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On the stability of a modified Nyström method for Mellin convolution equations in weighted spaces

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Abstract This paper deals with the numerical solution of second kind integral equations with fixed singularities of Mellin convolution type. The main difficulty in solving such equations is the proof of the stability of the chosen numerical method, being the noncompactness of the Mellin integral operator the chief theoretical barrier. Here, we propose a Nyström method suitably modified in order to achieve the theoretical stability under proper assumptions on the Mellin kernel. We also provide an error estimate in weighted uniform norm and prove the well-conditioning of the involved linear systems. Some numerical tests which confirm the efficiency of the method are shown.

Keywords Mellin kernel · Mellin convolution equations · Nyström method · Gaussian rule

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1 Introduction

In this paper we consider second kind integral equations of Mellin convolution type having the form

$$f(y) + \int_0^1 k(x,y)f(x)dx + \int_0^1 h(x,y)f(x)dx = g(y), \quad y \in (0,1], \tag{1}$$

where $f(y)$ is the unknown, $h(x,y)$ and $g(y)$ are given sufficiently smooth functions and

$$k(x,y) = \frac{1}{x}\tilde{k}\left(\frac{y}{x}\right) \tag{2}$$

is a Mellin kernel with $\tilde{k}$ a given function on $[0,+\infty)$ satisfying proper assumptions.

The equation (1) can be written in operator form as

$$(\mathcal{I} + \mathcal{K} + \mathcal{H})f = g, \tag{3}$$

where $\mathcal{K}$ and $\mathcal{H}$ are the integral operators defined as follows

$$(\mathcal{K}F)(y) = \int_0^1 k(x,y)F(x)dx, \tag{4}$$

$$(\mathcal{H}F)(y) = \int_0^1 h(x,y)F(x)dx, \tag{5}$$

and $\mathcal{I}$ denotes the identity operator.

The development of numerical methods for the solution of such kind of integral equations has a strong practical motivation due to the wide range of applications, particularly in engineering and physics.

In literature, several papers dealing with the numerical treatment of Mellin convolution integral equations are available. Numerical methods based on piecewise or global polynomial approximation were proposed in [14,21,8,9, 19,7,20,10,17,22,11,4,15,5]. A crucial challenge is the proof of the stability since, because of the fixed singularity of the Mellin kernel at the origin, the classical theory for the numerical analysis of second kind Fredholm integral equations cannot be applied. To this end, suitable regularization techniques as well as modifications of the classical methods in a neighbourhood of 0 are employed.

In particular, in [9] the Author assumed that the Mellin kernel $k(x,y)$ in (2) satisfies the following condition

$$\int_0^{+\infty} t^{\frac{1}{p}-1+\sigma}|\tilde{k}(t)|dt < +\infty, \tag{6}$$

with $1 \le p \le +\infty$ and $\sigma \le 0$. Galerkin, collocation and Nyström methods, employing spline approximation and suitable cutoff techniques, were introduced. Stability and exponential rates of convergence are obtained requiring some additional smoothness of the Mellin kernel and the right-hand side function.

Very recently, in [15] the case when $\tilde{k}(t)$ is non-negative and fulfills (6) with $\sigma = 0$ and $p = \infty$ was considered. In such a case the solution of the equation (1) is continuous on $[0, 1]$. A modified Nyström method based on the Gauss-Legendre quadrature rule was introduced and its stability and convergence were proved.

Subsequently in [5] we treated the case where condition (6) holds true for $1 \leq p < +\infty$ and $\sigma \in [-\frac{1}{p}, 1 - \frac{1}{p}]$. According to the corresponding mapping properties of the Mellin operator $\mathcal{K}$, we looked for the solution of the problem in suitable weighted L^p spaces. Then, for its approximation we proposed a Nyström type method based on a proper Gauss-Jacobi quadrature formula. A slight modification of the classical procedure together with a preconditioning strategy [16] allowed us to obtain satisfactory numerical results showing the stability and convergence of the method. A theoretical proof of the stability has remained an open challenge. Nevertheless, assuming it, the convergence was proved and an error estimate was given.

In this paper we address the concern over the stability of a numerical procedure for the solution of (3) in the case where the kernel $k(x, y)$ in (2) satisfies the condition (6) with $p = +\infty$ and $\sigma > 0$, i.e.

$$\int_0^{+\infty} t^{-1+\sigma} |\tilde{k}(t)| dt < +\infty, \quad \sigma > 0. \tag{7}$$

Under this assumption the Mellin integral operator $\mathcal{K}$ is not necessarily bounded with respect to the uniform norm. We study the integral equation in a suitable weighted space of continuous functions. Then, following an idea in [12], we consider an equivalent Mellin integral equation whose unknown is at least a continuous function. Finally, in order to approximate its solution, we apply a modified Nyström method.

Since the definition of the integral operators associated to the new equation involves a Jacobi weight, the proposed method uses a Gauss-Jacobi quadrature formula for their discretization. Unfortunately, due to the fixed singularity of the Mellin kernel at the point $x = y = 0$, such quadrature rule becomes inefficient for the approximation of the Mellin operator when y is very close to 0. Therefore, it becomes necessary to modify it in order to achieve stability and convergence results.

To this end, we follow an idea proposed in [19], where a modified Nyström method based on the Gauss-Legendre formula is proposed for the numerical solution of a special singular integral equation arising from a cruciform crack problem.

This approach led us to reach our goal of proving theoretically the stability and the convergence of the proposed method. Furthermore, we are able to provide an error estimate in weighted uniform norm and to prove the well-conditioning of the involved linear systems which is crucial for the computation of the approximate solution.

The paper is organized as follows. In Section 2 we collect some definitions and preliminary results useful in the sequel. Section 3 deals with the mapping properties of the operators $\mathcal{K}$ and $\mathcal{H}$. In Section 4 we introduce the new equivalent integral equation we go to solve. The description of the proposed numerical method and the results stating its stability and convergence are given in Section 5. Section 6 contains the proofs of the theoretical results. Finally, in Section 7 we show some numerical tests which confirm the efficiency of the method.

2 Preliminaries

2.1 Function spaces

In this section we introduce some notations and recall some results, which will be useful in the sequel.

With $v^\rho(x) = x^\rho$, $\rho > -\frac{1}{p}$, $1 \le p < +\infty$, we denote by $L^p_{v^\rho}$ the set of all functions such that

$$\|F\|_{L^p_{v^\rho}} = \|v^\rho F\|_p = \left(\int_0^1 |v^\rho(x) F(x)|^p dx \right)^{\frac{1}{p}} < +\infty.$$

When $p = +\infty$ we define, for $\rho > 0$,

$$L^\infty_{v^\rho} := C_{v^\rho} = \left\{ F \in C((0,1]) \ : \ \lim_{x \to 0} v^\rho(x) F(x) = 0 \right\},$$

where $C(A)$ is the set of the continuous functions in $A \subseteq [0,1]$. The space C_{v^ρ} equipped with the norm

$$\|F\|_{C_{v^\rho}} = \|v^\rho F\|_\infty = \sup_{0 \le x \le 1} |(v^\rho F)(x)|$$

is a Banach space. For $\rho = 0$, we set $C := C_{v^0} = C([0,1])$.

For smoother functions, we consider the following Sobolev type subspaces of $L^p_{v^\rho}$ of order $r \in \mathbb{N}$

$$W^p_r(v^\rho) = \left\{ F \in L^p_{v^\rho} : \ F^{(r-1)} \in AC(0,1), \ \|v^\rho F^{(r)} \varphi^r\|_p < +\infty \right\},$$

where $1 \le p \le +\infty$, $\varphi(x) = \sqrt{x(1-x)}$ and $AC(0,1)$ is the collection of all functions which are absolutely continuous on every closed subset of $(0,1)$. The space $W^p_r(v^\rho)$ equipped with the following norm

$$\|F\|_{W^p_r(v^\rho)} = \|v^\rho F\|_p + \|v^\rho F^{(r)} \varphi^r\|_p$$

is a Banach space, too.

Denoting by $\mathbb{P}_m$ the set of all polynomials of degree at most m, the error of best approximation

$$E_m(F)_{v^\rho,p} = \inf_{P \in \mathbb{P}_m} \|v^\rho (F - P)\|_p$$

satisfies the following estimate (see, for example, [18, (2.5.22), p. 172])

$$E_m(F)_{v^\rho,p} \le \frac{\mathcal{C}}{m^r} \|v^\rho F^{(r)} \varphi^r\|_p, \quad \mathcal{C} \ne \mathcal{C}(m,F), \tag{8}$$

for any function $F \in W_r^p(v^\rho)$ with $r \in \mathbb{N}$, $1 \le p \le +\infty$ and $\rho > -\frac{1}{p}$.

In (8) and in the following $\mathcal{C}$ denotes a positive constant which may assume different values in different formulas. We write $\mathcal{C}(a, b, \ldots)$ to say that $\mathcal{C}$ depends only on the parameters $a, b, \ldots$ and $\mathcal{C} \neq \mathcal{C}(a, b, \ldots)$ to say that $\mathcal{C}$ is independent of the parameters $a, b, \ldots$.

From now on we use the notation W_r^∞ and $E_m(F)_\infty$ for the case $\rho = 0$ and $p = +\infty$. Moreover, we consider the following subspaces

$$\mathring{C} = \{f \in C \ : \ f(0) = 0\} \quad \text{and} \quad \mathring{W}_r^\infty = W_r^\infty \cap \mathring{C}$$

of C and W_r^∞, respectively.

2.2 Quadrature formula

The Nyström type method we are going to introduce will be based on a quadrature rule of Gaussian type. For a Jacobi weight $v^\rho(x) = x^\rho$, $\rho > -1$, on the interval $[0, 1]$, we consider the following formula

$$\int_0^1 F(x) v^\rho(x) dx = \sum_{j=1}^m \lambda_{m,j}^\rho F(x_{m,j}^\rho) + e_m(F), \tag{9}$$

where $x_{m,j}^\rho, j = 1, \ldots, m$, are the zeros of the m-th Jacobi polynomial which is orthogonal w.r.t. the weight v^ρ, $\lambda_{m,j}^\rho, j = 1, \ldots, m$, are the corresponding Christoffel numbers and $e_m(F)$ is the remainder term.

For functions F belonging to $W_r^1(v^\rho)$ the following estimate of the quadrature error $e_m(F)$ is well known (see, for example, [18, Theorem 5.1.8, p. 338] and [18, (2.5.22), p. 172])

$$|e_m(F)| \le \frac{\mathcal{C}}{m^r} \int_0^1 |F^{(r)}(x)| \varphi^r(x) v^\rho(x) dx, \quad \mathcal{C} \neq \mathcal{C}(m, F). \tag{10}$$

where $r \in \mathbb{N}$ and $\rho > -1$.

3 The Mellin integral equation

In next theorems we state some results concerning the mapping properties of the operators $\mathcal{K}$ and $\mathcal{H}$ given in (4) and (5), respectively. They allow us to establish the solvability of equation (3) in suitable weighted spaces.

Theorem 1 *If the function $\tilde{k}(t)$ in (2) satisfies (7) for some $\sigma > 0$, then the operator $\mathcal{K} : C_{v^\sigma} \longrightarrow C_{v^\sigma}$ is continuous and*

$$\|\mathcal{K}\|_{C_{v^\sigma} \to C_{v^\sigma}} \le \int_0^{+\infty} t^{-1+\sigma} |\tilde{k}(t)| dt. \tag{11}$$

Theorem 2 *If the kernel $h(x,y)$ satisfies*

$$\lim_{y\to 0}\sup_{0\le x\le 1}|v^{\sigma}(y)h(x,y)|=0 \quad \textit{and} \quad \sup_{0\le x\le 1}\|h(x,\cdot)\|_{W_r^{\infty}(v^{\sigma})}<+\infty, \tag{12}$$

for some $0<\sigma<1$ and $r\ge 1$, then the operator $\mathcal{H}:C_{v^\sigma}\longrightarrow C_{v^\sigma}$ is compact.

Using the Neumann series theorem and [13, Corollary 3.8], the following result is a consequence of the above theorems.

Theorem 3 *Let us assume that $\mathrm{Ker}(\mathcal{I}+\mathcal{K}+\mathcal{H})=\{0\}$ in C_{v^σ}, with $0<\sigma<1$. If the function $\tilde{k}(t)$ in (2) is such that*

$$\int_0^{+\infty} t^{-1+\sigma}|\tilde{k}(t)|dt<1 \tag{13}$$

and the kernel $h(x,y)$ satisfies (12), then equation (3) admits a unique solution f in C_{v^σ} for each right-hand side $g\in C_{v^\sigma}$.

4 A new integral equation

From now on, we will assume that the hypotheses of Theorem 3 are fulfilled and that the function $\tilde{k}(t)$ and the kernel $h(x,y)$ satisfy the following further assumptions:

$$\int_0^{+\infty} t^{-1+\beta}|\tilde{k}(t)|dt<1, \quad \text{for some } 0<\sigma<\beta<1, \tag{14}$$

and

$$h(\cdot,y)\in C \text{ uniformly with respect to } y. \tag{15}$$

We remark that, since $C_{v^\sigma}\subset C_{v^\beta}$, from (14) it is possible to deduce that the equation (3) is unisolvent in C_{v^β}, too. Moreover, for a fixed $g\in C_{v^\sigma}$, there exists $\bar{g}\in \mathring{C}$ such that $\bar{g}=v^\beta g$. Then, if f is the unique solution of (3) corresponding to g, we can find a function $\bar{f}\in\mathring{C}$ such that $\bar{f}=v^\beta f$. Letting

$$(\bar{\mathcal{K}}F)(y)=\int_0^1 \bar{k}(x,y)F(x)v^{-\beta}(x)dx, \quad y\in(0,1], \tag{16}$$

with the kernel $\bar{k}(x,y)$ given by

$$\bar{k}(x,y)=k(x,y)v^\beta(y)=\frac{1}{x}\tilde{k}\left(\frac{y}{x}\right)v^\beta(y), \tag{17}$$

and

$$(\bar{\mathcal{H}}F)(y)=\int_0^1 \bar{h}(x,y)F(x)v^{-\beta}(x)dx, \quad y\in[0,1], \tag{18}$$

with the kernel $\bar{h}(x,y)$ defined as

$$\bar{h}(x,y)=h(x,y)v^\beta(y), \tag{19}$$

it is easy to show that the new integral equation

$$(\mathcal{I} + \bar{\mathcal{K}} + \bar{\mathcal{H}})\bar{f} = \bar{g}, \tag{20}$$

is equivalent to equation (3), in the sense that if f is a solution of $(\mathcal{I}+\mathcal{K}+\mathcal{H})f = g$ then $\bar{f} = v^\beta f$ is a solution of (20), with $\bar{g} = v^\beta g$, and vice versa.

Let us note that, when $F \in C$, it is possible to extend the definition of the function $(\bar{\mathcal{K}}F)(y)$ at the point $y = 0$ as follows

$$(\bar{\mathcal{K}}F)(y) = \begin{cases} \displaystyle\int_0^1 \bar{k}(x,y)F(x)v^{-\beta}(x)dx, & y \in (0,1], \\ \\ \displaystyle F(0)\int_0^{+\infty} t^{-1+\beta}\tilde{k}(t)dt, & y = 0. \end{cases} \tag{21}$$

In fact, we can write

$$\begin{aligned}(\bar{\mathcal{K}}F)(y) &= \int_0^1 \frac{1}{x}\tilde{k}\left(\frac{y}{x}\right)\left(\frac{y}{x}\right)^\beta F(x)dx \\ &= \int_0^{+\infty} t^{-1-\beta}\tilde{k}\left(\frac{1}{t}\right)F(yt)\chi_{[0,1/y]}(t)dt, \end{aligned} \tag{22}$$

where χ_A denotes the characteristic function of the subset $A \subseteq \mathbb{R}$, and then, taking into account (14) and applying the Lebesgue convergence theorem, we have

$$\begin{aligned}\lim_{y\to 0}(\bar{\mathcal{K}}F)(y) &= \lim_{y\to 0}\int_0^{+\infty} t^{-1-\beta}\tilde{k}\left(\frac{1}{t}\right)F(yt)\chi_{[0,1/y]}(t)dt \\ &= F(0)\int_0^{+\infty} t^{-1+\beta}\tilde{k}(t)dt. \end{aligned} \tag{23}$$

In the next theorem we establish the solvability of the new equation (20) in $\mathring{C}$.

Theorem 4 *Under the assumptions stated above, if* $\mathrm{Ker}(\mathcal{I} + \bar{\mathcal{K}} + \bar{\mathcal{H}}) = \{0\}$ *in* $\mathring{C}$*, then equation* (20) *admits a unique solution* $\bar{f}$ *in* $\mathring{C}$ *for each right-hand side* $\bar{g} \in \mathring{C}$.

5 A modified Nyström method

In this section we propose a Nyström type method for the numerical solution of equation (20). It consists in computing the solution $\bar{f}_m$ of the following approximating equation

$$(\mathcal{I} + \widetilde{\mathcal{K}}_m + \bar{\mathcal{H}}_m)\bar{f}_m = \bar{g}, \tag{24}$$

where the operators $\bar{\mathcal{H}}_m$ and $\widetilde{\mathcal{K}}_m$ are defined as follows

$$(\bar{\mathcal{H}}_m F)(y) = \sum_{j=1}^{m} \lambda_{m,j}^{-\beta} \bar{h}(x_{m,j}^{-\beta}, y) F(x_{m,j}^{-\beta}), \quad y \in [0,1],$$

and

$$(\widetilde{\mathcal{K}}_m F)(y) = \begin{cases} \dfrac{y}{y_m} (\bar{\mathcal{K}}_m F)(y_m), & 0 \le y < y_m \\ \\ (\bar{\mathcal{K}}_m F)(y), & y_m \le y \le 1 \end{cases}, \tag{25}$$

with $y_m = \frac{c}{m^{2-2\varepsilon}}$ a point chosen by fixing a positive constant c and an arbitrarily small positive quantity ε, and

$$(\bar{\mathcal{K}}_m F)(y) = \sum_{j=1}^{m} \lambda_{m,j}^{-\beta} \bar{k}(x_{m,j}^{-\beta}, y) F(x_{m,j}^{-\beta}), \quad y \in [y_m, 1]. \tag{26}$$

In this way, we use the Gauss-Jacobi quadrature rule (9) w.r.t. the Jacobi weight $v^{-\beta}$ in order to approximate $(\bar{\mathcal{H}}F)(y)$ in the whole interval $[0,1]$ and $(\bar{\mathcal{K}}F)(y)$ only when y is sufficiently far from the singularity point 0. For y very close to 0, $(\bar{\mathcal{K}}F)(y)$ is approximated by a linear polynomial assuming the value $(\bar{\mathcal{K}}F)(0) = 0$ at the point 0 and the value $(\bar{\mathcal{K}}_m F)(y_m)$ at the point y_m.

Consequently, the method defined by the discrete equation (24) can be regarded as a "modification" of the classical Nyström method just based on the Gauss-Jacobi quadrature rule. The introduction of such a modification for the discrete operator $\bar{\mathcal{K}}_m$ has been suggested by the following lemma.

Lemma 1 *Assuming that the kernel $\bar{k}(x,y)$ given in* (17) *satisfies*

$$\left\| v^{-\beta} \frac{\partial^j}{\partial x^j} \bar{k}(\cdot, y) \varphi^j \right\|_1 \le \mathcal{C} y^{-\frac{j}{2}}, \quad j = 0, 1, \ldots, r, \quad \mathcal{C} \ne \mathcal{C}(y), \tag{27}$$

for all $F \in \mathring{W}_r^\infty$, we have

$$|e_m(\bar{k}(\cdot, y)F)| \le \frac{\mathcal{C}}{m^r} \|F\|_{W_r^\infty} y^{-\frac{r}{2}}, \quad \mathcal{C} \ne \mathcal{C}(m, F, y). \tag{28}$$

We observe that (28) provides an estimate for the remainder term of the quadrature rule (9) applied, with $\rho = -\beta$, for the approximation of $(\bar{\mathcal{K}}F)(y)$. It shows that the behaviour of the Gaussian formula becomes worse and worse when y is closer and closer to 0.

Concerning the operators $\bar{\mathcal{H}}_m$ and $\widetilde{\mathcal{K}}_m$ we are able to state the following theorems that are crucial in order to prove the stability and the convergence of the proposed method in $\mathring{C}$.
From now on, for the sake of simplicity, we will set $\lambda_{m,j} := \lambda_{m,j}^{-\beta}$ and $x_{m,j} := x_{m,j}^{-\beta}$, $j = 1, \ldots, m$.

Theorem 5 *Let us assume that the kernel $h(x,y)$ in* (19) *satisfies* (12) *and* (15)*. Then*

$$\sup_m \|\bar{\mathcal{H}}_m\|_{\mathring{C}\to\mathring{C}} < +\infty, \tag{29}$$

the sequence of operators $\{\bar{\mathcal{H}}_m\}_m$, as maps from $\mathring{C}$ into $\mathring{C}$, is collectively compact, and

$$\lim_m \|(\bar{\mathcal{H}} - \bar{\mathcal{H}}_m)F\|_\infty = 0, \quad \forall F \in \mathring{C}. \tag{30}$$

Theorem 6 *Assuming that the kernel $\bar{k}(x,y)$ given in* (17) *satisfies* (14) *and* (27)*, we have*

$$\sup_m \|\widetilde{\mathcal{K}}_m\|_{\mathring{C}\to\mathring{C}} \le \mathcal{C}, \quad \mathcal{C} \ne \mathcal{C}(m), \tag{31}$$

$$\limsup_m \|\widetilde{\mathcal{K}}_m\|_{\mathring{C}\to\mathring{C}} \le \int_0^{+\infty} t^{-1+\beta}|\tilde{k}(t)|dt, \tag{32}$$

and

$$\lim_m \|(\bar{\mathcal{K}} - \widetilde{\mathcal{K}}_m)F\|_\infty = 0, \quad \forall F \in \mathring{C}. \tag{33}$$

The following theorem establishes that the proposed method is stable.

Theorem 7 *Let us assume that* $\mathrm{Ker}(\mathcal{I} + \bar{\mathcal{K}} + \bar{\mathcal{H}}) = \{0\}$ *in $\mathring{C}$, the kernel $\bar{k}(x,y)$ given in* (17) *satisfies* (13), (14) *and* (27)*, and the kernel $h(x,y)$ in* (19) *satisfies* (12) *and* (15)*. Then, for sufficiently large m, the operators $\mathcal{I} + \widetilde{\mathcal{K}}_m + \bar{\mathcal{H}}_m : \mathring{C} \to \mathring{C}$ are invertible and their inverses are uniformly bounded.*

Now, in order to compute the approximating solution $\bar{f}_m$ of equation (24), following the standard scheme of Nyström type methods, we derive a linear system in some sense equivalent to (24). We collocate it at the zeros $x_{m,i}$, $i = 1,\dots,m$, of the Jacobi polynomial of degree m that is orthogonal w.r.t. the weight function $v^{-\beta}$ and, letting

$$x_{m,s} = \min_{i=1,\dots,m}\{x_{m,i} \ : \ x_{m,i} > y_m\},$$

we obtain the following linear system of order m

$$\begin{cases} \displaystyle\sum_{j=1}^m \left[\delta_{i,j} + \lambda_{m,j}\left(\bar{k}(x_{m,j},x_{m,i}) + \bar{h}(x_{m,j},x_{m,i})\right)\right] a_j = \bar{g}(x_{m,i}), & i \ge s, \\ \displaystyle\sum_{j=1}^m \left[\delta_{i,j} + \lambda_{m,j}\left(\frac{x_{m,i}}{y_m}\bar{k}(x_{m,j},y_m) + \bar{h}(x_{m,j},x_{m,i})\right)\right] a_j = \bar{g}(x_{m,i}), & i < s, \end{cases} \tag{34}$$

with $i = 1,\dots,m$, in the unknowns $a_j = \bar{f}_m(x_{m,j})$, $j = 1,\dots,m$.

After solving the above system, we construct the Nyström interpolant

$$\bar{f}_m(y) = \begin{cases} \left[\bar{g}(y) - \sum_{j=1}^{m} \lambda_{m,j} \left(\bar{k}(x_{m,j}, y) + \bar{h}(x_{m,j}, y) \right) a_j \right], & y_m \leq y \leq 1, \\ \left[\bar{g}(y) - \sum_{j=1}^{m} \lambda_{m,j} \left(\frac{y}{y_m} \bar{k}(x_{m,j}, y_m) + \bar{h}(x_{m,j}, y) \right) a_j \right], & 0 \leq y < y_m. \end{cases} \tag{35}$$

Note that each solution $\bar{f}_m$ of (24) furnishes a solution of (34), merely evaluating it at the node points $(x_{m,j})_{j=1,\ldots,m}$. Conversely, if $\mathbf{a} = (a_1, \ldots, a_m)^T \in \mathbb{R}^m$ is solution of (34), then $\bar{f}_m$ given in (35) is the unique solution of (24) that agrees with $\mathbf{a}$ at the collocation knots.

Besides proving the stability, we are also able to show that the numerical procedure is convergent and leads to solve well conditioned linear systems.

Theorem 8 *Under the same assumptions of Theorem 7, if $\bar{g} \in \mathring{C}$, then the approximating functions $\bar{f}_m$ given in* (35) *converge to the exact solution $\bar{f}$ of* (20).
Moreover, denoting by A_m the matrices of the coefficients of systems (34) *and by* $\mathrm{cond}(A_m)$ *their condition numbers in uniform norm, we have*

$$\sup_m \mathrm{cond}(A_m) \leq \sup_m \mathrm{cond}(I + \tilde{\mathcal{K}}_m + \bar{\mathcal{H}}_m) < +\infty. \tag{36}$$

Concerning the convergence order, we are able to state the following result.

Theorem 9 *Under the assumptions of Theorem 7, if $\bar{h}(x, y)$ given in* (19) *also satisfies*

$$\sup_{0 \leq y \leq 1} \|\bar{h}(\cdot, y)\|_{W_r^1(v^{-\beta})} < +\infty \tag{37}$$

and the function $\bar{f}$ belongs to $\mathring{W}_r^\infty$, then, for sufficiently large m, the following error estimate

$$\|\bar{f} - \bar{f}_m\|_\infty \leq \frac{\mathcal{C}}{m^\mu}, \tag{38}$$

holds true, with $\mu = \min\{r\varepsilon, 2(1-\varepsilon)(\beta - \sigma)\}$ and $\mathcal{C} \neq \mathcal{C}(m)$.

We note that (38) actually provides an error estimate for the approximation of the solution f of the initial problem (3) by means of the function $f_m = v^{-\beta} \bar{f}_m$, since it is easily seen that

$$\|f - f_m\|_{C_{v^\beta}} = \|\bar{f} - \bar{f}_m\|_\infty. \tag{39}$$

Remark 1 *We note that the operator $\bar{\mathcal{K}}$ in* (16) *can be also defined as follows*

$$(\bar{\mathcal{K}} F)(y) = \int_0^1 \bar{\bar{k}}(x, y) F(x) dx,$$

where the kernel $\bar{\bar{k}}(x,y) = \frac{1}{x}\tilde{k}\left(\frac{y}{x}\right)\left(\frac{y}{x}\right)^{\beta} =: \frac{1}{x}\tilde{\tilde{k}}\left(\frac{y}{x}\right)$ *is of Mellin type. Then, equation* (20) *can be also regarded as a Mellin integral equation. Consequently, if the function* $\tilde{k}(t)$ *satisfies the assumptions of Theorem 7 and*

$$\int_0^\infty t^{-1+\sigma}\left|t^j\tilde{k}^{(j)}(t)\right| < \infty, \quad j = 1,\ldots,l, \quad l \in \mathbb{N}, \tag{40}$$

then $\tilde{\tilde{k}}(t) = \tilde{k}(t)t^{\beta}$ *fulfills*

$$\int_0^\infty t^{-1-(\beta-\sigma)}\left|t^j\tilde{\tilde{k}}^{(j)}(t)\right| < \infty, \quad j = 0,1,\ldots,l, \quad l \in \mathbb{N}_0,$$

and, following the same arguments in [8], we can conclude that the solution $\bar{f}$ *of* (20) *belongs to* $\mathring{W}_r^\infty$ *with* $r \geq \lfloor 2(\beta-\sigma)\rfloor$ *(see also [4, 5]).*

6 Proofs

Proof of Theorem 1
First, we prove that if $F \in C_{v^\sigma}$, then $\mathcal{K}F \in C_{v^\sigma}$. We observe that $\mathcal{K}F \in C((0,1])$. It remains to verify that

$$\lim_{y\to 0} v^\sigma(y)(\mathcal{K}F)(y) = 0. \tag{41}$$

For $y \in (0,1]$, we have

$$\begin{aligned} v^\sigma(y)(\mathcal{K}F)(y) &= v^\sigma(y)\int_0^1 \frac{1}{x}\tilde{k}\left(\frac{y}{x}\right)F(x)dx \\ &= \int_0^\infty t^{-1-\sigma}\tilde{k}\left(\frac{1}{t}\right)(yt)^\sigma F(yt)\,\chi_{[0,1]}(yt)dt, \end{aligned} \tag{42}$$

where χ_A denotes the characteristic function of the subset $A \subseteq \mathbb{R}$. Then, by assumption,

$$\lim_{y\to 0} v^\sigma(y)(\mathcal{K}F)(y) = \left(\lim_{z\to 0} v^\sigma(z)F(z)\right)\int_0^{+\infty} t^{-1+\sigma}\tilde{k}(t)dt = 0.$$

Now we prove (11). Using (42) again, one has

$$\begin{aligned} |v^\sigma(y)(\mathcal{K}F)(y)| &\leq \int_0^\infty t^{-1-\sigma}\left|\tilde{k}\left(\frac{1}{t}\right)\right|(yt)^\sigma\,|F(yt)|\chi_{[0,1]}(yt)dt \\ &\leq \|F\|_{C_{v^\sigma}}\int_0^\infty t^{-1+\sigma}|\tilde{k}(t)|dt \end{aligned}$$

and then

$$\|\mathcal{K}\|_{C_{v^\sigma}\to C_{v^\sigma}} \leq \int_0^{+\infty} t^{-1+\sigma}|\tilde{k}(t)|dt.$$

Proof of Theorem 2
Assuming $F \in C_{v^\sigma}$, with $0 < \sigma < 1$, for any $y \in [0,1]$ we get

$$\begin{aligned}|v^\sigma(y)(\mathcal{H}F)(y)| &= \int_0^1 |v^\sigma(y)h(x,y)||v^\sigma(x)F(x)|v^{-\sigma}(x)dx \\ &\le \|F\|_{C_{v^\sigma}} \sup_{0\le x\le 1} |v^\sigma(y)h(x,y)| \int_0^1 v^{-\sigma}(x)dx \\ &\le \mathcal{C}\|F\|_{C_{v^\sigma}} \sup_{0\le x\le 1} |v^\sigma(y)h(x,y)|,\end{aligned}$$

with $\mathcal{C} \neq \mathcal{C}(y,F)$. Then, under the assumption (12) we have

$$\|v^\sigma \mathcal{H}F\|_\infty \le \mathcal{C}\|F\|_{C_{v^\sigma}} \sup_{0\le x\le 1} \|v^\sigma h(x,\cdot)\|_\infty \tag{43}$$

and

$$\lim_{y\to 0} |v^\sigma(y)(\mathcal{H}F)(y)| \le \mathcal{C}\|F\|_{C_{v^\sigma}} \lim_{y\to 0} \sup_{0\le x\le 1} |v^\sigma(y)h(x,y)| = 0. \tag{44}$$

From (43) and (44) we can deduce that the operator $\mathcal{H}$ is a continuous map from C_{v^σ} into itself.
Now, since

$$|v^\sigma(y)(\mathcal{H}F)^{(r)}(y)\varphi^r(y)| \le \|F\|_{C_{v^\sigma}} \sup_{0\le x\le 1} \left\| v^\sigma \frac{\partial^r}{\partial y^r} h(x,\cdot)\varphi^r \right\|_\infty \int_0^1 v^{-\sigma}(x)dx,$$

in virtue of (12), the operator $\mathcal{H} : C_{v^\sigma} \to W_r^\infty(v^\sigma)$ is continuous, too. Moreover, using the inequality (8) we have

$$E_m(\mathcal{H}F)_{v^\sigma,\infty} \le \frac{\mathcal{C}}{m^r}\|v^\sigma(\mathcal{H}F)^{(r)}\varphi^r\|_\infty \le \frac{\mathcal{C}}{m^r}\|F\|_{C_{v^\sigma}}, \quad \mathcal{C} \neq \mathcal{C}(m,F),$$

and, then,

$$\lim_m \sup_{\|F\|_{C_{v^\sigma}}=1} E_m(\mathcal{H}F)_{v^\sigma,\infty} = 0. \tag{45}$$

Finally, using a result in [23, p. 44], from (45) we deduce the compactness of the operator $\mathcal{H}$ as a map from C_{v^σ} into itself.

Proof of Theorem 4
Assuming $F \in C$, by definition (21) and taking into account (23), $\bar{\mathcal{K}}F \in C([0,1])$. Moreover, when $F \in \mathring{C}$, it results $(\bar{\mathcal{K}}F)(0) = 0$. Consequently, $\bar{\mathcal{K}}$ is a map from $\mathring{C}$ into itself. Moreover, for $F \in \mathring{C}$, recalling (22), we have

$$\begin{aligned}|(\bar{\mathcal{K}}F)(y)| &\le \|F\|_\infty \int_0^{+\infty} t^{-1-\beta} \left|\tilde{k}\left(\frac{1}{t}\right)\right| \chi_{[0,1/y)}(t)dt \\ &\le \|F\|_\infty \int_0^{+\infty} t^{-1+\beta}|\tilde{k}(t)|dt,\end{aligned}$$

from which, under the assumption (14), we deduce

$$\|\bar{\mathcal{K}}\|_{\mathring{C}\to\mathring{C}} \leq \int_0^{+\infty} t^{-1+\beta}|\tilde{k}(t)|dt < 1. \tag{46}$$

Concerning the operator $\bar{\mathcal{H}}$, for any $F \in C$, from the hypotheses (12) and (15) we deduce that the function $\bar{\mathcal{H}}F$ belongs to C. Moreover, we have

$$\begin{aligned}\lim_{y\to 0}|(\bar{\mathcal{H}}F)(y)| &\leq \|F\|_\infty \lim_{y\to 0}\sup_{0\leq x\leq 1}|\bar{h}(x,y)|\int_0^1 v^{-\beta}(x)dx \\ &= \mathcal{C}\lim_{y\to 0}\sup_{0\leq x\leq 1}|v^\beta(y)h(x,y)| = 0.\end{aligned}$$

Then, $\bar{\mathcal{H}}$ is a map from $\mathring{C}$ into itself. Moreover, since by (12) and (15) the kernel $\bar{h}(x,y)$ is continuous for $0 \leq x,y \leq 1$, the operator $\bar{\mathcal{H}} : \mathring{C} \longrightarrow \mathring{C}$ is compact [2].

Consequently, as a consequence of the Neumann series theorem and [13, Corollary 3.8], the inverse operator $(\mathcal{I}+\bar{\mathcal{K}}+\bar{\mathcal{H}})^{-1} : \mathring{C} \longrightarrow \mathring{C}$ exists and is bounded.

Proof of Lemma 1
Using (10) and Lemma 2.1 in [6], we have

$$\begin{aligned}|e_m(\bar{k}(\cdot,y)F)| &\leq \frac{\mathcal{C}}{m^r}\int_0^1\left|\frac{\partial^r}{\partial x^r}\left[\bar{k}(x,y)F(x)\right]\right|\varphi^r(x)v^{-\beta}(x)dx \\ &\leq \frac{\mathcal{C}}{m^r}\sum_{j=0}^r\binom{r}{j}\int_0^1\left|\frac{\partial^j}{\partial x^j}\bar{k}(x,y)\varphi^j(x)\right|\left|F^{(r-j)}(x)\varphi^{r-j}(x)\right|v^{-\beta}(x)dx \\ &\leq \frac{\mathcal{C}}{m^r}\|F\|_{W_r^\infty}\sum_{j=0}^r\binom{r}{j}\left\|v^{-\beta}\frac{\partial^j}{\partial x^j}\bar{k}(\cdot,y)\varphi^j\right\|_1.\end{aligned} \tag{47}$$

and, under the assumption (27), (28) follows.

Proof of Theorem 5
Analysis similar to that done for the operator $\bar{\mathcal{H}}$ in the proof of Theorem 4 shows that, under our hypotheses, the $\bar{\mathcal{H}}_m$ are linear operators from $\mathring{C}$ into itself. Moreover, since the weights $\lambda_{m,j}$, $j = 1,\ldots,m$, are all positive (see, for example, [3, p. 97]), we have

$$\begin{aligned}|(\bar{\mathcal{H}}_mF)(y)| &\leq \sum_{j=1}^m\lambda_{m,j}|F(x_{m,j})||\bar{h}(x_{m,j},y)| \leq \|F\|_\infty\sup_{0\leq x\leq 1}\|\bar{h}(x,\cdot)\|_\infty\sum_{j=1}^m\lambda_{m,j} \\ &= \|F\|_\infty\sup_{0\leq x\leq 1}\|\bar{h}(x,\cdot)\|_\infty\int_0^1 v^{-\beta}(x)dx\end{aligned} \tag{48}$$

and from the hypothesis (12) we can deduce (29). Finally, since the Gauss-Jacobi quadrature rule (9) is convergent for any continuous function and the kernel $\bar{h}(x,y)$ is continuous for $0 \leq x,y \leq 1$ in virtue of (12) and (15), the collectively compactness of the sequence $\{\mathcal{H}_m\}_m$ and (30) follow (see, for instance, [1,13]).

Proof of Theorem 6
First, we note that, by definition (25) it is easily seen that $\widetilde{\mathcal{K}}_m F$ belongs to $\mathring{C}$, for any function $F \in \mathring{C}$.
Now, let us prove that (31) and (32) are satisfied. We have

$$\|\widetilde{\mathcal{K}}_m F\|_\infty = \max\left\{\sup_{y\in[0,y_m]} \frac{y}{y_m}|(\bar{\mathcal{K}}_m F)(y_m)|, \sup_{y\in[y_m,1]} |(\bar{\mathcal{K}}_m F)(y)|\right\}. \tag{49}$$

We start by estimating the second term in the braces. Recalling (26) and (9), we get

$$\begin{aligned}|(\bar{\mathcal{K}}_m F)(y)| &\le \sum_{j=1}^m \lambda_{m,j}|\bar{k}(x_{m,j},y)||F(x_{m,j})| \le \|F\|_\infty \sum_{j=1}^m \lambda_{m,j}|\bar{k}(x_{m,j},y)| \\ &= \|F\|_\infty \left[\int_0^1 |\bar{k}(x,y)|v^{-\beta}(x)dx - e_m(|\bar{k}(\cdot,y)|)\right] \\ &\le \|F\|_\infty \left[\int_0^1 |\bar{k}(x,y)|v^{-\beta}(x)dx + \left|e_m(|\bar{k}(\cdot,y)|)\right|\right].\end{aligned}$$

Then, taking into account (46) and observing that, under the assumption (27), the kernel $|\bar{k}(\cdot,y)|$ satisfies the condition

$$\left\|v^{-\beta}\frac{\partial}{\partial x}|\bar{k}(\cdot,y)|\varphi\right\|_1 \le \mathcal{C}y^{-\frac{1}{2}},$$

and, consequently, we can apply (28) in order to estimate the quadrature error $\left|e_m(|\bar{k}(\cdot,y)|)\right|$, we easily deduce that

$$\sup_{y\in[y_m,1]} |(\bar{\mathcal{K}}_m F)(y)| \le \|F\|_\infty \left\{\int_0^{+\infty} t^{-1+\beta}|\tilde{k}(t)|dt + \frac{\mathcal{C}}{m^\varepsilon}\right\}. \tag{50}$$

Concerning the first term in the braces in (49), we have

$$\sup_{y\in[0,y_m]} \frac{y}{y_m}|(\bar{\mathcal{K}}_m F)(y_m)| = |(\bar{\mathcal{K}}_m F)(y_m)| \le \sup_{y\in[y_m,1]} |(\bar{\mathcal{K}}_m F)(y)|. \tag{51}$$

Finally, combining (50) and (51) with (49), we have

$$\|\widetilde{\mathcal{K}}_m F\|_\infty \le \|F\|_\infty \left\{\int_0^{+\infty} t^{-1+\beta}|\tilde{k}(t)|dt + \frac{\mathcal{C}}{m^\varepsilon}\right\},$$

and (31) and (32) easily follow.
Since, by (32), the operators $\widetilde{\mathcal{K}}_m : \mathring{C} \to \mathring{C}$ are uniformly bounded with respect to m, if we prove that

$$\lim_m \|(\widetilde{\mathcal{K}}_m - \bar{\mathcal{K}})p\|_\infty = 0, \quad \forall p \in \mathring{\mathbb{P}}, \tag{52}$$

with $\mathring{\mathbb{P}} = \mathbb{P} \cap \mathring{C}$ and $\mathbb{P}$ the set of the algebraic polynomials on the interval $[0,1]$ ($\mathring{\mathbb{P}}$ is a dense subspace of $\mathring{C}$), applying the Banach-Steinhaus theorem (see, for instance, [2, p. 517]), we can deduce (33). In order to prove (52) we note that

$$\|(\widetilde{\mathcal{K}}_m-\bar{\mathcal{K}})p\|_\infty=\max\left\{\sup_{y\in[0,y_m]}\left|(\bar{\mathcal{K}}p)(y)-\frac{y}{y_m}(\bar{\mathcal{K}}_mp)(y_m)\right|,\right.$$
$$\left.\sup_{y\in[y_m,1]}\left|(\bar{\mathcal{K}}p)(y)-(\bar{\mathcal{K}}_mp)(y)\right|\right\}. \tag{53}$$

Using Lemma 1, under the assumption (27), for $y\in[y_m,1]$, we get

$$|(\bar{\mathcal{K}}p)(y)-(\bar{\mathcal{K}}_mp)(y)|=|e_m(\bar{k}(\cdot,y)p)|\le\frac{\mathcal{C}}{m^r}y^{-\frac{r}{2}},$$

and then

$$\sup_{y\in[y_m,1]}|(\bar{\mathcal{K}}p)(y)-(\bar{\mathcal{K}}_mp)(y)|\le\frac{\mathcal{C}}{m^{r\varepsilon}}. \tag{54}$$

In order to estimate the first term in the braces in (53), for $y\in[0,y_m]$, we write

$$(\bar{\mathcal{K}}p)(y)-\frac{y}{y_m}(\bar{\mathcal{K}}_mp)(y_m)=(\bar{\mathcal{K}}p)(y)-\frac{y}{y_m}(\bar{\mathcal{K}}p)(y_m)$$
$$+\frac{y}{y_m}\left[(\bar{\mathcal{K}}p)(y_m)-(\bar{\mathcal{K}}_mp)(y_m)\right]. \tag{55}$$

Then, we observe that, in virtue of (54),

$$\sup_{y\in[0,y_m]}\frac{y}{y_m}\left[(\bar{\mathcal{K}}p)(y_m)-(\bar{\mathcal{K}}_mp)(y_m)\right]=\left[(\bar{\mathcal{K}}p)(y_m)-(\bar{\mathcal{K}}_mp)(y_m)\right]\le\frac{\mathcal{C}}{m^{r\varepsilon}}. \tag{56}$$

Moreover, one has that

$$(\bar{\mathcal{K}}p)(y)-\frac{y}{y_m}(\bar{\mathcal{K}}p)(y_m)=\left[(\bar{\mathcal{K}}p)(y)-(\bar{\mathcal{K}}p)(0)\right]+(\bar{\mathcal{K}}p)(0)\left[1-\frac{y}{y_m}\right]$$
$$+\frac{y}{y_m}\left[(\bar{\mathcal{K}}p)(0)-(\bar{\mathcal{K}}p)(y_m)\right].$$

Taking into account that $\bar{\mathcal{K}}p\in\mathring{C}$, we get

$$\lim_m\sup_{y\in[0,y_m]}\left|(\bar{\mathcal{K}}p)(y)-\frac{y}{y_m}(\bar{\mathcal{K}}p)(y_m)\right|=0. \tag{57}$$

Combining (57) and (56) with (55), we conclude that

$$\lim_m\sup_{y\in[0,y_m]}\left|(\bar{\mathcal{K}}p)(y)-\frac{y}{y_m}(\bar{\mathcal{K}}_mp)(y_m)\right|=0. \tag{58}$$

Finally, combining (58) and (54) with (53), we deduce (52).

Proof of Theorem 7
In virtue of (32) and (33), the operators $I + \widetilde{\mathcal{K}}_m : \mathring{C} \to \mathring{C}$ are bounded and pointwise convergent to the operator $I + \bar{\mathcal{K}}$, under our assumptions. Moreover, using the Neumann series theorem together with (32) and (13), we deduce that, for sufficiently large m, the inverse operators $(I + \widetilde{\mathcal{K}}_m)^{-1} : \mathring{C} \to \mathring{C}$ exist and are uniformly bounded.
Since, in addition, the sequence $\{\bar{\mathcal{H}}_m\}_m$ is collectively compact and pointwise convergent to the compact operator $\bar{\mathcal{H}}$ (see Theorem 5), we can use [13, Problem 10.3, p. 153] and, then, deduce that, for sufficiently large m, the operators $I + \widetilde{\mathcal{K}}_m + \bar{\mathcal{H}}_m : \mathring{C} \to \mathring{C}$ are invertible and their inverses are uniformly bounded.

Proof of Theorem 8
The convergence of the sequence of the approximating functions $\bar{f}_m$ to the solution $\bar{f}$ of (24) can be immediately deduced taking into account that, by standard arguments,

$$\|\bar{f} - \bar{f}_m\|_\infty \leq \|(I + \widetilde{\mathcal{K}}_m + \bar{\mathcal{H}}_m)^{-1}\|_{\mathring{C} \to \mathring{C}} \left(\|(\bar{\mathcal{K}} - \widetilde{\mathcal{K}}_m)\bar{f}\|_\infty + \|(\bar{\mathcal{H}} - \bar{\mathcal{H}}_m)\bar{f}\|_\infty \right) \tag{59}$$

and using Theorem 7, (33) and (30).

Furthermore, following a standard scheme (see [2]), it is possible to prove that

$$\|A_m\|_\infty \leq \|I + \widetilde{\mathcal{K}}_m + \bar{\mathcal{H}}_m\|_{\mathring{C} \to \mathring{C}}$$

and

$$\|A_m^{-1}\|_\infty \leq \|(I + \widetilde{\mathcal{K}}_m + \bar{\mathcal{H}}_m)^{-1}\|_{\mathring{C} \to \mathring{C}},$$

from which (36) easily follows.

Proof of Theorem 9
We start from (59). Using (10) and Lemma 2.1 in [6], we have

$$\begin{aligned}
|(\bar{\mathcal{H}}\bar{f})(y) - (\bar{\mathcal{H}}_m\bar{f})(y)| &= |e_m(\bar{h}(\cdot, y)\bar{f})| \\
&\leq \frac{\mathcal{C}}{m^r} \int_0^1 \left| \frac{\partial^r}{\partial x^r} \left[\bar{h}(x, y)\bar{f}(x) \right] \right| \varphi^r(x) v^{-\beta}(x) dx \\
&\leq \frac{\mathcal{C}}{m^r} \sum_{j=0}^{r} \binom{r}{j} \int_0^1 \left| \frac{\partial^j}{\partial x^j} \bar{h}(x, y)\varphi^j(x) \right| \left| \bar{f}^{(r-j)}(x)\varphi^{r-j}(x) \right| v^{-\beta}(x) dx \\
&\leq \frac{\mathcal{C}}{m^r} \|\bar{f}\|_{W_r^\infty} \sum_{j=0}^{r} \binom{r}{j} \left\| v^{-\beta} \frac{\partial^j}{\partial x^j} \bar{h}(\cdot, y)\varphi^j \right\|_1 \\
&\leq \frac{\mathcal{C}}{m^r} \|\bar{f}\|_{W_r^\infty} \|\bar{h}(\cdot, y)\|_{W_r^1(v^{-\beta})},
\end{aligned}$$

where $\mathcal{C} \neq \mathcal{C}(m, \bar{h}(\cdot, y), \bar{f})$. Therefore, under the assumptions (37), we get

$$\|(\bar{\mathcal{H}} - \bar{\mathcal{H}}_m)\bar{f}\|_\infty \leq \frac{\mathcal{C}}{m^r}, \quad \mathcal{C} \neq \mathcal{C}(m). \tag{60}$$

By definition

$$\|(\bar{\mathcal{K}}-\widetilde{\mathcal{K}}_m)\bar{f}\|_\infty = \max\left\{ \sup_{y\in[0,y_m]} \left|(\bar{\mathcal{K}}\bar{f})(y) - \frac{y}{y_m}(\bar{\mathcal{K}}_m\bar{f})(y_m)\right|, \right.$$
$$\left. \sup_{y\in[y_m,1]} \left|(\bar{\mathcal{K}}\bar{f})(y) - (\bar{\mathcal{K}}_m\bar{f})(y)\right| \right\}. \tag{61}$$

In virtue of our assumptions, by (27), we get

$$\left|(\bar{\mathcal{K}}\bar{f})(y) - (\bar{\mathcal{K}}_m\bar{f})(y)\right| = |e_m(\bar{k}(\cdot,y)\bar{f})| \le \frac{\mathcal{C}}{m^r} y^{-\frac{r}{2}} \tag{62}$$

and then

$$\sup_{y\in[y_m,1]} |(\bar{\mathcal{K}}\bar{f})(y) - (\bar{\mathcal{K}}_m\bar{f})(y)| = \sup_{y\in[y_m,1]} |e_m(\bar{k}(\cdot,y)\bar{f})| \le \frac{\mathcal{C}}{m^{r\varepsilon}}. \tag{63}$$

Concerning the first term in the braces in (61), for $y\in[0,y_m]$, we have

$$\left|(\bar{\mathcal{K}}\bar{f})(y) - \frac{y}{y_m}(\bar{\mathcal{K}}_m\bar{f})(y_m)\right| \le \left|(\bar{\mathcal{K}}\bar{f})(y) - \frac{y}{y_m}(\bar{\mathcal{K}}\bar{f})(y_m)\right|$$
$$+ \frac{y}{y_m}\left|(\bar{\mathcal{K}}\bar{f})(y_m) - (\bar{\mathcal{K}}_m\bar{f})(y_m)\right|. \tag{64}$$

Now, we observe that

$$\sup_{y\in[0,y_m]} \frac{y}{y_m}\left|(\bar{\mathcal{K}}\bar{f})(y_m) - (\bar{\mathcal{K}}_m\bar{f})(y_m)\right| \le \left|(\bar{\mathcal{K}}\bar{f})(y_m) - (\bar{\mathcal{K}}_m\bar{f})(y_m)\right| \le \frac{\mathcal{C}}{m^{r\varepsilon}}, \tag{65}$$

according to (63), while

$$\begin{aligned}
&\left|(\bar{\mathcal{K}}\bar{f})(y) - \frac{y}{y_m}(\bar{\mathcal{K}}\bar{f})(y_m)\right| \le |(\bar{\mathcal{K}}\bar{f})(y)| + |(\bar{\mathcal{K}}\bar{f})(y_m)| \\
&\quad \le \int_0^{\frac{1}{y}} t^{-1-\beta}\left|\tilde{k}\left(\frac{1}{t}\right)\right| |\bar{f}(yt)|dt + \int_0^{\frac{1}{y_m}} t^{-1-\beta}\left|\tilde{k}\left(\frac{1}{t}\right)\right| |\bar{f}(y_m t)|dt \\
&\quad \le y^{\beta-\sigma}\int_0^{\frac{1}{y}} t^{-1-\sigma}\left|\tilde{k}\left(\frac{1}{t}\right)\right| |(yt)^\sigma f(yt)|dt \\
&\quad + y_m^{\beta-\sigma}\int_0^{\frac{1}{y_m}} t^{-1-\sigma}\left|\tilde{k}\left(\frac{1}{t}\right)\right| |(y_m t)^\sigma f(y_m t)|dt \\
&\quad \le y^{\beta-\sigma}\|f\|_{C_{v^\sigma}}\int_0^{\frac{1}{y}} t^{-1-\sigma}\left|\tilde{k}\left(\frac{1}{t}\right)\right| dt + y_m^{\beta-\sigma}\|f\|_{C_{v^\sigma}}\int_0^{\frac{1}{y}} t^{-1-\sigma}\left|\tilde{k}\left(\frac{1}{t}\right)\right| dt \\
&\quad \le 2y_m^{\beta-\sigma}\|f\|_{C_{v^\sigma}}\int_0^{\frac{1}{y}} t^{-1-\sigma}\left|\tilde{k}\left(\frac{1}{t}\right)\right| dt \\
&\quad \le 2y_m^{\beta-\sigma}\|f\|_{C_{v^\sigma}}\int_0^{\infty} t^{-1+\sigma}|\tilde{k}(t)|dt \le \frac{\mathcal{C}}{m^{2(1-\varepsilon)(\beta-\sigma)}}.
\end{aligned} \tag{66}$$

Therefore, combining (63)-(66) with (61), we get

$$\|(\bar{\mathcal{K}} - \widetilde{\mathcal{K}}_m)\bar{f}\|_\infty \leq \frac{\mathcal{C}}{m^{r\varepsilon}} + \frac{\mathcal{C}}{m^{2(1-\varepsilon)(\beta-\sigma)}}. \tag{67}$$

Finally, substituting (67) and (60) into (59) and taking into account Theorem 7, the estimate (38) follows.

Remark 2 *By looking at the estimates* (62)-(66) *in the proof of Theorem 9, we can note that the approximation error* $\left|(\bar{\mathcal{K}}\bar{f})(y) - (\widetilde{\mathcal{K}}_m\bar{f})(y)\right|$ *depends on the evaluation point* $y \in [0,1]$*. More precisely, the error is of order* $O(m^{-r})$ *in any interval* $[\delta, 1]$*, with fixed* $0 < \delta < 1$*, while it is of order* μ*, with* $\mu = \min\{r\varepsilon, 2(1-\varepsilon)(\beta-\sigma)\}$*, in the whole interval* $[0,1]$*.*

7 Numerical results

In this section we show by some numerical tests the effectiveness of the proposed method. For each example, in the tables we will report the maximum weighted absolute errors

$$err_m = \max_{i=1,\ldots,10^6} e_m(y_i),$$

where

$$e_m(y) = v^\beta(y)|f(y) - f_m(y)|$$

is the pointwise weighted absolute error at a point $y \in (0,1]$ and $y_0, y_1, \ldots, y_{10^6}$ are equispaced points in the interval $[0,1]$. When the exact solution f is not known we will retain as exact the approximating one f_{2048}.
In the following tables we will also show the estimated order of convergence

$$EOC_m = \frac{\log\left(err_m/err_{2m}\right)}{\log 2}.$$

Finally, we will report the condition numbers, in the matrix infinity norm, of the matrix A_m associated with the linear systems (34). As one will be able to see, these values do not increase with m.

The parameters c and ε involved in the definition of the point $y_m = \frac{c}{m^{2-2\varepsilon}}$ have been chosen taking into account the behavior of the solution f near the origin and in such a way that the convergence order $\mu = \min\{r\varepsilon, 2(1-\varepsilon)(\beta-\sigma)\}$ (see (38)) is maximized. Then, in our tests we have taken $c = 10^{(\beta-\sigma)-1}$ and ε such that $r\varepsilon = 2(1-\varepsilon)(\beta-\sigma)$, i.e.

$$\varepsilon = \frac{2(\beta-\sigma)}{r + 2(\beta-\sigma)}, \tag{68}$$

choosing the smallest value of σ and the largest value of β $(0 < \sigma < \beta < 1)$ which, respectively, make (13) and (14) satisfied. Note that, corresponding to large values of r, (68) provides small values of ε which optimize the convergence order μ.

For each example, we will specify the values of the parameters σ and β chosen according to the criterium described above, as well as the value of σ_0 such that $0 < \sigma_0 < \sigma < \beta < 1$. In the cases where the exact solution $\bar{f}$ is unknown (see Examples 1 and 2), in (68) we have taken the value of r for which all the conditions (12), (27) and (37) are fulfilled.

Example 1 *We consider the second kind equation of Mellin convolution type* (1) *with*

$$k(x,y) = \frac{1}{8\pi}\sqrt{\frac{x}{x^2y+y^3}}, \quad h(x,y) = xy^2+1, \quad g(y) = \frac{y^5e^{y+2}}{1+y^3},$$

whose exact solution is unknown. In Table 1 we report the results obtained by applying the proposed method with $\sigma = 0.55$ and $\beta = 0.99$, being $\sigma_0 \simeq 0.54$. Moreover, taking into account that conditions (12), (27) *and* (37) *hold true for any $r \in \mathbb{N}$, we select $r = 100$. The corresponding theoretical convergence order is $\mu = 8.72e-01$. The numerical evidence shows that the condition numbers of the linear system* (34) *do not increase when the dimension m grows up.*

Table 1 Example 1

m	err_m	EOC_m	cond(A_m)
16	1.11e-04	2.05e+00	3.4373e+03
32	2.66e-05	1.89e+00	3.4554e+03
64	7.15e-06	1.08e+00	3.4583e+03
128	3.36e-06	1.07e+00	3.4573e+03
256	1.59e-06	2.87e+00	3.4551e+03
512	2.17e-07	1.87e+00	3.4525e+03
1024	5.91e-08		3.4498e+03

Example 2 *Let us assume that the known functions in the Mellin integral equation* (1) *are the following*

$$k(x,y) = \frac{x^2}{(5x+y)^3}\left(\frac{x}{y}\right)^{\frac{4}{5}}, \quad h(x,y) = e^{x+y}, \quad g(y) = \frac{\sin^4 y}{(2+y^2)}.$$

The exact solution is unknown. According to the proposed criterium, since for the function $\tilde{k}(t) = \frac{1}{t^{4/5}(5+t)^3}$ one has $\sigma_0 \simeq 0.80$, we take $\sigma = 0.81$ and $\beta = 0.99$. Also in this case conditions (12), (27) *and* (37) *are fulfilled for any $r \in \mathbb{N}$, then we take $r = 100$, obtaining a theoretical convergence order $\mu = 3.58e-01$. In Table 2 we present the numerical results.*

Example 3 *In this example we consider an integral equation of type* (1) *with the Mellin kernel $k(x,y)$ having the form*

$$k(x,y) = \frac{1}{\pi}\frac{x\sin\alpha}{x^2-2xy\cos\alpha+y^2}. \tag{69}$$

Table 2 Example 2

m	err_m	EOC_m	$\mathrm{cond}(A_m)$
16	5.74e-06	1.42e+00	1.8211e+04
32	2.14e-06	5.31e-01	1.8524e+04
64	1.48e-06	3.98e-01	1.8598e+04
128	1.12e-06	4.57e-01	1.8611e+04
256	8.19e-07	2.01e+00	1.8609e+04
512	2.02e-07	1.21e+00	1.8603e+04
1024	8.73e-08		1.8595e+04

Such type of kernel occurs when boundary integral methods are used in order to solve the exterior Neumann problem for the Laplace equation in planar domain with corners. Representing the harmonic solution in the form of a single layer potential leads to a system of integral equations involving Mellin integral operators with kernel given by (69)*, where* α *is the interior angle at the corner point of the piecewise smooth boundary of the domain. Here we take* $\alpha = \frac{\pi}{3}$ *and the other known functions involved in* (1) *given by*

$$h(x,y) = x^{\frac{9}{2}}(x^2+y^2)$$

and

$$g(y) = y^{\frac{5}{2}} + \frac{1}{10} + \frac{y^2}{8} + \frac{6+10y}{10\sqrt{3}\pi} + \frac{(5+5i)(-1)^{\frac{1}{12}}\sqrt{6}y^{\frac{5}{2}}\left((-1)^{\frac{5}{6}}\arctan\left(\frac{1+i\sqrt{3}}{2\sqrt{y}}\right)+\arctan\left(\frac{(-1)^{\frac{1}{6}}}{\sqrt{y}}\right)\right)}{10\sqrt{3}\pi}.$$

We observe that the right hand side $g(y)$ *takes real values in the interval* $[0,1]$ *and is such that the function* $f(y) = y^{\frac{5}{2}}$ *is the exact solution of* (1)*.*
In Table 3 we show the numerical results obtained applying the described method with $\beta = 0.99$*,* $\sigma = 0.41$ *and* $r = 6$*, being* $\sigma_0 \simeq 0.40$ *and* $\bar{f} \in W_6^{\infty}$*, for which* $\mu = 9.72e-01$*.*

Table 3 Example 3

m	err_m	EOC_m	$\mathrm{cond}(A_m)$
16	1.07e-04	1.65e+00	2.35337e+00
32	3.38e-05	1.65e+00	2.41644e+00
64	1.07e-05	1.65e+00	2.43279e+00
128	3.39e-06	1.65e+00	2.43681e+00
256	1.07e-06	1.66e+00	2.43797e+00
512	3.38e-07	1.69e+00	2.43813e+00
1024	1.04e-07		2.43826e+00

In Table 4 we further present the following error

$$\widetilde{err}_m = \max_{i=10^4,\dots,10^6} e_m(y_i),$$

and the corresponding estimated order of convergence

$$\widetilde{EOC}_m = \frac{\log(\widetilde{err}_m/\widetilde{err}_{2m})}{\log 2}.$$

As one can see, the errors $\widetilde{err}_m$ appear to be smaller than err_m in Table 3 as well as the order of convergence appears to be higher. These results confirm that, if one approximates the solution in a interval $[\delta, 1]$ with $0 < \delta < 1$, the larger δ the better the performance of the method (see Remark 2).

Table 4 Example 3

m	$\widetilde{err}_m$	$\widetilde{EOC}_m$
16	4.59e-08	5.53e+00
32	9.93e-10	4.83e+00
64	3.47e-11	4.34e+00
128	1.71e-12	4.33e+00
256	8.47e-14	4.34e+00
512	4.17e-15	1.42e+00
1024	1.55e-15	

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