta_1}}, \quad \mathcal{C} = \mathcal{C}(\zeta,\nu), \quad \mathcal{C} \neq \mathcal{C}(m) \tag{2.5}$$

be the Mhaskar–Rachmanov–Saff (from now on M-R-S number) w.r.t. ω_2. We recall (see [7]), that for any function $h \in C_{\omega_2}(I_2)$, s.t. $\|h^{(r)}\varphi^r\omega_2\|_\infty < \infty$, $\varphi(x) = \sqrt{x}$, the following Favard-type estimate holds true

$$E_m(h)_{\omega_2} \le \mathcal{C}\left(\frac{\sqrt{\hat{a}_m}}{m}\right)^r \|h^{(r)}\varphi^r\omega_2\|_\infty, \quad \mathcal{C} \neq \mathcal{C}(m,h). \tag{2.6}$$

Let

$$\bar{a}_m = \bar{a}_m(\omega_3) \sim \left[\frac{2^{\delta_2-1}\Gamma^2(\delta_2/2)}{\Gamma(\delta_2)}\right]^{\frac{1}{\delta_2}} \left(1+\frac{\xi+1}{2m}\right)^{\frac{1}{\delta_2}} m^{\frac{1}{\delta_2}} = \mathcal{C} m^{\frac{1}{\delta_2}}, \quad \mathcal{C} = \mathcal{C}(\xi,\eta), \quad \mathcal{C} \neq \mathcal{C}(m) \tag{2.7}$$

be the M-R-S number w.r.t. ω_3. For any function $h \in C_{\omega_3}(I_3)$, s.t. $\|h^{(r)}\omega_3\|_\infty < \infty$ and if the exponent ξ is not an integer, the following Favard-type estimate holds true (see [8,9])

$$E_m(h)_{\omega_3} \le \mathcal{C}\left(\frac{\bar{a}_m}{m}\right)^r \left(\|h\omega_3\|_\infty + \|h^{(r)}\omega_3\|_\infty\right), \quad \mathcal{C} \neq \mathcal{C}(m,h). \tag{2.8}$$

Now we recall that for any function $h \in C_{\omega_1}(I_1)$, s.t. $\|h^{(r)}\varphi^r\omega_1\|_\infty < \infty$, where $\varphi(x) = \sqrt{1-x^2}$, the following estimate holds true

$$E_m(h)_{\omega_1} \le \mathcal{C}\frac{\|h^{(r)}\varphi^r\omega_1\|_\infty}{m^r}, \quad \mathcal{C} \neq \mathcal{C}(m,h). \tag{2.9}$$

Setting $\mathbf{u}(\mathbf{P}) = \omega_k(x)\omega_j(y)$, where $\omega_k(x)$, $\omega_j(y)$ are two weights functions among those defined in (2.1) and $\mathbf{P} \in \mathrm{S} = I_k \times I_j$, we define the following space of weighted continuous bivariate functions

$$C_{\mathbf{u}}(\mathrm{S}) = \left\{ f \in C(\mathrm{S}) : \lim_{\mathbf{P}\to\partial S} f(\mathbf{P})\mathbf{u}(\mathbf{P}) = 0 \right\} \tag{2.10}$$

where the limit condition holds true independently from the directions. Moreover another limit condition for $\mathbf{P}$ tending to the line $x = 0$ (or y=0) must be added if one (or both) the weight(s) is(are) of the type ω_3.

Therefore for any $f \in C_{\mathbf{u}}(\mathrm{S})$,

$$E_{m,m}(f)_{\mathbf{u}} = \inf_{P\in\mathbb{P}_{m,m}} \|(f-P)\mathbf{u}\|_\infty = \inf_{P\in\mathbb{P}_{m,m}} \max_{\mathbf{P}\in S} |(f(\mathbf{P})-P(\mathbf{P}))\mathbf{u}|$$

will denote the error of best bivariate polynomial approximation of f in $C_{\mathbf{u}}(\mathrm{S})$.

The next theorem allows us to estimate $E_{m,m}(f)_{\mathbf{u}}$ by means of the univariate best approximation errors w.r.t. the single variables x and y.

Theorem 2.1. *With k, j ∈ {1, 2, 3}, let* $\mathbf{u}(\mathbf{P}) = \omega_k(x)\omega_j(y)$, *where* $\omega_k(x)$, $\omega_j(y)$ *are two weights functions chosen among those defined in (2.1). Then,*

$$E_{m,m}(f)_{\mathbf{u}} \le \mathcal{C}\left[\sup_{x\in I_k} \omega_k(x) E_m(f_x)_{\omega_j} + \sup_{y\in I_j} \omega_j(y) E_m(f_y)_{\omega_k}\right], \tag{2.11}$$

where $0 < \mathcal{C} \neq \mathcal{C}(m,f)$.

Remark 2.1. In the case $I_k = I_j = [-1,1]$ and ω_k, ω_j are both Jacobi weights, we proved (2.11) in [4]. We remark that the proof that we give here is based on a simpler and more general approach, which can be used also in other norms.

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3. The case $S = (0, +\infty) \times (0, +\infty)$

Consider the Eq. (1.1) with $S := (0, +\infty) \times (0, +\infty)$ and $\mathbf{w}(x, y) = w_1(x)w_2(y)$ where $w_1(x) = e^{-x^b} x^{\alpha_1}$, $w_2(y) = e^{-y^b} y^{\alpha_2}$, $\alpha_1, \alpha_2 > -1$, $b > \frac{1}{2}$.

w_1, w_2 are nonclassical weight functions, known as *Generalized Laguerre weights*. Many properties of the corresponding orthonormal polynomials and the connected results in the univariate approximation of functions can be found in [7,10–12].

We start by introducing the spaces of functions in which we will study the equation and then we determine some estimates of the corresponding best approximation error. Moreover, we introduce the "truncated" Gaussian cubature formula and we study the convergence. Then we show the Nyström method based on the above cubature rule, by determining the assumptions under which the convergence and the stability of the method hold true.

3.1. Functional spaces and preliminary results

Fixing the weight $\mathbf{u}(\mathbf{P}) = u_1(x)u_2(y)$ with $u_1(x) = x^{\sigma_1} e^{-\frac{x^b}{2}} (1+x)^{\eta_1}$, $u_2(y) = y^{\sigma_2} e^{-\frac{y^b}{2}} (1+y)^{\eta_2}$, $\sigma_1, \sigma_2, \eta_1, \eta_2 \geq 0$, $b > \frac{1}{2}$, let $C_{\mathbf{u}}$ be the space defined in (2.10) with $\mathbf{u}(\mathbf{P}) = u_1(x)u_2(y)$, which means that the product $f\mathbf{u}$ vanishes as $\mathbf{P}$ approaches the boundaries of S, independently from the direction. Equip $C_{\mathbf{u}}$ with the weighted uniform norm on S

$$\|f\|_{C_{\mathbf{u}}} = \|f\mathbf{u}\|_\infty = \max_{\mathbf{P}\in S} |f(\mathbf{P})\mathbf{u}(\mathbf{P})|.$$

Whenever one or more of the parameters σ_1, σ_2 is greater than 0, the functions in $C_{\mathbf{u}}$ can be singular on one or both the positive $x-$axis and /or on the positive $y-$ axis. Moreover the functions can have an exponential growth at infinity at most.

For smoother functions, we introduce the following Sobolev–type space

$$W_r(\mathbf{u}) = \left\{ f \in C_{\mathbf{u}} : M_r(f, \mathbf{u}) := \max\left\{ \left\| f_y^{(r)} \varphi_1^r \mathbf{u} \right\|_\infty, \left\| f_x^{(r)} \varphi_2^r \mathbf{u} \right\|_\infty \right\} < \infty \right\},$$

where $\varphi_1(x) = \sqrt{x}$, $\varphi_2(y) = \sqrt{y}$ and the superscript (r) denotes the rth derivative of the one-dimensional function f_y or f_x. We equip $W_r(\mathbf{u})$ with the norm

$$\|f\|_{W_r(\mathbf{u})} = \|f\mathbf{u}\|_\infty + M_r(f, \mathbf{u}).$$

One or more of the r partial derivatives of functions in $W_r(\mathbf{u})$ can present singularities on the positive $x-$axis and /or on the positive $y-$ axis.

We remark that u_1 and u_2 are of the same type of ω_2 in (2.1) and then setting $a_m = \max(\hat{a}_m(u_1), \hat{a}_m(u_2))$ where $\hat{a}_m(u_1), \hat{a}_m(u_2)$ are defined as in (2.5), by (2.11) and (2.6) we get

$$E_{m,m}(f)_{\mathbf{u}} \leq \mathcal{C} M_r(f, \mathbf{u}) \left(\frac{\sqrt{a_m}}{m} \right)^r, \quad \forall f \in W_r(\mathbf{u}), \tag{3.1}$$

where $0 < \mathcal{C} \neq \mathcal{C}(m, f)$ and $\frac{\sqrt{a_m}}{m} \sim m^{\frac{1}{2b}-1}$. Therefore a Favard-type inequality holds true also for the bivariate error of best approximation in $C_{\mathbf{u}}$.

3.2. The "Truncated" Gauss–Laguerre cubature formula

Let $\{p_m(w_1)\}_m$ and $\{p_m(w_2)\}_m$ be the sequences of the orthonormal polynomials, with positive leading coefficients, w.r.t. the weight functions $w_1(x) = x^{\alpha_1} e^{-x^b}$ and $w_2(y) = y^{\alpha_2} e^{-y^b}$, $\alpha_1, \alpha_2 > -1, b > \frac{1}{2}$. Denote by $\{x_i\}_{i=1}^m$, $\{y_j\}_{j=1}^m$ the zeros of $p_m(w_1)$ and $p_m(w_2)$, respectively.

For any fixed $0 < \theta < 1$, q_1 and q_2 will denote the indexes of the minimal zeros of $p_m(w_1)$ and $p_m(w_2)$ greater than $a_m\theta$

$$x_{q_1} = x_{q_1(m)} = \min_{1\leq i\leq m} \{x_i : x_i \geq a_m\theta\},$$

$$y_{q_2} = y_{q_2(m)} = \min_{1\leq j\leq m} \left\{y_j : y_j \geq a_m\theta\right\}.$$

Hence let

$$q = q(m) = \max\{q_1, q_2\}. \tag{3.2}$$

Setting $\mathbf{w}(\mathbf{P}) = \mathbf{w}(x, y) = w_1(x)w_2(y)$, consider the following Gauss-type cubature rule

$$\int_S F(\mathbf{P})\mathbf{w}(\mathbf{P})\, d\mathbf{P} = \sum_{i=1}^{q} \sum_{j=1}^{q} \lambda_i^{w_1} \lambda_j^{w_2} F(\mathbf{P}_{i,j}) + \mathcal{E}_m(f), \tag{3.3}$$

where $\mathbf{P}_{i,j} \equiv (x_i, y_j)$, $i, j = 1, 2, \ldots, m$ and $\{\lambda_i^{w_1}\}_{i=1}^m$, $\{\lambda_j^{w_2}\}_{j=1}^m$ denote the Christoffel numbers related to the weights w_1 and w_2, respectively. We recall that the univariate Truncated Gauss–Laguerre rules were introduced in [6,13] and [14] and that (3.3) has been obtained as the tensor product of two univariate Truncated Gauss–Laguerre rules.

About the error estimate, the following Proposition holds true:

Proposition 3.1. *For any $F \in C_{\mathbf{u}^2}$, under the assumption*

$$\int_S \frac{\mathbf{w}(\mathbf{P})}{\mathbf{u}^2(\mathbf{P})}\, d\mathbf{P} < +\infty, \tag{3.4}$$

it results

$$|\mathcal{E}_m(F)| \le \mathcal{C}\left(E_{M,M}(F)_{\mathbf{u}^2} + e^{-\mathcal{A}m}\|F\mathbf{u}^2\|_\infty\right), \tag{3.5}$$

where $M = \left\lfloor \left(\frac{\theta}{1+\theta}\right)^b m \right\rfloor \sim m$ and the positive constants $\mathcal{C}, \mathcal{A}$ are independent of m and F.

3.3. A Nyström method

Now we are able to introduce the Nyström method based on the above defined Gauss–Laguerre cubature formula. Denoting by

$$Kf(x,y) = \mu \int_S k(x,y,s,t) f(s,t) \mathbf{w}(s,t)\, ds\, dt$$

and by I the identity operator on $C_{\mathbf{u}}$, the Eq. (1.1) can be rewritten as

$$(I - K)f = g. \tag{3.6}$$

At first we determine suitable assumptions on the kernel k such that the Fredholm Alternative holds true in $C_{\mathbf{u}}$.

Proposition 3.2. *Assume that (3.4) holds true and that for some $r \in \mathbb{N}$*

$$\sup_{(s,t)\in S} \mathbf{u}(s,t)\|k_{(s,t)}\|_{W_r(\mathbf{u})} < +\infty. \tag{3.7}$$

Then the operator K: $C_{\mathbf{u}} \to C_{\mathbf{u}}$ is compact and $Kf \in W_r(\mathbf{u})$ for any $f \in C_{\mathbf{u}}$.

Now starting with the Gaussian formula (3.3) and using the same notation, we define the following discrete operator

$$K_m f(\mathbf{P}) = \mu \sum_{i=1}^{q}\sum_{j=1}^{q} \lambda_i^{w_1}\lambda_j^{w_2} k(\mathbf{P}, \mathbf{P}_{i,j}) f(\mathbf{P}_{i,j}), \tag{3.8}$$

and consequently the operator equation

$$(I - K_m) f_m = g, \tag{3.9}$$

where f_m is unknown.

Setting $a_{ij} = f(\mathbf{P}_{i,j})\mathbf{u}(\mathbf{P}_{i,j}),\ i,j = 1,\dots,q$, by multiplying both sides of Eq. (3.9) by the weight $\mathbf{u}$ and by collocating it on the pairs (x_h, y_ℓ): $\equiv \mathbf{P}_{h,\ell}$, $h, \ell = 1,\dots,q$, we achieve the following linear system in the unknowns a_{ij}, $i,j = 1,\dots,q$,

$$a_{h\ell} - \mu \mathbf{u}(\mathbf{P}_{h,\ell}) \sum_{i=1}^{q}\sum_{j=1}^{q} \frac{\lambda_i^{w_1}\lambda_j^{w_2}}{\mathbf{u}(\mathbf{P}_{i,j})} k(\mathbf{P}_{h,\ell}, \mathbf{P}_{i,j}) a_{ij} = (g\mathbf{u})(\mathbf{P}_{h,\ell}), \quad h,\ell = 1,\dots,q. \tag{3.10}$$

The matrix solution $(a^*_{ij})_{i,j=1\dots,q}$ of this system (if it exists) allows us to construct the weighted Nyström interpolant in two variables

$$(f_m\mathbf{u})(\mathbf{P}) = \mu \mathbf{u}(\mathbf{P}) \sum_{i=1}^{q}\sum_{j=1}^{q} \frac{\lambda_i^{w_1}\lambda_j^{w_2}}{\mathbf{u}(\mathbf{P}_{i,j})} k(\mathbf{P}, \mathbf{P}_{i,j}) a^*_{ij} + (g\mathbf{u})(\mathbf{P}) \tag{3.11}$$

which will approximate the unknown $f\mathbf{u}$. Denoting by $\mathbf{A}_m$ the coefficient matrix of system in (3.10) and by $cond(\mathbf{A}_m)$ its condition number in infinity norm, we can state the following result about the convergence and stability of the method.

Theorem 3.1. *Assume that k satisfies (3.7), that $Ker\{I - K\} = \{0\}$ in $C_{\mathbf{u}}$ and the weights $\mathbf{w}$ and $\mathbf{u}$ verify (3.4). Denote by f^* the unique solution of (3.6) in $C_{\mathbf{u}}$ for a given $g \in C_{\mathbf{u}}$. If in addition, with the same $r \in \mathbb{N}$ in (3.7), it results*

$$g \in W_r(\mathbf{u}) \tag{3.12}$$

and

$$\sup_{(x,y)\in S} \mathbf{u}(x,y)\|k_{(x,y)}\|_{W_r(\mathbf{u})} < +\infty, \tag{3.13}$$

then, for m sufficiently large, the system (3.10) is uniquely solvable and well conditioned, since

$$cond(\mathbf{A}_m) \le \mathcal{C}, \qquad \mathcal{C} \ne \mathcal{C}(m), \tag{3.14}$$

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and there results

$$\|(f^* - f_m)\mathbf{u}\|_\infty \le \mathcal{C}\left(\frac{\sqrt{a_m}}{m}\right)^r \|f^*\|_{W_r(\mathbf{u})}, \tag{3.15}$$

where $\mathcal{C} \neq \mathcal{C}(m, f^*)$.

Comparing (3.15) with (3.1) we remark that the proposed Nyström method converges as the best polynomial approximation of the solution $f^* \in W_r(\mathbf{u})$.

Moreover since we are using a "truncation" of a complete tensor product cubature formula, this allow to use, for a fixed m, a linear system of reduced size with an evident computational saving.

Finally the condition numbers of the involved linear systems are bounded in virtue of (3.14). The magnitude of the constant $\mathcal{C}$ depends on the condition number of the operator $I - K$ in (3.6) as an operator defined in $C_{\mathbf{u}}$.

4. Computational details and numerical tests

First we observe that for $b \neq 1$ the coefficients in the three-term recurrence relation for the polynomial sequences $\{p_m(w_1)\}_m, \{p_m(w_2)\}_m$ are not always known. However, the computation of the zeros of the Generalized Laguerre polynomials, as well as the Christoffel numbers can be performed by using the *Mathematica* package *Orthogonal Polynomials* (see [15]).

4.1. The matrix $\mathbf{A_m}$

Let q be defined in (3.2). The matrix $\mathbf{A_m}$ is a $q \times q$ block matrix, the entries of which are $q \times q$ matrices:

$$\mathbf{A_m} = \begin{pmatrix} \mathbf{A}^{(1,1)} & \mathbf{A}^{(1,2)} & \dots & \mathbf{A}^{(1,q)} \\ \mathbf{A}^{(2,1)} & \mathbf{A}^{(2,2)} & \dots & \mathbf{A}^{(2,q)} \\ \dots & \dots & \dots & \dots \\ \mathbf{A}^{(q,1)} & \mathbf{A}^{(q,2)} & \dots & \mathbf{A}^{(q,q)} \end{pmatrix}$$

and the blocks are defined as

$$\mathbf{A}^{(h,i)} = \delta_{h,i}\mathbf{I_q} - \mu \mathbf{D}_h \mathbf{K}_q^{(h,i)}\mathbf{U}_i, \quad h, i = 1, 2 \dots, q,$$

where

$$\mathbf{D}_h = diag(\mathbf{u}(x_h, y_1), \mathbf{u}(x_h, y_2), \dots, \mathbf{u}(x_h, y_q)),$$

$$\mathbf{U}_i = diag\left(\frac{\lambda_i^{w_1}\lambda_1^{w_2}}{\mathbf{u}(x_i, y_1)}, \frac{\lambda_i^{w_1}\lambda_2^{w_2}}{\mathbf{u}(x_i, y_2)}, \dots, \frac{\lambda_i^{w_1}\lambda_q^{w_2}}{\mathbf{u}(x_i, y_q)}\right),$$

$\mathbf{I_q}$ denotes the identity matrix of order q and $\mathbf{K}_q^{(h,i)}$ are matrices defined as follows

$$\mathbf{K}_q^{(h,i)}(\ell, j) = k(x_h, y_\ell, x_i, y_j), \quad \ell, j = 1, 2, \dots, q.$$

4.2. Numerical tests

Now we propose some numerical tests in order to show the performance of our method. The test equations were constructed to underline the efficiency of the method in the case of functions that can be singular along the boundaries of the domain S (see Ex.2–3). We point out that in all the examples the involved parameter are chosen such that the assumptions in Theorem 3.1 are satisfied. We remark that in particular the choice of the parameters of the weight u is made according to the smoothness of the known functions. Usually the exponents of u which satisfy (3.4) can vary in an interval. There is no particular reason to choose a value instead of another because the order of convergence is exactly the same. Nevertheless we note that the norm of the function f in (3.15) usually is smaller for smaller value of the exponents of u.

In any test we approximate the weighted solution $f\mathbf{u}$ by the weighted Nyström interpolant $f_m\mathbf{u}$ given in (3.11). The q^2-square linear system needed in order to construct $f_m\mathbf{u}$ was solved by the Gaussian elimination, so the major computational effort is of order $\frac{q^6}{3}$. In the tables, for each m, we give the maximum relative error attained in the computation of $f_m\mathbf{u}$ at a grid of equally spaced points. We remark that all the computations were performed in 16−digits precision.

Example 1. Consider the equation

$$f(x, y) - \int_S \left(\frac{x + y}{(15 + s^2 + t^2)^7} + \frac{|\sin(x - y)|^{4.8}}{(15 + s^2 + t^2)^5}\right) f(s, t)e^{-s-t}\, ds\, dt$$
$$= 1 + x + y.$$

In this case the involved parameters are

$$b = 1, \quad \mu = 1, \quad \alpha_1 = \alpha_2 = 0$$

and the weight $\mathbf{u}$, according to (3.4), is chosen with

$$\sigma_1 = \sigma_2 = \frac{1}{4}, \quad \eta_1 = \eta_2 = 0.3.$$

Therefore the known functions belong to $W_4(\mathbf{u})$ and the convergence rate of the method is $\frac{1}{m^2}$.
The numerical results, for $\theta = 0.7$, are as follows:

m	q	Max relative error
4	4	0.55×10^{-8}
8	8	0.10×10^{-9}
16	14	0.75×10^{-12}
32	27	0.42×10^{-15}
64	53	*eps*

that shows a fast speed of convergence, higher than the expected one.
The condition numbers in infinity norm of the matrices are less than 1.0006.

Example 2. Consider the equation

$$f(x,y) - \frac{1}{2}\int_S \left[\frac{1}{(25+s^3+t^3)^5} + \frac{\pi}{(36+x^3+y^3)^7}\right] \frac{e^{-s-t}}{s^{\frac{1}{5}} t^{\frac{3}{10}}} f(s,t)\, ds\, dt$$
$$= \sin(x+y).$$

In this case the integrand has singularities for $s = 0$ and for $t = 0$, i.e. it is unbounded along the positive semi-axes. The involved parameters are

$$b = 1, \quad \mu = \frac{1}{2}, \quad \alpha_1 = -\frac{1}{5}, \ \alpha_2 = -\frac{3}{10}$$

and the weight $\mathbf{u}$ is chosen with

$$\sigma_1 = \sigma_2 = \frac{1}{4}, \quad \eta_1 = \eta_2 = 0.2.$$

In this way the known functions belong to $W_r(\mathbf{u})$ for any $r > 0$. From the theoretical results a fast convergence is expected.
The numerical results, for $\theta = 0.5$, are as follows:

m	q	Max relative error
4	4	0.45×10^{-7}
8	8	0.31×10^{-8}
16	14	0.56×10^{-10}
32	27	0.73×10^{-11}
64	53	0.22×10^{-14}
80	66	0.27×10^{-15}
100	82	*eps*

Hence with a system of order 66^2 the machine precision is attained.
Moreover we remark that the condition numbers of the matrices are less than 1.0000004.

Example 3.

$$f(x,y) - \frac{1}{2}\int_S \frac{\sqrt{1+y}}{\sqrt[4]{x}(25+s^3+t^3)^4} \frac{e^{-s-t}}{s^{\frac{1}{10}} t^{\frac{1}{5}}} f(s,t) ds dt = \frac{\frac{1}{\sqrt[4]{x}} + \frac{1}{\sqrt[4]{y}}}{1+x^2+y^2}.$$

In this case

$$b = 1, \quad \mu = \frac{1}{2}, \quad \alpha_1 = -\frac{1}{10}, \ \alpha_2 = -\frac{1}{5},$$

and $\mathbf{u}$ is chosen with

$$\sigma_1 = \sigma_2 = \frac{1}{4}, \quad \eta_1 = \eta_2 = 0.3.$$

Therefore the solution is only in $C_{\mathbf{u}}$. We expect a very low convergence, but the numerical test give nice results

m	q	Max relative error
4	4	0.46×10^{-6}
8	8	0.35×10^{-6}
16	14	0.20×10^{-6}
32	27	0.10×10^{-6}
64	43	0.39×10^{-7}
128	106	0.33×10^{-8}

Moreover we remark that the condition numbers of the matrices are less than 1.0001 and hence, if an higher accuracy is needed, it is possible to increase the dimension of the linear system, without occurring in stability problems.

5. The case $S = (-\infty, +\infty)\backslash\{0\} \times (-1, 1)$

Consider the Eq. (1.1) with $S := \{(x, y) : x \in \mathbb{R} - \{0\}, y \in (-1, 1)\}$ and $\mathbf{w}(\mathbf{P}) = w_1(x)w_2(y)$ where $w_1(x) = |x|^\gamma e^{-|x|^d}$, $w_2(y) = (1-y)^\alpha(1+y)^\beta$, $\alpha, \beta, \gamma > -1$, $d > 1$. Hence we are considering the case in which w_1 is a Generalized Freud weight, while w_2 is a Jacobi weight.

Following the same scheme used in the previous case, first we introduce the spaces of functions and some estimates of the corresponding best approximation error. Then we derive a cubature formula constructed as the tensor product of a "truncated" Gaussian rule w.r.t. a Generalized Freud weight and a Gauss–Jacobi rule. Finally we discuss a Nyström method based on this formula. The convergence and stability of the method will be showed. We remark that for a better readability we have employed the same symbols S, to denote the domain, and $\mathbf{u}$ to denote the weight of the space of functions, that we used in Sections 3 and 4.

5.1. Functional spaces and preliminary results

Fixing the weight $\mathbf{u}(\mathbf{P}) = u_1(x)u_2(y)$ with $u_1(x) = |x|^\xi(1+|x|)^\eta e^{-\frac{|x|^d}{2}}$, $u_2(y) = (1-y)^\rho(1+y)^\sigma$, $\xi, \eta, \rho, \sigma \geq 0$, ξ not an integer, and setting $\mathbf{L} = \{(0, y),\ y \in (-1, 1)\}$, we define the space $C_\mathbf{u}$,

$$C_\mathbf{u} = \left\{ f \in C(S) : \lim_{\mathbf{P}\to\partial S} f(\mathbf{P})\mathbf{u}(\mathbf{P}) = 0 = \lim_{\mathbf{P}\to\mathbf{L}^\pm} f(\mathbf{P})\mathbf{u}(\mathbf{P}) \right\}.$$

Equip $C_\mathbf{u}$ as usual with the weighted uniform norm

$$\|f\|_{C_\mathbf{u}} = \|f\mathbf{u}\|_\infty = \max_{\mathbf{P}\in S} |f(\mathbf{P})\mathbf{u}(\mathbf{P})|.$$

Whenever one or more of the parameters ρ, σ, ξ is greater than 0, the functions in $C_\mathbf{u}$ can be singular on one or both the axes $y = \pm 1$ or along the segment $\{(0, y),\ -1 < y < 1\}$, with a possible exponential growth for $x \to \pm\infty$.

For smoother functions, i.e. for functions having some derivatives which can be discontinuous on the finite boundaries of S or along the segment $\{(0, y),\ -1 < y < 1\}$, we introduce the following Sobolev–type space

$$W_r(\mathbf{u}) = \left\{ f \in C_\mathbf{u} : M_r(f, \mathbf{u}) := \max\left\{ \left\| f_y^{(r)}\mathbf{u} \right\|_\infty, \left\| f_x^{(r)}\varphi^r\mathbf{u} \right\|_\infty \right\} < \infty \right\},$$

where $\varphi(y) = \sqrt{1-y^2}$ and the superscript (r) denotes the rth derivative of the one-dimensional function f_y or f_x. We equip $W_r(\mathbf{u})$ with the norm

$$\|f\|_{W_r(\mathbf{u})} = \|f\mathbf{u}\|_\infty + M_r(f, \mathbf{u}).$$

We remark that u_1 is of the same type of ω_3 in (2.1) and in what follows we set $a_m = \bar{a}_m(u_1)$, where $\bar{a}_m(u_1)$ is defined as in (2.7). Taking into account (2.11), using (2.8) and (2.9), we get

$$E_{m,m}(f)_\mathbf{u} \leq \mathcal{C}\left(\frac{a_m}{m}\right)^r \|f\|_{W_\mathbf{u}^r}, \quad \forall f \in W_r(\mathbf{u}), \tag{5.1}$$

where $0 < \mathcal{C} \neq \mathcal{C}(m, f)$ and $\frac{a_m}{m} \sim m^{\frac{1}{d}-1}$. Therefore a Favard-type inequality holds true also in strips of the plane.

5.2. The "Truncated" Gauss–Freud–Jacobi cubature formula

Fix $w_1(x) = |x|^\gamma e^{-|x|^d}$ and denote by $\{x_k\}_{k=1}^{\lfloor \frac{m}{2} \rfloor}$ the positive zeros of $p_m(w_1)$ and by $x_{m,-k}$ the negative ones. If m is odd, then $x_0 = 0$ is a zero of $p_m(w_1)$. From now on we assume m even.

For any fixed $0 < \theta < 1$ define

$$x_{\bar{q}} = x_{m,\bar{q}(\theta)} := \min\left\{ x_k : x_k \geq \theta a_m, \quad k = 1, 2, \ldots, \frac{m}{2} \right\} \tag{5.2}$$

Now set $w_2(y)=(1+y)^\alpha(1+y)^\beta$ and denote by $\{y_i\}_{i=1}^m$ the zeros of $p_m(w_2)$. Setting $\mathbf{w}(x,y)=w_1(x)w_2(y)$ we consider the following semi-truncated Gaussian rule

$$\int_S F(\mathbf{P})\mathbf{w}(\mathbf{P})\, d\mathbf{P} = \sum_{1\le|i|\le\bar q}\sum_{j=1}^m \lambda_i^{w_1}\lambda_j^{w_2} f(\mathbf{P}_{i,j}) + \mathcal{E}_m(f), \tag{5.3}$$

where $\{\lambda_i^{w_1}\}_{i=-\frac{m}{2}}^{\frac{m}{2}}$ and $\{\lambda_j^{w_2}\}_{i=1}^m$ denote the Christoffel numbers related to the weights w_1 and w_2 respectively.

Truncated Gaussian rules for Freud weights appeared for the first time in [9,16]. In the next proposition an estimate of the cubature error is given.

Proposition 5.1. *For any $F\in C_{\mathbf{u}^2}$, under the assumption*

$$\int_S \frac{\mathbf{w}(\mathbf{P})}{\mathbf{u}^2(\mathbf{P})}\, d\mathbf{P} < +\infty \tag{5.4}$$

it results

$$|\mathcal{E}_m(F)| \le \mathcal{C}\left(E_{M,m}(F)_{\mathbf{u}^2} + e^{-\mathcal{A}m}\|F\mathbf{u}^2\|_\infty\right) \tag{5.5}$$

where

$$M = \left\lfloor \left(\frac{\theta}{1+\theta}\right)^d \frac{m}{2} \right\rfloor \sim m \tag{5.6}$$

and $0<\mathcal{C}\neq\mathcal{C}(m,f)$, $0<\mathcal{A}\neq\mathcal{A}(m,f)$.

5.3. A Nyström method

In this section we describe a Nyström method, based on the cubature formula described in the previous section. As done in the case when S is the first quadrant of the plane, we can immediately prove that the Fredholm Alternative holds true for the integral equation rewritten as in (3.6), under the following assumption for the function kernel $k(x,y,s,t)$

$$\sup_{(s,t)} \mathbf{u}(s,t)\|k_{(s,t)}\|_{W_r(\mathbf{u})} < \infty. \tag{5.7}$$

Indeed mutatis mutandis, Proposition 3.2 can be reformulated for the case under consideration and holds true if we assume that (5.7) is satisfied.

Now starting with the Gaussian formula (5.3) and recalling the definition (5.2), we can define the following discrete operator

$$K_m f(\mathbf{P}) = \mu \sum_{1\le|i|\le\bar q}\sum_{j=1}^m \lambda_i^{w_1}\lambda_j^{w_2} k(\mathbf{P},\mathbf{P}_{i,j}) f(\mathbf{P}_{i,j}), \tag{5.8}$$

where we recall that $P_{i,j}=(x_i,y_j)$ and $x_i,\lambda_i^{w_1}$, $i=-\frac{m}{2},\dots,\frac{m}{2}$, are the zeros of the mth orthonormal polynomial w.r.t. the weight w_1 and the corresponding Christoffel numbers, while $y_j,\lambda_j^{w_2}$, $j=1,\dots,m$, are the zeros of the mth orthonormal polynomial w.r.t. the weight w_2 and the corresponding Christoffel numbers.

Then we consider the operator equation

$$(I-K_m)f_m = g, \tag{5.9}$$

where f_m is unknown. Since we are considering Eq. (3.6) in the weighted space $C_{\mathbf{u}}$ we do the same with (5.9). Therefore we multiply both sides of Eq. (5.9) by the weight $\mathbf{u}$ and then we collocate it on the pairs (x_h, y_ℓ), $|h|=1,\dots,\bar q$, $\ell=1,\dots,m$. In this way we have that the quantities $a_{ij}=f(\mathbf{P}_{i,j})\mathbf{u}(\mathbf{P}_{i,j})$, $|i|=1,\dots,\bar q$, $j=1,\dots,m$, are the unknowns of the linear system

$$a_{h\ell} - \mu\mathbf{u}(\mathbf{P}_{h,\ell}) \sum_{1\le|i|\le\bar q}\sum_{j=1}^m \frac{\lambda_i^{w_1}\lambda_j^{w_2}}{\mathbf{u}(\mathbf{P}_{i,j})} k(\mathbf{P}_{h,\ell},\mathbf{P}_{i,j}) a_{ij} = (g\mathbf{u})(\mathbf{P}_{h,\ell})$$
$$|h|=1,\dots,\bar q, \qquad \ell=1,\dots,m \tag{5.10}$$

The matrix solution $(a_{ij}^*)_{|i|=1,\dots,\bar q, j=1,\dots,m}$ of this system (if it exists) allows us to construct the weighted Nyström interpolant in two variables

$$(f_m\mathbf{u})(\mathbf{P}) = \mu\mathbf{u}(\mathbf{P}) \sum_{1\le|i|\le\bar q}\sum_{j=1}^m \frac{\lambda_i^{w_1}\lambda_j^{w_2}}{\mathbf{u}(\mathbf{P}_{i,j})} k(\mathbf{P},\mathbf{P}_{i,j}) a_{ij}^* + (g\mathbf{u})(\mathbf{P})$$

which will approximate the unknown $f\mathbf{u}$. Now denote by $\mathbf{B}_m$ the coefficient matrix of system (3.10) and by $cond(\mathbf{B}_m)$ its condition number in infinity norm. We can state the following theorem about the convergence and stability of the proposed Nyström method.

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Theorem 5.1. *Assume that k satisfies (5.7), that* $Ker\{I-K\}=\{0\}$ *in* $C_{\mathbf{u}}$ *and* $\mathbf{w}$ *and* $\mathbf{u}$ *verify (5.4). Denote by* f^* *the unique solution of (3.6) in* $C_{\mathbf{u}}$ *for a given* $g\in C_{\mathbf{u}}$. *If in addition, for the same* $r\in\mathbb{N}$ *in (5.7), it results*

$$g\in W_r(\mathbf{u}) \tag{5.11}$$

and

$$\sup_{(x,y)\in S}\mathbf{u}(x,y)\|k_{(x,y)}\|_{W_r(\mathbf{u})}<+\infty, \tag{5.12}$$

then, for m sufficiently large, the system (5.10) is uniquely solvable and well conditioned, since

$$cond(\mathbf{B}_m)\le\mathcal{C},\qquad \mathcal{C}\ne\mathcal{C}(m) \tag{5.13}$$

and in addition

$$\|(f^*-f_m)\mathbf{u}\|_\infty\le\mathcal{C}\left(\frac{a_m}{m}\right)^r\|f^*\|_{W_r(\mathbf{u})}, \tag{5.14}$$

where $\mathcal{C}\ne\mathcal{C}(m,f^*)$.

Comparing (5.14) with (5.1) we remark that the rate of convergence of the proposed Nyström method is of the same order of the error of best polynomial approximation of the solution $f^*\in W_r(\mathbf{u})$.

Also in this case the condition numbers of the involved linear systems are uniformly bounded in virtue of (5.13), where the magnitude of the constant $\mathcal{C}$ depends on the condition number of the integral operator $I-K$ in $C_{\mathbf{u}}$.

6. Computational details and numerical tests

6.1. The matrix of the linear system

We note that the matrix $\mathbf{B_m}$ is a $2\bar{q}-$blocks matrix, with blocks of order m, which takes the following expression

$$\mathbf{B_m}=\begin{pmatrix}\mathbf{B}^{(-\bar q,-\bar q)} & \mathbf{B}^{(-\bar q,-\bar q+1)} & \dots & \mathbf{B}^{(-\bar q,-1)} & \mathbf{B}^{(-\bar q,1)} & \dots & \mathbf{B}^{(-\bar q,\bar q)}\\ \mathbf{B}^{(-\bar q+1,-\bar q)} & \mathbf{B}^{(-\bar q+1,-\bar q+1)} & \dots & \mathbf{B}^{(-\bar q+1,-1)} & \mathbf{B}^{(-\bar q+1,1)} & \dots & \mathbf{B}^{(-\bar q+1,\bar q)}\\ \dots & \dots & \dots & \dots & \dots & \dots & \dots\\ \mathbf{B}^{(-1,-\bar q)} & \mathbf{B}^{(-1,-\bar q+1)} & \dots & \mathbf{B}^{(-1,-1)} & \mathbf{B}^{(-1,1)} & \dots & \mathbf{B}^{(-1,\bar q)}\\ \mathbf{B}^{(1,-\bar q)} & \mathbf{B}^{(1,-\bar q+1)} & \dots & \mathbf{B}^{(1,-1)} & \mathbf{B}^{(1,1)} & \dots & \mathbf{B}^{(1,\bar q)}\\ \dots & \dots & \dots & \dots & \dots & \dots & \dots\\ \mathbf{B}^{(\bar q,-\bar q)} & \mathbf{B}^{(\bar q,-\bar q+1)} & \dots & \mathbf{B}^{(\bar q,-1)} & \mathbf{B}^{(\bar q,1)} & \dots & \mathbf{B}^{(\bar q,\bar q)}\end{pmatrix}$$

with the blocks defined as

$$\mathbf{B}^{(h,i)}=\delta_{h,i}\mathbf{I_m}-\mu\bar{\mathbf{D}}_h\bar{\mathbf{K}}_m^{(h,i)}\bar{\mathbf{U}}_i,\quad |h|,|i|=1,2\dots,\bar q,$$

where

$$\bar{\mathbf{D}}_h=diag(\mathbf{u}(x_h,y_1),\mathbf{u}(x_h,y_2),\dots,\mathbf{u}(x_h,y_m)),$$

$$\bar{\mathbf{U}}_i=diag\left(\frac{\lambda_i^{w_1}\lambda_1^{w_2}}{\mathbf{u}(x_i,y_1)},\frac{\lambda_i^{w_1}\lambda_2^{w_2}}{\mathbf{u}(x_i,y_2)},\dots,\frac{\lambda_i^{w_1}\lambda_m^{w_2}}{\mathbf{u}(x_i,y_m)}\right),$$

$\mathbf{I_m}$ denotes the identity matrix of order m and $\bar{\mathbf{K}}_m^{(h,i)}$ is defined as

$$\bar{\mathbf{K}}_m^{(h,i)}(\ell,j)=k(x_h,y_\ell,x_i,y_j),\quad \ell,j=1,2,\dots,m.$$

6.2. Numerical tests

We propose here three numerical examples, confirming the theoretical results stated in Section 5. We point out that in all the examples the involved parameter satisfy the assumptions in Theorem 5.1. For the choice of the parameters the same considerations made at the beginning of Subsection 4.2 hold true.

Example 4. Consider the equation

$$f(x,y)-\tfrac{1}{2}\int_S\left(\frac{\cos(x+y)}{(15+s^2+t^2)^5}+\frac{\sin(x-y)}{(15+s^2+t^2)^3}\right)f(s,t)e^{-s^2}\,ds\,dt$$
$$=\frac{1}{\pi^2+x^2+y^2}.$$

In this case the involved parameters are the following

$$\mu = \frac{1}{2}, \quad \alpha = \beta = \gamma = 0, \quad d = 2$$

and the weight $\mathbf{u}$ can be chosen with

$$\rho = 0.1, \quad \sigma = 0.2, \quad \xi = 0.4 \quad \eta = 0.2.$$

According to the theoretical results we expect a fast convergence.

The numerical results, for $\theta = 0.7$, are as follows:

m	$2\bar{p}$	Max relative error
4	4	0.85×10^{-4}
8	8	0.77×10^{-6}
16	16	0.29×10^{-9}
32	28	0.12×10^{-14}
64	54	*eps*

The condition numbers of the matrices are less than 1.0006.

Example 5. Consider the equation

$$f(x,y) \; -0.3 \int_S \left(\frac{\cosh(x+y)|\sin(s+t)|}{(\pi^4+s^2+t^2)^5} \right) f(s,t) \frac{e^{-s^2}}{\sqrt{1-t^2}} |s|^{\frac{1}{2}} \; ds \; dt$$

$$= \quad \sin(x+y) e^{\frac{x}{2}}.$$

In this case the involved parameters are the following

$$\mu = 0.3, \quad \alpha = \beta = -0.5, \quad \gamma = 0.5, \quad d = 2,$$

and the weight $\mathbf{u}$ can be chosen with

$$\rho = \sigma = 0, \quad \xi = 0.5, \quad \eta = 0.3.$$

In this case we expect a very low convergence since the solution is only in $C_{\mathbf{u}}$. Nevertheless the numerical results, for $\theta = 0.7$, are as follows:

m	$2\bar{p}$	Max relative error
4	4	0.14×10^{-8}
8	8	0.31×10^{-10}
16	16	0.12×10^{-10}
32	28	0.18×10^{-10}
64	54	0.71×10^{-11}

Also in this case the condition numbers of the matrices are less than 1.00000001, that means that if we are interested in an higher precision we can improve m without occurring in instability problems.

Example 6. Consider the equation

$$f(x,y) \; -0.4 \int_S \left(\frac{\arctan(x+y)\sin(s^2+t^2)}{(225+s^2+t^2)^8} \right) f(s,t) \frac{e^{-s^2}}{\sqrt{1-t}|s|^{\frac{1}{2}}} \; ds \; dt$$

$$= \quad \frac{1}{\pi^2+x^2+y^2}.$$

In this case the integrand has singularities for $s = 0$ and for $t = 1$, i.e. it is unbounded along the segment $\{(0,y), -1 < y < 1\}$ and on the upper boundary of the strip. Hence the involved parameters are the following

$$\mu = 0.4, \quad \alpha = -0.5, \quad \beta = 0, \quad \gamma = -0.5, \quad d = 2$$

and the weight $\mathbf{u}$ can be chosen with

$$\rho = \sigma = 0, \quad \xi = 0.2, \quad \eta = 0.4.$$

According to the theoretical results we expect a very fast convergence.

The numerical results, for $\theta = 0.75$, are as follows:

m	$2\bar{p}$	Max relative error
4	4	0.14×10^{-13}
8	8	0.66×10^{-14}
16	16	0.49×10^{-14}
32	32	eps

The condition numbers of the matrices are less than 1.00000001.

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7. Proofs

Proof of Theorem 2.1. For a given function $f \in C_{\mathbf{u}}$, fix (x, y) in $I_k \times I_j$. Denote by $P_{m,x}$ and $P_{m,y}$ the polynomials of best approximation of the functions f_x and f_y w.r.t. the weights ω_k and ω_j respectively, i.e.

$$E_m(f_x)_{\omega_j} = \max_{z \in I_j} |(f_x(z) - P_{m,x}(z))\omega_j(z)|,$$

$$E_m(f_y)_{\omega_k} = \max_{z \in I_k} |(f_y(z) - P_{m,y}(z))\omega_k(z)|.$$

Then consider the bivariate polynomial $P_{m,m} = \frac{P_{m,x}}{2} + \frac{P_{m,y}}{2}$. Thus, for any fixed (x, y)

$$\begin{aligned} & |[f(x,y) - P_{m,m}(x,y)]\mathbf{u}(x,y)| \\ \le & \frac{\omega_j(y)}{2}|(f_y(x) - P_{m,y}(x))\omega_k(x)| + \frac{\omega_k(x)}{2}|(f_x(y) - P_{m,x}(y))\omega_j(y)| \\ = & \frac{\omega_j(y)}{2}E_m(f_y)_{\omega_k} + \frac{\omega_k(x)}{2}E_m(f_x)_{\omega_j}. \end{aligned}$$

Therefore, taking the maximum on $(x, y) \in I_k \times I_j$ and then taking the infimum on $\mathbb{P}_{m,m}$ the Theorem follows. □

In order to prove the main results of Section 3, first of all we recall some results of the univariate case. Let ω_2 a Generalized Laguerre weight defined as in (2.1) and let $\hat{a}_m(\omega_2)$ be the M-R-S number defined in (2.5).

If $\tilde{p} \in \mathbb{P}_m$ is a polynomial of degree m it results [7]

$$\max_{x \ge 0} |\tilde{p}(x)\omega_2(x)| \le \mathcal{C} \max_{0 \le x \le \hat{a}_m(\omega_2)} |\tilde{p}(x)\omega_2(x)|, \tag{7.1}$$

and, for any $\delta > 0$,

$$\max_{x \ge \hat{a}_m(\omega_2)(1+\delta)} |\tilde{p}(x)\omega_2(x)| \le \mathcal{C}e^{-\mathcal{A}m} \max_{0 \le x \le \hat{a}_m(\omega_2)} |\tilde{p}(x)\omega_2(x)|, \tag{7.2}$$

where $\mathcal{C}$ and $\mathcal{A}$ are positive constants independent of m.

Now let $\mathbf{u}(\mathbf{P}) = u_1(x)u_2(y)$ with $u_1(x) = x^{\sigma_1}e^{-\frac{x^b}{2}}(1+x)^{\eta_1}$, $u_2(y) = y^{\sigma_2}e^{-\frac{y^b}{2}}(1+y)^{\eta_2}$, $\sigma_1, \sigma_2, \eta_1, \eta_2 \ge 0$, $b > \frac{1}{2}$. We remark that u_1 and u_2 are of the same type of ω_2.

Moreover let $a_m = \max\{\hat{a}_m(u_1), \hat{a}_m(u_2)\}$ and set $D_0 = [0, a_m]^2$. By (7.1), for any $Q \in \mathbb{P}_{m,m}$, it follows

$$\|Q\mathbf{u}\|_\infty \le \mathcal{C} \max_{\mathbf{P} \in D_0} |Q(\mathbf{P})\mathbf{u}(\mathbf{P})|, \quad \mathcal{C} \ne \mathcal{C}(m). \tag{7.3}$$

Next lemma extends the univariate infinite-finite range inequality (7.2) to the bivariate case.

Lemma 7.1. *Let $\delta > 0$ be fixed and $D_\delta = [0, a_m(1+\delta)]^2$, $S = (0, +\infty) \times (0, +\infty)$, $\mathbf{u}(\mathbf{P}) = u_1(x)u_2(y)$ with u_1 and u_2 defined above. For any polynomial $Q \in \mathbb{P}_{m,m}$,*

$$\max_{\mathbf{P} \in S \setminus D_\delta} |Q(\mathbf{P})\mathbf{u}(\mathbf{P})| \le \mathcal{C}e^{-\mathcal{A}m} \max_{\mathbf{P} \in D_0} |Q(\mathbf{P})\mathbf{u}(\mathbf{P})|, \tag{7.4}$$

where $0 < \mathcal{C} \ne \mathcal{C}(m, f)$, $0 < \mathcal{A} \ne \mathcal{A}(m, f)$.

Proof. Let $\mathbf{P}_0 \equiv (x_0, y_0)$ s.t.

$$\max_{\mathbf{P} \in S \setminus D_\delta} |Q(\mathbf{P})u(\mathbf{P})| = |Q(\mathbf{P}_0)u(\mathbf{P}_0)|.$$

Using (7.2) we have

$$\begin{aligned} |Q(\mathbf{P}_0)\mathbf{u}(\mathbf{P}_0)| &\le \mathcal{C}e^{-\mathcal{A}_1 m} \max_{y \le a_m} |Q(x_0, y)|u_1(x_0)u_2(y) \\ &\le \mathcal{C}e^{-(\mathcal{A}_1+\mathcal{A}_2)m} \max_{x \le a_m} \max_{y \le a_m} |Q(x, y)|u_1(x)u_2(y) \\ &= \mathcal{C}e^{-\mathcal{A}m} \max_{\mathbf{P} \in D_0} |Q(\mathbf{P})\mathbf{u}(\mathbf{P})|, \end{aligned}$$

and the Lemma follows. □

Proof of Proposition 3.1. Let $M = \left\lfloor \left(\frac{\theta}{\theta+1}\right)^b m \right\rfloor$ and $Q_{M,M} \in \mathbb{P}_{M,M}$ be the polynomial of best approximation of $F \in C_{\mathbf{u}^2}$. We can write

$$\mathcal{E}_m(f) = \mathcal{E}_m(F - Q_{M,M}) + \mathcal{E}_m(Q_{M,M}).$$

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First we get

$$|\mathcal{E}_m(F-Q_{M,M})| \le \|(F-Q_{M,M})\mathbf{u}^2\|_\infty \left[\int_S \frac{\mathbf{w}(\mathbf{P})}{\mathbf{u}^2(\mathbf{P})}d\mathbf{P} + \sum_{i=1}^q\sum_{j=1}^q \frac{\lambda_i^{w_1}\lambda_j^{w_2}}{\mathbf{u}^2(\mathbf{P}_{i,j})}\right]$$

Since

$$\lambda_i^{w_1} \sim w_1(x_i)\Delta x_i, \quad \lambda_j^{w_2} \sim w_2(y_j)\Delta y_j, \quad \Delta x_i = x_{i+1}-x_i, \quad \Delta y_j = y_{j+1}-y_j$$

it can be easily deduced that

$$\sum_{i=1}^q\sum_{j=1}^q \frac{\lambda_i^{w_1}\lambda_j^{w_2}}{\mathbf{u}^2(\mathbf{P}_{i,j})} \le \mathcal{C}\sum_{i=1}^q \Delta x_i \frac{w_1(x_i)}{u_1^2(x_i)} \sum_{j=1}^q \Delta y_j \frac{w_2(y_j)}{u_2^2(y_j)} \le \mathcal{C}\int_S \frac{\mathbf{w}(\mathbf{P})}{\mathbf{u}^2(\mathbf{P})}d\mathbf{P}, \tag{7.5}$$

and then we obtain

$$|\mathcal{E}_m(F-Q_{M,M})| \le \mathcal{C}E_{M,M}(F)_{\mathbf{u}^2}\int_S \frac{\mathbf{w}(\mathbf{P})}{\mathbf{u}^2(\mathbf{P})}d\mathbf{P}. \tag{7.6}$$

On the other hand

$$|\mathcal{E}_m(Q_{M,M})| = \left|\int_S Q_{M,M}(\mathbf{P})\mathbf{w}(\mathbf{P})d\mathbf{P} - \sum_{i=1}^q\sum_{j=1}^q \lambda_i^{w_1}\lambda_j^{w_2}Q_{M,M}(\mathbf{P}_{i,j})\right|$$
$$= \left|\sum_{i=q+1}^m\sum_{j=q+1}^m \lambda_i^{w_1}\lambda_j^{w_2}Q_{M,M}(\mathbf{P}_{i,j})\right|$$
$$\le \max_{\mathbf{P}\in[\theta a_m,+\infty)\times[\theta a_m,+\infty)} |Q_{M,M}(\mathbf{P})\mathbf{u}^2(\mathbf{P})| \int_S \frac{\mathbf{w}(\mathbf{P})}{\mathbf{u}^2(\mathbf{P})}d\mathbf{P}.$$

Hence, we apply (7.4) to $Q_{M,M}$, by replacing u_1, u_2 with u_1^2, u_2^2 respectively and with $\delta=\theta$ and taking into account that $a_m \sim a_m(\mathbf{u}^2)$ and that M is the maximal value s.t. $a_M(1+\theta) \le a_m\theta$, we deduce

$$|\mathcal{E}_m(Q_{M,M})| \le \mathcal{C}\left(E_{M,M}(F)_{\mathbf{u}^2} + e^{-\mathcal{A}m}\|F\mathbf{u}^2\|_\infty\right)\int_S \frac{\mathbf{w}(\mathbf{P})}{\mathbf{u}^2(\mathbf{P})}d\mathbf{P} \tag{7.7}$$

and consequently the Proposition follows by (7.6) and (7.7), under the assumption (3.4). □

Proof of Proposition 3.2. Following [17] it is sufficient to show that

$$\lim_m \sup_{\|f\mathbf{u}\|_\infty\le 1} E_{m,m}(Kf)_{\mathbf{u}} = 0.$$

Let $Q_m(x, y, s, t)$ be a polynomial of degree m in each variable, coinciding for any fixed (s, t) in S, with the polynomial of best approximation of $k_{(s,t)}(x, y)$ in $C_{\mathbf{u}}$. For any $f \in C_{\mathbf{u}}$, define the bivariate polynomial

$$K^{Q_m}f(x,y) = \int_S Q_m(x,y,s,t)f(s,t)\mathbf{w}(s,t)\ ds\ dt.$$

Then

$$\left|Kf(x,y) - K^{Q_m}f(x,y)\right|\mathbf{u}(x,y) =$$
$$= \left|\int_S [k(x,y,s,t) - Q_m(x,y,s,t)]f(s,t)\mathbf{w}(s,t)\ ds\ dt\right|\mathbf{u}(x,y)$$
$$\le \|f\mathbf{u}\|_\infty \int_S \left|k_{(s,t)}(x,y) - Q_m(x,y,s,t)\right|\mathbf{u}(x,y)\mathbf{u}(s,t)\frac{\mathbf{w}(s,t)}{\mathbf{u}^2(s,t)}\ ds\ dt$$
$$\le \|f\mathbf{u}\|_\infty \sup_{(s,t)\in S}\mathbf{u}(s,t)E_m(k_{(s,t)})_{\mathbf{u}}\int_S \frac{\mathbf{w}(s,t)}{\mathbf{u}^2(s,t)}\ ds\ dt.$$

Therefore

$$E_{m,m}(Kf)_{\mathbf{u}} \le \|[Kf - K^{Q_m}f]\mathbf{u}\|_\infty$$
$$\le \|f\mathbf{u}\|_\infty \sup_{(s,t)\in S}\mathbf{u}(s,t)E_m(k_{(s,t)})_{\mathbf{u}}\int_S \frac{\mathbf{w}(s,t)}{\mathbf{u}^2(s,t)}\ ds\ dt.$$

Hence the proposition follows by the assumptions (3.4) and (3.7). □

From now on, when the context is clear, we will omit the subscript in the operator norms, i.e. we will simply denote $\|\cdot\|$ instead of $\|\cdot\|_{C_{\mathbf{u}}\to C_{\mathbf{u}}}$.

Proof of Theorem 3.1. First we prove that the Nyström method is convergent and stable in $C_{\mathbf{u}}$. This can be done by showing that,

1. $\lim_m \|[Kf - K_m f]\mathbf{u}\|_\infty = 0$ for any $f \in C_{\mathbf{u}}$;
2. $\sup_m \lim_n \sup_{\|f\mathbf{u}\|_\infty \le 1} E_{n,n}(K_m f)_{\mathbf{u}} = 0.$

By virtue of the principle of uniform boundedness, if Step (1) holds, then

$$\sup_m \|K_m\| < +\infty.$$

On the other hand, Step (2) is equivalent to prove that the sequence $\{K_m\}_m$ is collectively compact and consequently that $\|(K - K_m)K_m\| \to 0$. Therefore (see for instance [1]), under these assumptions and for m sufficiently large, $I - K_m$ is invertible in $C_{\mathbf{u}}$ and uniformly bounded, i.e. the method is stable, since

$$\|(I - K_m)^{-1}\| \le \frac{1 + \|(I-K)^{-1}\| \ \|K_m\|}{1 - \|(I-K)^{-1}\| \|(K - K_m)K_m\|},$$

and, in addition,

$$\|[f^* - f_m]\mathbf{u}\|_\infty \sim \|[Kf - K_m f]\mathbf{u}\|_\infty, \tag{7.8}$$

where the constants in $\sim$ are independent of m and f^*.

Thus, we start proving Step (1). We note that

$$\|[Kf - K_m f]\mathbf{u}\|_\infty = \sup_{(x,y)\in S} |\mathcal{E}_m(k_{(x,y)} f)|\mathbf{u}(x,y).$$

Therefore by (3.5) it follows

$$\begin{aligned} \|[Kf - K_m f]\mathbf{u}\|_\infty &\le \mathcal{C} \sup_{(x,y)\in S} \mathbf{u}(x,y) E_{M,M}(k_{(x,y)} f)_{\mathbf{u}^2} \\ &+ e^{-\mathcal{A}m} \sup_{(x,y)\in S} \mathbf{u}(x,y) \|k_{(x,y)} f\mathbf{u}^2\|_\infty \\ &\le \mathcal{C} \sup_{(x,y)\in S} \mathbf{u}(x,y) E_{M,M}(k_{(x,y)})_{\mathbf{u}} \|f\mathbf{u}\|_\infty \\ &+ \mathcal{C} \sup_{(x,y)\in S} \mathbf{u}(x,y) \|k_{(x,y)}\mathbf{u}\|_\infty \left(E_{M,M}(f)_{\mathbf{u}} + e^{-\mathcal{A}m}\|f\mathbf{u}\|_\infty\right) \end{aligned} \tag{7.9}$$

and hence Step (1) follows by (3.13). Estimate (3.15) follows by (7.8) and (7.9), taking into account (3.1).

In order to prove Step (2), we have to estimate $E_{n,\,n}(K_m f)_{\mathbf{u}}$, for all n. Let $Q_n(x, y, s, t)$ be a polynomial of degree n in each variable, coinciding for any fixed (s, t) in S, with the polynomial of best approximation of $k_{(s,\,t)}(x, y)$ in $C_{\mathbf{u}}$. Define, for any $f \in C_{\mathbf{u}}$, the polynomial

$$K^{Q_n} f(x,y) = \mu \sum_{i=1}^q \sum_{j=1}^q \frac{\lambda_i^{w_1}\lambda_j^{w_2}}{\mathbf{u}(x_i,y_j)} Q_n(x,y,x_i,y_j) f(x_i,y_j)\mathbf{u}(x_i,y_j).$$

Then

$$\begin{aligned} &\left|K_m f(x,y) - K^{Q_n} f(x,y)\right|\mathbf{u}(x,y) = \\ &= \mathbf{u}(x,y)\left|\mu \sum_{i=1}^q \sum_{j=1}^q \frac{\lambda_i^{w_1}\lambda_j^{w_2}}{\mathbf{u}(x_i,y_j)}\left[k(x,y,x_i,y_j) - Q_n(x,y,x_i,y_j)\right] f(x_i,y_j)\mathbf{u}(x_i,y_j)\right| \\ &\le \|f\mathbf{u}\|_\infty |\mu| \sum_{i=1}^q \sum_{j=1}^q \frac{\lambda_i^{w_1}\lambda_j^{w_2}}{\mathbf{u}^2(x_i,y_j)} \left|k(x,y,x_i,y_j) - Q_n(x,y,x_i,y_j)\right| \mathbf{u}(x,y)\mathbf{u}(x_i,y_j) \\ &\le \|f\mathbf{u}\|_\infty |\mu| \sum_{i=1}^q \sum_{j=1}^q \frac{\lambda_i^{w_1}\lambda_j^{w_2}}{\mathbf{u}^2(x_i,y_j)} \mathbf{u}(x_i,y_j) E_{n,n}(k_{(x_i,y_j)})_{\mathbf{u}}. \end{aligned}$$

Taking into account (7.5), we get

$$\begin{aligned} &E_{n,n}(K_m f)_{\mathbf{u}} \le \|[K_m f - K^{Q_n} f]\mathbf{u}\|_\infty \\ &\le \mathcal{C}\|f\mathbf{u}\|_\infty \sup_{(s,t)\in S} E_{n,n}(k_{(s,t)})_{\mathbf{u}} \int_S \frac{\mathbf{w}(s,t)}{\mathbf{u}^2(s,t)}\, ds\, dt, \quad \mathcal{C} \ne \mathcal{C}(m,n,f) \end{aligned}$$

Therefore Step (2) follows by (3.4) and (3.7).

Finally (3.14) follows by standard arguments. The reader can consult the general framework in [1] or, referring to the bivariate case and mutatis mutandis, the proof of Theorem 3.1 in [4]. □

Now we conclude the section with the proofs of the main results of Section 5.

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Let ω_3 be defined in (2.1) and $\bar{a}_m = \bar{a}_m(\omega_3)$ the corresponding M-R-S number. For any polynomial $\tilde{p} \in \mathbb{P}_m$ and for any fixed $d > 0$, it results (see [18, (4.3), (4.4)]),

$$\max_{x\in\mathbb{R}} |\widetilde{p}(x)\omega_3(x)| \le \mathcal{C} \max_{[-\bar{a}_m,\bar{a}_m]\setminus\left[-d\frac{\bar{a}_m}{m},d\frac{\bar{a}_m}{m}\right]} |\widetilde{p}(x)\omega_3(x)|, \tag{7.10}$$

and, for any $\delta > 0$,

$$\max_{|x|\ge \bar{a}_m(1+\delta)} |\widetilde{p}(x)\omega_3(x)| \le \mathcal{C}e^{-\mathcal{A}m} \max_{x\in\mathbb{R}} |\widetilde{p}(x)\omega_3(x)|, \tag{7.11}$$

where $\mathcal{C}$ and $\mathcal{A}$ are positive constants independent of m.

Now let $\mathbf{u}(\mathbf{P}) = u_1(x)u_2(y)$ with $u_1(x) = |x|^\xi(1+|x|)^\eta e^{-\frac{|x|^d}{2}}$, $u_2(y) = (1-y)^\rho(1+y)^\sigma$, $\xi, \eta, \rho, \sigma \ge 0$. We remark that u_1 is of the same type of ω_3.

Morever let $a_m := \bar{a}_m(u_1)$ and set $D_0 = [-a_m, a_m]^2$. By (7.10), for any $Q \in \mathbb{P}_{m,m}$, it follows

$$\|Q\mathbf{u}\|_\infty \le \mathcal{C} \max_{\mathbf{P}\in D_0} |Q(\mathbf{P})\mathbf{u}(\mathbf{P})|, \quad \mathcal{C} \ne \mathcal{C}(m). \tag{7.12}$$

The next Lemma extends the univariate infinite–finite range inequality (7.11) to the case of the strip.

Lemma 7.2. *Let $\delta > 0$ be fixed and $D_\delta = [-a_m(1+\delta), a_m(1+\delta)] \times [-1, 1]$. Moreover let $\mathbf{u}(\mathbf{P}) = u_1(x)u_2(y)$ be defined as above. Then, for any polynomial $Q \in \mathbb{P}_{m,m}$*

$$\max_{\mathbf{P}\in S\setminus\bar{D}_\delta} |Q(\mathbf{P})u(\mathbf{P})| \le \mathcal{C}e^{-\mathcal{A}m} \max_{\mathbf{P}\in D_0} |Q(\mathbf{P})u(\mathbf{P})|, \tag{7.13}$$

where $0 < \mathcal{C} \ne \mathcal{C}(m, f)$, $0 < \mathcal{A} \ne \mathcal{A}(m, f)$.

Proof of Proposition 5.1. Let $M = \left\lfloor \left(\frac{\theta}{\theta+1}\right)^d \frac{m}{2} \right\rfloor$ and $Q_{M,m} \in \mathbb{P}_{M,m}$ be the polynomial of best approximation of $F \in C_{\mathbf{u}^2}$.

We can write

$$\mathcal{E}_m(F) = \mathcal{E}_m(F - Q_{M,m}) + \mathcal{E}_m(Q_{M,m}).$$

First we get

$$|\mathcal{E}_m(F - Q_{M,m})| \le \|(F - Q_{M,m})\mathbf{u}^2\|_\infty \left[\int_S \frac{\mathbf{w}(\mathbf{P})}{\mathbf{u}^2(\mathbf{P})} d\mathbf{P} \right.$$
$$\left. + \sum_{1\le|i|\le\bar{q}} \sum_{j=1}^m \frac{\lambda_i^{w_1}\lambda_j^{w_2}}{\mathbf{u}^2(\mathbf{P}_{i,j})} \right] \le \mathcal{C}E_{M,m}(F)_{\mathbf{u}^2} \int_S \frac{\mathbf{w}(\mathbf{P})}{\mathbf{u}^2(\mathbf{P})} d\mathbf{P}. \tag{7.14}$$

On the other hand, setting $D_{m,\theta} = \{(x, y) : |x| \ge \theta a_m, \ -1 \le y \le 1\}$

$$\begin{aligned} |\mathcal{E}_m(Q_{M,m})| &= \left| \int_S Q_{M,m}(\mathbf{P})\mathbf{w}(\mathbf{P})d\mathbf{P} - \sum_{1\le|i|\le\bar{q}} \sum_{j=1}^m \lambda_i^{w_1}\lambda_j^{w_2} Q_{M,m}(\mathbf{P}_{i,j}) \right| \\ &= \left| \sum_{\bar{q}+1\le|i|\le m} \sum_{j=1}^m \lambda_i^{w_1}\lambda_j^{w_2} Q_{M,m}(\mathbf{P}_{i,j}) \right| \\ &\le \max_{\mathbf{P}\in\bar{D}_{m,\theta}} |Q_{M,m}(\mathbf{P})\mathbf{u}^2(\mathbf{P})| \int_S \frac{\mathbf{w}(\mathbf{P})}{\mathbf{u}^2(\mathbf{P})} d\mathbf{P}. \end{aligned}$$

Hence we apply (7.13) to $Q_{M,\ m}$ replacing u_1, u_2 with u_1^2, u_2^2 and $\delta = \theta$, and taking into account that $a_m \sim a_m(u_1^2)$ and that M is the maximal value s.t. $a_M(1+\theta) \le a_m\theta$. Therefore we get

$$|\mathcal{E}_m(Q_{M,m})| \le \mathcal{C}\left[E_{M,m}(F)_{\mathbf{u}^2} + e^{-\mathcal{A}m}\|F\mathbf{u}^2\|_\infty\right] \int_S \frac{\mathbf{w}(\mathbf{P})}{\mathbf{u}^2(\mathbf{P})} d\mathbf{P} \tag{7.15}$$

and consequently the Proposition follows by (7.14) and (7.15) and assumption (5.4). □

Proof of Theorem 5.1. The proof can be lead as the proof of Theorem 3.1, by using assumptions (5.7), (5.11) and (5.12) instead of (3.7), (3.12) and (3.13) respectively and the estimate (5.5) instead of (3.5). □

8. Conclusions

We presented a global approximation method to approximate the solution of bivariate Fredholm equations defined on unbounded domains by means of a suitable Nyström method. Among all the possible unbounded domains we have treated in details the half-plane $x \ge 0$, $y \ge 0$ and the strip $x \in \mathbf{R}, y \in [-1, 1]$. The procedure essentially uses the Gaussian cubature rules, w.r.t. Generalized Laguerre and Freud weights which are of non standard type. According to the theoretical error estimates, accurate results have been achieved for "small" value of m also when the involved functions have derivatives presenting algebraic singularities along the finite borders of the domain. Moreover smoother are the functions, faster is the rate of convergence. Finally, the additional difficulties due to the unboundedness of the domains have been overcome by using truncated Gaussian rules [6], which lead to linear systems of reduced order (from order m^2 to j^2, being $j = j(m) \ll m$).

In conclusion we remark that, since in the previous paper [4] the analogous problem on the domain $[-1,1] \times [-1,1]$ was treated, we have now a complete framework for a wide class of finite, semi-infinite and infinite rectangular domains.

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