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Second Variation Formula and Stability of Exponentially Subelliptic Harmonic Maps

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Abstract

We study the stability of exponentially subelliptic harmonic (e.s.h.) maps from a Carnot–Carathéodory complete strictly pseudoconvex pseudohermitian manifold (M, θ) into a Riemannian manifold (N, h). E.s.h. maps are C^∞ solutions $\phi : M \to N$ to the nonlinear PDE system $\tau_b(\phi) + \phi_* \nabla^H e_b(\phi) = 0$ [the Euler–Lagrange equations of the variational principle $\delta E_b(\phi) = 0$ where $E_b(\phi) = \int_\Omega \exp\left[e_b(\phi)\right] \Psi$ and $e_b(\phi) = \frac{1}{2} \operatorname{trace}_{G_\theta} \{\Pi_H \phi^* h\}$ and $\Omega \subset M$ is a Carnot–Carathéodory bounded domain]. We derive the second variation formula about an e.s.h. map, leading to a pseudohermitian analog to the Hessian (of an ordinary exponentially harmonic map between Riemannian manifolds)

$$
\begin{aligned}
H(E_b)_\phi(V, W) &= \int_\Omega h^\phi\big(J^\phi_{b,\exp} V,\ W\big)\ \Psi \\
&\quad + \int_M \exp\left[e_b(\phi)\right] (h^\phi)^*(D^\phi V,\ \Pi_H \phi_*)\, (h^\phi)^*(D^\phi W,\ \Pi_H \phi_*)\ \Psi, \\
J^\phi_{b,\exp} V &\equiv \big(D^\phi\big)^*\big(\exp\left[e_b(\phi)\right]\, D^\phi V\big) \\
&\quad - \exp\left[e_b(\phi)\right] \operatorname{trace}_{G_\theta} \left\{\Pi_H \left(R^h\right)^\phi\big(V,\ \phi_* \cdot\ \big)\phi_* \cdot\right\},
\end{aligned}
$$

$[\Psi = \theta \wedge (d\theta)^n]$. Given a bounded domain $\Omega \subset M$ and an e.s.h. map $\phi \in C^\infty\big(\overline{\Omega},\ N\big)$ with values in a Riemannian manifold $N = N^m(k)$ of nonpositive constant sectional curvature $k \le 0$, we solve the generalized Dirichlet eigenvalue problem $J^\phi_{b,\exp} V = \lambda V$ in Ω and $V = 0$ on $\partial\Omega$ for the degenerate elliptic operator $J^\phi_{b,\exp}$, provided that Ω supports Poincaré inequality

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$$\|V\|_{L^2} \leq C\|D^\phi V\|_{L^2}, \quad V \in C_0^\infty\big(\Omega,\ \phi^{-1}TN\big),$$

and the embedding $\mathring{W}_H^{1,2}(\Omega,\ \phi^{-1}TN) \hookrightarrow L^2(\Omega,\ \phi^{-1}TN)$ is compact.

Keywords CR manifold · Tanaka–Webster connection · Fefferman's metric · Exponentially subelliptic harmonic map · Stability

1 Introduction and a Glimpse to the Main Results

Let $(\mathfrak{M}, g)$ be a n-dimensional semi-Riemannian manifold with the semi-Riemannian metric g of signature $(\nu,\ n-\nu)$. A C^∞ map $\Phi : \mathfrak{M} \to N$ into a Riemannian manifold (N, h) is **exponentially harmonic** (e.h.) if Φ is a critical point of

$$\mathbb{E}_{\exp}(\Phi) = \int_\Omega \exp\big[e(\Phi)\big]\, d\,\mathrm{v}_g$$

for any relatively compact domain $\Omega \subset\subset \mathfrak{M}$, where the **energy density** $e(\Phi) : \mathfrak{M} \to \mathbb{R}$ is

$$e(\Phi) = \frac{1}{2}\,\mathrm{trace}_g\,\big(\Phi^* h\big).$$

When g is a Lorentzian metric (i.e. $\nu = 1$) an e.h. map is also referred to as an **exponential wave map**. Eells and Lemaire started (cf. [23]) a theory of e.h. maps (though solely from Riemannian manifolds i.e. $\nu = 0$) as one of the many ramifications of harmonic map theory (cf. [22]) got by considering the energy functionals

$$\mathbb{E}_F(\Phi) = \int_\Omega F\big[e(\Phi)\big]\, dv_g \tag{1}$$

where

$$F(t) \in \left\{t,\ \ \frac{1}{p}\,(2t)^{p/2},\quad (1+2t)^\alpha,\quad e^t\right\},\quad t > 0,\quad p \geq 4,\quad \alpha > 1, \tag{2}$$

leading respectively to ordinary harmonic maps, p-harmonic maps, α-energy functionals (cf. Sacks and Uhlenbeck [30], where $n = 2$) and of course exponentially harmonic maps. In a series of papers Ara unified (cf. [1–3]) the treatment of the main geometric properties of F-**harmonic maps** [critical points of (1)] by working with arbitrary C^2 functions $F : [0, +\infty) \to [0, +\infty)$ such that $F'(t) > 0$ for every $t > 0$. However the PDEs side of the subject matter is not investigated in [1–3].

Exponential wave maps $\Phi : \mathfrak{M} \to N$ from the total space $\mathfrak{M} = C(M)$ of the canonical[1] circle bundle $S^1 \to C(M) \xrightarrow{\pi} M$ over a compact strictly pseudoconvex CR manifold M, of CR dimension n, endowed with the Fefferman metric $g = F_\theta$ (a

[1] The relevant notions and basic results (of CR and pseudohermitian geometry) are recalled in Sect. 2.1.

Lorentzian metric on $\mathfrak{M}$, associated to a given positively oriented contact form θ on M), where considered by us in [16]. When such Φ is S^1-invariant one could integrate over the fibres in

$$\mathbb{E}_{\exp}(\Phi) = \int_{C(M)} \exp\left[\frac{1}{2}\operatorname{trace}_{F_\theta}\big(\Phi^* h\big)\right] d\,v_{F_\theta}$$

to discover that $\mathbb{E}_{\exp}(\Phi) = 2\pi\, E_b(\phi)$ where[2] $\phi : M \to N$ is the base map associated to Φ (i.e. $\phi \circ \pi = \Phi$) and

$$E_b(\phi) = \int_M \exp\left[e_b(\phi)\right]\ \theta \wedge (d\theta)^n,$$
$$e_b(\phi) = \frac{1}{2}\operatorname{trace}_{G_\theta}\big(\Pi_H\, \phi^* h\big). \tag{3}$$

Critical points of E_b are referred to as **exponentially subelliptic harmonic** (e.s.h.) maps. The motivation for the (apparently rather involved) chosen terminology is given in Sect. 2.3 and relies on the classification of the Euler–Lagrange equations of the variational principle $\delta\, E_b(\phi) = 0$ i.e.

$$\tau_b(\phi) + \phi_* \nabla^H e_b(\phi) = 0. \tag{4}$$

There are a few notable differences in the pseudohermitian analog (3) to the ordinary energy density $e(\phi)$ that we briefly comment upon. First, one doesn't compute the full trace of $\phi^* h$ but rather the trace of $\Pi_H\, \phi^* h$, the restriction of $\phi^* h$ to $H(M) \otimes H(M)$ where $H(M)$ is the Levi, or maximally complex, distribution of M as a CR manifold. As explained in [19], omitting a direction [here the Reeb vector field T of (M, θ)] in the calculation of the trace of $\phi^* h$ has far reaching consequences: the second order differential operator Δ_b appearing in

$$\tau_b(\phi)^A \equiv -\Delta_b \phi^A + \sum_{a=1}^{2n} \left(\left\{ \begin{array}{c} A \\ BC \end{array} \right\} \circ \phi \right) X_a(\phi^B) X_a(\phi^C)$$

is degenerate elliptic and its ellipticity degenerates precisely at the cotangent directions spanned by θ. Second, in the spirit of complex analysis in several complex variables, the inner product used to compute traces springs from the Levi form G_θ (determined by the given nondegenerate CR structure up to a conformal factor) rather than any other metric on M e.g. the first fundamental form of $\mathbf{j} : M \hookrightarrow \mathbb{C}^{n+1}$ when, say, M is a real hypersurface in $\mathbb{C}^{n+1}$ [and M does embed, at least locally, as a CR submanifold of $\mathbb{C}^{n+1}$, by the positive solution to the CR embedding problem in the compact strictly pseudoconvex case (due to Boutet de Monvel [13])].

The stability problem for F-harmonic maps has been studied for all ramifications (3) starting with the case $F = 1_{\mathbb{R}}$, cf. Smith [31]. The main ingredient in any stability

[2] Here $\pi \in \mathbb{R} \setminus \mathbb{Q}$ (the irrational number π).

theory is the second variation formula for $\mathbb{E}_F$ e.g. for any smooth 2-parameter variation $\{\Phi_{s,t}\}_{|s|<\epsilon,\ |t|<\epsilon}$ of a given harmonic map $\Phi_{0,0} = \Phi$

$$\frac{\partial^2}{\partial s\,\partial t}\left\{\mathbb{E}_{1_{\mathbb{R}}}(\phi_{s,t})\right\}_{s=t=0} = H(\mathbb{E}_{1_{\mathbb{R}}})_{\phi}(V, W),$$
$$V = \left\{\frac{\partial\Phi_{s,t}}{\partial t}\right\}_{s=t=0}, \quad W = \left\{\frac{\partial\Phi_{s,t}}{\partial s}\right\}_{s=t=0},$$

where $H(\mathbb{E}_{1_{\mathbb{R}}})_{\Phi}$ (the **Hessian** of $E_{1_{\mathbb{R}}}$ at Φ) is

$$H(\mathbb{E}_{1_{\mathbb{R}}})_{\Phi}(V, W) = \int_{\mathfrak{M}} h^{\Phi}\big(J^{\phi}V,\ W\big)\, dv_g,$$

while J^{Φ} (the **Jacobi operator**) is

$$J^{\Phi}V = \Delta^{\Phi}V - \mathrm{trace}_g\left\{(R^h)^{\Phi}(V, \phi_*\cdot)\phi_*\cdot\right\},$$
$$V,\ W \in C^{\infty}\left(\Phi^{-1}T(N)\right),$$

and Δ^{Φ} is the rough Laplacian, thus leading to natural notions of index and nullity for every harmonic map $\Phi : \mathfrak{M} \to N$

$$\mathrm{ind}(\Phi) = \Big\{\dim_{\mathbb{R}}(F)\ :\ F \subset C^{\infty}(\Phi^{-1}TN)\ \text{ subspace such that}$$
$$H(\mathbb{E}_{1_{\mathbb{R}}})_{\Phi}(V, V) \le 0, \quad \forall\ V \in F\Big\},$$
$$\mathrm{null}(\Phi) = \dim_{\mathbb{R}}\Big\{V \in C^{\infty}(\Phi^{-1}TN)\ :\ H(\mathbb{E}_{1_{\mathbb{R}}})_{\Phi}(V, W) = 0,$$
$$\forall\ W \in C^{\infty}(\Phi^{-1}TN)\Big\}.$$

When g is Riemannian J^{Φ} is elliptic and hence, as $\mathfrak{M}$ is assumed to be compact, the spectrum $\mathrm{Spec}(J^{\Phi})$ consists of infinitely many eigenvalues with finite multiplicities and without accumulation points

$$\lambda_1(\Phi) \le \lambda_2(\Phi) \le \cdots \le \lambda_k(\Phi) \le \cdots\ \uparrow\ +\infty.$$

Each eigenvalue $\lambda_k(\Phi)$ has multiplicity $m\big[\lambda_k(\Phi)\big] = \dim_{\mathbb{R}} \mathrm{Eigen}\big(J^{\Phi}\,;\,\lambda\big)$ and the index and nullity of ϕ are

$$\mathrm{ind}(\Phi) = \sum_{\substack{\lambda \in \mathrm{Spec}(J^{\Phi}) \\ \lambda > 0}} m(\lambda), \quad \mathrm{null}(\Phi) = \dim_{\mathbb{R}} \mathrm{Ker}(J^{\Phi}).$$

Only partial results in this sense are known for arbitrary F-harmonic maps. The second variation formula for e.h. maps was obtained by Eells and Lemaire (cf. [23]) and the same for exponential wave maps is due to Chiang [14,15]. The arbitrary case $F \in C^2$, $F' > 0$, was treated by Ara (cf. Theorem 6.1 in [1–3], p. 256) yet confined to the Riemannian category. A second variation formula and a theory of stability about a given subelliptic harmonic map were derived by Barletta et al. [8]. The generalized Dirichlet eigenvalue problem for the subelliptic Jacobi operator J_b^ϕ, on a domain $\Omega \subset M$ in a particular strictly pseudoconvex CR manifold M, was solved by Magliaro et al. [11]. The study of partial regularity of weak e.s.h. maps was started in [16] yet up to this work no stability theory for e.s.h. maps was available.

Let ρ_θ be the Canrnot–Carathéodory metric [associated to the sub-Riemannian structure $(H(M),\ G_\theta)$]. Let us assume that $(M,\ \rho_\theta)$ is a complete metric space and let $\Omega \subset M$ be a ρ_θ-bounded domain. In the present paper we obtain the second variation formula

$$\begin{aligned}
&\frac{\partial^2}{\partial s\,\partial t}\left\{E_b(\phi_{s,t})\right\}_{s=t=0} = \int_\Omega h^\phi\big(\mathcal{J}^\phi_{b,\,\exp} V,\ W\big)\,\theta \wedge (d\theta)^n,\\
&\mathcal{J}^\phi_{b,\,\exp} \equiv J^\phi_{b,\,\exp} + Q^\phi,\\
&J^\phi_{b,\exp} V \equiv (D^\phi)^*\left(\exp\left[e_b(\phi)\right] D^\phi V\right)\\
&\quad - \exp\left[e_b(\phi)\right] \mathrm{trace}_{G_\theta}\left\{\pi_H (R^h)^\phi(V,\ \phi_*\,\cdot)\phi_*\,\cdot\right\},\\
&Q^\phi V \equiv (D^\phi)^*\left[\exp\left[e_b(\phi)\right](h^\phi)^*(D^\phi V,\ \phi_*)\,\phi_*\right],
\end{aligned} \tag{5}$$

and start a stability theory for e.s.h. maps. Let M be a compact strictly pseudoconvex CR manifold, endowed with the positively oriented contact form θ. Among other results we show that

Theorem 1 *Let $\phi : M \to H^m$ be an exponentially subelliptic harmonic map of (M, θ) into the m-dimensional hyperbolic space H^m. Then ϕ is weakly stable.*

Theorem 2 *Let $\phi : M \to S^m$ be an exponentially subelliptic harmonic map of (M, θ) into the m-dimensional $(m \geq 3)$ sphere S^m. If $e_b(\phi) < (m-2)/2$ then ϕ is unstable.*

Theorems 1 and 2 are respectively corollaries of our (more general) Theorems 6 and 7.

As a long range consequence of the construction of the energy density $e_b(\phi)$ [one doesn't compute the trace of the whole bilinear form $\phi^* h$, but only the trace (with respect to the Levi form G_θ) of the restriction of $\phi^* h$ to $H(M) \otimes H(M)$, thus missing a direction i.e. the cotangent direction spanned by θ] the (pseudohermitian) analog $J^\phi_{b,\,\exp}$ of the Jacobi operator (for an ordinary exponentially harmonic mapping of Riemannian manifolds) is a degenerate elliptic operator and the solution to the Dirichlet eigenvalue problem

$$J^\phi_{b,\,\exp} V = \lambda V \quad \text{in } \Omega, \qquad V = 0 \quad \text{on } \partial\Omega, \tag{6}$$

is a priori unknown. Let $N = N^m(k)$ be a Riemannian manifold of nonpositive constant sectional curvature $k \leq 0$. When M is ρ_θ-complete and $\Omega \subset M$ is a bounded domain supporting a version of Poincaré inequality

$$\int_\Omega h^\phi(V, V)\, \theta \wedge (d\theta)^n \leq C^2 \int_\Omega \left(h^\phi\right)^* \left(D^\phi V, D^\phi V\right) \theta \wedge (d\theta)^n, \tag{7}$$

$$V \in C^\infty\left(\Omega, \phi^{-1} T N\right),$$

we solve the generalized Dirichlet problem

$$J^\phi_{b,\,\exp} V = F \quad \text{in } \Omega, \qquad V = 0 \quad \text{on } \partial\Omega, \tag{8}$$

for any $F \in L^2(\Omega, \phi^{-1} T N)$, and establish existence and uniqueness of the solution $V = V_F \in \mathring{W}^{1,2}_H(\Omega, \phi^{-1} T N)$ to (8). When additionally Ω supports a form of Kondrakov compactness i.e. the embedding

$$\mathring{W}^{1,2}_H(\Omega, \phi^{-1} T N) \hookrightarrow L^2(\Omega, \phi^{-1} T N)$$

is compact, we use the resulting Green operator

$$G : L^2\left(\Omega, \phi^{-1} T N\right) \to L^2\left(\Omega, \phi^{-1} T N\right), \quad G V = V_F,$$

to solve the Dirichlet eigenvalue problem (6) and establish

Theorem 3 *Let $\phi : M \to N^m(k)$ be a non constant exponentially subelliptic harmonic map from the Carnot–Carathèodory complete pseudohermitian manifold (M, θ) into the space form $N^m(k)$ of nonpositive sectional curvature $k \leq 0$. Let $\Omega \subset M$ be a Carnot–Carathèodory bounded domain supporting Poincaré inequality (7) and Kondrakov compactness. Then there is an infinite sequence*

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_\nu \leq \cdots \uparrow +\infty$$

and an infinite sequence $\{V_\nu\}_{\nu \geq 1} \subset \mathring{W}^{1,2}_H\left(\Omega, \phi^{-1} T N^m(k)\right)$ such that $\{\lambda_\nu \ : \ \nu \geq 1\}$ is the generalized Dirichlet spectrum of $J^\phi_{b,\,\exp}$ i.e.

$$a_\phi\left(V_\nu, S\right) = \lambda_\nu \int_\Omega h^\phi(V_\nu, S)\, \theta \wedge (d\theta)^n$$

for any $\nu \geq 1$ and any $S \in \mathring{W}^{1,2}_H\left(\Omega, \phi^{-1} T N^m(k)\right)$.

Here the bilinear form a_ϕ is

$$a_\phi(V, W) = \int_\Omega \exp\left[e_b(\phi)\right] \left\{ \left(h^\phi\right)^* \left(D^\phi V,\ D^\phi W\right) \right.$$
$$\left. -h^\phi\left(\mathrm{trace}_{G_\theta}\left[\Pi_H\ (R^h)^\phi(V,\ \phi_* \cdot)\phi_* \ \cdot\ \right],\ W\right)\right\}\ \theta \wedge (d\theta)^n.$$

Theorem 3 is a corollary of the more general Theorem 9 in Sect. 3.5.

The paper is organized as follows. In Sect. 2.1 we recall the needed notions of CR and pseudohermitian geometry (the Tanaka–Webster connection, Fefferman's metric, the Graham connection, etc.) by following mainly the reference [18].

In Sect. 2.2 we recall the basics on exponential wave maps.

In Sect. 2.3 we introduce exponentially subelliptic harmonic maps, appearing as base maps associated to S^1 invariant exponential wave maps from the total space of the canonical circle bundle endowed with Fefferman's metric, and give a few examples.

In Sect. 3 we derive the second variation formula for E_b about an e.s.h. map ϕ and hence the Hessian $H(E_b)_\phi$ allowing us to introduce natural pseudohermitian analogs $\mathrm{ind}_{b,\,\exp}(\phi)$ and $\mathrm{null}_{b,\,\exp}(\phi)$ to the index and nullity of a harmonic map.

The stability of e.s.h. maps into either a Riemannian manifold N of nonpositive sectional curvature, or a totally umbilical real hypersurface N of a space form $M^{m+1}(c)$ is examined in Sects. 3.3.2 and 3.4.

In Sect. 3.5 we introduce Sobolev type spaces $W_H^{1,2}(\Omega,\ \phi^{-1}TN\big)$ and $\mathring{W}_H^{1,2}$ $(\Omega,\ \phi^{-1}TN)$ for every e.s.h. map $\phi \in C^\infty\big(\Omega,\ \phi^{-1}TN)$, and solve the Dirichlet problem (8) and the Dirichlet eingenvalue problem (6) for target Riemannian manifolds (N, h) satisfying the curvature estimate

$$\|R^h(A, B)C\| \leq \gamma\ \|A\|\ \|B\|\ \|C\|$$

for some constant $\gamma > 0$ and any $A, B, C \in \mathfrak{X}(N)$.

2 Exponentially Subelliptic Harmonic Maps

2.1 CR and Pseudohermitian Geometry

2.1.1 Tangential Cauchy–Riemann Equations

Let $(M, T_{1,0}(M))$ be a compact strictly pseudoconvex CR manifold, of CR dimension n, carrying the CR structure $T_{1,0}(M) \subset T(M) \otimes \mathbb{C}$. The first order differential operator

$$\overline{\partial}_b : C^1\big(M, \mathbb{C}\big) \to C\big(T_{0,1}(M)^*\big), \quad \big(\overline{\partial}_b u\big)\overline{Z} = \overline{Z}(u),$$

is the **tangential Cauchy–Riemann operator** and

$$\overline{\partial}_b u = 0 \tag{9}$$

are the **tangential C–R equations**. A C^1 solution to (9) is a **CR function** on M.

2.1.2 Canonical Circle Bundle Over a CR Manifold

A complex valued differential p-form η on M is of **type** $(p, 0)$, or a $(p, 0)$-**form**, if $T_{1,0}(M) \rfloor \eta = 0$. Let $\Lambda^{p,0}(M) \to M$ be the relevant vector bundle i.e. cross-sections in $\Lambda^{p,0}(M)$ are $(p, 0)$-forms on M. Top degree forms of type $(p, 0)$ are $(n + 1, 0)$-forms. There is a natural free action of $\mathbb{R}_+ = \mathrm{GL}^+(1, \mathbb{R})$ (the multiplicative positive reals) on

$$K_0(M) = \Lambda^{n+1,0}(M) \setminus \{\text{zero section}\}$$

and the quotient space

$$C(M) = K_0(M)/\mathbb{R}_+$$

is the total space of a principal S^1-bundle over M. The construction makes sense for an arbitrary CR manifold of hypersurface type, without any nondegeneracy assumptions on the CR structure $T_{1,0}(M)$, and the resulting principal bundle

$$S^1 \to C(M) \xrightarrow{\pi} M$$

is referred to as the **canonical circle bundle** over M. As M is compact and the projection π has compact fibres, $C(M)$ is compact as well.

2.1.3 Levi Distribution, Levi Form, Graham's Connection, Fefferman's Metric

Let $H(M)$ be the **Levi**, or maximally complex, **distribution** on M i.e.

$$H(M) = \mathrm{Re}\big\{T_{1,0}(M) \oplus T_{0,1}(M)\big\}, \quad T_{0,1}(M) \equiv \overline{T_{1,0}(M)},$$

and let

$$J : H(M) \to H(M), \quad J\big(Z + \overline{Z}\big) = i\big(Z - \overline{Z}\big), \quad Z \in T_{1,0}(M),$$

be the complex structure on $H(M)$. Let θ be a positively oriented contact form on M i.e., i) $\mathrm{Ker}(\theta) = H(M)$, ii) the **Levi form**

$$G_\theta(X, Y) = (d\theta)(X, JY), \quad X, Y \in H(M),$$

is positive definite, and iii) $\Psi = \Psi_\theta = \theta \wedge (d\theta)^n$ is a volume form on M. Property (iii) is commonly associated with the label "contact form" and is a nontrivial consequence of (ii). For the main notions and basic results in CR and pseudohermitian geometry we rely on [18]. By a result of Lee (cf. [26]) $C(M)$ carries a Lorentzian metric F_θ, the **Fefferman metric** of (M, θ), given by

$$
\begin{aligned}
F_\theta &= \pi^* \tilde{G}_\theta + 2\left(\pi^*\theta\right) \odot \sigma, \\
\sigma &= \frac{1}{n+2}\left\{d\mathbf{s} + \pi^*\left(i\,\omega_\alpha{}^\alpha - \frac{i}{2}\, g^{\alpha\overline{\beta}}\, dg_{\alpha\overline{\beta}} - \frac{\rho}{4(n+1)}\,\theta\right)\right\}, \\
\nabla T_\beta &= \omega_\beta{}^\alpha\, T_\alpha, \quad \rho = g^{\alpha\overline{\beta}} R_{\alpha\overline{\beta}}, \\
\left[g^{\alpha\overline{\beta}}\right] &= \left[g_{\alpha\overline{\beta}}\right]^{-1}, \quad g_{\alpha\overline{\beta}} = G_\theta\left(T_\alpha,\, T_{\overline{\beta}}\right),
\end{aligned} \tag{10}
$$

with respect to a local frame $\{T_\alpha \ : \ 1 \le \alpha \le n\}$ of $T_{1,0}(M)$, defined on some open subset $U \subset M$. Here $\mathbf{s}$ is a local fibre coordinate on $C(M)$, ∇ is the **Tanaka–Webster connection** of (M, θ), $R_{\mu\overline{\nu}}$ is the **pseudohermitian Ricci tensor**, and ρ is the **pseudohermitian scalar curvature**.

2.2 Exponential Wave Maps

Let N be a m-dimensional Riemannian manifold, with the Riemannian metric h, and let $\Phi : C(M) \to N$ be a C^∞ map. We set

$$
\begin{aligned}
\mathbb{E}(\Phi) &= \mathbb{E}_{\exp}(\Phi) = \int_{C(M)} \exp\left[e(\Phi)\right] d\mathrm{vol}(F_\theta), \\
e(\Phi) &= \frac{1}{2}\, g^{\mu\nu}\, \frac{\partial \Phi^A}{\partial x^\mu}\, \frac{\partial \Phi^B}{\partial x^\nu}\, G_{AB}(\Phi), \\
d\,\mathrm{vol}(F_\theta) &= \sqrt{-\mathfrak{g}}\, dx^1 \wedge \cdots \wedge dx^{2n+2}, \\
g_{\mu\nu} &= F_\theta\big(\partial_\mu,\, \partial_\nu\big), \quad \partial_\mu \equiv \frac{\partial}{\partial x^\mu}, \quad \mathfrak{g} = \det\left[g_{\mu\nu}\right], \quad \left[g^{\mu\nu}\right] = \left[g_{\mu\nu}\right]^{-1}, \\
\Phi^A &= y^A \circ \Phi, \quad G_{AB} = h\big(\partial_A,\, \partial_B\big), \quad \partial_A \equiv \frac{\partial}{\partial y^A}.
\end{aligned}
$$

Here $(U,\ \tilde{x}^j)$ and $(\mathcal{U},\ y^A)$ are local coordinate systems on M and N such that $\Phi\big(\pi^{-1}(U)\big) \subset \mathcal{U}$ and one makes use of the induced local coordinates $\big(\pi^{-1}(U),\ x^j\big)$ on $C(M)$ i.e.

$$
x^j = \tilde{x}^j \circ \pi, \quad x^{2n+2} = \mathbf{s}.
$$

The range of the indices is

$$
\begin{aligned}
&\alpha,\ \beta,\ \ldots \in \{1,\ \ldots,\ n\}, \quad A,\ B,\ \ldots \in \{1,\ \ldots,\ m\}, \\
&\mu,\ \nu,\ \ldots \in \{0,\ \ldots, 2n+1\}, \quad j, k,\ \ldots \in \{1,\ \ldots,\ 2n+1\}.
\end{aligned}
$$

We willingly adopt the classical tensor notation familiar in the general relativity and gravity theory, for $(C(M),\ F_\theta)$ is a space-time, for any 3-dimensional strictly pseudoconvex CR manifold M. Indeed if $n = 1$ then $(C(M), F_\theta)$ is a 4-dimensional

Lorentzian manifold. On the other hand, by a result of Graham (cf. [24]) σ is a connection 1-form in the principal bundle $S^1 \to C(M) \xrightarrow{\pi} M$. Let $X^\uparrow \in \mathfrak{X}(C(M))$ denote the horizontal lift of $X \in \mathfrak{X}(M)$ with respect to σ i.e.

$$X_z^\uparrow \in \mathrm{Ker}\big(\sigma_z\big), \quad (d_z\pi)X_z^\uparrow = X_{\pi(z)}, \quad z \in C(M).$$

Moreover let $S \in \mathfrak{X}(C(M))$ be the tangent to the S^1 action and let $T \in \mathfrak{X}(M)$ be the **Reeb vector** field of (M, θ) i.e. the globally defined, nowhere zero, vector field transverse to the Levi distribution $H(M)$, uniquely determined by

$$\theta(T) = 1, \quad T \rfloor d\theta = 0.$$

Then $T^\uparrow - S$ is a globally defined time-like vector field on $(C(M), F_\theta)$. Hence $C(M)$ is **time oriented**, and $(C(M),\ F_\theta,\ T^\uparrow - S)$ is a genuine **space-time**. The resulting relationship between CR geometry, as a chapter of complex analysis of functions of several complex variables, and space-time physics was pursued by Barletta et al. (cf. [9]) and Barletta et al. (cf. [10]).

A C^∞ map $\Phi : C(M) \to N$ is an **exponential wave map** if Φ is a critical point of $\mathbb{E}$ i.e.

$$\frac{d}{ds}\left\{\mathbb{E}\big(\Phi_s\big)\right\}_{s=0} = 0$$

for any smooth 1-parameter variation $\{\Phi_s\}_{|s|<\epsilon} \subset C^\infty(C(M), N)$ of $\Phi_0 = \Phi$. The Euler–Lagrange equations of the variational principle $\delta\, \mathbb{E}(\Phi) = 0$ are

$$\begin{aligned} &\tau\big(\Phi\big) + \Phi_* \, De(\Phi) = 0, \\ &\tau(\Phi)^A = -\Box\Phi^A + \left\{ \begin{array}{c} A \\ BC \end{array} \right\} (\Phi)\, \frac{\partial \Phi^B}{\partial x^\mu} \frac{\partial \Phi^C}{\partial x^\nu}\, g^{\mu\nu}, \\ &Df = g^{\mu\nu}\, \frac{\partial f}{\partial x^\mu} \frac{\partial}{\partial x^\nu}, \quad \Box f = -\frac{1}{\sqrt{-\mathfrak{g}}} \frac{\partial}{\partial x^\mu} \left(\sqrt{-\mathfrak{g}}\, g^{\mu\nu}\, \frac{\partial f}{\partial x^\nu} \right), \end{aligned} \tag{11}$$

for any $f \in C^2(C(M))$. Exponential wave maps were introduced by Eells and Lemaire (cf. [23]), though confined to the Riemannian category [i.e. as maps $\Phi : \mathfrak{M} \to N$ from a Riemannian manifold $(\mathfrak{M}, g)$] under the name **exponentially harmonic maps**. Their notion carries over to the semi-Riemannian case, and in particular to the Lorentzian case, in a rather obvious manner and the Euler–Lagrange equations in the Riemannian case are formally identical to (11) [except that, of course, the geometric wave operator $\Box$ is replaced there by the Laplace–Beltrami operator of $(\mathfrak{M},\ g)$].

Let $\nu \in \mathbb{Z}$, $0 \le \nu \le n$, and let $\mathbb{R}^n_\nu \equiv \big(\mathbb{R}^n,\ g_{n,\nu}\big)$ be the n-dimensional semi-Euclidean space i.e.

$$g_{n,\nu} = \sum_{i=1}^{n} \epsilon_i \, (dx^i)^2,$$

$$\epsilon_1 = \cdots = \epsilon_\nu = -1 = -\epsilon_{nu+1} = \cdots = -\epsilon_n.$$

Even for maps $\Phi : \mathbb{R}^n_1 \to \mathbb{E}^m$ from the Minkowski space $\mathbb{R}^n_1$ into Euclidean space $\mathbb{E}^m = \mathbb{R}^m_0$ the e.h. map equations are nonlinear and the second order terms are coupled. Indeed for any $u \in C^2(\mathbb{R}^n)$

$$\Box u = -\sum_{i=1}^n \epsilon^i \frac{\partial^2 u}{\partial (x^i)^2}, \quad \left\{ \begin{matrix} A \\ BC \end{matrix} \right\} = 0,$$

$$\tau(\Phi)^A = \sum_{i=1}^n \epsilon^i \frac{\partial^2 \Phi^A}{\partial (x^i)^2}, \quad e(\Phi) = \frac{1}{2} \sum_{A=1}^m \sum_{i=1}^n \epsilon^i \left(\frac{\partial \Phi^A}{\partial x^i} \right)^2,$$

$$De(\Phi) = \sum_{A=1}^m \sum_{i,j=1}^n \epsilon_i \epsilon_j \frac{\partial \Phi^A}{\partial x^j} \frac{\partial^2 \Phi^A}{\partial x^i \, \partial x^j} \frac{\partial}{\partial x^i},$$

hence Eq. (11) become

$$\sum_{i=1}^n \epsilon_i \frac{\partial^2 \Phi^A}{\partial (x^i)^2} + \sum_{B=1}^m \sum_{i,j=1}^n \epsilon_i \epsilon_j \frac{\partial \Phi^A}{\partial x^i} \frac{\partial \Phi^B}{\partial x^j} \frac{\partial^2 \Phi^B}{\partial x^i \, \partial x^j} = 0. \tag{12}$$

When $n = 2$ and $m = 1$ the system (12) reduces to the single PDE (with $u = \Phi^1$ and $x^1 = x, x^2 = y$)

$$\left[-1 + \left(\frac{\partial u}{\partial x} \right)^2 \right] \frac{\partial^2 u}{\partial x^2} - 2 \frac{\partial u}{\partial x} \frac{\partial u}{\partial y} \frac{\partial^2 u}{\partial x \, \partial y} + \left[1 + \left(\frac{\partial u}{\partial y} \right)^2 \right] \frac{\partial^2 u}{\partial y^2} = 0. \tag{13}$$

Compare (13) to

$$\left[1 + \left(\frac{\partial u}{\partial x} \right)^2 \right] \frac{\partial^2 u}{\partial x^2} + 2 \frac{\partial u}{\partial x} \frac{\partial u}{\partial y} \frac{\partial^2 u}{\partial x \, \partial y} + \left[1 + \left(\frac{\partial u}{\partial y} \right)^2 \right] \frac{\partial^2 u}{\partial y^2} = 0 \tag{14}$$

which is the e.h. map equation for maps $u = \Phi^1 : \mathbb{E}^2 \to \mathbb{E}^1$. Equation (14) is provided by J. Serrin (cf. [27, p. 431]) as an example of a non-uniformly elliptic equation which is regularly elliptic. With the classical Monge notations

$$p_i = \frac{\partial u}{\partial x_i}, \quad p_{ij} = \frac{\partial^2 u}{\partial x_i \, \partial x_j}, \quad 1 \le i, j, \le 2,$$

Equation (13) [governing exponential wave functions $u : \mathbb{R}^2_1 \to \mathbb{E}^1$] reads

$$u_{yy} = \frac{p_1^2 - 1}{p_2^2 + 1} p_{11} - \frac{2 p_1 p_2}{p_2^2 + 1} p_{12} \tag{15}$$

and the right hand side of (15) doesn't involve u_{yy} any more. Thus given $\varphi,\ \psi \in C^\omega(\mathbb{R})$ the Cauchy problem

$$u(x,0) = \varphi(x), \quad u_y(x,0) = \psi(x), \tag{16}$$

for the Eq. (15) admits a unique C^ω solution defined in a neighbourhood of the origin. The line $y = 0$ is noncharacteristic [for the Cauchy problem (16)] and timelike. If in turn one prescribes the Cauchy data along the spacelike line $x = 0$

$$u(0,y) = \varphi(y), \quad u_x(0,y) = \psi(y), \tag{17}$$

then the points $\{(0, y) \in \mathbb{R}^2 \ : \ \psi(y) \in \{\pm 1\}\}$ are characteristic [for the Cauchy problem (17)]. If $x = 0$ is a characteristic line then a necessary condition [for the existence of the solution to the Cauchy problem (17)] is that φ be affine.

By a colloquial[3] remark of Eells and Lemaire (cf. [23, p. 130]) the equation

$$\Phi_* De(\Phi) = 0 \tag{18}$$

has a "life of its own". A C^∞ map of Riemannian manifolds $\Phi : \mathfrak{M} \to N$ satisfying (18) is e.h. if and only if it is harmonic. Given a domain $\Omega \subset \mathbb{R}^n$ we set as customary

$$\|u\|_p = \left(\int_\Omega |u(x)|^p \, dx \right)^{1/p}$$

for any $u \in L^p(\Omega)$, $1 \leq p < \infty$. Also for every essentially bounded function $u : \Omega \to \mathbb{R}$ we set

$$\|u\|_\infty = \operatorname{ess\,sup}_{x\in\Omega} |u(x)| = \sup \left\{ K > 0 \ : \ |u(x)| \leq K \ \text{ a.e. in } \ \Omega \right\}.$$

Moreover for $\mathfrak{M} = \mathbb{E}^n$ and $N = \mathbb{E}^1$ let us consider (together with [6, p. 557]) the functionals

$$H(\Phi) = \operatorname{ess\,sup}_{x\in\Omega} \left| D\Phi(x) \right|,$$
$$I_N(\Phi) = \left(\int_\Omega \left| D\Phi \right|^{2N} d\, v_g \right)^{1/(2N)}, \quad N \in \mathbb{N},$$

where $g = g_{n,0}$ is the Euclidean metric. Then

$$H(\Phi) = \big\| |D\Phi| \big\|_\infty, \quad I_N(\Phi) = \big\| |D\Phi| \big\|_{2N},$$

and

$$\lim_{N\to\infty} I_N(\Phi) = H(\Phi)$$

[3] As observed by Valli (Italian mathematician, † 1999, see [21]) at the time [23] was written the e.h. maps theory was quite new and the results in [23] somewhat patchy, yet the adopted expository style made [23] a piece of very enjoyable reading [cf. MR1205818 (94d:58045)].

provided that $\mathrm{Vol}(\Omega) = \int_\Omega 1\, dx < \infty$. On the other hand the Eq. (18) may be formally derived (cf. [6, p. 557]) by a limiting process from the Euler–Lagrange equations of the variational principle $\delta I_N(\Phi) = 0$

$$(N-1)\left|D\Phi\right|^{2(N-1)} \left\{ -\frac{1}{N-1} \left|D\Phi\right|^2 \Delta\Phi + \Phi_* D\left(\left|D\Phi\right|^2\right) \right\} = 0 \tag{19}$$

[with $\Delta \equiv -\sum_{i=1}^n \partial^2/\partial x_i^2$]. Indeed, removing $(N-1)\left|D\Phi\right|^{2(N-1)}$ and tending with $N \to \infty$ in (19) leads to Eq. (18).

G. Aronsson's arguments (cf. *op. cit.*) carry over to maps $\phi : M \to N$ from a strictly pseudoconvex CR manifold M (endowed with a contact form θ) into a Riemannian manifold (as shown in Sect. 2.3).

2.3 Exponentially Subelliptic Harmonic Maps

Let $\Phi : C(M) \to N$ be S^1 invariant and let $\phi : M \to N$ be the corresponding base map. By a result in [16] integration along the fibres in $\mathbb{E}(\Phi)$ leads to

$$\mathbb{E}(\Phi) = 2\pi\, E_b(\phi) \tag{20}$$

where we have set

$$E_b(\phi) = E_{b,\,\exp}(\phi) = \int_M \exp\left[e_b(\phi)\right]\, \Psi,$$
$$e_b(\phi) = \frac{1}{2}\, \mathrm{trace}_{G_\theta}\left(\Pi_H\, \phi^* h\right),$$

where $\Pi_H B$ denotes the restriction of the bilinear form B to $H(M) \otimes H(M)$. By (20) every critical point of $\mathbb{E}$ is a critical point of E as well. Conversely, again by a result in [16], the PDE system (11) projects on

$$\tau_b(\phi) + \phi_*\, \nabla^H e_b(\phi) = 0 \tag{21}$$

and (21) are precisely the Euler–Lagrange equations of the variational principle $\delta\, E_b(\phi) = 0$. A C^2 solution to (21) is an **exponentially subelliptic harmonic** (e.s.h.) map. As to the notations in (21)

$$\tau_b(\phi)^A = -\Delta_b \phi^A + \sum_{a=1}^{2n} \left\{ \begin{matrix} A \\ BC \end{matrix} \right\} X_a(\phi^B)\, X_a(\phi^C),$$
$$\Delta_b u = -\mathrm{div}\left(\nabla^H u\right), \quad u \in C^2(M),$$

with respect to a G_θ orthonormal local frame $\{X_a : 1 \le a \le 2n\}$ of $H(M)$. Also

$$\nabla^H u = \Pi_H\, \nabla u, \quad u \in C^1(M),$$

is the **horizontal gradient** of u, where $\Pi_H : T(M) \to H(M)$ is the projection with respect to the direct sum decomposition

$$T(M) = H(M) \oplus \mathbb{R}T \tag{22}$$

and ∇u is the ordinary gradient of u with respect to the Riemannian metric g_θ [the **Webster metric** of (M, θ)] extending the Levi form G_θ to the whole of $T(M)$ by requesting that the Reeb vector field T be orthogonal to $H(M)$ and adding the normalization requirement $g_\theta(T, T) = 1$. The Webster metric g_θ is customarily referred to as a **contraction** of G_θ: indeed one may observe that $\big(M,\ H(M),\ G_\theta)$ is a sub-Riemannian manifold (in the sense of Strichartz [32]) and that the Carnot–Carathéodory and Riemannian distance functions [springing respectively from the sub-Riemannian structure $(H(M),\ G_\theta)$ and the Riemannian metric g_θ]

$$\rho_\theta, \quad d_\theta \ : \ M \times M \to [0, +\infty)$$

are related by

$$d_\theta(x, y) \le \rho_\theta(x, y), \quad x, y \in M.$$

Finally the divergence operator above $\operatorname{div} : \mathfrak{X}(M) \to C^\infty(M)$ is meant with respect to the volume form $\Psi = \theta \wedge (d\theta)^n$ i.e.

$$\mathcal{L}_X \Psi = \operatorname{div}(X)\, \Psi$$

for any $X \in C^1$, where $\mathcal{L}_X$ is the Lie derivative at the direction X. There is a constant $C_n > 0$ (depending only on the CR dimension n) such that

$$\Psi = C_n\, d\, \operatorname{vol}(g_\theta)$$

so the divergence operator one works with coincides with the ordinary Riemannian divergence on $(M,\ g_\theta)$.

The **sublaplacian** Δ_b is a formally self-adjoint second order differential operator, similar to the Laplace–Beltrami operator on a Riemannian manifold, yet degenerate elliptic (in the sense of Bony [12]) and **subelliptic** of order $\epsilon = 1/2$ i.e. for every point $x \in M$ there is an open neighborhood $U \subset M$ of x and a constant $C > 0$ such that

$$\|u\|_{1/2}^2 \le C \left(\|\Delta_b u\|^2 + \|u\|^2 \right)$$

for any $u \in C_0^\infty(U)$. Here $\|\cdot\|_\epsilon$ and $\|\cdot\|$ are respectively the Sobolev norm of order ϵ and the L^2 norm i.e.

$$\|u\|_\epsilon = \left(\int \left(1 + |\xi|^2\right)^\epsilon |\hat{u}(\xi)|^2 \, d\xi \right)^{\frac{1}{2}}, \quad \|u\| = \left(\int_M |u|^2\, \Psi \right)^{\frac{1}{2}}.$$

This motivates our rather involved adopted terminology (i.e. **subelliptic** h. maps). *A fortiori*, by a result of L. Hörmander (cf. [25]) subelliptic operators such as Δ_b are hypoelliptic (i.e. if u is a distribution solution to $\Delta_b u = f$ and $f \in C^\infty$ then $u \in C^\infty$). This prompted the Jost and Xu program (cf. [33]) aiming to recover results, known within elliptic theory, on local properties of weak solutions to certain non linear PDE systems of variational origin, to the case where the principal part of said PDE systems is at least hypoelliptic. In performing this task subelliptic theory (cf. e.g. Danielli et al. [17]) played the strong role played by elliptic theory in Riemannian geometry.

Let $F : [0, +\infty) \to [0, +\infty)$ be a C^2 function such that $F'(t) > 0$. By a result of E. Barletta (cf. Theorem 1 in [7], pp. 34–35) the Euler–Lagrange equations of the variational principle $\delta \int F(Q)\, \theta \wedge (d\theta)^n = 0$ are $\tau_F(\phi) = 0$ where

$$\begin{aligned}\tau_F(\phi) = \tau_F(\phi\,;\,\theta,\,h) \equiv \Big\{&\operatorname{div}\big(\rho(Q)\,\nabla^H \phi^A\big) \\ &+ \sum_{a=1}^{2n} \rho(Q) \left(\left\{ \begin{matrix} A \\ BC \end{matrix} \right\} \circ \phi \right) X_a(\phi^B)\, X_a(\phi^C) \Big\} Y_A, \\ Q = 2\, e_b(\phi), \quad \rho(t) = F'&\left(\frac{t}{2}\right).\end{aligned}$$

Hence for $F(t) = (1/p)\,(2t)^{p/2}$ and $p = 2\mathrm{N}$ with $\mathrm{N} \in \mathbb{N}$

$$\begin{aligned}\tau_F(\phi) &= (\mathrm{N}-1)2^{\mathrm{N}-1} e_b(\phi)^{\mathrm{N}-2}\, \varphi_{\mathrm{N}} \\ \varphi_{\mathrm{N}} &\equiv \frac{1}{\mathrm{N}-1}\, e_b(\phi)\, \tau_b(\phi) + \phi_*\, \nabla^H e_b(\phi).\end{aligned}$$

Given a domain $\Omega \subset M$, a compact subset $K \subset \Omega$, and a continuous section $\varphi \in C(\Omega,\ \phi^{-1} TN)$ we set

$$p_K(\varphi) = \sup_{x \in K} \left(h^\phi\right)^* (\varphi,\ \varphi)_x^{1/2}$$

so that $C(\Omega,\ \phi^{-1} TN)$ is a Fréchet space with the locally convex topology τ determined by the family of seminorms $\{p_K\ :\ K \subset\subset \Omega\}$. The sets

$$\begin{aligned}&V(p_K,\ m) = \left\{ \varphi \in C^\infty(\Omega,\ \phi^{-1} TN)\ :\ p_K(\varphi) < \frac{1}{m} \right\}, \\ &K \subset\subset \Omega, \quad m \in \mathbb{N},\end{aligned}$$

form a local basis for τ and for any pair (K, m) there is $\mathrm{M} \in \mathbb{N}$ such that $\varphi_{\mathrm{N}} \in \phi_*\, \nabla^H e_b(\phi) + V(p_K,\ m)$ for any $\mathrm{N} \geq \mathrm{M}$ which is the limiting process devised in [6, p. 557], adapted to C^∞ maps $\phi : M \to N$. The problem of relating the equation

$$\phi_*\, \nabla^H e_b(\phi) = 0 \tag{23}$$

to Aronson's problem (of producing extensions of Lipschitz functions, cf. [6, p. 551]) is open.

We close the section by giving a few examples of e.s.h. maps.

1) (*Constant maps*) Let $q \in N$ and the us consider the constant map $\phi : M \to N$ given by $\phi(x) = q$ for any $x \in M$. Then ϕ is e.s.h. with respect to the data $(\theta,\ h)$, of energy $E_b(\phi) = \mathrm{Vol}(M,\ \Psi)$.

2) (*Identity map*) Let $N = M$ and $h = g_\theta$ and let $\phi : M \to N$ be the identity map $\phi = 1_M$. Then 1_M is e.s.h. with respect to the data $(\theta,\ g_\theta)$, of energy $E_b(1_M) = e^n\,\mathrm{Vol}(M,\ \Psi)$.

3) (*E.s.h. functions*) Let $\phi : M \to \mathbb{R}$ be an e.s.h. map of (M, θ) into $(\mathbb{R},\ dt \otimes dt)$. Then $\left\{ \begin{matrix} 1 \\ 11 \end{matrix} \right\} = 0$ hence $\tau_b(\phi)^1 = -\Delta_b \phi$ and

$$\phi_* X = X(\phi) \left(\frac{\partial}{\partial t} \right)^{\phi}, \quad X \in H(M),$$

$$e_b(\phi) = \frac{1}{2} \sum_{a=1}^{2n} h^\phi \big(\phi_* X_a,\ \phi_* X_a \big) = \frac{1}{2} \sum_{a=1}^{2n} X_a(\phi)^2 = \frac{1}{2} \big\| \nabla^H \phi \big\|^2,$$

so that the e.s.h. map equation is

$$-\Delta_b \phi + \frac{1}{2} G_\theta \Big(\nabla^H \big(\|\nabla^H \phi\|^2 \big),\ \nabla^H \phi \Big) = 0. \tag{24}$$

4) (*Duan's construction*) Let $\gamma : I \to N$ be a geodesic in (N, h), parametrized by arc-length, and $s : M \to \mathbb{R}$ an e.s.h. function. Then $\phi = \gamma \circ s$ is an e.s.h. map. To prove the statement note that

$$X(\phi^A) = \frac{d\gamma^A}{ds}(s)\, X(s), \quad \gamma^A = y^A \circ \gamma,$$

for any C^1 curve γ in N hence

$$e_b(\phi) = E(\gamma)\, \|\nabla^H s\|^2 \tag{25}$$

where $E(\gamma) = \frac{1}{2} h\big(\dot\gamma,\ \dot\gamma\big)$. If γ is a geodesic then $E(\gamma) \in \mathbb{R}$ [and $E(\gamma) = \frac{1}{2}$ provided that γ is parametrized by arc-lengh]. Moreover

$$\Delta_b \phi^A = -\frac{d^2\gamma^A}{ds^2}(s)\, \|\nabla^H s\|^2 + \frac{d\gamma^A}{ds}(s)\, \Delta_b s, \tag{26}$$

$$\begin{aligned} &\sum_{a=1}^{2n} X_a\big(\phi^B\big)\, X_a\big(\phi^C\big) \left\{ \begin{matrix} A \\ BC \end{matrix} \right\} (\phi) = \\ &= \frac{d\gamma^B}{ds}(s)\, \frac{d\gamma^C}{ds}(s)\, \|\nabla^H s\|^2 \left\{ \begin{matrix} A \\ BC \end{matrix} \right\} \big(\gamma(s)\big), \end{aligned} \tag{27}$$

hence

$$\tau_b(\phi)^A = -\frac{d\gamma^A}{ds}(s)\,\Delta_b s$$
$$+\,\|\nabla^H s\|^2\left[\frac{d^2\gamma^A}{ds}(s) + \left\{\begin{matrix} A \\ BC \end{matrix}\right\}\big(\gamma(s)\big)\,\frac{d\gamma^B}{ds}(s)\,\frac{d\gamma^C}{ds}(s)\right].$$

Next, as γ is a geodesic

$$\frac{d^2\gamma^A}{ds} + \left\{\begin{matrix} A \\ BC \end{matrix}\right\}(\gamma)\,\frac{d\gamma^B}{ds}\,\frac{d\gamma^C}{ds} = 0$$

one has

$$\tau_b(\phi) = -\big(\Delta_b s\big)\,\dot\gamma(s). \tag{28}$$

Moreover

$$\phi_*\,\nabla^H e_b(\phi) = E(\gamma)\,\big(\nabla^H s\big)\big(\|\nabla^H s\|^2\big)\,\dot\gamma(s). \tag{29}$$

Finally [by (28)–(29) and (24) with $\phi = s$]

$$\tau_b(\phi) + \phi_*\,\nabla^H e_b(\phi) = \left\{-\Delta_b s + \frac{1}{2}\,(\nabla^H s)\,\Big(\|\nabla^H s\|^2\Big)\right\}\,\dot\gamma(s) = 0.$$

i.e. ϕ is e.s.h. □

An alternative proof relies on a result by Duan (cf. [20]) together with the geometric interpretation of exponential wave maps in [16]. One has[4]

Theorem 4 *Let* $(\mathfrak{M}, g)$ *be a Lorentzian manifold and* $\sigma : \mathfrak{M} \to \mathbb{R}$ *an exponential wave function. If* $\gamma : I = [0, \ell(\gamma)] \to N$ *is a geodesic in* (N, h)*, parametrized by arc-length, then* $\Phi = \gamma \circ \sigma : \mathfrak{M} \to N$ *is an exponential wave map.*

Let $\mathfrak{M} = C(M)$, $h = F_\theta$ and $\sigma = s \circ \pi$ for some C^∞ e.s.h. function $s : M \to \mathbb{R}$. By a result of Lee (cf. [26]) $\pi_*\Box = \Delta_b$ and by a result in [16] $\pi_* D(u \circ \pi) = \nabla^H u$ for any $u \in C^1(M)$. Hence (24) [with $\phi = s$] yields

$$-\Box\sigma + g\big(D_{D\sigma} D\sigma,\ D\sigma\big) = 0$$

i.e. σ is an exponential wave function and Duan's Theorem 4 applies. Yet $\phi = \gamma \circ s$ is the base map associated to $\Phi = \gamma \circ \sigma$ hence (by Theorem 1 in [16]) ϕ is e.s.h.

[4] The result in [20] is about ordinary wave maps, yet the proof of Theorem 4 is a *verbatim* repetition of the arguments in [20] (hence Theorem 4 is attributed to Duan, cf. *op. cit.*).

3 Second Variation Formula

3.1 Rough Sublaplacians

Let $\phi : M \to N$ be an e.s.h. map. The map

$$\phi_* : \mathfrak{X}(M) \to C^\infty\big(\phi^{-1} TN\big)$$

may be thought of as a section in $T^*(M) \otimes \phi^{-1} T(N)$. Let $\Pi_H \phi_*$ be the restriction of ϕ_* to $C^\infty(H(M))$ [a section in $H(M)^* \otimes \phi^{-1} T(N)$]. Given two sections $\varphi, \psi \in C^\infty\big(H(M)^* \otimes \phi^{-1} TN\big)$ a pointwise inner product

$$(h^\phi)^*(\varphi, \psi) : M \to \mathbb{R}$$

may be defined as follows. Let $x \in M$ and let $\{X_a : 1 \le a \le 2n\}$ be a local G_θ orthonormal frame of $H(M)$, defined on an open neighbourhood $U \subset M$ of x. Then

$$\big(h^\phi\big)^*(\varphi,\ \psi)_x = \sum_{a=1}^{2n} h^\phi\big(\varphi X_a,\ \psi X_a\big)_x .$$

L^2 inner products on sections in $\phi^{-1} T(N)$ and $H(M)^* \otimes \phi^{-1} T(N)$ are respectively given by

$$\begin{aligned} &(V, W)_\phi = \int_M h^\phi(V, W)\ \Psi, \quad (\varphi, \psi)_\phi = \int_M (h^\phi)^*(\varphi, \psi)\ \Psi, \\ &V, W \in C^\infty(\phi^{-1} TN), \quad \varphi, \psi \in C^\infty\big(H(M)^* \otimes \phi^{-1} TN\big). \end{aligned}$$

Let ∇^h be the Levi–Civita connection of (N, h). The covariant derivative associated to the pullback of ∇^h by ϕ may be thought of as a linear operator

$$\big(\nabla^h\big)^\phi : C^\infty(\phi^{-1} TN) \to C^\infty\big(T^*(M) \otimes \phi^{-1} TN\big).$$

For every $V \in C^\infty(\phi^{-1} TN)$ let

$$D^\phi V = \big((\nabla^h)^\phi\big)^H V$$

denote the restriction of $(\nabla^h)^\phi V$ to $H(M)$. Next let

$$\big(D^\phi\big)^* : C^\infty\big(H(M)^* \otimes \phi^{-1} TN\big) \to C^\infty\big(\phi^{-1} TN\big)$$

be the formal adjoint of D^ϕ i.e.

$$\big((D^\phi)^* \varphi,\ V\big)_\phi = \big(\varphi,\ D^\phi V\big)_\phi ,$$

$$\varphi \in C^\infty\big(H(M)^* \otimes \phi^{-1}TN\big), \quad V \in C^\infty(\phi^{-1}TN).$$

The **rough sublaplacian** is the second order differential operator

$$\Delta_b^\phi = \left(D^\phi\right)^* \circ D^\phi .$$

Let R^h be the curvature tensor field of ∇^h i.e.

$$R^h(Y, Z) = \left[\nabla^h_Y,\ \nabla^h_Z\right] - \nabla^h_{[Y,Z]}, \quad Y, Z \in \mathfrak{X}(N).$$

Let $(R^h)^\phi \in C^\infty\big(\otimes^3 \phi^{-1}TN\big)$ be given by

$$(R^h)^\phi_x\big(\mathbf{r},\ \mathbf{s}\big)\mathbf{t} = R^h_{\phi(x)}\big(\mathbf{r}, \mathbf{s}\big)\mathbf{t},$$
$$\mathbf{r},\ \mathbf{s},\ \mathbf{t} \in T_{\phi(x)}(N), \quad x \in M.$$

The **subelliptic Jacobi operator** is

$$J_b^\phi V = \Delta_b^\phi V - \operatorname{trace}_{G_\theta}\left\{\Pi_H (R^h)^\phi (V, \phi_* \cdot)\phi_* \cdot\right\}$$

for any $V \in C^\infty(\phi^{-1}TN)$.

Let $\Omega \subset M$ be a bounded domain. The **Hessian** of $E_b(\phi) = \int_\Omega \exp\left[e_b(\phi)\right] \Psi$ at the point ϕ is

$$H(E_b)_\phi(V, W) = \left(\exp\left[e_b(\phi)\right]\left[J_b^\phi V - D^\phi_{\nabla^H e_b(\phi)} V\right],\ W\right)_\phi +$$
$$+ \int_\Omega \exp\left[e_b(\phi)\right] (h^\phi)^*\left(D^\phi V,\ \Pi_H \phi_*\right)\,(h^\phi)^*\left(D^\phi W,\ \Pi_H \phi_*\right)\,\Psi \tag{30}$$

for any $V, W \in C^\infty(\phi^{-1}TN)$.

3.2 Second Variation Formula

3.2.1 Hessian of E_b at an e.s.h. Map

Of course the Hessian $H(E_b)$ isn't postulated as in (30) but derived by computing the second variation of E_b. Precisely let $\{\phi_{s,t}\}_{|s|<\epsilon,\ |t|<\epsilon} \subset C^\infty(M, N)$ be a smooth 2-parameter variation of $\phi = \phi_{0,0}$ and let us consider

$$f : \tilde{M} = M \times (-\epsilon, \epsilon)^2 \to N,$$
$$f(x, s, t) = \phi_{s,t}(x), \quad x \in M, \quad |s| < \epsilon, \quad |t| < \epsilon,$$
$$V_s,\ V,\ W \in C^\infty(\phi^{-1}TN), \quad |s| < \epsilon,$$
$$V_s(x) = (d_{(x,s,0)} f)(\partial/\partial t)_{(x,s,0)}, \quad V = V_0,$$

$$W(x) = (d_{(x,0,0)} f)(\partial/\partial s)_{(x,0,0)}, \quad x \in M,$$
$$\mathrm{Supp}(V_s) \subset \Omega, \quad |s| < \epsilon.$$

We may state

Theorem 5 *Let $\phi : M \to N$ be an e.s.h. map of (M, θ) into (N, h). The second variation formula for E_b about ϕ is*

$$\frac{\partial^2}{\partial s\, \partial t} \left\{ E_b\big(\phi_{s,t}\big) \right\}_{s=t=0} = H(E_b)_\phi(V, W). \tag{31}$$

Proof Let $\varphi : \tilde{M} \to \mathbb{R}$ be given by

$$\varphi(x, s, t) = e_b(\phi_{s,t}), \quad x \in M, \ \ |s| < \epsilon, \ \ |t| < \epsilon.$$

If $\alpha_{s,t}$ is the injection

$$\alpha_{s,t} : M \to \tilde{M}, \quad \alpha_{s,t}(x) = (x, s, t), \quad x \in M,$$

then $\phi_{s,t} = f \circ \alpha_{s,t}$. Also for all $X \in \mathfrak{X}(M)$ and $A \in \mathfrak{X}(\tilde{M})$ we define $\tilde{X} \in \mathfrak{X}(\tilde{M})$ and $f_* A \in C^\infty\big(f^{-1} T N\big)$ by

$$\tilde{X}_{(x,s,t)} = (d_x \alpha_{s,t}) X_x, \quad (f_* A)_{(x,s,t)} = (d_{(x,s,t)} f) A_{(x,s,t)}.$$

Consequently

$$\varphi = \frac{1}{2} \sum_{a=1}^{2n} h^f\big(f_* \tilde{X}_a, \ f_* \tilde{X}_a\big) \tag{32}$$

everywhere in the open set $\tilde{U} = U \times (-\epsilon, \epsilon)^2$. Then [by (32) together with $(\nabla^h)^f_{\partial/\partial t} h^f = 0$ and $\big[\tilde{X}_a, \ \partial/\partial t\big] = 0$]

$$\frac{\partial \varphi}{\partial t} = \sum_a h^f \left((\nabla^h)^f_{\tilde{X}_a} \ f_* \, \partial/\partial t, \ f_* \tilde{X}_a \right) =$$

[by $(\nabla^h)^f_{\tilde{X}_a} h^f = 0$]

$$= \sum_a \left\{ \tilde{X}_a \left(h^f \left(f_* \, \partial/\partial t, \ f_* \tilde{X}_a \right) \right) - h^f \left(f_* \, \partial/\partial t, \ (\nabla^h)^f_{\tilde{X}_a} f_* \tilde{X}_a \right) \right\}.$$

For every $(s, t) \in (-\epsilon, \epsilon)^2$ we consider the vector field $X_{s,t} \in H(M)$ determined by

$$h^f \left(f_* \, \partial/\partial t, \ f_* \tilde{Y} \right) \circ \alpha_{s,t} = G_\theta\big(X_{s,t}, \ Y\big), \tag{33}$$

for any $Y \in H(M)$. As $\nabla\Psi = 0$ the divergence of $X_{s,t}$ may be computed as the trace of $Y \mapsto \nabla_Y X_{s,t}$ i.e.

$$\operatorname{div}\big(X_{s,t}\big) = \sum_{j=0}^{2n} g_\theta\big(\nabla_{X_j} X_{s,t},\ X_j\big)$$

where

$$\big\{X_j\ :\ 0 \le j \le 2n\big\} = \{X_a,\ T\ :\ 1 \le a \le 2n\}, \quad X_0 = T.$$

Next [by $\nabla_T X_{s,t} \in H(M) \Longrightarrow g_\theta\big(\nabla_T X_{s,t},\ T\big) = 0$]

$$\operatorname{div}\big(X_{s,t}\big) = \sum_{a=1}^{2n} g_\theta\big(\nabla_{X_a} X_{s,t},\ X_a\big) =$$

[by $\nabla g_\theta = 0$ and $g_\theta = G_\theta$ on $H(M) \otimes H(M)$]

$$= \sum_a \big\{X_a\big(G_\theta(X_{s,t},\ X_a)\big) - G_\theta\big(X_{s,t},\ \nabla_{X_a} X_a\big)\big\} =$$

[by (33)]

$$= \sum_a \big\{X_a\big(h^f(f_*\frac{\partial}{\partial t},\ f_*\tilde{X}_a) \circ \alpha_{s,t}\big) - h^f\big(f_*\frac{\partial}{\partial t},\ f_*\tilde{X}_a\big) \circ \alpha_{s,t}\big\}$$

that is

$$\begin{aligned}&\operatorname{div}(X_{s,t})\\ &\quad = \sum_{a=1}^{2n}\Big\{\tilde{X}_a\big(h^f(f_*\frac{\partial}{\partial t},\ f_*\tilde{X}_a)\big) - h^f\big(f_*\frac{\partial}{\partial t},\ f_*\widetilde{\nabla_{X_a} X_a}\big)\Big\} \circ \alpha_{s,t}\end{aligned}$$

everywhere in $\tilde{U}$. Consequently

$$\begin{aligned}&\frac{\partial\varphi}{\partial t} \circ \alpha_{s,t} = \operatorname{div}\big(X_{s,t}\big) - \\ &\quad - \sum_{a=1}^{2n} h^f\big(f_*\frac{\partial}{\partial t},\ \big(\nabla^h\big)^f_{\tilde{X}_a} f_*\tilde{X}_a + f_*\widetilde{\nabla_{X_a} X_a}\big) \circ \alpha_{s,t}.\end{aligned} \tag{34}$$

To obtain a frame-free, and in particular global, version of (34) we introduce the bilinear form

$$B(f)(X,Y) = \big(\nabla^h\big)^f_{\tilde{X}} f_*\tilde{Y} - f_*\widetilde{\nabla_X Y}, \quad X, Y \in \mathfrak{X}(M), \tag{35}$$

Then (34) reads

$$\frac{\partial \varphi}{\partial t} \circ \alpha_{s,t} = \operatorname{div}\big(X_{s,t}\big) - h^f\Big(f_* \frac{\partial}{\partial t},\ \operatorname{trace}_{G_\theta}\big\{\Pi_H\, B(f)\big\}\Big) \circ \alpha_{s,t}. \tag{36}$$

By the sub-Riemannian analog to Hopf–Rinow theorem (in Riemannian geometry) if $(M,\ \rho_\theta)$ is a complete metric space then every closed ball of finite radius in $(M,\ \rho_\theta)$ is compact (cf. [29]). In particular for every ρ_θ-bounded domain $\Omega \subset M$ the closure $\overline{\Omega}$ is a compact set. Hence the relevant integrals are convergent i.e. $E_b(\phi) = \int_\Omega \exp\big[e_b(\phi)\big]\ \Psi < \infty$. Then let us consider the function

$$\psi : (-\epsilon,\ \epsilon)^2 \to \mathbb{R}, \quad \psi(s,t) = E_b(\phi_{s,t}), \quad |s| < \epsilon, \quad |t| < \epsilon.$$

Then

$$\begin{aligned}\frac{\partial \psi}{\partial t}(s,t) &= \int_\Omega e^{\varphi(x,s,t)}\, \frac{\partial \varphi}{\partial t}(x,s,t)\ \Psi(x)\\ &= \int_\Omega e^{\varphi\circ\alpha_{s,t}} \Big\{\operatorname{div}\big(X_{s,t}\big) - h^f\Big(f_* \frac{\partial}{\partial t},\ \operatorname{trace}_{G_\theta}\big\{\Pi_H B(f)\big\}\Big) \circ \alpha_{s,t}\Big\}\ \Psi\end{aligned}$$

and

$$\begin{aligned}&\int_\Omega e^{\varphi(\cdot,s,t)} \operatorname{div}\big(X_{s,t}\big)\ \Psi\\ &\quad = \int_\Omega \Big\{\operatorname{div}\big(e^{\varphi(\cdot,s,t)}\, X_{s,t}\big) - X_{s,t}\big(e^{\varphi\circ\alpha_{s,t}}\big)\Big\}\ \Psi =\end{aligned}$$

(by Green's lemma)

$$= \int_{\partial\Omega} e^{\varphi\circ\alpha_{s,t}}\, g_\theta\big(X_{s,t},\ \nu\big)\, dA - \int_\Omega G_\theta\Big(X_{s,t},\ \nabla^H e^{\varphi\circ\alpha_{s,t}}\Big)\ \Psi =$$

[by (33)]

$$\begin{aligned}&= \int_{\partial\Omega} e^{\varphi\circ\alpha_{s,t}}\, g_\theta\big(X_{s,t},\ \nu\big)\, dA\\ &- \int_\Omega e^{\varphi\circ\alpha_{s,t}} \Big\{h^f\Big(f_* \frac{\partial}{\partial t},\ f_*\nabla^H \widetilde{(\varphi\circ\alpha_{s,t})}\Big)\circ\alpha_{s,t}\Big\}\ \Psi\end{aligned}$$

where ν is the outward pointing unit normal vector field on $\partial\Omega$ and dA is short for the canonical volume form $d\operatorname{vol}\big(\iota^* g_\theta\big)$ [with $\iota : \partial\Omega \to M$ the inclusion]. Consequently

$$\begin{aligned}\frac{\partial \psi}{\partial t}(s,t) &= \int_{\partial\Omega} e^{\varphi\circ\alpha_{s,t}}\, g_\theta\big(X_{s,t},\ \nu\big)\, dA +\\ &- \int_\Omega e^{\varphi\circ\alpha_{s,t}}\Big\{h^f\Big(f_* \frac{\partial}{\partial t},\ f_*\nabla^H \widetilde{(\varphi\circ\alpha_{s,t})} + \operatorname{trace}_{G_\theta}\big\{\Pi_H B(f)\big\}\Big)\circ\alpha_{s,t}\Big\}\ \Psi.\end{aligned} \tag{37}$$

Note that for any $x \in U$

$$
\begin{aligned}
\nabla^H \big(\varphi \circ \alpha_{s,t}\big)_x &= \sum_a X_a \big(\varphi \circ \alpha_{s,t}\big) X_{a,x} \\
&= \sum_a \big[(d_x \alpha_{s,t}) X_{a,x}\big] (\varphi) X_{a,x} = \sum_a \tilde{X}_a(\varphi)_{(x,s,t)} X_{a,x}
\end{aligned}
$$

that is

$$
\nabla^H \big(\varphi \circ \alpha_{s,t}\big) = \sum_{a=1}^{2n} \big\{\tilde{X}_a(\varphi) \circ \alpha_{s,t}\big\} X_a. \tag{38}
$$

Let $\pi : \tilde{M} \to M$ be the projection i.e. $\pi(x, \sigma, \tau) = x$ for any $x \in M$ and any $(\sigma, \tau) \in (-\epsilon, \epsilon)^2$. Then

$$
\big\{\nabla^H \widetilde{(\varphi \circ \alpha_{s,t})}\big\}_{(x,\sigma,\tau)} = (d_x \alpha_{\sigma,\tau}) \big\{\nabla^H (\varphi \circ \alpha_{s,t})\big\}_x =
$$

[by (38)]

$$
= \sum_a \tilde{X}_a(\varphi)_{(x,s,t)} (d_x \alpha_{\sigma,\tau}) X_{a,x} = \sum_a \tilde{X}_a(\varphi)_{(x,s,t)} \tilde{X}_{a,\,(x,\sigma,\tau)}
$$

so that

$$
\nabla^H \widetilde{(\varphi \circ \alpha_{s,t})} = \sum_{a=1}^{2n} \big\{\tilde{X}_a(\varphi) \circ \alpha_{s,t} \circ \pi\big\} \tilde{X}_a. \tag{39}
$$

Therefore

$$
\big\{f_* \nabla^H \widetilde{(\varphi \circ \alpha_{s,t})}\big\}_{(x,\sigma,\tau)} = (d_{(x,\sigma,\tau)} f) \big\{\nabla^H \widetilde{(\varphi \circ \alpha_{s,t})}\big\}_{(x,\sigma,\tau)}
$$

[by (39)]

$$
= \sum_a \tilde{X}_a(\varphi)_{(x,s,t)} (d_{(x,\sigma,\tau)} f) \tilde{X}_{a,(x,\sigma,\tau)} = \sum_a \tilde{X}_a(\varphi)_{(x,s,t)} (f_* \tilde{X}_a)_{(x,\sigma,\tau)}
$$

or

$$
f_* \nabla^H \widetilde{(\varphi \circ \alpha_{s,t})} = \sum_{a=1}^{2n} \big\{\tilde{X}_a(\varphi) \circ \alpha_{s,t} \circ \pi\big\} f_* \tilde{X}_a \tag{40}
$$

in $\tilde{U}$. The elementary identity $\alpha_{s,t} \circ \pi \circ \alpha_{s,t} = \alpha_{s,t}$ together with (40) yields

$$
\big\{f_* \nabla^H \widetilde{(\varphi \circ \alpha_{s,t})}\big\} \circ \alpha_{s,t} = \left\{\sum_{a=1}^{2n} \tilde{X}_a(\varphi) f_* \tilde{X}_a\right\} \circ \alpha_{s,t}. \tag{41}
$$

Let us set

$$\tau_b(f) = \operatorname{trace}_{G_\theta}\big(\Pi_H B(f)\big)$$

for brevity. Also note that $\sum_{a=1}^{2n} \tilde{X}_a(\varphi)\,\tilde{X}_a$ is the local manifestation (on $\tilde{U}$) of a (globally defined) tangent vector field on $\tilde{M}$ which we denote by $\mathfrak{X}_\varphi$. With these ingredients we may compute the integrand function in the right hand side of (37) as follows

$$\begin{aligned} & h^f\big(f_* \frac{\partial}{\partial t},\ f_* \nabla^H \widetilde{(\varphi \circ \alpha_{s,t})} + \tau_b(f)\big) \circ \alpha_{s,t} \\ &= \Big\{ h^f\big(f_* \frac{\partial}{\partial t},\ \tau_b(f) + f_* \mathfrak{X}_\varphi\big) \Big\} \circ \alpha_{s,t} \end{aligned}$$

and hence (37) reads

$$\begin{aligned} \frac{\partial \psi}{\partial t}(s,t) &= \int_{\partial\Omega} e^{\varphi \circ \alpha_{s,t}}\, g_\theta\big(X_{s,t},\ \nu\big)\, dA + \\ & - \int_\Omega e^{\varphi(x,s,t)} \Big\{ h^f\big(f_* \frac{\partial}{\partial t},\ \tau_b(f) + f_* \mathfrak{X}_\varphi\big) \circ \alpha_{s,t} \Big\}\ \Psi. \end{aligned} \tag{42}$$

Let us differentiate (42) with respect to s and evaluate at $s = t = 0$. For the first term [on the right hand side of (42)] one has

$$\begin{aligned} & \Big\{ \frac{\partial}{\partial s} \int_{\partial\Omega} e^{\varphi(x,s,t)}\, g_\theta(X_{s,t},\ \nu)_x\, dA(x) \Big\}_{s=t=0} \\ &= \Big\{ \int_{\partial\Omega} \frac{\partial \varphi}{\partial s}(x,s,t) e^{\varphi(x,s,t)}\, G_\theta\big(X_{s,t},\ \Pi_H \nu\big)_x\, dA(x) \Big\}_{s=t=0} \\ &\quad + \Big\{ \int_{\partial\Omega} e^{\varphi(x,s,t)}\, \frac{\partial}{\partial s}\big[G_\theta\big(X_{s,t},\ \Pi_H \nu\big)_x\big]\, dA(x) \Big\}_{s=t=0} \end{aligned}$$

and [by (33) with $s = t = 0$]

$$\begin{aligned} G_\theta\big(X_{0,0},\ \Pi_H \nu\big)_x &= h^f\left(f_* \frac{\partial}{\partial t},\ f_* \widetilde{\Pi_H \nu} \right)_{(x,0,0)} \\ &= h_{\phi(x)}\big(V_x,\ \big\{ f_* \widetilde{\Pi_H \nu} \big\}_{(x,0,0)}\big) = 0 \end{aligned}$$

for any $x \in \partial\Omega$, because of $\operatorname{Supp}(V) \subset \Omega$. Also

$$\frac{\partial}{\partial s}\big[G_\theta\big(X_{s,t},\ \Pi_H \nu\big)\big]_{s=t=0} =$$

[again by (33)]

$$= \left\{ \frac{\partial}{\partial s} \left[h^f \left(f_* \frac{\partial}{\partial t}, \; f_* \widetilde{\Pi_H \nu} \right) \circ \alpha_{s,t} \right] \right\}_{s=t=0} =$$

[by $\left(\nabla^h\right)^f_{\partial/\partial s} h^f = 0$]

$$= h^f \left(\left(\nabla^h\right)^f_{\partial/\partial s} f_* \frac{\partial}{\partial t}, \; f_* \widetilde{\Pi_H \nu} \right)_{s=t=0} + \\ + h^f \left(f_* \frac{\partial}{\partial t}, \; \left(\nabla^h\right)^f_{\partial/\partial s} f_* \widetilde{\Pi_H \nu} \right)_{s=t=0} \tag{43}$$

and the second term in (43) vanishes at the boundary (as $\mathrm{Supp}(V) \subset \Omega$). On the other hand if we set $f^A = y^A \circ f$ then

$$f_* \frac{\partial}{\partial t} = \frac{\partial f^A}{\partial t} \left(\frac{\partial}{\partial y^A} \right) \circ f, \quad f_* \frac{\partial}{\partial s} = \frac{\partial f^A}{\partial s} \left(\frac{\partial}{\partial y^A} \right) \circ f,$$
$$\left(\nabla^h\right)^f_{\partial/\partial s} f_* \frac{\partial}{\partial t} = \left\{ \frac{\partial f^A}{\partial s \, \partial t} + \left(\left\{ \begin{matrix} A \\ BC \end{matrix} \right\} \circ f \right) \frac{\partial f^B}{\partial s} \frac{\partial f^C}{\partial t} \right\} \left(\frac{\partial}{\partial y^A} \right) \circ f,$$
$$\mathrm{Supp} \left[\frac{\partial f^A}{\partial t} (\cdot, \, 0, \, 0) \right] \subset \mathrm{Supp}(V),$$
$$\mathrm{Supp} \left[\frac{\partial f^A}{\partial s \, \partial t} (\cdot, \, 0, \, 0) \right] \subset \mathrm{Supp}\big(V_s\big), \quad |s| < \epsilon,$$

so the first term in (43) vanishes at the boundary, as well. Moreover, as

$$\varphi \circ \alpha_{0,0} = e_b(\phi), \quad \tau_b(f) \circ \alpha_{0,0} = \tau_b(\phi), \quad (f_* \, \mathfrak{X}_\varphi) \circ \alpha_{0,0} = \phi_* \, \nabla^H e_b(\phi),$$

we obtain

$$\frac{\partial^2 \psi}{\partial s \, \partial t}(0, 0) \\ = - \int_\Omega \exp\left[e_b(\phi)\right] \left(\frac{\partial \varphi}{\partial s} \circ \alpha_{0,0} \right) h^\phi\big(V, \; \tau_b(\phi) + \phi_* \, \nabla^H e_b(\phi)\big) \; \Psi \\ - \int_\Omega \exp\left[e_b(\phi)\right] \left\{ \frac{\partial}{\partial s} \left[h^f \big(f_* \frac{\partial}{\partial t}, \; \tau_b(f) + f_* \, \mathfrak{X}_\varphi \big) \right] \right\}_{s=t=0} \Psi$$

or [by $\tau_b(\phi) + \phi_* \, \nabla^H e_b(\phi) = 0$]

$$\frac{\partial^2 \psi}{\partial s \, \partial t}(0, 0) = \\ = - \int_\Omega \exp\left[e_b(\phi)\right] \left\{ \frac{\partial}{\partial s} \left[h^f \big(f_* \frac{\partial}{\partial t}, \; \tau_b(f) + f_* \mathfrak{X}_\varphi \big) \right] \right\}_{s=t=0} \Psi. \tag{44}$$

3.2.2 The "Classical Term"

The term $\frac{\partial}{\partial s}\left[h^f\left(f_*\frac{\partial}{\partial t},\ \tau_b(f)\right)\right]$ in (44) may be dealt with as follows

$$\frac{\partial}{\partial s}\left[h^f\left(f_*\frac{\partial}{\partial t},\ \tau_b(f)\right)\right] =$$

[by $(\nabla^h)^f_{\partial/\partial s}h^f = 0$]

$$= h^f\left((\nabla^h)^f_{\partial/\partial s}\, f_*\frac{\partial}{\partial t},\ \tau_b(f)\right) + h^f\left(f_*\frac{\partial}{\partial t},\ (\nabla^h)^f_{\partial/\partial s}\,\tau_b(f)\right). \tag{45}$$

The second term in (45) requires interchanging the covariant derivatives $(\nabla^h)^f_{\partial/\partial s}$ and $(\nabla^h)^f_{\tilde{X}_a}$ and therefore involves curvature terms. Let $R^{(\nabla^h)^f}$ be the curvature tensor field of $(\nabla^h)^f$

$$R^{(\nabla^h)^f}(A, B) = \left[(\nabla^h)^f_A,\ (\nabla^h)^f_B\right] - (\nabla^h)^f_{[A,B]},\quad A, B \in \mathfrak{X}(\tilde{M}).$$

Then (by $\left[\partial/\partial s,\ \tilde{X}_a\right] = 0$)

$$\begin{aligned}(\nabla^h)^f_{\partial/\partial s}(\nabla^h)^f_{\tilde{X}_a} f_*\tilde{X}_a &= (\nabla^h)^f_{\tilde{X}_a}(\nabla^h)^f_{\partial/\partial s} f_*\tilde{X}_a + \\ &+ R^{(\nabla^h)^f}\left(\frac{\partial}{\partial s},\ \tilde{X}_a\right) f_*\tilde{X}_a\end{aligned} \tag{46}$$

so that (locally)

$$\begin{aligned}&h^f\left(f_*\frac{\partial}{\partial t},\ \left(\nabla^h\right)^f_{\partial/\partial s}\tau_b(f)\right)\\ &\quad= h^f\left(f_*\frac{\partial}{\partial t},\ (\nabla^h)^f_{\partial/\partial s}(\nabla^h)^f_{\tilde{X}_a} f_*\tilde{X}_a\right)\\ &\qquad- h^f\left(f_*\frac{\partial}{\partial t},\ (\nabla^h)^f_{\partial/\partial s}\, f_*\,\widetilde{(\nabla^h)^f_{\tilde{X}_a}\tilde{X}_a}\right) =\end{aligned}$$

[by (46) and $\left[\partial/\partial s,\ \widetilde{\nabla_{X_a}X_a}\right] = 0$]

$$\begin{aligned}&= h^f\left(f_*\frac{\partial}{\partial t},\ (\nabla^h)^f_{\tilde{X}_a}(\nabla^h)^f_{\partial/\partial s} f_*\tilde{X}_a\right)\\ &\quad+ h^f\left(f_*\frac{\partial}{\partial t},\ R^{(\nabla^h)^f}\left(\frac{\partial}{\partial s},\ \tilde{X}_a\right) f_*\tilde{X}_a\right) - h^f\left(f_*\frac{\partial}{\partial t},\ (\nabla^h)^f_{\widetilde{\nabla_{X_a}X_a}}\, f_*\frac{\partial}{\partial s}\right)\end{aligned}$$

that is

$$h^f\left(f_*\frac{\partial}{\partial t},\ \left(\nabla^h\right)^f_{\partial/\partial s}\tau_b(f)\right)$$

$$= h^f\big(f_* \frac{\partial}{\partial t}, \sum_a R^{(\nabla^h)^f}\big(\frac{\partial}{\partial s}, \tilde{X}_a\big) f_* \tilde{X}_a\big)$$
$$+ h^f\big(f_* \frac{\partial}{\partial t}, \sum_a \Big\{(\nabla^h)^f_{\tilde{X}_a}(\nabla^h)^f_{\tilde{X}_a} - (\nabla^h)^f_{\widetilde{\nabla_{X_a} X_a}}\Big\} f_* \frac{\partial}{\partial s}\big)$$

or [by evaluating at $s = t = 0$]

$$\begin{aligned} & h^f\big(f_* \frac{\partial}{\partial t}, \big(\nabla^h\big)^f_{\partial/\partial s} \tau_b(f)\big) \circ \alpha_{0,0} = \\ & = h^\phi\big(V, \sum_a (R^h)^\phi\big(W, \phi_* X_a\big) \phi_* X_a\big) + \\ & + h^\phi\big(V, \sum_a \Big\{(\nabla^h)^\phi_{X_a}(\nabla^h)^\phi_{X_a} W - (\nabla^h)^\phi_{\nabla_{X_a} X_a} W\Big\}\big). \end{aligned} \tag{47}$$

Let $X_f \in H(M)$ be determined by

$$G_\theta(X_f, Y) = h^\phi\big(V, (\nabla^h)^\phi_Y W\big) \tag{48}$$

for any $Y \in H(M)$. Then

$$h^\phi\big(V, \sum_a (\nabla^h)^\phi_{X_a}(\nabla^h)^\phi_{X_a} W\big) =$$

[as $(\nabla^h)^\phi_{X_a} h^\phi = 0$]

$$= \sum_a \Big\{X_a\Big(h^\phi\big(V, (\nabla^h)^\phi_{X_a} W\big)\Big) - h^\phi\big((\nabla^h)^\phi_{X_a} V, (\nabla^h)^\phi_{X_a} W\big)\Big\} =$$

[by (48)]

$$= \sum_a X_a\big(G_\theta\big(X_f, X_a\big)\big) - (h^\phi)^*\Big(\big((\nabla^h)^\phi\big)^H V, \big((\nabla^h)^\phi\big)^H W\Big) =$$

[by $\nabla_{X_a} G_\theta = 0$]

$$\begin{aligned} & = \sum_a \big\{G_\theta\big(\nabla_{X_a} X_f, X_a\big) + G_\theta\big(X_f, \nabla_{X_a} X_a\big)\big\} \\ & \quad - (h^\phi)^*(D^\phi V, D^\phi W) = \end{aligned}$$

[again by (48)]

$$= \mathrm{div}(X_f) + \sum_a h^\phi\Big(V, (\nabla^h)^\phi_{\nabla_{X_a} X_a} W\Big) - (h^\phi)^*\big(D^\phi V, D^\phi W\big)$$

that is

$$
\begin{aligned}
& h^\phi\big(V,\ \sum_a \Big\{(\nabla^h)^\phi_{X_a}(\nabla^h)^\phi_{X_a} W - (\nabla^h)^\phi_{\nabla_{X_a} X_a} W\Big\}\big) = \\
& = \operatorname{div}(X_f) - (h^\phi)^*\big(D^\phi V,\ D^\phi W\big).
\end{aligned} \tag{49}
$$

Let K^h be the Riemann–Christoffel (0, 4)-tensor field of N

$$
K^h(A, B, C, D) = h\big(R^h(C, D)B,\ A\big), \quad A, B, C, D \in \mathfrak{X}(N).
$$

Then

$$
K^h(A, B, C, D) = -K^h(A, B, D, C)
$$

(as R^h is a 2-form) and

$$
K^h(A, B, C, D) = -K^h(B, A, C, D)
$$

(as $\nabla^h h = 0$). Also

$$
K^h(A, B, C, D) = K^h(C, D, A, B)
$$

(as a consequence of the first Bianchi identity). Therefore

$$
\begin{aligned}
& h^\phi\big(V,\ (R^h)^\phi(W,\ \phi_* X_a)\phi_* X_a\big) = (K^h)^\phi\big(V,\ \phi_* X_a,\ W,\ \phi_* X_a\big) \\
& = (K^h)^\phi\big(W,\ \phi_* X_a,\ V,\ \phi_* X_a\big) = h^\phi\big((R^h)^\phi(V, \phi_* X_a)\phi_* X_a,\ W\big).
\end{aligned}
$$

Let us substitute from (49) into (47). We obtain

$$
\begin{aligned}
& h^f\big(f_* \frac{\partial}{\partial t},\ (\nabla^h)^f_{\partial/\partial s}\, \tau_b(f)\big) \circ \alpha_{0,0} = \\
& = h^\phi\big(\operatorname{trace}_{G_\theta}\big\{(R^h)^\phi(V,\ \phi_* \cdot)\,\phi_* \cdot\ \big\},\ W\big) + \\
& + \operatorname{div}(X_f) - (h^\phi)^*\big(D^\phi V,\ D^\phi W\big).
\end{aligned} \tag{50}
$$

As to the first term in (45) [evaluated at $s = t = 0$]

$$
\begin{aligned}
& h^f\big((\nabla^h)^f_{\partial/\partial s}\, f_* \frac{\partial}{\partial t},\ \tau_b(f)\big) \circ \alpha_{0,0} \\
& = h^\phi\big(\Big\{(\nabla^h)^f_{\partial/\partial s}\, f_* \frac{\partial}{\partial t}\Big\} \circ \alpha_{0,0},\ \tau_b(\phi)\big)
\end{aligned}
$$

so that [by $\tau_b(\phi) = -\phi_* \nabla^H e_b(\phi)$]

$$
h^f\big((\nabla^h)^f_{\partial/\partial s}\, f_* \frac{\partial}{\partial t},\ \tau_b(f)\big) \circ \alpha_{0,0} =
$$

$$= -h^{\phi}\left(\left\{(\nabla^h)^f_{\partial/\partial s}\, f_*\frac{\partial}{\partial t}\right\}\circ\alpha_{0,0},\ \phi_*\,\nabla^H e_b(\phi)\right). \tag{51}$$

Let us substitute from (50)–(51) into (45). We obtain

$$\begin{aligned}
&\int_\Omega \exp\left[e_b(\phi)\right]\frac{\partial}{\partial s}\left[h^f\left(f_*\frac{\partial}{\partial t},\ \tau_b(f)\right)\right]_{s=t=0}\Psi = \\
&= \int_\Omega \exp\left[e_b(\phi)\right]\Big\{\mathrm{div}\big(X_f\big) - \big(h^{\phi}\big)^*(D^{\phi}V,\ D^{\phi}W) + \\
&\quad + h^{\phi}\Big(\mathrm{trace}_{G_\theta}\left[\big(R^h\big)^{\phi}(V,\ \phi_*\cdot)\phi_*\cdot\right],\ W\Big) + \\
&\quad - h^{\phi}\left(\left[(\nabla^h)^f_{\partial/\partial s}\, f_*\frac{\partial}{\partial t}\right]\circ\alpha_{0,0},\ \phi_*\nabla^H e_b(\phi)\right)\Big\}\,\Psi.
\end{aligned} \tag{52}$$

The divergence term may be computed as follows

$$\begin{aligned}
&\int_\Omega \exp\left[e_b(\phi)\right]\,\mathrm{div}(X_f)\,\Psi \\
&= \int_\Omega \left\{\mathrm{div}\left(\exp\left[e_b(\phi)\right]X_f\right) - X_f\big(\exp\left[e_b(\phi)\right]\big)\right\}\Psi =
\end{aligned}$$

(by Green's lemma, as $\mathrm{Supp}(X_f)\subset\mathrm{Supp}(V)\subset\Omega$)

$$= -\int_\Omega \exp\left[e_b(\phi)\right]G_\theta\big(X_f,\ \nabla^H e_b(\phi)\big)\,\Psi =$$

[by (48)]

$$= -\int_\Omega \exp\left[e_b(\phi)\right]h^{\phi}\big(V,\ (\nabla^h)^{\phi}_{\nabla^H e_b(\phi)}W\big)\,\Psi$$

hence the "classical" term in (44) is

$$\begin{aligned}
&\int_\Omega \exp\left[e_b(\phi)\right]\frac{\partial}{\partial s}\left\{h^f\left(f_*\frac{\partial}{\partial t},\ \tau_b(f)\right)\right\}_{s=t=0}\Psi = \\
&= -\Big(\big(D^{\phi}\big)^*\big(\exp\left[e_b(\phi)\right]D^{\phi}V\big),\ W\Big)_{\phi} + \\
&\quad - \int_\Omega \exp\left[e_b(\phi)\right]\Big\{h^{\phi}\big(V,\ D^{\phi}_{\nabla^H e_b(\phi)}W\big) + \\
&\quad - h^{\phi}\big(\mathrm{trace}_{G_\theta}\left[(R^h)^{\phi}(V,\ \phi_*\cdot\right],\ W\big) + \\
&\quad + h^{\phi}\left(\left[(\nabla^h)^f_{\partial/\partial s}\, f_*\frac{\partial}{\partial t}\right]\circ\alpha_{0,0},\ \phi_*\,\nabla^H e_b(\phi)\right)\Big\}\,\Psi.
\end{aligned} \tag{53}$$

3.2.3 The "New Term"

The "new" [i.e. not appearing in the second variation formula for E_b about an ordinary subelliptic harmonic map] term in (44) may be locally calculated as follows

$$\left\{\frac{\partial}{\partial s}\left[h^f\big(f_*\frac{\partial}{\partial t},\ f_*\mathfrak{X}_\varphi\big)\right]\right\}\circ\alpha_{0,0}$$
$$=\left\{\frac{\partial}{\partial s}\left[h^f\big(f_*\frac{\partial}{\partial t},\ \sum_a\tilde{X}_a(\varphi)\,f_*\,\tilde{X}_a\big)\right]\right\}\circ\alpha_{0,0}=$$

[by $(\nabla^h)^f_{\partial/\partial s}h^f=0$]

$$=\left\{h^f\left((\nabla^h)^f_{\partial/\partial s}\,f_*\,\frac{\partial}{\partial t},\ \sum_a\tilde{X}_a(\varphi)\,f_*\,\tilde{X}_a\right)+\right.$$
$$\left.+h^f\left(f_*\frac{\partial}{\partial t},\ \sum_a(\nabla^h)^f_{\partial/\partial s}\big[\tilde{X}_a(\varphi)\,f_*\,\tilde{X}_a\big]\right)\right\}\circ\alpha_{0,0}$$
$$=h^\phi\left(\left[(\nabla^h)^f_{\partial/\partial s}\,f_*\,\frac{\partial}{\partial t}\right]\circ\alpha_{0,0},\ \phi_*\,\nabla^H e_b(\phi)\right)+\mathcal{T}$$

where the term $\mathcal{T}$ is

$$\mathcal{T}=h^f\big(f_*\frac{\partial}{\partial t},\ \sum_a\big[\frac{\partial\tilde{X}_a(\varphi)}{\partial s}\,f_*\,\tilde{X}_a$$
$$+\tilde{X}_a(\varphi)\,\big(\nabla^h\big)^f_{\partial/\partial s}\,f_*\,\tilde{X}_a\big]\big)\circ\alpha_{0,0}.$$

Next (by $\big[\tilde{X}_a,\ \partial/\partial s\big]=0$)

$$\mathcal{T}=\sum_a\left[\left\{\tilde{X}_a\left(\frac{\partial\varphi}{\partial s}\right)\right\}_{s=t=0}h^\phi\big(V,\ \phi_*\,X_a\big)+\right.$$
$$\left.+\left\{\tilde{X}_a(\varphi)\,h^f\left(f_*\,\frac{\partial}{\partial t},\ (\nabla^f)^f_{\tilde{X}_a}\,f_*\,\frac{\partial}{\partial s}\right)\right\}\circ\alpha_{0,0}\right]$$

hence

$$\frac{\partial}{\partial s}\left[h^f\big(f_*\frac{\partial}{\partial t},\ f_*\mathfrak{X}_\varphi\big)\right]_{s=t=0}=$$
$$=h^\phi\left(\left[(\nabla^h)^f_{\partial/\partial s}\,f_*\,\frac{\partial}{\partial t}\right]\circ\alpha_{0,0},\ \phi_*\nabla^H e_b(\phi)\right)+$$
$$+h^\phi\big(V,\ (\nabla^h)^\phi_{\nabla^H e_b(\phi)}W\big)+\sum_a\left[\tilde{X}_a\left(\frac{\partial\varphi}{\partial s}\right)\right]_{s=t=0}h^\phi\big(V,\ \phi_*X_a\big).\tag{54}$$

Let us substitute from (53)–(54) into (44) and observe the cancellation of terms. We obtain

$$\begin{aligned}
\frac{\partial^2 \psi}{\partial s\, \partial t}(0,0) &= \big((D^\phi)^*\big(\exp\big[e_b(\phi)\big]\, D^\phi V\big),\ W\big)_\phi + \\
&- \int_\Omega \exp\big[e_b(\phi)\big]\, h^\phi \left(\mathrm{trace}_{G_\theta}\left[\Pi_H (R^h)^\phi (V,\ \phi_* \cdot)\phi_* \cdot\right],\ W\right)\ \Psi + \\
&- \int_\Omega \exp\big[e_b(\phi)\big]\, \big(\Gamma \circ \alpha_{0,0}\big)\ \Psi
\end{aligned} \tag{55}$$

where $\Gamma : \tilde{M} \to \mathbb{R}$ is locally given by

$$\Gamma = \sum_{a=1}^{2n} h^f\big(f_* \frac{\partial}{\partial t},\ f_* \tilde{X}_a\big)\, \tilde{X}_a \left(\frac{\partial \varphi}{\partial s}\right)$$

in $\tilde{U}$. By the very definition of φ [i.e. by differentiating in (32) with respect to s and using the commutation formula $[\tilde{X}_a,\ \partial/\partial s] = 0$]

$$\frac{\partial \varphi}{\partial s} = \sum_b h^f \left((\nabla^h)^f_{\tilde{X}_b}\, f_* \frac{\partial}{\partial s},\ f_* \tilde{X}_b\right) =$$

[by $(\nabla^h)^f_{\tilde{X}_a} h^f = 0$]

$$= \sum_b \left\{ \tilde{X}_b \left(h^f\big(f_* \frac{\partial}{\partial s},\ f_* \tilde{X}_b\big)\right) - h^f\big(f_* \frac{\partial}{\partial s},\ (\nabla^h)^f_{\tilde{X}_a}\, f_* \tilde{X}_b\big)\right\}$$

hence

$$\begin{aligned}
&\left\{\frac{\partial \varphi}{\partial s} \circ \alpha_{0,0}\right\}(x) = \\
&= \sum_b \left\{ \tilde{X}_{b,\,(x,0,0)} \left(h^f\big(f_* \frac{\partial}{\partial s},\ f_* \tilde{X}_b\big)\right) - h^\phi \left(W,\ (\nabla^h)^\phi_{X_b} \phi_* X_b\right)_x \right\} = \\
&= \sum_b \left\{ X_{b,x} \left(h^f\big(f_* \frac{\partial}{\partial s},\ f_* \tilde{X}_b\big) \circ \alpha_{0,0}\right) - h^\phi \left(W,\ \nabla^\phi_{X_b} \phi_* X_b\right)_x \right\} = \\
&= \sum_b h^\phi\big((\nabla^h)^\phi_{X_b} W,\ \phi_* X_b\big)_x
\end{aligned}$$

that is

$$\frac{\partial \varphi}{\partial s} \circ \alpha_{0,0} = \big(h^\phi\big)^* \big(D^\phi W,\ \phi_*\big). \tag{56}$$

Moreover

$$\tilde{X}_a\left(\frac{\partial\varphi}{\partial s}\right)\circ\alpha_{0,0} = X_a\left(\frac{\partial\varphi}{\partial s}\circ\alpha_{0,0}\right). \tag{57}$$

Hence [by (57)]

$$\Gamma\circ\alpha_{0,0} = \sum_a h^\phi\big(V,\ \phi_* X_a\big)\ X_a\left(\frac{\partial\varphi}{\partial s}\circ\varphi_{0,0}\right) =$$

[by (56)]

$$= \sum_a h^\phi\big(V,\ \phi_* X_a\big)\, X_a\left((h^\phi)^*(D^\phi W,\ \phi_*)\right)$$

that is

$$\Gamma\circ\alpha_{0,0} = h^\phi\left(V,\ \phi_*\nabla^H\big[(h^\phi)^*(D^\phi W,\ \phi_*)\big]\right). \tag{58}$$

To calculate the term $\int_M \exp[e_b(\phi)]\,(\Gamma\circ\alpha_{0,0})\ \Psi$ in (55) we use (58) and integration by parts. Precisely let $Y_f \in H(M)$ be determined by

$$G_\theta\big(Y_f,\ X\big) = h^\phi(V, \phi_* X), \quad X\in H(M), \tag{59}$$

[so that $\mathrm{Supp}(Y_f)\subset\mathrm{Supp}(V)\subset\Omega$] and let us consider the functions $v, w\in C^\infty(M)$ given by

$$v = (h^\phi)^*\big(D^\phi V,\ \phi_*\big), \quad w = (h^\phi)^*\big(D^\phi W,\ \phi_*\big).$$

Then [by (58)]

$$\int_\Omega \exp\big[e_b(\phi)\big]\ \big(\Gamma\circ\alpha_{0,0}\big)\ \Psi = \int_\Omega \exp\big[e_b(\phi)\big]\, h^\phi\left(V,\ \phi_*\,\nabla^H w\right)\ \Psi =$$

[by (59)]

$$\begin{aligned}
&= \int_\Omega \exp\big[e_b(\phi)\big]\, G_\theta\big(Y_f,\ \nabla^H w\big)\,\Psi = \int_\Omega \exp\big[e_b(\phi)\big]\, Y_f(w)\ \Psi \\
&= \int_\Omega \big\{Y_f\left(w\ \exp\big[e_b(\phi)\big]\right) - w\, Y_f\left(\exp\big[e_b(\phi)\big]\right)\big\}\ \Psi \\
&= \int_\Omega \big\{\mathrm{div}\left(w\ \exp\big[e_b(\phi)\big]\, Y_f\right) - \\
&\quad - w\ \exp\big[e_b(\phi)\big]\,\mathrm{div}(Y_f) - w\, Y_f\left(\exp\big[e_b(\phi)\big]\right)\big\}\ \Psi
\end{aligned}$$

that is

$$\begin{aligned}&\int_{\Omega} \exp\left[e_b(\phi)\right]\, \left(\Gamma \circ \alpha_{0,0}\right)\, \Psi = \\ &= -\int_{\Omega} w\, \exp\left[e_b(\phi)\right]\, \left\{\mathrm{div}(Y_f) + Y_f\big(e_b(\phi)\big)\right\}\, \Psi. \end{aligned} \tag{60}$$

On the other hand

$$\mathrm{div}\big(Y_f\big) = \sum_a G_\theta\big(\nabla_{X_a} Y_f,\, X_a\big) =$$

[by $\nabla_{X_a} G_\theta = 0$]

$$= \sum_a \left\{X_a \left(G_\theta(Y_f,\, X_a)\right) - G_\theta\left(Y_f,\, \nabla_{X_a} X_a\right)\right\} =$$

[by (59)]

$$= \sum_a \left\{X_a \left(h^\phi(V,\, \phi_* X_a)\right) - h^\phi\left(V,\, \phi_* \nabla_{X_a} X_a\right)\right\} =$$

[by $(\nabla^h)^\phi_{X_a} h^\phi = 0$]

$$\begin{aligned} &= \sum_a \Big\{ h^\phi\left((\nabla^h)^\phi_{X_a} V,\, \phi_* X_a\right) + \\ &\quad + h^\phi\left(V,\, (\nabla^h)^\phi_{X_a} \phi_* X_a - \phi_* \nabla_{X_a} X_a\right)\Big\} \\ &= (h^\phi)^* \left(\left((\nabla^h)^\phi\right)^H V,\, \phi_*\right) + h^\phi\left(V,\, \mathrm{trace}_{G_\theta} \Pi_H B(\phi)\right) \end{aligned}$$

that is

$$\mathrm{div}(Y_f) = v + h^\phi\big(V,\, \tau_b(\phi)\big). \tag{61}$$

Also

$$Y_f\big(e_b(\phi)\big) = G_\theta\left(Y_f,\, \nabla^H e_b(\phi)\right) =$$

[by (59)]

$$= h^\phi\left(V,\, \phi_* \nabla^H e_b(\phi)\right)$$

which [together with (60)–(61)] yields [as ϕ is e.s.h.]

$$\int_{\Omega} \exp\left[e_b(\phi)\right]\, \left(\Gamma \circ \alpha_{0,0}\right)\, \Psi = -\int_{\Omega} v\, w\, \exp\left[e_b(\phi)\right] \Psi. \tag{62}$$

Let us substitute from (62) into (55). We obtain

$$\frac{\partial^2 \psi}{\partial s\, \partial t}(0,0) = \big((D^\phi)^*\big(\exp\left[e_b(\phi)\right] D^\phi V\big),\ W\big)_\phi +$$
$$- \int_\Omega \exp\left[e_b(\phi)\right] h^\phi \left(\operatorname{trace}_{G_\theta} \left\{\Pi_H (R^h)^\phi (V,\ \phi_* \cdot)\phi_* \cdot\right\},\ W\right) \Psi +$$
$$+ \int_\Omega v\, w\, \exp\left[e_b(\phi)\right] \Psi. \tag{63}$$

3.2.4 The Operator $J^\phi_{b,\,\exp}$

Let us consider the linear operator

$$J^\phi_{b,\exp} : C^\infty\big(\phi^{-1}TN\big) \to C^\infty\big(\phi^{-1}TN\big),$$
$$J^\phi_{b,\exp} V \equiv (D^\phi)^* \left(\exp\left[e_b(\phi)\right] D^\phi V\right)$$
$$- \exp\left[e_b(\phi)\right] \operatorname{trace}_{G_\theta} \left\{\pi_H (R^h)^\phi (V,\ \phi_* \cdot)\phi_* \cdot\right\},$$

so that (63) reads

$$\frac{\partial^2 \psi}{\partial s\, \partial t}(0,0) = \left(J^\phi_{b,\,\exp} V,\ W\right)_\phi +$$
$$+ \int_\Omega \exp\left[e_b(\phi)\right] (h^\phi)^* \left(D^\phi V,\ \Pi_H \phi_*\right) (h^\phi)^* \left(D^\phi W,\ \Pi_H \phi_*\right) \Psi. \tag{64}$$

To check that (31) and (64) coincide [thus proving Theorem 5] one needs to relate $J^\phi_{b,\,\exp}$ to the subelliptic Jacobi operator J^ϕ_b. We start by establishing

Lemma 1 *The formal adjoint $(D^\phi)^*$ of D^ϕ is locally given by*

$$(D^\phi)^* \varphi = -\operatorname{Trace}_{G_\theta} \left\{\Pi_H\, D^\phi \varphi\right\} =$$
$$= -\sum_{a=1}^{2n} \left\{(\nabla^h)^\phi_{X_a} \varphi X_a - \varphi \nabla_{X_a} X_a\right\} \tag{65}$$

for any $\varphi \in C^\infty\big(H(M)^* \otimes \phi^{-1}TN\big)$. In particular

$$\Delta^\phi_b V = -\sum_{a=1}^{2n} \left\{D^\phi_{X_a} D^\phi_{X_a} V - D^\phi_{\nabla_{X_a} X_a} V\right\}$$

for any $V \in C^\infty(\phi^{-1}TN)$.

Proof A covariant derivative of $\varphi \in C^\infty(T^*(M) \otimes \phi^{-1}TN)$ is defined in terms of ∇^h and ∇ [the Levi–Civita and Tanaka–Webster connections of (N, h) and (M, θ), respectively]

$$(D^\phi_X \varphi)Y = (\nabla^h)^\phi_X \varphi Y - \varphi \nabla_X Y, \quad X, Y \in \mathfrak{X}(M).$$

For any

$$V \in C^\infty(\phi^{-1}TN), \quad \varphi \in C^\infty\big(H(M)^* \otimes \phi^{-1}TN\big),$$

let $Y_\varphi \in H(M)$ be determined by

$$h^\phi\big(V,\ \varphi X\big) = G_\theta\big(Y_\varphi,\ X\big), \quad X \in H(M). \tag{66}$$

Then (locally)

$$\begin{aligned} \operatorname{div}(Y_\varphi) &= \sum_a G_\theta\big(\nabla_{X_a} Y_\varphi,\ X_a\big) \\ &= \sum_a \big\{X_a\big(G_\theta(Y_\varphi,\ X_a)\big) - G_\theta\big(Y_\varphi,\ \nabla_{X_a} X_a\big)\big\} = \end{aligned}$$

[by (66)]

$$\begin{aligned} &= \sum_a \big\{X_a\big(h^\phi(V, \varphi X_a)\big) - h^\phi\big(V,\ \varphi \nabla_{X_a} X_a\big)\big\} \\ &= \sum_a \Big\{h^\phi\left((\nabla^h)^\phi_{X_a} V,\ \varphi X_a\right) + h^\phi\left(V,\ (\nabla^h)^\phi_{X_a} \varphi X_a - \varphi \nabla_{X_a} X_a\right)\Big\} \end{aligned}$$

that is

$$\operatorname{div}(Y_\varphi) = (h^\phi)^*\big(D^\phi V,\ \varphi\big) + h^\phi\left(V,\ \operatorname{trace}_{G_\theta}\left\{\Pi_H\, D^\phi \varphi\right\}\right). \tag{67}$$

Next

$$\big((D^\phi)^* \varphi,\ V\big)_\phi = \big(\varphi,\ D^\phi V\big)_\phi = \int_\Omega (h^\phi)^*\big(\varphi,\ D^\phi V\big)\ \Psi =$$

[by (67) and Green's lemma]

$$\begin{aligned} &= -\int_\Omega h^\phi\left(V,\ \operatorname{trace}_{G_\theta}\left\{\Pi_H\, D^\phi \varphi\right\}\right)\ \Psi \\ &= -\big(V,\ \operatorname{trace}_{G_\theta}\left\{\Pi_H\, D^\phi \varphi\right\}\big)_\phi \end{aligned}$$

yielding (65). $\square$

The second statement in Lemma 1 follows from (65) for $\varphi = D^\phi V$.
Moreover for any $V \in C^\infty(\phi^{-1}TN)$

$$(D^\phi)^*\big(\exp\left[e_b(\phi)\right] D^\phi\big) =$$

[by (65) for $\varphi = \exp\left[e_b(\phi)\right] D^\phi V$]

$$\begin{aligned} &= (D^\phi)^*\varphi = -\sum_a \left\{D^\phi_{X_a}\varphi X_a - \varphi \nabla_{X_a} X_a\right\} \\ &= -\exp\left[e_b(\phi)\right]\sum_a \left\{X_a\big(e_b(\phi)\big)\, D^\phi_{X_a} V + D^\phi_{X_a} D^\phi_{X_a} V - D^\phi_{\nabla_{X_a} X_a} V\right\} \end{aligned}$$

that is

$$(D^\phi)^*\left(\exp\left[e_b(\phi)\right] D^\phi V\right) = \exp\left[e_b(\phi)\right]\left\{\Delta^\phi_b V - D^\phi_{\nabla^H e_b(\phi)} V\right\}. \tag{68}$$

Finally

$$\begin{aligned} J^\phi_{b,\exp} V &= (D^\phi)^*\left(\exp\left[e_b(\phi)\right] D^\phi V\right) \\ &\quad - \exp\left[e_b(\phi)\right] \mathrm{trace}_{G_\theta}\left\{\Pi_H (R^h)^\phi (V,\ \phi_* \cdot\right\} \end{aligned}$$

so that [by (68)]

$$J^\phi_{b,\exp} = \exp\left[e_b(\phi)\right]\left\{J^\phi_b - D^\phi_{\nabla^H e_b(\phi)}\right\} \tag{69}$$

and substitution from (69) into (64) yields (31). □

3.2.5 Symbol of $L^\phi V = (D^\phi)^*\big(\exp\left[e_b(\phi)\right] D^\phi V\big)$

Let us set

$$T'(M) = T^*(M)\backslash\{\text{zero section}\}$$

and let $\pi : T'(M) \to M$ be the projection. Let $E = \phi^{-1}T(N)$ and let $\pi^{-1}E \to T'(M)$ be the pullback of $E \to M$ by π. As customary let $\mathrm{Smbl}_k(E, E)$ consist of all $\sigma \in \mathrm{Hom}\big(\pi^{-1}E,\ \pi^{-1}E\big)$ such that

$$\sigma_{\rho\,\omega} = \rho^k\,\sigma_\omega, \quad \omega \in T'(M), \quad \rho > 0.$$

Moreover let

$$\begin{aligned} &\sigma_k : \mathrm{Diff}_k(E, E) \to \mathrm{Smbl}_k(E, E), \\ &\sigma_k(L)_\omega v = L\left(\frac{i^k}{k!}\left[f - f(x)\right]^k V\right)_x \in E_x, \end{aligned}$$

$$\omega \in T'(M), \quad x = \pi(\omega), \quad v \in E_x,$$
$$V \in C^\infty\big(E\big), \quad V_x = v, \quad f \in C^\infty(M), \quad (df)_x = \omega.$$

Lemma 2 *If* $L^\phi V \equiv (D^\phi)^*\big(\exp\big[e_b(\phi)\big]\, D^\phi V\big)$ *then*

$$L^\phi\big(u\, V\big) = u\, L^\phi V + \\ + \exp\big[e_b(\phi)\big]\, \Big\{(\Delta_b u)\, V - 2\, D^\phi_{\nabla^H u} V\Big\} - (\nabla^H u)\, \big(\exp\big[e_b(\phi)\big]\big)\, V \tag{70}$$

for any $u \in C^\infty(M)$ *and any* $V \in C^\infty(\phi^{-1}TN)$.

Proof If $\varphi = \exp\big[e_b(\phi)\big]\, D^\phi(u\, V)$ then (locally, by Lemma 1)

$$L^\phi(u\, V) = (D^\phi)^*\varphi = -\sum_{a=1}^{2n}\Big\{(\nabla^h)^\phi_{X_a}\varphi X_a - \varphi\nabla_{X_a}X_a\Big\}$$
$$= -\sum_a \Big\{D^\phi_{X_a}\Big(e^{e_b(\phi)}\, D^\phi_{X_a}(u\, V)\Big) - e^{e_b(\phi)}\, D^\phi_{\nabla_{X_a}X_a}(u\, V)\Big\}$$

and one makes use of

$$\Delta_b u = -\sum_{a=1}^{2n}\Big\{X_a^2(u) - (\nabla_{X_a}X_a)(u)\Big\}, \quad \nabla^H u = \sum_{a=1}^{2n} X_a(u)\, X_a.$$

□

To compute the symbol

$$\sigma_2(L^\phi)_\omega v = -\frac{1}{2}\, L^\phi\Big([f - f(x)]^2\, V\Big)_x$$

we use Lemma 2 with $u = v^2$ and $v = f - f(x)$ so that

$$\Delta_b(v^2) = 2\, v\, \Delta_b v - 2\, \|\nabla^H v\|^2\,,$$
$$u(x) = 0, \quad \big(\nabla^H u\big)_x = 0, \quad \big(D^\phi_{\nabla^H u} V\big)_x = 0, \quad \big(\Delta_b u\big)_x = -2\|\nabla^H f\|^2_x,$$

hence [by (70)]

$$L^\phi(u\, V)_x = -2\, \exp\big[e_b(\phi)_x\big]\, \|\nabla^H f\|^2_x\, V_x.$$

The decomposition

$$\nabla^H f = \nabla f - \theta\big(\nabla f\big)\, T$$

yields [by $\omega = (df)_x$]

$$\|\nabla^H f\|_x^2 = \|\omega\|^2 - \omega(T_x)^2.$$

Finally

$$\sigma_2(L^\phi)_\omega v = \exp\left[e_b(\phi)_x\right] \left\{\omega(T_x)^2 - \|\omega\|^2\right\} v. \tag{71}$$

A symbol $s \in \mathrm{Smbl}_k(E, F)$ is **elliptic** if $s_\omega : E_\omega \to F_\omega$ is an isomorphism for any $\omega \in T'(M)$. Note that

$$\begin{aligned} &\omega(T_x)^2 - \|\omega\|^2 = 0 \Longleftrightarrow (\nabla^H f)_x = 0 \Longleftrightarrow \omega\big(X_{a,x}\big) = 0 \\ &\Longleftrightarrow \mathrm{Ker}(\omega) \supset H(M)_x \Longleftrightarrow \omega \in H(M)_x^\perp = \mathbb{R}\theta_x \end{aligned}$$

hence [by (71)] the ellipticity of $\sigma_2(L^\phi)$ degenerates at the cotangent directions spanned by θ_x, $x \in M$.

3.3 Stability Theory

3.3.1 Index and Nullity of an e.s.h. Map

Let

$$\mathfrak{har}_{b,\,\exp}(M, N) \equiv \mathfrak{har}_{b,\,\exp}\big[(M, \theta),\ (N, h)\big]$$

consist of all $\phi \in C^\infty(M, N)$ such that ϕ is an e.s.h. map with respect to the data (θ, h). Given $\phi \in \mathfrak{har}_{b,\,\exp}(M, N)$ let

$$\mathrm{ind}^\Omega(\phi) \equiv \mathrm{ind}^\Omega_{b,\,\exp}(\phi)$$

denote the upper bound of $\dim(F)$ over all finite dimensional subspaces $F \subset C^\infty\big(\Omega,\ \phi^{-1}TN\big)$ on which the Hessian $H\big(E_b\big)_\phi$ is negative definite. Also let

$$\mathrm{null}^\Omega(\phi) = \mathrm{null}^\Omega_{b,\,\exp}(\phi)$$

be the dimension of the space

$$\Big\{V \in C^\infty\big(\phi^{-1}TN\big)\ :\ H(E_b)_\phi(V, W) = 0,\ \ \forall\, W \in C^\infty\big(\Omega,\ \phi^{-1}TN\big)\Big\}.$$

When M is compact and $\Omega = M$ we write merely $\mathrm{ind}(\phi) = \mathrm{ind}^M(\phi)$ and $\mathrm{null}(\phi) = \mathrm{null}^M(\phi)$. A map $\phi \in \mathfrak{har}_{b,\,\exp}(M, N)$ is **weakly stable** if $\mathrm{ind}(\phi) = 0$ i.e.

$$H(E_b)_\phi(V, V) \geq 0, \quad \forall\, V \in C^\infty\big(\Omega,\, \phi^{-1}TN\big).$$

Otherwise ϕ is **unstable**.

As shown above the operator $J^\phi_{b,\,\exp}$ fails to be elliptic [by (71) $J^\phi_{b,\,\exp}$ is only degenerate elliptic] so the behaviour of its spectrum is *a priori* unknown (even if M is compact). However if the eigenvalues of $J^\phi_{b,\,\exp}$ form a discrete set

$$\lambda_1(\phi) \leq \lambda_2(\phi) \leq \cdots \leq \lambda_\nu(\phi) \leq \cdots \uparrow +\infty$$

with finite multiplicities and without accumulation points then, by (64) i.e.

$$\begin{aligned} H(E_b)_\phi(V, V) &= \big(J^\phi_{b,\,\exp} V,\, V\big)_\phi \\ &+ \int_\Omega \exp\big[e_b(\phi)\big]\,\big(h^\phi\big)^*\big(D^\phi V,\, \Pi_H \phi_*\big)^2\,\Psi, \end{aligned}$$

it follows that ϕ is weakly stable provided that $\lambda_\nu(\phi) \geq 0$ for any $\nu \geq 1$. The question as to when the spectrum of $J^\phi_{b,\,\exp}$ is discrete is examined in Sect. 3.5 for a particular class of domains $\Omega \subset M$.

3.3.2 E.s.h. Maps into Spaces of Nonpositive Curvature

In this section M is a compact strictly pseudoconvex CR manifold, and the findings in the previous sections are tacitly used with $\Omega = M$.

Theorem 6 *If (N, h) has nonpositive sectional curvature then any $\phi \in \mathfrak{har}_{b,\,\exp}(M, N)$ is weakly stable.*

Proof For every $V \in C^\infty\big(\phi^{-1}TN\big)$

$$\begin{aligned} H(E_b)_\phi(V, V) &= \int_M \exp\big[e_b(\phi)\big]\,\|D^\phi V\|^2\,\Psi \\ &- \int_M \exp\big[e_b(\phi)\big]\,h^\phi\Big(\mathrm{trace}_{G_\theta}\Big\{\Pi_H\,(R^h)^\phi(V,\,\phi_*\cdot)\phi_*\cdot\Big\},\; V\Big)\,\Psi \\ &+ \int_M \exp\big[e_b(\phi)\big]\,(h^\phi)^*\big(D^\phi V,\, \phi_*\big)^2\,\Psi \end{aligned}$$

and for any $x \in U$

$$\begin{aligned} &h^\phi\big(\mathrm{trace}_{G_\theta}\big\{\Pi_H(R^h)^\phi(V,\,\phi_*\cdot)\phi_*\,\cdot\,\big\},\; V)_x \\ &= h^\phi\Big(\sum_{a=1}^{2n}\Pi_H(R^h)^\phi(V,\,\phi_* X_a)\phi_* X_a,\; V\Big)_x \\ &= \sum_a (K^h)^\phi\big(V,\,\phi_* X_a,\; V,\,\phi_* X_a\big)_x \end{aligned}$$

$$= \sum_a K^h_{\phi(x)}\big(V_x,\ (d_x\phi)X_{a,x},\ V_x,\ (d_x\phi)X_{a,x}\big)$$
$$= \sum_a \Big[\|V\|^2\,\|\phi_* X_a\|^2 - h^\phi(V,\ \phi_* X_a)^2\Big]_x\ k\big(\sigma_a(x)\big)$$

where $k\big(\sigma_a(x)\big) \leq 0$ is the sectional curvature of the 2-plane

$$\sigma_a(x) = \mathbb{R}V_x + \mathbb{R}X_{a,x} \subset T_{\phi(x)}(N).$$

Hence $H(E_b)_\phi(V, V) \geq 0$. □

Let $(\mathbb{R}^{m+1},\ q)$ be the Minkowski space i.e.

$$q(x, y) = -x_1 y_1 + x_2 y_2 + \cdots + x_{m+1} y_{m+1}, \quad x, y \in \mathbb{R}^{m+1},$$

and let $g_{m,1}$ be the (flat) Lorentzian metric (determined by q) on $\mathbb{R}^{m+1}$. Let

$$H^n = \big\{x \in \mathbb{R}^{m+1}\ :\ q(x, x) = -1,\ \ x_{m+1} > 0\big\}$$

be the hypersurface of $\mathbb{R}^{m+1}_1 = \big(\mathbb{R}^{m+1},\ g_{m,1}\big)$ endowed with the induced metric $h = \mathbf{j}^* g_{m,1}$ where $\mathbf{j} : H^m \hookrightarrow \mathbb{R}^{m+1}_1$. Theorem 6 applies to $N = (H^m,\ h)$ and yields Theorem 1.

3.3.3 Tangent Space to $\mathfrak{har}_{b,\,\exp}(M, N)$ and Nullity of e.s.h. Maps

Assume M to be compact. The tangent space

$$T_\phi\big(\mathfrak{har}_{b,\,\exp}(M, N)\big)$$

consists of all $V \in C^\infty(\phi^{-1}TN)$ such that

$$\exists\ \ \{\phi_s\}_{|s|<\epsilon} \subset \mathfrak{har}_{b,\,\exp}(M, N)\ \ :\ \ \phi_0 = \phi, \quad V = \frac{\partial \phi_s}{\partial s}.$$

Let $\phi_{s,t} \in C^\infty(M, N)$ be an arbitrary smooth 1-parameter variation of ϕ_s i.e. $\phi_{s,0} = \phi_s$. If one sets as customary

$$\Phi(x,\ s,\ t) = \phi_{s,t}(x), \quad x \in M,\ \ |s| > \epsilon,\ \ |t| < \epsilon,$$
$$V_x = (d_{(x,0,0)}\Phi)\left(\frac{\partial}{\partial s}\right)_{(x,0,0)}, \quad W_x = (d_{(x,0,0)}\Phi)\left(\frac{\partial}{\partial t}\right)_{(x,0,0)},$$

then [as ϕ_s is an e.s.h. map]

$$\frac{d}{dt}\big\{E_b\big(\phi_{s,t}\big)\big\}_{t=0} = 0, \quad \forall\ |s| < \epsilon,$$

hence

$$
\begin{aligned}
0 &= \frac{\partial^2}{\partial s\,\partial t}\left\{E_b(\phi_{s,t})\right\}_{s=t=0} = H(E_b)_\phi(V, W) = \\
&= \int_M h^\phi\left(J^\phi_{b,\,\exp} V,\ W\right)\Psi + \\
&\quad + \int_M \exp\left[e_b(\phi)\right](h^\phi)^*\left(D^\phi V,\ \phi_*\right)(h^\phi)^*\left(D^\phi W,\ \phi_*\right)\Psi
\end{aligned} \tag{72}
$$

with $W \in C^\infty(\phi^{-1}TN)$ arbitrary. By taking into account

$$
\begin{aligned}
&\int_M \exp\left[e_b(\phi)\right](h^\phi)^*\left(D^\phi V,\ \phi_*\right)(h^\phi)^*\left(D^\phi W,\ \phi_*\right)\Psi \\
&\quad = \left(\exp\left[e_b(\phi)\right](h^\phi)^*(D^\phi V,\ \phi_*)\,\phi_*,\ D^\phi W\right)_\phi \\
&\quad = \left((D^\phi)^*\left[\exp\left[e_b(\phi)\right](h^\phi)^*(D^\phi V,\ \phi_*)\,\phi_*\right],\ W\right)_\phi
\end{aligned}
$$

the relation (72) implies

$$
\mathcal{J}^\phi_{b,\,\exp} V = 0 \tag{73}
$$

where we have set

$$
\begin{aligned}
&\mathcal{J}^\phi_{b,\,\exp} \equiv J^\phi_{b,\,\exp} + Q^\phi, \\
&Q^\phi V \equiv (D^\phi)^*\left[\exp\left[e_b(\phi)\right](h^\phi)^*(D^\phi V,\ \phi_*)\,\phi_*\right].
\end{aligned}
$$

Then [by (73)]

$$
T_\phi\left(\mathfrak{har}_{b,\,\exp}(M, N)\right) \subset \operatorname{Ker}\left(\mathcal{J}^\phi_{b,\,\exp}\right)
$$

so that

$$
\dim_{\mathbb{R}} T_\phi\left(\mathfrak{har}_{b,\,\exp}(M, N)\right) \le \operatorname{null}(\phi).
$$

3.3.4 Constant Maps

Assume M to be compact. Let $q \in N$ and $\phi : M \to N$ the constant map $\phi(x) = q$. Let $(\mathcal{U},\ y^A)$ be a local coordinate system about $q \in \mathcal{U} \subset N$ so that each $V \in C^\infty\left(\phi^{-1}TN\right)$ may be represented as

$$
V = f^A\left(\frac{\partial}{\partial y^A}\right)^\phi
$$

for some $f^A \in C^\infty(M)$. As ϕ is constant

$$e_b(\phi) = 0, \quad \phi_* X = 0, \quad \left(\nabla^h\right)^\phi_X V = X(f^A)\left(\frac{\partial}{\partial y^A}\right)^\phi,$$
$$L^\phi V = (D^\phi)^* D^\phi V,$$

hence

$$\mathcal{J}^\phi_{b,\,\exp} V = (\Delta_b f^A)\left(\frac{\partial}{\partial y^A}\right)^\phi. \tag{74}$$

Proposition 1 *For every constant mapping $\phi : (M, \theta) \to (N, h)$ the spectrum* Spec$\left(\mathcal{J}^\phi_{b,\,\exp}\right)$ *is the set of eigenvalues of the sublaplacian Δ_b of (M, θ) acting on $C^\infty(M)$ counted m times. In particular* Spec$\left(\mathcal{J}^\phi_{b,\,\exp}\right)$ *doesn't depend upon $\{q\} = \phi(M) \subset N$.*

By a result of Menikoff and Sjöstrand (cf. [28]) the sublaplacian Δ_b has a discrete spectrum

$$\mathrm{Spec}(\Delta_b) = \left\{\lambda_0(\theta) = 0 < \lambda_1(\theta) < \cdots < \lambda_k(\theta) < \cdots \uparrow +\infty\right\}.$$

The continuity of the eigenvalues $\lambda_k(\theta)$ as functions of the contact form θ was established in [4]. The effect of deformations $\hat{\theta} = e^u\,\theta$, $u \in C^\infty(M)$, on the eigenvalues $\lambda_k(\theta)$ was studied in [5]. For an arbitrary e.s.h. map ϕ we conjecture that J^ϕ_b, $J^\phi_{b,\exp}$ and $\mathcal{J}^\phi_{b,\,\exp}$ have discrete spectra on domains $\Omega \subset M$ enjoying appropriate[5] properties. This would of course lead to further developments of stability theory for e.s.h. maps.

Let $F \subset C^\infty(\phi^{-1}TN)$ be a finite dimensional subspace and $V \in F$. Then

$$H\big(E_b\big)_\phi(V, V) = \int_M h^\phi\big(\mathcal{J}^\phi_{b,\,\exp} V,\ V\big)\ \Psi$$
$$= G_{AB}(q)\int_M (\Delta_b f^A)\ f^B\ \Psi$$

where $G_{AB} = h\big(\partial_A,\ \partial_B\big)$ and $\partial_A \equiv \partial/\partial y^A$. We may start with an orthonormal linear basis $\{w_A\ :\ 1 \le A \le m\} \subset T_q(N)$ and consider the normal coordinate system $(\mathcal{U},\ y^A)$ with center at q i.e. $(\partial/\partial y^A)_q = w_A$. Then [as Δ_b is a positive operator]

[5] For instance, discreteness of the spectrum of the operator J^ϕ_b associated to a subelliptic harmonic map ϕ is established (cf. [11]) for a class of CR structures arising as orbit spaces M^3 of null Killing vector fields on a space-time (Gödel's universe in [11]), on a domain $\Omega \subset M^3$ supporting a form of Poincaré's inequality and a form of Kondrakov compactness involving $L^2(\Omega,\ \phi^{-1}TN)$. The approach in [11] carries over *verbatim* to arbitrary subelliptic harmonic maps.

$$H\big(E_b\big)_\phi(V,V) = \sum_{A=1}^{m} \Big(\Delta_b f^A,\ f^A\Big)_{L^2} \ge 0$$

thus yielding $\operatorname{ind}(\phi) = 0$ and $\operatorname{null}(\phi) = m$.

3.4 E.s.h. Maps into Real Hypersurfaces of Space Forms

Let $\big(M^{m+1}(c),\ \overline{h}\big)$ be a real $(m+1)$-dimensional space form i.e. a Riemannian manifold, with the Riemannian metric $\overline{h}$, of (constant) sectional curvature $c \in \mathbb{R}$. Let $\mathbf{j} : N \subset M^{m+1}(c)$ be an orientable totally umbilical real hypersurface and let $T(N)^\perp \to N$ be the normal bundle of the given immersion. Let $\tan_x$ and nor_x be the canonical projections associated to the direct sum decomposition

$$T_x\big[M^{m+1}(c)\big] = T_x(N) \oplus T_x(N)^\perp, \quad x \in N.$$

We shall need the Gauss and Weingarten formulas

$$\begin{aligned}
&\overline{\nabla}_X Y = \nabla^N_X Y + B(X,Y), \quad \overline{\nabla}_X \eta = -A_\eta X + \nabla^\perp_X \eta,\\
&X, Y \in \mathfrak{X}(N), \quad \eta \in C^\infty\big(T(N)^\perp\big),
\end{aligned}$$

where $\overline{\nabla}$, ∇, B, A_η and $\nabla^\perp$ are respectively the Levi–Civita connections of $\overline{h}$ and $h = \mathbf{j}^* h$, the second fundamental form, the Weingarten operator (associated to the normal vector field η), and the normal connection of the given immersion. Then $B = H \otimes h$ where $H = \operatorname{trace}_h B$ is the mean curvature vector of $\mathbf{j}$. Let $\Omega \subset M^{m+1}(c)$ be a simple and convex open subset such that $N \cap \Omega \neq \emptyset$ and let $\phi : M \to N \cap \Omega$ be a nonconstant e.s.h. map of a compact strictly pseudoconvex CR manifold M, endowed with the positively oriented contact form θ, into $\big(N \cap \Omega,\ h\big)$.
Let $\{Z_A : 1 \le A \le m+1\}$ be a local $\overline{h}$-orthonormal frame of $T\big[M^{m+1}(c)\big]$ such that $\overline{\nabla} Z_A = 0$. If $V_A = \tan(Z_A)$ then

$$\begin{aligned}
&H(E_b)_\phi\big(V_A,\ V_A\big) = \\
&= \Big(\exp\big[e_b(\phi)\big]\, D^\phi V_A,\ D^\phi V_A\Big)_\phi + \\
&\quad - \int_M \exp\big[e_b(\phi)\big]\, h^\phi \Big(\operatorname{trace}_{G_\theta} \Big\{\Pi_H (R^h)^\phi\, (V_A,\ \phi_* \cdot)\, \phi_* \cdot\Big\},\ V_A\Big)\, \Psi + \\
&\quad + \int_M \exp\big[e_b(\phi)\big]\, (h^\phi)^*\big(D^\phi V_A,\ \phi_*\big)^2\, \Psi. \qquad (75)
\end{aligned}$$

To evaluate the terms in (75) note first that

$$\begin{aligned}
&D^\phi_{X_a} V_A = \nabla^h_{(d\phi)X_a} V_A = \quad \text{[by the Gauss formula]}\\
&\quad = \tan\big[\overline{\nabla}_{(d\phi)X_a} \tan(Z_A)\big] = \tan\big[\overline{\nabla}_{(d\phi)X_a}\big(Z_A - \mathrm{nor}\,(Z_A)\big)\big] =
\end{aligned}$$

[by $\overline{\nabla} Z_A = 0$]

$$= -\tan \overline{\nabla}_{(d\phi)X_a} \mathrm{nor}(Z_A) = \quad \text{[by the Weingarten formula]}$$
$$= A_{\mathrm{nor}(Z_A)}(d\phi)X_a$$

so that

$$D^{\phi}_{X_a} V_A = A_{\mathrm{nor}(Z_A)}(d\phi)X_a. \tag{76}$$

Moreover using

$$\|X\|^2 = \overline{h}(X, X) = \sum_{A=1}^{m+1} \overline{h}(X,\ Z_A)^2, \quad X \in \mathfrak{X}(M^{m+1}),$$

together with (76) one may conduct the following calculation

$$\begin{aligned} \left\| D^{\phi}_{X_a} V_A \right\|^2 &= \left\| A_{\mathrm{nor}(Z_A)}(d\phi)X_a \right\|^2 \\ &= \sum_{B=1}^{m+1} \overline{h} \left(A_{\mathrm{nor}(Z_A)}(d\phi)X_a,\ Z_B \right)^2 \\ &= \sum_{B=1}^{m+1} h^{\phi} \left(A_{\mathrm{nor}(Z_A)}(d\phi)X_a,\ \tan(Z_B) \right)^2 = \end{aligned}$$

[using

$$\overline{h}(B(X, Y), \eta) = h(A_\eta X, Y)$$

for any $X, Y \in \mathfrak{X}(N)$ and any $\eta \in C^\infty(T(N)^\perp)$]

$$= \sum_B \overline{h} \left(B\big((d\phi)X_a,\ V_B\big),\ \mathrm{nor}(Z_A) \right)^2$$

so that

$$\left\| D^{\phi}_{X_a} V_A \right\|^2 = \sum_{B=1}^{m+1} \overline{h} \left(B\big((d\phi)X_a,\ V_B\big),\ \mathrm{nor}(Z_A) \right)^2. \tag{77}$$

Let us substitute $B = H \otimes h$ into (77). We have

$$\left\| D^{\phi}_{X_a} V_A \right\|^2 = \sum_B h^{\phi} \left((d\phi)X_a,\ V_B \right)^2 \overline{h} \left(H,\ \mathrm{nor}(Z_A) \right)^2$$

hence

$$\left\| D^{\phi}_{X_a} V_A \right\|^2 = \|(d\phi)X_a\|^2 \, \overline{h}\big(H, \, \mathrm{nor}(Z_A)\big)^2. \tag{78}$$

Let $\xi \in C^{\infty}(T(N)^{\perp})$ be a unit normal vector field on N. One has

$$\|\eta\|^2 = \overline{h}\big(\eta, \, \xi\big)^2, \quad \eta \in C^{\infty}(T(N)^{\perp}),$$
$$\mathrm{nor}(Z_A) = f_A \, \xi,$$

for some $f_A \in C^{\infty}(N)$. Then

$$\overline{h}\big(H, \, \mathrm{nor}(Z_A)\big)^2 = f_A^2 \, \overline{h}(H, \xi)^2 = f_A^2 \, \|H\|^2$$

yielding

$$\overline{h}\big(H, \, \mathrm{nor}(Z_A)\big)^2 = \|\mathrm{nor}(Z_A)\|^2 \; \|H\|^2. \tag{79}$$

Finally substitution from (79) into (78) gives

$$\left\| D^{\phi}_{X_a} Z_A \right\|^2 = \|(d\phi)X_a\|^2 \; \|\mathrm{nor}(Z_A)\|^2 \; \|H\|^2. \tag{80}$$

We may now compute the first term in the right hand side of (75) as follows

$$\begin{aligned}
&(h^{\phi})^* \left(\exp\left[e_b(\phi)\right] D^{\phi} V_A, \; D^{\phi} V_A\right) \\
&= \exp\left[e_b(\phi)\right] \sum_{a=1}^{2n} \left\| D^{\phi}_{X_a} V_A \right\|^2 = \quad [\text{by } (80)] \\
&= \exp\left[e_b(\phi)\right] \sum_{a=1}^{2n} \|(d\phi)X_a\|^2 \; \|\mathrm{nor}(Z_A)\|^2 \, \|H\|^2 \\
&= 2 \exp\left[e_b(\phi)\right] e_b(\phi) \, \|\mathrm{nor}(Z_A)\|^2 \, \|H\|^2
\end{aligned}$$

so that

$$\begin{aligned}
&\left(\exp\left[e_b(\phi)\right] D^{\phi} V_A, \; D^{\phi} V_A\right)_{\phi} = \\
&= 2 \, \|H\|^2 \int_M \exp\left[e_b(\phi)\right] e_b(\phi) \, \|\mathrm{nor}(Z_A)\|^2 \; \Psi.
\end{aligned} \tag{81}$$

Indeed $\|H\| \in \mathbb{R}$ as a consequence of the Codazzi equation

$$(\nabla_X A_{\xi})Y - (\nabla_Y A_{\xi})X = 0, \quad X, Y \in \mathfrak{X}(N),$$

together with $A_\xi X = \pm\|H\|\, X$. If $\xi = g^A Z_A$ for some $g^A \in C^\infty(N)$ then

$$\xi - g^A \operatorname{nor}(Z_A) = g^A\, V_A$$

yields $g^A\, V_A = 0$ and then

$$\xi = g^A\, Z_A = g^A\, f_A\, \xi$$

implying

$$f \cdot g = 1 \tag{82}$$

where $f = (f_1, \cdots, f_{m+1})$ and $g = (g^1, \cdots, g^{m+1})$. Moreover

$$\begin{aligned} \sum_A f_A^2 = \sum_A \|\operatorname{nor}(Z_A)\|^2 = \sum_A \overline{h}\,(\operatorname{nor}(Z_A),\ Z_A) \\ = \sum_A f_A\, \overline{h}\big(\xi,\ Z_A\big) = \sum_A f_A g^A \end{aligned}$$

implying

$$\|f\|^2 = f \cdot g \tag{83}$$

and summing up [by (82)–(83)]

$$\sum_{A=1}^{m+1} \big\|\operatorname{nor}(Z_A)\big\|^2 = 1. \tag{84}$$

Let us sum over $1 \le A \le m+1$ in (81) and take into account (84). We obtain

$$\begin{aligned} \sum_{A=1}^{m+1} \big(\exp\big[e_b(\phi)\big]\, D^\phi V_A,\ D^\phi V_A\big)_\phi = \\ = 2\,\|H\|^2 \int_M \exp\big[e_b(\phi)\big]\, e_b(\phi)\ \Psi. \end{aligned} \tag{85}$$

To compute the curvature term in (75) we need to recall the Gauss equation [of a totally umbilical submanifold in a real space form $N \subset M^{m+1}(c)$]

$$\begin{aligned} & R^h(X, Y)Z = \Big(c + \|H\|^2\Big)\, \big\{h(Y, Z)X - h(X, Z)Y\big\}, \\ & X, Y, Z \in \mathfrak{X}(N), \end{aligned}$$

so that

$$
\begin{aligned}
&\mathrm{trace}_{G_\theta}\left\{\Pi_H\,(R^h)^\phi\,(V_A,\phi_*\,\cdot\,)\,\phi_*\,\cdot\right\} \\
&=\sum_{a=1}^{2n}(R^h)^\phi\big(V_A,\ \phi_*X_a\big)\phi_*X_a \\
&=\left(c+\|H\|^2\right)\sum_a\left\{h^\phi\big(\phi_*X_a,\ \phi_*X_a\big)V_A-h^\phi\big(V_A,\ \phi_*X_a\big)\phi_*X_a\right\} \\
&=\left(c+\|H\|^2\right)\left\{2\,e_b(\phi)\,V_A-\sum_a h^\phi\big(V_A,\ \phi_*X_a\big)\,\phi_*X_a\right\}
\end{aligned}
$$

hence

$$
\begin{aligned}
&h^\phi\left(\mathrm{trace}_{G_\theta}\left\{\Pi_H\,(R^h)^\phi\,(V_A,\phi_*\,\cdot\,)\,\phi_*\,\cdot\right\},\ V_A\right) \\
&=\left(c+\|H\|^2\right)\left\{2\,e_b(\phi)\,\|V_A\|^2-\sum_a h^\phi\big(V_A,\ \phi_*X_a\big)^2\right\} \\
&=\left(c+\|H\|^2\right)\left\{2\,e_b(\phi)\,\|V_A\|^2-\sum_a \overline{h}\,(\phi_*X_a,\ V_A)^2\right\}
\end{aligned}
$$

and since

$$
\|V_A\|^2+\|\mathrm{nor}(Z_A)\|^2=1\Longrightarrow\sum_{A=1}^{m+1}\|V_A\|^2=m
$$

one has

$$
\begin{aligned}
&\sum_{A=1}^{m+1}h^\phi\left(\mathrm{trace}_{G_\theta}\left\{\Pi_H\,(R^h)^\phi\,(V_A,\phi_*\,\cdot\,)\,\phi_*\,\cdot\right\},\ V_A\right) \\
&=\left(c+\|H\|^2\right)\left\{2\,m\,e_b(\phi)-\sum_{a=1}^{2n}\|\phi_*X_a\|^2\right\}
\end{aligned}
$$

hence one may conclude

$$
\begin{aligned}
&\sum_{A=1}^{m+1}\int_M\exp\left[e_b(\phi)\right]\,\times \\
&\quad\times h^\phi\left(\mathrm{trace}_{G_\theta}\left\{\Pi_H\,(R^h)^\phi\,(V_A,\phi_*\,\cdot\,)\,\phi_*\,\cdot\right\},\ V_A\right)\,\Psi= \\
&=2(m-1)\left(c+\|H\|^2\right)\int_M e_b(\phi)\,\exp\left[e_b(\phi)\right]\,\Psi. \qquad (86)
\end{aligned}
$$

We are left with the calculation of the last term in (75) i.e.

$$(h^\phi)^* \left(D^\phi V_A,\, \phi_*\right)^2 = \left[\sum_{a=1}^{2n} h^\phi \left(D^\phi_{X_a} V_A,\, \phi_* X_a\right)\right]^2 =$$

[by (76)]

$$= \left[\sum_a h^\phi \left(A_{\mathrm{nor}(Z_A)}\, \phi_* X_a,\, \phi_* X_a\right)\right]^2$$

$$= \left[\sum_a \overline{h} \left(B\left(\phi_* X_a,\, \phi_* X_a\right),\, \mathrm{nor}(Z_A)\right)\right]^2 =$$

(again by umbilicity)

$$= \left[\sum_a h^\phi\big(\phi_* X_a,\, \phi_* X_a\big)\, \overline{h}\big(H,\, \mathrm{nor}(Z_A)\big)\right]^2 = 4\, e_b(\phi)^2\, \overline{h}\big(H,\, Z_A\big)^2$$

so that

$$\sum_{A=1}^{m+1} \int_M \exp\left[e_b(\phi)\right] (h^\phi)^* \left(D^\phi V_A,\, \phi_*\right)^2 \;\Psi = \\ = 4\|H\|^2 \int_M e_b(\phi)^2 \,\exp\left[e_b(\phi)\right]\, \Psi. \tag{87}$$

Summing up [by (85)–(87)]

$$\sum_{A=1}^{m+1} H(E_b)_\phi\big(V_A,\, V_A\big) = 2\int_M e_b(\phi)\, \exp\left[e_b(\phi)\right] \times \\ \times \left\{-(m-1)c + \|H\|^2\big[2\, e_b(\phi) - (m-2)\big]\right\}\; \Psi. \tag{88}$$

We obtain

Theorem 7 *Let* $\mathbf{j} : N \subset M^{m+1}(c)$ *be an orientable real hypersurface, of mean curvature vector* H*, of a real space form* $\big(M^{m+1}(c),\, \overline{h}\big)$*. Let* $\Omega \subset M^{m+1}(c)$ *be a simple and convex open subset such that* $\Omega \cap N \neq \emptyset$*. Let* $\phi : M \to \Omega \cap N$ *be an exponentially subelliptic harmonic map of a compact strictly pseudoconvex CR manifold, endowed with a positively oriented contact form* θ*, into* $(\Omega \cap N,\, h)$ *with* $h = \mathbf{j}^* \overline{h}$*.*

i) *If* $m \geq 2$*,* $c > 0$*, and* $\mathbf{j}$ *is totally geodesic, then* ϕ *is unstable.*

ii) *If* $m \geq 3$*,* $c \geq 0$*, and* $\mathbf{j}$ *is totally umbilical with* $H \neq 0$*, and*

$$2\, e_b(\phi) < (m-1)\, \frac{c}{\|H\|^2} + m - 2 \tag{89}$$

then ϕ is unstable. In particular if $N = S^m \subset \mathbb{R}^{m+1}$ $(c = 0)$ and $e_b(\phi) < (m-2)/2$ then ϕ is unstable.

3.5 Spectrum of $J^{\phi}_{b,\,\exp}$

3.5.1 Sobolev Spaces of Sections in $\phi^{-1}T(N)$

Let $\phi \in \mathfrak{har}_{b,\,\exp}(M, N)$ be an e.s.h. map from a Carnot–Carathèodory complete pseudohermitian manifold (M, θ) into the Riemannian manifold (N, h). Let $L^2(\Omega) = L^2(\Omega,\ \Psi)$ consist of all measurable functions $u : \Omega \to \mathbb{R}$ such that $\int_\Omega u^2\, \Psi < \infty$. Also let $L^2(\Omega,\ \phi^{-1}TN)$ consist of all sections $V : \Omega \to \phi^{-1}T(N)$ such that

$$\|V\| = h^\phi\big(V,\ V\big)^{1/2} \in L^2(\Omega)$$

i.e. $\int_\Omega h^\phi(V, V)\ \Psi < \infty$. Then $L^2(\Omega,\ \phi^{-1}TN)$ is a Hilbert space with the inner product

$$\big(V,\ W)_{L^2} = \int_\Omega h^\phi(V,\ W)\ \Psi.$$

If $\varphi,\ \psi$ are sections in $H(M)^* \otimes \phi^{-1}T(N)$ and $x \in M$ we set as customary

$$\big(h^\phi)^*\big(\varphi, \psi\big)_x = \sum_{a=1}^{2n} h^\phi\big(\varphi(E_a),\ \psi(E_a)\big)_x$$

where $\{E_a\ :\ 1 \le a \le 2n\}$ is a (local) G_θ-orthonormal frame in $H(M)$, defined on an open neighbourhood $U \subset M$ of x. Let $L^2\big(\Omega,\ H(M)^* \otimes \phi^{-1}TN\big)$ consist of all sections φ in $H(M)^* \otimes \phi^{-1}T(N)$ such that

$$(h^\phi)^*(\varphi,\ \varphi)^{1/2} \in L^2(\Omega).$$

Then $L^2\big(\Omega,\ H(M)^* \otimes \phi^{-1}TN\big)$ is a Hilbert space with the scalar product

$$\big(\varphi,\ \psi\big)_{L^2} = \int_\Omega (h^\phi)^*(\varphi,\ \psi)\ \Psi.$$

Let

$$D^\phi = \big[\big(\nabla^h\big)^\phi\big]^H : C^\infty(\Omega,\ \phi^{-1}TN) \to C^\infty(\Omega,\ H(M)^* \otimes \phi^{-1}TN),$$
$$D^\phi V = \big[\big(\nabla^h\big)^\phi\big]\Big|_{C^\infty(\Omega, H(M))}.$$

Let

$$\big(D^\phi\big)^* : C^\infty_0(\Omega,\ H(M)^* \otimes \phi^{-1}TN) \to C^\infty(\Omega,\ \phi^{-1}TN)$$

be the formal adjoint of D^ϕ i.e.

$$\big((D^\phi)^*\varphi,\ V\big)_{L^2} = \big(\varphi,\ D^\phi V\big)_{L^2}.$$

A section $V \in L^2(\Omega,\ \phi^{-1}TN)$ is **weakly differentiable** along $H(M)$ if there is a section $\varphi_V : \Omega \to H(M)^* \otimes \phi^{-1}T(N)$ such that

$$\big(h^\phi\big)^*\big(\varphi_V,\ \varphi_V\big)^{1/2} \in L^1_{\mathrm{loc}}(\Omega),$$
$$\int_\Omega \big(h^\phi\big)^*\big(\varphi_V,\ \psi\big)\,\Psi = \int_\Omega h^\phi\big(V,\ (D^\phi)^*\psi\big)\,\Psi,$$

for any $\psi \in C_0^\infty(\Omega,\ H(M)^* \otimes \phi^{-1}TN)$. Such φ_V is unique up to a set of measure zero and then denoted by $\varphi_V = D^\phi V$.

Let $W_H^{1,2}(\Omega,\ \phi^{-1}TN)$ consist of all $V \in L^2(\Omega,\ \phi^{-1}TN)$ such that V is weakly differentiable along $H(M)$ and $D^\phi V \in L^2(\Omega,\ H(M)^* \otimes \phi^{-1}TN)$. Then $W_H^{1,2}(\Omega,\ \phi^{-1}TN)$ is a Hilbert space with the scalar product

$$\big(V,\ W\big)_{W^{1,2}} = \big(V,\ W\big)_{L^2} + \big(D^\phi V,\ D^\phi W\big)_{L^2}.$$

Let $\|V\|_{W^{1,2}} = \big(V,\ V\big)^{1/2}_{W^{1,2}}$ be the norm on $W_H^{1,2}(\Omega,\ \phi^{-1}TN)$. Let $\mathring{W}_H^{1,2}(\Omega,\ \phi^{-1}TN)$ be the completion of $C_0^\infty(\Omega, \phi^{-1}TN)$ with respect to the norm $\|\cdot\|_{W^{1,2}}$. Through this section we shall work with domains $\Omega \subset M$ supporting the **Poincaré inequality**

$$\|V\|_{L^2} \le C\,\big\|D^\phi V\big\|_{L^2}$$

for any $V \in C_0^\infty(\Omega,\ \phi^{-1}TN)$ and some constant $C > 0$ not depending on V.

Lemma 3 *$\mathring{W}_H^{1,2}(\Omega,\ \phi^{-1}TN)$ is a Hilbert space with*

$$\big(V,\ W\big)_{\mathring{W}^{1,2}} = \big(D^\phi V,\ D^\phi W\big)_{L^2}\,. \tag{90}$$

Proof Poincaré inequality implies that (90) is a scalar product on $\mathring{W}_H^{1,2}(\Omega,\ \phi^{-1}TN)$. Let us set $\|V\|_{\mathring{W}^{1,2}} = \big(V,\ V\big)^{1/2}_{\mathring{W}^{1,2}}$. The statement follows from the fact that $\|\cdot\|_{\mathring{W}^{1,2}}$ and $\|\cdot\|_{W^{1,2}}$ are equivalent norms on $C_0^\infty(\Omega, \phi^{-1}TN)$ i.e.

$$\|V\|_{\mathring{W}^{1,2}} \le \|V\|_{W^{1,2}} \le \big(1 + C\big)\,\|V\|_{\mathring{W}^{1,2}}$$

for any $V \in C_0^\infty(\Omega, \phi^{-1}TN)$. □

This is the main feature of domains supporting Poincaré inequality: the failure of the rough sublaplacian Δ^ϕ to be elliptic is compensated by $\|\cdot\|_{\mathring{W}^{1,2}}$ being already a norm on $\mathring{W}_H^{1,2}(\Omega,\ \phi^{-1}TN)$.

3.5.2 Generalized Dirichlet Problem for $J^{\phi}_{b,\,\exp}$

The **Dirichlet problem** for $J^{\phi}_{b,\,\exp}$ is

$$J^{\phi}_{b,\,\exp} V = F \quad \text{in} \ \ \Omega, \qquad V = 0 \quad \text{on} \ \ \partial\Omega. \tag{91}$$

In this section we solve a weak, or generalized, version of (91) i.e. for every $F \in L^2(\Omega,\ \phi^{-1}TN)$ we prove existence and uniqueness of $V \in \mathring{W}^{1,2}_H(\Omega,\ \phi^{-1}TN)$ such that

$$\begin{aligned} &\big(\exp\left[e_b(\phi)\right] D^{\phi} V,\ D^{\phi} S\big)_{L^2} + \\ &- \int_{\Omega} \exp\left[e_b(\phi)\right] h^{\phi} \Big(\mathrm{trace}_{G_\theta}\big\{\Pi_H (R^h)^{\phi}(V,\ \phi_* \cdot)\phi_* \ \cdot\ \big\},\ S\Big) = (F,\, S)_{L^2} \end{aligned} \tag{92}$$

for any $S \in \mathring{W}^{1,2}_H(\Omega,\ \phi^{-1}TN)$.

We adopt the following notations

$$\begin{aligned} &\mathcal{F}_{\phi}(V) = \frac{1}{2}\, q(V) - \big(F,\ V\big)_{L^2}, \quad q(V) = a_{\phi}(V,\, V), \\ &a_{\phi}(V,\, W) = \int_{\Omega} \exp\left[e_b(\phi)\right] \big\{ (h^{\phi})^{*} \big(D^{\phi} V,\ D^{\phi} W\big) + \\ &-h^{\phi} \Big(\mathrm{trace}_{G_\theta}\big[\Pi_H (R^h)^{\phi}(V,\ \phi_* \cdot)\phi_* \ \cdot\ \big],\ W\Big)\Big\}\ \Psi. \end{aligned}$$

Lemma 4 *Let us assume that*

$$\left\| R^h(A,\, B)C \right\| \leq \gamma\, \|A\|\, \|B\|\, \|C\| \tag{93}$$

for some constant $\gamma > 0$ *and any* $A, B, C \in \mathfrak{X}(N)$*, where* $\|A\| = h(A,\, A)^{1/2}$*. Then for each* $W \in \mathring{W}^{1,2}_H(\Omega,\ \phi^{-1}TN)$ *the function*

$$V \in \mathring{W}^{1,2}_H(\Omega,\ \phi^{-1}TN) \longmapsto a_{\phi}(V,\, W) \in \mathbb{R}$$

is continuous.

Proof We have

$$\begin{aligned} \left|a_{\phi}(V,\, W)\right| &\leq \left|\big(\exp\left[e_b(\phi)\right] D^{\phi} V,\ D^{\phi} W\big)_{L^2}\right| \\ &+ \left|\Big(\exp\left[e_b(\phi)\right] \mathrm{trace}_{G_\theta}\Big[\Pi_H (R^h)^{\phi}(V,\ \phi_* \cdot)\phi_* \cdot\Big],\ W\Big)_{L^2}\right| \leq \end{aligned}$$

[by distributing a $\exp\left[\frac{1}{2}\, e_b(\phi)\right]$ factor to each term of the L^2 scalar products]

$$
\begin{aligned}
&\leq \left\| \exp\left[\frac{1}{2}\, e_b(\phi)\right] D^\phi V \right\|_{L^2} \left\| \exp\left[\frac{1}{2}\, e_b(\phi)\right] D^\phi W \right\|_{L^2} \\
&\quad + \left\| \exp\left[\frac{1}{2}\, e_b(\phi)\right] \mathrm{trace}_{G_\theta} \left[\Pi_H (R^h)^\phi (V,\ \phi_* \cdot)\phi_* \cdot \right] \right\|_{L^2} \left\| \exp\left[\frac{1}{2}\, e_b(\phi)\right] W \right\|_{L^2}
\end{aligned}
$$

and

$$
\begin{aligned}
&\left\| \exp\left[\frac{1}{2}\, e_b(\phi)\right] D^\phi V \right\|_{L^2}^2 \\
&\quad = \int_\Omega \exp\left[e_b(\phi)\right] \left(h^\phi\right)^* \left(D^\phi V,\ D^\phi V\right) \Psi \leq \exp\left[C(\Omega, \phi)\right] \left\| D^\phi V \right\|_{L^2}^2
\end{aligned}
$$

where

$$
C(\Omega, \phi) = \sup_{x \in \overline{\Omega}} e_b(\phi)_x \, .
$$

Then $C(\Omega, \phi) < \infty$ [as $\overline{\Omega}$ is compact] and

$$
C(\Omega, \phi) > 0 \Longleftrightarrow \phi : \overline{\Omega} \to \mathbb{R} \ \text{ is non constant.}
$$

Indeed if $C(\Omega, \phi) = 0$ then $e_b(\phi) = 0$ in $\overline{\Omega}$. For each point $x_0 \in \Omega$ we consider a local orthonormal frame $\{E_a \,:\, 1 \leq a \leq 2n\} \subset C^\infty(U,\ H(M))$ defined on an open neighbourhood $U \subset \Omega$ of x_0, and a local coordinate system $(V,\ y^A)$ about the point $\phi(x_0)$, such that $\phi(U) \subset V$. Let us set $\phi^A = y^A \circ \phi$. Then

$$
\begin{aligned}
&0 = e_b(\phi)|_U = \frac{1}{2} \sum_{a=1}^{2n} h^\phi\left(\phi_* E_a,\ \phi_* E_a\right) \Longrightarrow \\
&\Longrightarrow 0 = \phi_* E_a = E_a\left(\phi^A\right) \left(\frac{\partial}{\partial y^A}\right) \circ \phi \Longrightarrow \overline{\partial}_b \phi^A = 0
\end{aligned}
$$

i.e. ϕ^A is a real valued CR function on U. Yet M is strictly pseudoconvex and in particular nondegenerate, so that U is a nondegenerate CR manifold, with the induced CR structure. Hence real valued CR functions are constants, and then ϕ^A is constant on U. That is $\phi : \Omega \to N$ is locally constant, and then constant in Ω. Yet $\phi : \overline{\Omega} \to N$ is continuous, so $\phi : \overline{\Omega} \to N$ is constant. □

We have established the estimates

$$
\left\| \exp\left[\frac{1}{2}\, e_b(\phi)\right] D^\phi V \right\|_{L^2} \leq \exp\left[\frac{1}{2}\, C(\Omega, \phi)\right] \left\| D^\phi V \right\|_{L^2} , \tag{94}
$$

$$
\left\| \exp\left[\frac{1}{2}\, e_b(\phi)\right] \mathrm{trace}_{G_\theta} \left[\Pi_H (R^h)^\phi (V,\ \phi_* \cdot)\phi_* \cdot \right] \right\|_{L^2} \leq
$$

$$\leq \exp\Big[\frac{1}{2}\,C(\Omega,\phi)\Big]\,\Big\|\mathrm{trace}_{G_\theta}\Big[\Pi_H(R^h)^\phi(V,\,\phi_*\cdot)\phi_*\cdot\Big]\Big\|_{L^2}, \tag{95}$$

$$\Big\|\exp\Big[\frac{1}{2}\,e_b(\phi)\Big]\,W\Big\|_{L^2} \leq \exp\Big[\frac{1}{2}\,C(\Omega,\phi)\Big]\,\|W\|_{L^2}. \tag{96}$$

On the other hand

$$\mathrm{trace}_{G_\theta}\Big[\Pi_H(R^h)^\phi(V,\,\phi_*\cdot)\phi_*\cdot\Big]\Big|_U = \sum_{a=1}^{2n}(R^h)^\phi(V,\,\phi_* E_a)\phi_* E_a$$

on U. Let $\{(U_\alpha,\ f_\alpha)\ :\ \alpha\in I\}$ be a family of local orthonormal frames $f_\alpha=\{E_a^\alpha\ :\ 1\leq a\leq 2n\}\subset C^\infty(U_\alpha,\ H(M))$ such that $\{U_\alpha\}_{\alpha\in I}$ is an open cover of Ω. Let $\{\psi_\alpha\}_{\alpha\in I}$ be a smooth partition of unity subordinated to $\{U_\alpha\}_{\alpha\in I}$ i.e.

i) $0\leq\psi_\alpha(x)\leq 1$ for any $\alpha\in I$ and $x\in\Omega$,

ii) $\{\mathrm{Supp}(\psi_\alpha)\}_{\alpha\in I}$ is a locally finite family,

iii) $\sum_{\alpha\in I}\psi_\alpha(x)=1$ for any $x\in\Omega$.

By (93)

$$\Big\|\mathrm{trace}_{G_\theta}\Big[\Pi_H(R^h)^\phi(V,\,\phi_*\cdot)\phi_*\cdot\Big]\Big\|\Big|_{U_\alpha} \leq \sum_{a=1}^{2n}\Big\|(R^h)^\phi(V,\,\phi_* E_a^\alpha)\phi_* E_a^\alpha\Big\| \leq$$
$$\leq \sum_a \gamma\,\|V\|\,\|\phi_* E_a^\alpha\|^2 = 2\,\gamma\,\|V\|\,e_b(\phi)|_{U_\alpha}$$

hence

$$\begin{aligned}
&\Big\|\mathrm{trace}_{G_\theta}\Big[\Pi_H(R^h)^\phi(V,\,\phi_*\cdot)\phi_*\cdot\Big]\Big\|_{L^2}^2\\
&= \sum_{\alpha\in I}\int_{U_\alpha}\psi_\alpha\ \Big\|\mathrm{trace}_{G_\theta}\Big[\Pi_H(R^h)^\phi(V,\,\phi_*\cdot)\phi_*\cdot\Big]\Big\|^2\Big|_{U_\alpha}\ \Psi \leq\\
&\leq 4\gamma^2\sum_{\alpha\in I}\int_{U_\alpha}\psi_\alpha\,\|V\|^2\ e_b(\phi)^2\Big|_{U_\alpha}\ \Psi = 4\gamma^2\int_\Omega \|V\|^2\,e_b(\phi)^2\,\Psi \leq\\
&\leq 4\,\gamma^2\,C(\Omega,\phi)^2\int_\Omega\|V\|^2\,\Psi.
\end{aligned}$$

Thus

$$\Big\|\mathrm{trace}_{G_\theta}\Big[\Pi_H(R^h)^\phi(V,\,\phi_*\cdot)\phi_*\cdot\Big]\Big\|_{L^2} \leq 2\,\gamma\,C(\Omega,\phi)\,\|V\|_{L^2} \leq$$

(by Poincaré inequality)

$$\leq 2\,C\,\gamma\,C(\Omega,\phi)\,\|D^\phi V\|_{L^2}. \tag{97}$$

Finally [by (94)–(96)]

$$\left|a_\phi(V,W)\right| \leq e^{C(\Omega,\phi)}\ \Big\{\|D^\phi V\|_{L^2}\,\|D^\phi W\|_{L^2}+ \\ +\|\text{trace}_{G_\theta}\Pi_H(R^h)^\phi(V,\ \phi_*\cdot)\phi_*\ \cdot\ \|_{L^2}\|W\|_{L^2}\Big\} \leq$$

[by (97) and Poincaré inequality]

$$\leq \left[1+2C^2\gamma C(\Omega,\varphi)\right]e^{C(\Omega,\phi)}\|D^\phi V\|_{L^2}\|D^\phi W\|_{L^2}.$$

□

Lemma 5 *The function*

$$V\in \mathring{W}^{1,2}_H(\Omega,\phi^{-1}TN)\longmapsto (F,V)_{L^2}\in\mathbb{R}$$

is continuous for every $F\in L^2(\Omega,\phi^{-1}TN)$.

Proof By Cauchy and Poincaré inequalities

$$\left|(F,V)_{L^2}\right| \leq \|F\|_{L^2}\|V\|_{L^2}\leq C\|F\|_{L^2}\|V\|_{\mathring{W}^{1,2}}.$$

□

Lemma 6 *Under the assumptions in Lemmas 4–5 the functional $\mathcal{F}_\phi$ is Gâteaux differentiable and its Gâteaux derivative is*

$$\mathcal{F}'_\phi(V)W = a_\phi(V,W)-(F,W)_{L^2},\quad W\in \mathring{W}^{1,2}_H(\Omega,\phi^{-1}TN).$$

Proof By the symmetry of the Riemann–Christoffel tensor field $K^h(A,B,C,D)$ [in the pairs (A,B) and (C,D)] the bilinear form a_ϕ is symmetric, and hence q is a quadratic form. Then [by Lemma 4] q is Gâteaux differentiable and its Gâteaux derivative is

$$q(V)W = 2\,a_\phi(V,W). \tag{98}$$

Also [by Lemma 5]

$$\left(F,\ \cdot\right)'_{L^2}V = (F,V)_{L^2}. \tag{99}$$

and (98)–(99) yield Lemma 6. □

Lemma 7 *Let us assume that (N,h) has nonpositive sectional curvature. Then a_ϕ is coercive. Consequently $\mathcal{F}_\phi$ is strictly convex and*

$$\lim_{\|V\|_{\mathring{W}^{1,2}}}\mathcal{F}_\phi(V)=+\infty. \tag{100}$$

Proof As (N, h) has nonpositive sectional curvature

$$h^{\phi}\left(\mathrm{trace}_{G_\theta}\Pi_H(R^h)^{\phi}(V,\ \phi_*\cdot)\phi_*\cdot,\ V\right)\le 0$$

so that

$$a_\phi(V,V)\ge\int_\Omega \exp\left[e_b(\phi)\right]\|D^\phi V\|^2\,\Psi\ge\alpha\,\|V\|^2_{\mathring{W}^{1,2}}$$

where we have set

$$c(\Omega,\phi)=\inf_{x\in\overline{\Omega}}e_b(\phi),\quad \alpha=\exp\left[c(\Omega,\phi)\right].$$

Hence a_ϕ is coercive with coercivity constant $\alpha\ge 1$. Coercivity yields

$$a_\phi(V,W)\le\frac{1}{2}\left[q(V)+q(W)\right]\tag{101}$$

with equality if and only if $V=W$. Let $t\in(0,1)$ and $V,W\in\mathring{W}^{1,2}(\Omega,\phi^{-1}TN)$ with $V\ne W$. Then [by (101)]

$$\begin{aligned}&\mathcal{F}_\phi\left[tV+(1-t)W\right]<\\&<\frac{1}{2}\left[tq(V)+(1-t)q(W)\right]-t\,(F,V)_{L^2}-(1-t)\,(F,W)_{L^2}\\&=t\,\mathcal{F}_\phi(V)+(1-t)\,\mathcal{F}_\phi(W)\end{aligned}$$

i.e. $\mathcal{F}_\phi$ is strictly convex. The last statement in Lemma 7 follows by coercivity and Poincaré inequality

$$\begin{aligned}\mathcal{F}_\phi(V)&=\frac{1}{2}\,q(V)-(F,V)_{L^2}\ge\frac{1}{2}\,\|V\|^2_{\mathring{W}^{1,2}}-C\|F\|_{L^2}\|V\|_{\mathring{W}^{1,2}}\\&=\frac{1}{2}\,t^2-C\|F\|_{L^2}\,t\to+\infty,\quad t\to+\infty,\end{aligned}$$

where we have set $t=\|V\|_{\mathring{W}^{1,2}}$. □

Theorem 8 *Let (N, h) be a Riemannian manifold of nonpositive sectional curvature such that*

$$\|R^h(A,B)C\|\le\gamma\,\|A\|\,\|B\|\,\|C\|,\quad A,B,C\in\mathfrak{X}(N).$$

Let $\phi: M\to N$ be an e.s.h. map of the Carnot–Carathéodory complete pseudohermitian manifold (M,θ) into (N,h). Let $\Omega\subset M$ be a bounded domain supporting the Poincaré inequality

$$\left(\int_\Omega h^\phi(V,V)\,\Psi\right)^{\frac{1}{2}}\le C\left(\int_\Omega\left(h^\phi\right)^*\left(D^\phi V,\ D^\phi V\right)\,\Psi\right)^{\frac{1}{2}},$$

$$V \in C_0^\infty(\Omega,\ \phi^{-1}TN).$$

Then for every $F \in L^2(\Omega, \phi^{-1}TN)$ *there is a unique generalized solution* $V_F \in \mathring{W}^{1,2}(\Omega, \phi^{-1}TN)$ *to the Dirichlet problem* $J^{\phi}_{b,\exp} V = F$ *in* Ω *and* $V = 0$ *on* $\partial\Omega$.

Proof As

$$\mathcal{F}_\phi : \mathring{W}^{1,2}_H(\Omega, \phi^{-1}TN) \to \mathbb{R}$$

is strictly convex, Gâteaux differentiable, and obeys to (100), there is a unique global minimum point $V_F \in \mathring{W}^{1,2}_H(\Omega, \phi^{-1}TN)$ for $\mathcal{F}_\phi$. In particular V_F is a critical point for $\mathcal{F}_\phi$ hence [by Lemma 6]

$$0 = \mathcal{F}'_\phi(V_F)W = a_\phi\big(V_F,\ W\big) - (F, W)_{L^2}$$

so that V_F is a solution to the generalized Dirichlet problem for $J^{\phi}_{b,\exp}$ on Ω. If $\hat{V}_F$ is another generalized solution

$$a_\phi\big(\hat{V}_F,\ W\big) = (F, W), \quad W \in \mathring{W}^{1,2}_H(\Omega, \phi^{-1}TN),$$

then [again by Lemma 7]

$$\mathcal{F}'_\phi\big(\hat{V}_F\big)W = 0. \tag{102}$$

Yet $\mathcal{F}_\phi$ is convex, so (102) is a sufficient condition for $\hat{V}_F$ to be a global minimum point for $\mathcal{F}_\phi$. Uniqueness of global minima then yields $\hat{V}_F = V_F$. □

3.5.3 Dirichlet Eigenvalue Problem for $J^{\phi}_{b,\exp}$

Let (N, h) be a Riemannian manifold obeying to the assumptions in Theorem 8 and let $\Omega \subset M$ be a Carnot–Carathéodory bounded domain. Let $\phi : M \to N$ be an e.s.h. map from (M, θ) into (N, h). Solving (according to Theorem 8) the generalized Dirichlet problem for $J^{\phi}_{b\ \exp}$ on Ω produces the **Green operator**

$$G_\phi : L^2(\Omega, \phi^{-1}TN) \to L^2(\Omega, \phi^{-1}TN), \quad G_\phi(V) = F_V.$$

It should be observed that [by the proof of Theorem 8] the range $\mathcal{R}(G_\phi)$ is a subspace of $\mathring{W}^{1,2}_H(\Omega, \phi^{-1}TN)$.

The domain $\Omega \subset M$ is said to satisfy the **Kondrakov condition** if the embedding

$$\mathring{W}^{1,2}_H\big(\Omega,\ \phi^{-1}TN\big) \hookrightarrow L^2\big(\Omega,\ \phi^{-1}TN\big)$$

is compact.

Lemma 8 *Under the hypothesis of Theorem* 8 *let us additionally assume that* Ω *satisfies the Kondrakov condition. Then the Green operator* G_ϕ *of* $J^\phi_{b,\,\exp}$ *is* i) *linear,* ii) *continuous,* iii) *self-adjoint, and* iv) *compact.*

Proof i) Linearity of G_ϕ follows from that of $J^\phi_{b,\exp}$ and the uniqueness statement in Theorem 8.

ii) By Poincaré inequality and coercivity

$$\begin{aligned}\|G_\phi F\|^2_{L^2} &= \|V_F\|^2_{L^2} \le C^2 \|V_F\|^2_{\mathring{W}^{1,2}} \le \\ &\le \left(\frac{C}{\alpha}\right)^2 a_\phi\big(V_F,\ V_F\big) = \left(\frac{C}{\alpha}\right)^2 a_\phi\big(V_F,\ G_\phi F\big) =\end{aligned}$$

[as $G_\phi F$ is the solution to the generalized Dirichlet problem]

$$= \left(\frac{C}{\alpha}\right)^2 \big(F,\ G_\phi F\big)_{L^2} \le$$

[by the Cauchy inequality]

$$\le \left(\frac{C}{\alpha}\right)^2 \|F\|_{L^2} \|G_\phi F\|_{L^2}$$

and then either $GF = 0$ or

$$\|G_\phi F\|_{L^2} \le C^2 \exp\big[-2\,c(\Omega, \phi)\big] \, \|F\|_{L^2}.$$

□

iii) The proof is organized in three steps, as follows.

Step 1. $G_\phi : L^2(\Omega,\ \phi^{-1}TN) \to \mathcal{R}(G_\phi)$ *is invertible.*

Proof If $G_\phi F = 0$ then $V_F = 0$ so that

$$0 = a_\phi\big(V_F,\ W\big) = \big(F, W\big)_{L^2}$$

thus [as $\mathring{W}^{1,2}_H(\Omega, \phi^{-1}TN)$ is dense in $L^2(\Omega, \phi^{-1}TN)$] it must be $F = 0$ i.e. G_ϕ is injective. □

Step 2. $G_\phi^{-1} \equiv \big[G_\phi : L^2(\Omega,\ \phi^{-1}TN) \to \mathcal{R}(G_\phi)\big]^{-1}$ *is self-adjoint.*

Proof Let $V, W \in \mathcal{D}(G_\phi^{-1}) \equiv \mathcal{R}(G_\phi)$ and let us set $F = G_\phi^{-1}(V)$. Then [by $\mathcal{R}(G_\phi) \subset \mathring{W}^{1,2}_H(\Omega, \phi^{-1}TN)$]

$$\begin{aligned}\big(G_\phi^{-1}V,\ W\big)_{L^2} &= \big(F,\ W\big)_{L^2} = a_\phi\big(G_\phi F,\ W\big) = a_\phi(V, W) \\ &= a_\phi(W, V) = \big(G_\phi^{-1}W,\ V\big)_{L^2} = \big(V,\ G_\phi^{-1}W\big)_{L^2}.\end{aligned}$$

Step 3. *G_ϕ is self-adjoint.*

Proof Let $F_a \in L^2(\Omega,\ \phi^{-1}TN)$, $a \in \{1, 2\}$. Then

$$\begin{aligned}\big(G_\phi F_1,\ F_2\big)_{L^2} &= \big(G_\phi F_1,\ G_\phi^{-1} G_\phi F_2\big)_{L^2} = \quad \text{[by Step 2]}\\ &= \big(G_\phi^{-1} G_\phi F_1,\ G_\phi F_2\big)_{L^2} = \big(F_1,\ G_\phi F_2\big)_{L^2}.\end{aligned}$$ □

iv) To show that G_ϕ is a bounded operator, let $B \subset L^2(\Omega, \phi^{-1}TN)$ be a bounded set i.e. $\|F\|_{L^2} \leq C_B$ for some constant $C_B > 0$ and any $F \in B$. Then [by coercivity]

$$\|G_\phi F\|^2_{\mathring{W}^{1,2}} \leq \exp\big[-c(\Omega,\phi)\big]\, a_\phi\big(G_\phi F,\ G_\phi F\big) =$$

[as $G_\phi F$ is the solution to the generalized Dirichlet problem]

$$= \exp\big[-c(\Omega,\phi)\big]\, \big(F,\ G_\phi F\big)_{L^2} \leq$$

[by Cauchy inequality]

$$\leq \exp\big[-c(\Omega,\phi)\big]\, \|F\|_{L^2}\, \|G_\phi F\|_{L^2} \leq C_B\ \exp\big[-c(\Omega,\phi)\big]\, \|G_\phi F\|_{L^2} \leq$$

[by Poincaré inequality]

$$\leq C\, C_B\ \exp\big[-c(\Omega,\phi)\big]\, \|GF\|_{\mathring{W}^{1,2}}$$

hence either $G_\phi F = 0$ or

$$\|G_\phi F\|_{\mathring{W}^{1,2}} \leq C\, C_B\ \exp\big[-c(\Omega,\phi)\big]$$

i.e. $G-\phi(B)$ is a bounded set of $\mathring{W}^{1,2}_H(\Omega, \phi^{-1}TN)$. Then, by the Kondrakov condition, $G_\phi(B)$ is a bounded subset in $L^2(\Omega, \phi^{-1}TN)$. So G_ϕ maps bounded sets onto bounded sets. □

The **Dirichlet eigenvalue problem** is

$$J^\phi_{b,\,\exp} V = \lambda F \quad \text{in}\ \ \Omega, \qquad V = 0 \quad \text{on}\ \ \partial\Omega. \tag{103}$$

A weak version of (103), the **generalized Dirichlet eigenvalue problem**, is to determine $\lambda \in \mathbb{R}$ and $V \in \mathring{W}^{1,2}_H(\Omega, \phi^{-1}TN)$ such that

$$a_\phi\big(V,\ S\big) = \lambda\, \big(V,\ S\big)_{L^2}, \quad \forall\ \ S \in \mathring{W}^{1,2}_H(\Omega, \phi^{-1}TN).$$

Let $\sigma_{\text{gen}}\big(J^\phi_{b,\,\exp}\big)$ and $\sigma(G_\phi)$ be respectively the spectrae of the generalized Dirichlet eigenvalue problem and of G_ϕ. Note that $\text{Ker}(G_\phi) = (0)$ so that $0 \notin \sigma(G_\phi)$. Also [by the linearity of a_ϕ and G_ϕ] $0 \notin \sigma_{\text{gen}}\big(J^\phi_{b,\,\exp}\big)$.

Theorem 9 *Let $\Omega \subset M$ and $\phi \in \mathfrak{har}_{b,\,\exp}\big(M, N\big)$ satisfy the assumptions*[6] *in Lemma 8. Then there is an infinite sequence*

$$0 < \lambda_1(\phi) \leq \lambda_2(\phi) \leq \cdots \leq \lambda_\nu(\phi) \leq \cdots \uparrow +\infty$$

and an infinite sequence $\{V_\nu\}_{\nu \geq 1} \subset \mathring{W}^{1,2}_H(\Omega, \phi^{-1}TN)$ such that

$$\sigma_{\text{gen}}\big(J^{\phi}_{b,\,\exp}\big) = \big\{\lambda_\nu(\phi)\ :\ \nu \geq 1\big\},$$
$$a_\phi\big(V_\nu,\ S\big) = \lambda_\nu(\phi)\,\big(V_\nu,\ S\big)_{L^2}, \quad \nu \geq 1,$$

for any $S \in \mathring{W}^{1,2}_H(\Omega,\ \phi^{-1}TN)$.

The regularity problem for the eigen-sections V_ν will be addressed in a further paper. There is an obvious relationship between $\sigma_{\text{gen}}\big(J^{\phi}_{b,\,\exp}\big)$ and $\sigma(G_\phi)$, for given the bijection $f : \mathbb{R}\backslash\{0\} \to \mathbb{R}\backslash\{0\}$, $f(\lambda) = 1/\lambda$, one has $f\big[\sigma_{\text{gen}}\big(J^{\phi}_{b,\,\exp}\big)\big] = \sigma(G_\phi)$. The proof of Theorem 9 relies on that, together with a few standard results in functional analysis.

Step 1. $\emptyset \neq \sigma(G_\phi) \subset \big(0,\ +\infty\big)$.

Proof Green's operator G_ϕ is self-adjoint, continuous, and compact, so $\sigma(G_\phi) \neq \emptyset$. Let $\mu \in \sigma(G_\phi)$ so that there is $V \in L^2(\Omega, \phi^{-1}TN)$, $V \neq 0$, such that $G_\phi V = \mu V$. Let us take the L^2 inner product with V. We have

$$\mu\,\|V\|^2_{L^2} = \big(V,\ G_\phi V\big)_{L^2} =$$

[by solving the (generalized) Dirichlet problem $J^{\phi}_{b,\,\exp}U = V$ in Ω and $U = 0$ on $\partial\Omega$]

$$= a_\phi\big(G_\phi V,\ G_\phi V\big) \geq$$

[by coercivity]

$$\geq \exp\big[-c(\Omega, \phi)\big]\,\|G_\phi V\|^2_{\mathring{W}^{1,2}} > 0$$

hence $\mu > 0$. □

Step 2. $\dim_{\mathbb{R}} \mathcal{R}(G_\phi) = \infty$.

Proof Step 2 follows from

$$C^\infty_0(\Omega, \phi^{-1}TN) \subset \mathcal{R}(G_\phi), \quad \dim_{\mathbb{R}} C^\infty_0(\Omega, \phi^{-1}TN) = \infty.$$

To check the claimed inclusion let $V \in C^\infty_0(\Omega, \phi^{-1}TN)$ and let us set

$$F \equiv J^{\phi}_{b,\,\exp}(V) \tag{104}$$

[6] That is the assumptions in Theorem 8, including the curvature requirements on the Riemannian manifold (N, h), together with the Kondrakov condition.

so that $F \in C_0^\infty(\Omega, \phi^{-1}TN)$. Let us take the L^2 inner product of (104) with $S \in C_0^\infty(\Omega, \phi^{-1}TN)$ arbitrary i.e.

$$\left(J^\phi_{b,\exp} V,\ S\right)_{L^2} = \left(F, S\right)_{L^2}$$

or (integrating by parts)

$$a_\phi\left(V,\ S\right) = \left(F, S\right)_{L^2}. \tag{105}$$

Let $S \in \mathring{W}_H^{1,2}(\Omega, \phi^{-1}TN)$ and let $\{S_\nu\}_{\nu\geq 1} \subset C_0^\infty(\Omega, \phi^{-1}TN)$ be a sequence converging to S in the $\|\cdot\|_{W^{1,2}}$ norm. By passing to the limit with $\nu \to \infty$ in

$$a_\phi\left(V,\ S_\nu\right) = \left(F, S_\nu\right)_{L^2}$$

it follows that (105) holds for arbitrary $S \in \mathring{W}_H^{1,2}(\Omega, \phi^{-1}TN)$ hence V is the solution to the (generalized) Dirichlet problem $J^\phi_{b,\exp} V = F$ in Ω and $V = 0$ on $\partial\Omega$ i.e. $V = G_\phi(F) \in \mathcal{R}(G_\phi)$. □

Step 3. *The set $\sigma(G_\phi)$ is countable.*

Proof Green's operator G_ϕ is self-adjoint, continuous, and compact, hence $\sigma(G_\phi)$ is at most countable. We ought to show that $\sigma(G_\phi)$ isn't a finite set. One argues by contradiction. If $\sigma(G_\phi)$ is a finite set, say

$$\sigma(G_\phi) = \{\mu_1, \cdots, \mu_p\}$$

then (again because G_ϕ is self-adjoint, continuous, and compact) the system $\mathcal{E}(\mu_\nu)$ of all linearly independent eigenvectors of G_ϕ corresponding to the eigenvalue μ_ν consists of finitely many vectors, hence $\bigcup_{\nu=1}^p \mathcal{E}(\mu_\nu)$ is a finite set, say

$$\bigcup_{\nu=1}^{p} \mathcal{E}(\mu_\nu) = \left\{V_1, \cdots, V_q\right\}.$$

Finally

$$G_\phi F = \sum_{j=1}^{q} \left(GF,\ V_j\right)_{L^2} V_j, \quad \forall\ F \in L^2(\Omega, \phi^{-1}TN)$$

implying that the range $\mathcal{R}(G_\phi)$ is finite dimensional, in contradiction with Step 2. □

At this point one may end the proof of Theorem 9. By Steps 1 and 3 there is an infinite sequence

$$\{\mu_\nu(\phi)\}_{\nu\geq 1} \subset \left(0,\ +\infty\right)$$

such that $\sigma(G-\phi)=\{\mu_\nu(\phi)\ :\ \nu\geq 1\}$. If we set $\lambda_\nu(\phi)=1/\big[\mu_\nu(\phi)\big]$ then

$$\sigma_{\text{gen}}\big(J^{\phi}_{b,\,\exp}\big)=\big\{\lambda_\nu(\phi)\ :\ \nu\geq 1\big\}.$$

Once again as Green's operator G is self-adjoint, continuous, and compact, 0 is the only accumulation point of the set $\sigma(G)$. Hence

$$\lim_{\nu\to\infty}\mu_\nu(\phi)=0$$

and one may relabel the eigenvalues $\mu_\nu(\phi)$ so that to have

$$\mu_1(\phi)\geq\mu_2(\phi)\geq\cdots\geq\mu_\nu(\phi)\geq\cdots\ \downarrow\ 0.$$

$\square$

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