On the positive semigroups generated by Fleming-Viot type differential operators

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Abstract

In this paper we study a class of degenerate second-order elliptic differential operators, often referred to as Fleming-Viot type operators, in the framework of function spaces defined on the d-dimensional hypercube Q_d of $\mathbf{R}^d$, $d \geq 1$.

By making mainly use of techniques arising from approximation theory, we show that their closures generate positive semigroups both in the space of all continuous functions and in weighted L^p-spaces.

In addition, we show that the semigroups are approximated by iterates of certain polynomial type positive linear operators, which we introduce and study in this paper and which generalize the Bernstein-Durrmeyer operators with Jacobi weights on $[0, 1]$.

As a consequence, after determining the unique invariant measure for the approximating operators and for the semigroups, we establish some regularity properties of them along with their asymptotic behaviours.

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1 Introduction

The main aim of this paper is to study the class of degenerate second-order elliptic differential operators defined on the d-dimensional hypercube Q_d of $\mathbf{R}^d$, $d \geq 1$, by

$$A(u)(x) = \sum_{i=1}^{d} x_i(1 - x_i)\frac{\partial^2 u}{\partial x_i^2}(x) + (a_i + 1 - (a_i + b_i + 2)x_i)\frac{\partial u}{\partial x_i}(x)$$

for every $u \in C^2(Q_d)$ and $x = (x_1, \ldots, x_d) \in Q_d$, where $a_1, \ldots, a_d, b_1, \ldots, b_d \in \mathbf{R}$, $a_i > -1$ and $b_i > -1$ for all $i = 1, \ldots, d$.

The operators defined above arise in the theory of Fleming-Viot processes applied to some models of population dynamics which, however, usually take places in the framework of d-dimensional simplices, $d \geq 1$ (see, e.g., [1], [2], [5, Section 5.8], [7], [8], [9], [14] and the references therein).

Due to their intrinsic interest, more recently an increasing attention has been turned to them also in the setting of hypercubes (see, e.g., [5, Section 5.8], [6], [8], [13], [16] and the references therein).

In this paper we give some further contributions to this research area by heavily using techniques arising from approximation theory. The methods we employ allow to show that these operators generate positive semigroups both in the space of all continuous functions and in weighted L^p-spaces with respect to the Jacobi weights determined by the coefficients a_i and b_i, $i = 1, \dots, d$.

In addition, we disclose several qualitative properties of the generated semigroups.

As a first step we introduce a sequence of polynomial type positive linear operators which generalize the Bernstein-Durrmeyer operators with Jacobi weights on $[0, 1]$. Among other things, we show that these operators constitute an approximation process for continuous functions as well as for weighted L^p-functions.

By using the Trotter-Schnabl approximation theorem, we show that their closures generate positive semigroups which, in turn, are approximated by iterates of the above mentioned positive operators.

As a consequence, after determining the unique invariant measure for the approximating operators and for the semigroups, we describe their asymptotic behaviour by also evaluating the rate of convergence. Finally, we show that they preserve the class of Hölder continuous functions and the one of those continuous functions which are convex with respect to each variable.

2 Preliminaries and notation

We begin by recalling some basic notions about invariant measures which will play a key role within the whole paper. For more details on such a subject, and on its relationship with ergodic theory and asymptotic formulae, we refer the interested reader to [8], [15].

Let X be a compact Hausdorff space and let B_X be the σ-algebra of all Borel subsets of X; we denote by $M^+(X)$ (resp., $M_1^+(X)$) the cone of all regular Borel measures on X (resp., the cone of all regular Borel probability measures on X).

If $\mu \in M^+(X)$ and $1 \leq p < +\infty$, let us denote by $L^p(X, \mu)$ the space of all (the equivalence classes of) Borel measurable real-valued functions on X which are μ-integrable in the p^{th} power. The space $L^p(X, \mu)$ is endowed with the natural norm

$$\|f\|_p := \left(\int_X |f|^p \, d\mu\right)^{1/p} \qquad (f \in L^p(X,\mu)).$$

As usual, the symbol $C(X)$ indicates the linear space of all continuous real-valued functions on X. $C(X)$ will be endowed with the uniform norm $\|\cdot\|_\infty$, with respect to which it is a Banach space.

A Markov operator T on $C(X)$ is a positive linear operator $T : C(X) \to C(X)$ such that $T(\mathbf{1}) = \mathbf{1}$, where the symbol $\mathbf{1}$ stands for the constant function of constant value 1 on X.

By the Riesz representation theorem, for every $x \in X$ there exists $\mu_x \in M_1^+(X)$ such that

$$T(f)(x) = \int_X f \, d\mu_x \quad (f \in C(X)).$$

By applying the Hölder inequality to each μ_x, it follows that

$$|T(f)|^p \leq T(|f|^p) \quad (f \in C(X), p \in [1, +\infty[). \tag{2.1}$$

It is well-known that every Markov operator T on $C(X)$ admits at least one invariant probability measure, i.e., a measure $\mu \in M_1^+(X)$ such that

$$\int_X T(f) \, d\mu = \int_X f \, d\mu \quad \text{ for every } f \in C(X) \tag{2.2}$$

(see [15, Section 5.1, p. 178]).

Accordingly, on account of (2.1), if μ is an invariant measure for T, then for every $f \in C(X)$ and $p \in [1, +\infty[$,

$$\int_X |T(f)|^p \, d\mu \leq \int_X T(|f|^p) \, d\mu = \int_X |f|^p \, d\mu; \tag{2.3}$$

hence, T extends to a unique bounded linear operator $T_p : L^p(X,\mu) \to L^p(X,\mu)$ such that $\|T_p\| \leq 1$. Furthermore, T_p is a positive operator, since $C(X)$ is a sublattice of $L^p(X,\mu)$; moreover, if $1 \leq p < q < +\infty$, then $T_p = T_q$ on $L^q(X,\mu)$.

If X is a compact subset of $\mathbf{R}^d$, $d \geq 1$, the symbol $C^2(X)$ stands for the space of all real-valued continuous functions on X which are twice-continuously differentiable on the interior of X and whose partial derivatives of order ≤ 2 can be continuously extended to X. For $u \in C^2(X)$ and $i, j = 1, \ldots, d$, we shall continue to denote by $\frac{\partial u}{\partial x_i}$ and $\frac{\partial^2 u}{\partial x_i \partial x_j}$ the continuous extensions to X of $\frac{\partial u}{\partial x_i}$ and $\frac{\partial^2 u}{\partial x_i \partial x_j}$.

If A is a differential operator on $C^2(X)$, a measure $\mu \in M_1^+(X)$ is said to be infinitesimally invariant for A if, for every $u \in C^2(X)$,

$$\int_X A(u) \; d\mu = 0.$$

In what follows, we shall also fix some additional notation.

Let $\gamma = (\gamma_1, \dots, \gamma_d) \in \mathbf{R}^d$, $d \geq 1$. If $x = (x_1, \dots, x_d) \in \mathbf{R}^d$, $x_i > 0$ for every $i = 1, \dots, d$, we set

$$x^\gamma := \prod_{i=1}^{d} x_i^{\gamma_i}.$$

For $x = (x_1, \dots, x_d)$, $y = (y_1, \dots, y_d) \in \mathbf{R}^d$, we write $x \leq y$ if $x_i \leq y_i$ for every $i = 1, \dots, d$.

Let $j = (j_1, \dots, j_d)$, $k = (k_1, \dots, k_d) \in \mathbf{N}^d$ be two multi-indices such that $k \leq j$; we set

$$\binom{j}{k} := \prod_{i=1}^{d} \binom{j_i}{k_i}.$$

We also set $0_d := (0, \dots, 0)$, for every $n \geq 1$, $n_d := (n, \dots, n)$ and

$$v_1 := (1, 0, \dots, 0), \dots, v_d := (0, \dots, 0, 1). \tag{2.4}$$

All the results of this paper concern function spaces defined on the d-dimensional hypercube $Q_d := [0,1]^d$, $d \geq 1$.

In particular, we consider the space

$$Lip(Q_d) := \left\{ f \in C(Q_d) \ \middle| \ |f|_{Lip} := \sup_{\substack{x,y \in Q_d \\ x \neq y}} \frac{|f(x) - f(y)|}{\|x - y\|_1} < +\infty \right\} \tag{2.5}$$

and, for $M > 0$,

$$Lip(M, 1) := \{ f \in Lip(Q_d) \mid |f(x) - f(y)| \leq M \|x - y\|_1 \}, \tag{2.6}$$

where $\| \cdot \|_1$ is the norm on $\mathbf{R}^d$ defined by $\|x\|_1 := \sum_{i=1}^{d} |x_i|$, for every $x = (x_1, \dots, x_d) \in \mathbf{R}^d$.

More generally, given $0 < \alpha \leq 1$, we shall denote by $Lip(M, \alpha)$ the subset of all Hölder continuous functions on Q_d with exponent α and constant M, i.e., those $f \in C(Q_d)$ such that

$$|f(x) - f(y)| \leq M \|x - y\|_1^\alpha \quad \text{for every } x, y \in Q_d. \tag{2.7}$$

Finally, we denote by $\mathbb{P}_m$ the linear subspace generated by the polynomials on Q_d of degree $\leq m$.

3 A generalization of Bernstein-Durrmeyer operators with Jacobi weights on the hypercube

In this section we introduce and study a sequence of positive linear operators acting on weighted L^p-spaces. These operators map weighted L^p-functions into polynomials on Q_d and generalize the Bernstein-Durrmeyer operators with Jacobi weights on $[0,1]$ (see [7], [8], [11], [12], [18], [20]).

Although we are mainly interested in the role which they play in the approximation of the semigroups we shall investigate in the subsequent section, it seems that these operators also have an interest on their own as an approximation process for continuous functions as well as for weighted L^p-functions.

From now on fix $a = (a_1, \dots, a_d)$, $b = (b_1, \dots, b_d) \in \mathbf{R}^d$ with $a_i > -1$ and $b_i > -1$ for all $i = 1, \dots, d$. Let us denote by $\mu_{a,b} \in M_1^+(Q_d)$ the absolutely continuous measure with respect to the Borel-Lebesgue measure λ_d on Q_d with density the normalized Jacobi weight

$$w_{a,b}(x) := \frac{x^a(1-x)^b}{\int_{Q_d} y^a(1-y)^b\,dy} \qquad (x \in Q_d). \tag{3.1}$$

Moreover, for every $n \geq 1$, consider the operator $M_n : L^1(Q_d, \mu_{a,b}) \to C(Q_d)$ defined by setting, for every $f \in L^1(Q_d, \mu_{a,b})$ and $x \in Q_d$,

$$M_n(f)(x) := \sum_{\substack{h \in \mathbf{N}^d \\ 0_d \leq h \leq n_d}} \omega_{n_d,h}(f) \binom{n_d}{h} x^h (1_d - x)^{n_d - h}, \tag{3.2}$$

where, for every $n \geq 1$ and $h = (h_1, \dots h_d) \in \mathbf{N}^d$, $0_d \leq h \leq n_d$,

$$\begin{aligned}\omega_{n_d,h}(f) &:= \frac{1}{\int_{Q_d} y^{h+a}(1_d - y)^{n_d-h+b}\,dy} \int_{Q_d} y^{h+a}(1_d - y)^{n_d-h+b} f(y)\,dy \\ &= \prod_{i=1}^{d} \frac{\Gamma(n + a_i + b_i + 2)}{\Gamma(h_i + a_i + 1)\Gamma(n - h_i + b_i + 1)} \int_{Q_d} y^{h+a}(1_d - y)^{n_d-h+b} f(y)\,dy,\end{aligned} \tag{3.3}$$

$\Gamma(u)$ $(u \geq 0)$ being the classical Euler Gamma function.

Clearly, the restriction of each M_n to $C(Q_d)$ is a Markov operator on $C(Q_d)$.

In order to discuss the main properties of the operators M_n, we briefly examine the case $d = 1$, i.e., the classical Bernstein-Durrmeyer operators with Jacobi weights on $[0,1]$ (see [11]).

Consider $a > -1$, $b > -1$. Then, for every $\varphi \in L^1([0,1], \mu_{a,b})$, $n \geq 1$ and $x \in [0,1]$, set

$$M_{n,a,b}(\varphi)(x) := \sum_{h=0}^{n} \omega_{n,h}(\varphi) \binom{n}{h} x^h (1-x)^{n-h}, \tag{3.4}$$

where

$$\omega_{n,h}(\varphi) := \frac{\Gamma(n + a + b + 2)}{\Gamma(h + a + 1)\Gamma(n - h + b + 1)} \int_0^1 t^{h+a}(1-t)^{n-h+b}\varphi(t)\,dt. \tag{3.5}$$

For every $n \geq 1$, consider the positive linear operator $D_{n,a,b} : L^1([0,1], \mu_{a,b}) \to C([0,1])$ defined, for every $\varphi \in C([0,1])$ and $x \in [0,1]$, as

$$D_{n,a,b}(\varphi)(x) := \frac{\Gamma(n+a+b+2)}{\Gamma(nx+a+1)\Gamma(n-nx+b+1)} \int_0^1 t^{nx+a}(1-t)^{n-nx+b}\varphi(t)\,dt \tag{3.6}$$

(see [9, formula (4.6)]). Then

$$M_{n,a,b}(\varphi) = B_n(D_{n,a,b}(\varphi)), \tag{3.7}$$

where

$$B_n(\psi)(x) := \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} \psi\left(\frac{k}{n}\right)$$

($\psi \in C([0,1])$, $0 \leq x \leq 1$) is the classical Bernstein polynomial operator of order n on $C([0,1])$ (see, e.g. [4, pp. 218-220]).

In particular, if $m, n \geq 1$ and $e_m(t) = t^m$, $t \in [0,1]$, it is easy to prove that

$$D_{n,a,b}(e_m) = \frac{\Gamma(n+a+b+2)}{\Gamma(m+n+a+b+2)}(a+1+ne_1)\cdots(a+m+ne_1), \tag{3.8}$$

i.e., $D_{n,a,b}(e_m)$ is a polynomial of degree at most m.

Hence

$$M_{n,a,b}(e_m) = B_n(D_{n,a,b}(e_m)) \tag{3.9}$$

is a polynomial of degree at most m, since it is well-known that the Bernstein operators map polynomials into polynomials of the same degree.

In particular (see [19, Section 25]), the following result holds true.

Proposition 3.1. *For every $n \geq 1$,*

$$M_{n,a,b}(e_1) = \frac{a+1+ne_1}{n+a+b+2}, \tag{3.10}$$

and

$$M_{n,a,b}(e_2) = \frac{(a+1)(a+2)+n(2a+3)e_1+n(n-1)e_2}{(n+a+b+2)(n+a+b+3)}. \tag{3.11}$$

Moreover, for every $\varphi \in C([0,1])$,

$$\lim_{n\to\infty} M_{n,a,b}(\varphi) = \varphi \quad \textit{uniformly on } [0,1]. \tag{3.12}$$

Finally, an asymptotic formula holds for the operators $M_{n,a,b}$, $n \geq 1$. More precisely (see [19, formula (25-13)]), for every $u \in C^2([0,1])$,

$$\lim_{n\to\infty} n(M_{n,a,b}(u)(x) - u(x)) = (a+1-(a+b+2)x)u'(x) + x(1-x)u''(x) \tag{3.13}$$

uniformly w.r.t. $x \in [0,1]$.

Coming back to the general case $d \geq 1$ and to operators (3.2), we remark that, if $f = \prod_{i=1}^{d} f_i \circ pr_i$, $f_i \in C([0,1])$ for every $i = 1, \dots, d$, then (see (3.3) and (3.4))

$$M_n(f) = \prod_{i=1}^{d} M_{n,a_i,b_i}(f_i) \circ pr_i; \tag{3.14}$$

here, for every $i = 1, \dots, d$, pr_i stands for the i^{th} coordinate function on Q_d, i.e., $pr_i(x) := x_i$ for every $x = (x_1, \dots, x_d) \in Q_d$.

In particular, if $m_1, \dots, m_d \in \mathbf{N}$,

$$M_n\left(\prod_{i=1}^{d} pr_i^{m_i}\right) = M_n\left(\prod_{i=1}^{d} e_{m_i} \circ pr_i\right) = \prod_{i=1}^{d} M_{n,a_i,b_i}(e_{m_i}) \circ pr_i. \tag{3.15}$$

According to (3.9), $M_{n,a_i,b_i}(e_{m_i})$ is a polynomial in $[0,1]$ of degree $\leq m_i$ and, hence, $M_n\left(\prod_{i=1}^{d} pr_i^{m_i}\right)$ is a polynomial in Q_d of degree $\leq \sum_{i=1}^{d} m_i$. Thus,

$$M_n(\mathbb{P}_m) \subset \mathbb{P}_m \tag{3.16}$$

for every $n, m \geq 1$.

Next we discuss some approximation properties of the operators M_n in the spaces $C(Q_d)$ and $L^p(Q_d, \mu_{a,b})$, $1 \leq p < +\infty$. To this respect it is useful to point out that the measure $\mu_{a,b} \in M_1^+(Q_d)$ defined by (3.1) is an invariant measure for the operators M_n, $n \geq 1$, on $L^1(Q_d, \mu_{a,b})$ and, in particular, for their restrictions to $C(Q_d)$.

This can be easily verified because, for every $n \geq 1$ and $f \in L^1(Q_d, \mu_{a,b})$,

$$\int_{Q_d} M_n(f)(x)\, d\mu_{a,b}(x) = \frac{1}{\int_{Q_d} y^a(1-y)^b\, dy} \sum_{\substack{h \in \mathbf{N}^d \\ 0_d \leq h \leq n_d}} \omega_{n_d,h}(f) \binom{n_d}{h} \int_{Q_d} x^{h+a}(1_d - x)^{n_d - h + b}\, dx$$

$$= \frac{1}{\int_{Q_d} y^a(1-y)^b\, dy} \sum_{\substack{h \in \mathbf{N}^d \\ 0_d \leq h \leq n_d}} \binom{n_d}{h} \int_{Q_d} y^{h+a}(1_d - y)^{n_d - h + b} f(y)\, dy$$

$$= \frac{1}{\int_{Q_d} y^a(1-y)^b\, dy} \int_{Q_d} \left[\sum_{\substack{h \in \mathbf{N}^d \\ 0_d \leq h \leq n_d}} \binom{n_d}{h} y^h (1-y)^{n_d - h} \right] y^a(1-y)^b f(y)\, dy$$

$$= \int_{Q_d} f(y)\, d\mu_{a,b}(y).$$

We also remark that each M_n is a contraction from $L^p(Q_d, \mu_{a,b})$ into $L^p(Q_d, \mu_{a,b})$. By using the convexity of the function $|t|^p (t \in \mathbf{R})$ and the integral Jensen inequality, we get indeed that, if $f \in L^p(Q_d, \mu_{a,b})$, then $|M_n(f)|^p \leq M_n(|f|^p)$ and hence

$$\int_{Q_d} |M_n(f)|^p \, d\mu_{a,b} \leq \int_{Q_d} M_n(|f|^p) \, d\mu_{a,b} = \int_{Q_d} |f|^p \, d\mu_{a,b}. \tag{3.17}$$

From this remark in particular it follows that each restriction $M_n|_{L^p(Q_d,\mu_{a,b})}$ coincides with the extension of $M_n|_{C(Q_d)}$ to $L^p(Q_d, \mu_{a,b})$ as discussed in Section 2.

Theorem 3.2. *The following statements hold true:*

(a) For every $f \in C(Q_d)$, $\lim\limits_{n\to\infty} M_n(f) = f$ uniformly on Q_d.

(b) If $f \in L^p(Q_d, \mu_{a,b})$, $1 \leq p < +\infty$, then $\lim\limits_{n\to\infty} M_n(f) = f$ in $L^p(Q_d, \mu_{a,b})$.

(c) If $f : Q_d \to \mathbf{R}$ is Borel-measurable and bounded, then $\lim\limits_{n\to\infty} M_n(f)(x) = f(x)$ for every continuity point $x \in Q_d$ for f.

Proof. In order to prove statement (a) we shall use the Korovkin type theorem due to Volkov (see, e.g., [4, (4.4.22), p. 245]), from which it follows that

$$\left\{\mathbf{1}, pr_1, \ldots, pr_d, \sum_{i=1}^{d} pr_i^2\right\}$$

is a Korovkin set in $C(Q_d)$. Therefore, it is enough to verify the approximation formula only for these $d+2$ functions.

Obviously, $M_n(\mathbf{1}) = \mathbf{1}$ for every $n \geq 1$. Taking (3.10) and (3.15) into account, for every $x = (x_1, \ldots, x_d) \in Q_d$, $n \geq 1$ and $i = 1, \ldots, d$, we get

$$M_n(pr_i)(x) = M_{n,a_i,b_i}(e_1)(x_i) = \frac{a_i + 1 + nx_i}{n + a_i + b_i + 2} \to x_i \quad \text{uniformly in } Q_d.$$

Analogously,

$$M_n(pr_i^2)(x) = M_{n,a_i,b_i}(e_2)(x_i) = \frac{(a_i+1)(a_i+2) + n(2a_i+3)x_i + n(n-1)x_i^2}{(n + a_i + b_i + 2)(n + a_i + b_i + 3)}$$

so that $\lim_{n\to\infty} M_n(pr_i^2) = pr_i^2$ uniformly in Q_d, and this completes the proof of (a).

As regards statement (b), since $C(Q_d)$ is dense in $L^p(Q_d, \mu_{a,b})$ (see, e.g., [10, Lemma 26.2 and Theorem 29.14]) and since, on account of part (a), $\lim_{n\to\infty} M_n(f) = f$ in $L^p(Q_d, \mu_{a,b})$ for every $f \in C(Q_d)$, it is enough to show that the sequence $(M_n)_{n\geq 1}$ is equibounded from $L^p(Q_d, \mu_{a,b})$ into $L^p(Q_d, \mu_{a,b})$. This, indeed, is a consequence of (3.17).

Finally, statement (c) is a direct consequence of the previous formulas and [3, Theorem 3.3 and formula (4.3)]. $\square$

Remark 3.3. As already pointed out in the previous proof, $C(Q_d)$ is dense in $L^p(Q_d, \mu_{a,b})$ and, in addition, $\|\cdot\|_p \leq \|\cdot\|_\infty$ on $C(Q_d)$. Therefore, taking the Weierstrass-Stone theorem into account, the subalgebra of all (restrictions of) polynomials on Q_d is dense in $L^p(Q_d, \mu_{a,b})$ for the norm $\|\cdot\|_p$. Theorem 3.2, part (b), furnishes indeed a constructive method showing how each function $f \in L^p(Q_d, \mu_{a,b})$ can be approximated by a sequence of polynomials with respect to $\|\cdot\|_p$.

Now we present some shape preserving properties of the operators M_n.

First of all, we prove that they preserve the Lipschitz-continuity. To this end it is useful to evaluate the partial derivatives of $M_n(f)$ ($f \in C(Q_d)$). We point out, indeed, that, for every $n \geq 1$ and for every family $(\alpha_k)_{0\leq k\leq n} \in \mathbf{R}^{n+1}$ and $x \in [0,1]$, one has

$$\frac{d}{dx}\sum_{k=0}^{n}\binom{n}{k}\alpha_k x^k(1-x)^{n-k} = n\sum_{k=0}^{n-1}\binom{n-1}{k}(\alpha_{k+1}-\alpha_k)\,x^k(1-x)^{n-k-1}. \tag{3.18}$$

Hence, for every $f \in C(Q_d)$, $x = (x_1, \ldots, x_d) \in Q_d$ and $i = 1, \ldots, d$,

$$\frac{\partial M_n(f)}{\partial x_i}(x) = \sum_{h_1=0}^{n}\sum_{h_2=0}^{n}\cdots\sum_{h_{i-1}=0}^{n}\sum_{h_{i+1}=0}^{n}\cdots\sum_{h_d=0}^{n}\prod_{\substack{j=1\\ j\neq i}}^{d}\binom{n}{h_j}x_j^{h_j}(1-x_j)^{n-h_j}$$
$$\times\left\{n\sum_{h_i=0}^{n-1}(\omega_{n_d,h+v_i}(f)-\omega_{n_d,h}(f))\binom{n-1}{h_i}x_i^{h_i}(1-x_i)^{n-h_i-1}\right\}, \tag{3.19}$$

where, for every $i = 1, \ldots d$, the vector v_i is given by (2.4).

The next result shows the behaviour of the operators M_n on the Lipschitz-continuous functions (see (2.5) and (2.6)).

Theorem 3.4. *$M_n(f) \in Lip(Q_d)$ for every $n \geq 1$ and $f \in Lip(Q_d)$; moreover*

$$|M_n(f)|_{Lip} \leq \max_{1\leq i\leq d}\frac{n}{n+a_i+b_i+2}|f|_{Lip} \leq \left(1+\frac{\omega}{n}\right)|f|_{Lip} \leq |f|_{Lip}, \tag{3.20}$$

where

$$\omega := -\min_{1\leq i\leq d}\frac{a_i+b_i+2}{a_i+b_i+3} < 0. \tag{3.21}$$

In particular,

$$M_n(Lip(M,1)) \subset Lip(N_n,1) \subset Lip(M,1), \tag{3.22}$$

where $N_n := M\max_{1\leq i\leq d}\dfrac{n}{n+a_i+b_i+2} \leq M\left(1+\dfrac{\omega}{n}\right)$.

Proof. We shall apply the mean value theorem and, to this end, we prove that

$$\sup_{x\in Q_d}\left(\max_{1\le i\le d}\left|\frac{\partial M_n(f)}{\partial x_i}(x)\right|\right)\le \max_{1\le i\le d}\frac{n}{n+a_i+b_i+2}|f|_{Lip}.$$

Let $x\in Q_d$. Taking (3.19) into account, for a given $i=1,\dots,d$ and $f\in Lip(Q_d)$, fix $n\ge 1$, and set $h=(h_1,\dots,h_d)$, with $h_1,\dots,h_{i-1},h_{i+1},\dots,h_d=0,\dots,n$ and $h_i=0,\dots,n-1$. Then

$$\begin{aligned}
&\omega_{n_d,h+v_i}(f)-\omega_{n_d,h}(f)\\
&=\prod_{\substack{j=1\\ j\ne i}}^{d}\frac{\Gamma(n+a_j+b_j+2)}{\Gamma(h_j+a_j+1)\Gamma(n-h_j+b_j+1)}\int_{Q_{d-1}}dy_1\dots dy_{i-1}dy_{i+1}\dots dy_d\\
&\times\prod_{\substack{j=1\\ j\ne i}}^{d}y_j^{h_j+a_j}(1-y_j)^{n-h_j+b_j}\Bigg(\frac{\Gamma(n+a_i+b_i+2)}{\Gamma(h_i+a_i+2)\Gamma(n-h_i+b_i+1)}\\
&\times\int_0^1\Big((n-h_i+b_i)y_i^{h_i+a_i+1}(1-y_i)^{n-h_i+b_i-1}-(h_i+a_i+1)y_i^{h_i+a_i}(1-y_i)^{n-h_i+b_i}\Big)\\
&f(y_1,\dots,y_i,\dots,y_d)\,dy_i\Bigg).
\end{aligned}$$

If we fix $(y_1\dots y_{i-1},y_{i+1}\dots y_d)\in Q_{d-1}$ and we consider the function

$$\varphi(s)=f(y_1,\dots,y_{i-1},s,y_{i+1},\dots,y_d)\qquad 0\le s\le 1,$$

we have that $\varphi\in Lip([0,1])$ and $|\varphi|_{Lip}\le|f|_{Lip}$.

Accordingly, taking (3.5) into account,

$$\begin{aligned}
&\Bigg|\frac{\Gamma(n+a_i+b_i+2)}{\Gamma(h_i+a_i+2)\Gamma(n-h_i+b_i+1)}\\
&\times\int_0^1\Big((n-h_i+b_i)y_i^{h_i+a_i+1}(1-y_i)^{n-h_i+b_i-1}-(h_i+a_i+1)y_i^{h_i+a_i}(1-y_i)^{n-h_i+b_i}\Big)\\
&f(y_1,\dots,y_i,\dots,y_d)\,dy_i\Bigg|=|\omega_{n,h_i+1}(\varphi)-\omega_{n,h_i}(\varphi)|.
\end{aligned}$$

By means of [8, formula (3.15)],

$$|\omega_{n,h_i+1}(\varphi)-\omega_{n,h_i}(\varphi)|\le\frac{1}{n+a_i+b_i+2}|\varphi|_{Lip}\le\frac{1}{n+a_i+b_i+2}|f|_{Lip}.$$

From this and (3.19), we get that, for every $i=1,\dots,d$,

$$\left|\frac{\partial M_n(f)}{\partial x_i}(x)\right|\le\frac{n}{n+a_i+b_i+2}|f|_{Lip}\le\left(1-\frac{a_i+b_i+2}{n(a_i+b_i+3)}\right)|f|_{Lip}$$

and this completes the proof of (3.20).

Finally, (3.22) follows from the previous formula and from the fact that, if $f\in Lip(M,1)$, then $|f|_{Lip}\le M$.

□

Thanks to Theorem 3.4, it is possible to obtain some further information about the preservation of the Hölder continuity by the operators M_n.

To this end, consider the usual modulus of continuity $\Omega(f, \delta)$, defined, for every bounded function $f : Q_d \to \mathbf{R}$ and $\delta > 0$, by

$$\Omega(f,\delta) := \sup\{|f(x) - f(y)| \ \mid x, y \in Q_d,\ \|x-y\|_1 \le \delta\}. \tag{3.23}$$

The next result is a direct consequence of Theorem 3.4 and [4, Corollary 6.1.20].

Corollary 3.5. *If $f \in C(Q_d)$, then, for every $n \ge 1$ and $\delta > 0$,*

$$\Omega(M_n(f),\delta) \le \left(2 + \frac{\omega}{n}\right) \Omega(f,\delta), \tag{3.24}$$

where ω is defined in (3.21)*.*

Moreover, if $f \in Lip(M, \alpha)$ for some $M > 0$ and $0 < \alpha \le 1$ (cf. (2.7)*), then, for every $n \ge 1$,*

$$M_n(f) \in Lip\left(M\left(1+\frac{\omega}{n}\right)^{\alpha}, \alpha\right) \subset Lip(M,\alpha). \tag{3.25}$$

We proceed to investigate whether the operators M_n preserve convexity. First of all we consider the case $d = 1$ and, thus, we shall refer to operators (3.4).

From (3.18) it follows that

$$\frac{d^2}{dx^2}\sum_{k=0}^{n}\binom{n}{k}\alpha_k x^k(1-x)^{n-k} = n(n-1)\sum_{k=0}^{n-2}\binom{n-2}{k}(\alpha_{k+2}-2\alpha_{k+1}+\alpha_k)\, x^k(1-x)^{n-k-2} \tag{3.26}$$

($x \in [0, 1]$), and hence, for every $\varphi \in C([0, 1])$, $n \ge 1$ and $x \in [0, 1]$,

$$\frac{d^2}{dx^2}M_{n,a,b}(\varphi)(x) = n(n-1)\sum_{h=0}^{n-2}(\omega_{n,h+2}(\varphi) - 2\omega_{n,h+1}(\varphi) + \omega_{n,h}(\varphi))\, x^k(1-x)^{n-h-2}. \tag{3.27}$$

From (3.27) we infer that the operators $M_{n,a,b}$ preserve convexity, as stated in the following result.

Proposition 3.6. *If $\varphi \in C([0, 1])$ is convex, then $M_{n,a,b}(\varphi)$ is convex for every $n \ge 1$.*

Proof. Consider a convex function $\varphi \in C([0, 1])$. From (3.27), it follows that the statement will be proved once we show that, for every $h = 0, \dots, n-2$,

$$\omega_{n,h+2}(\varphi) - 2\omega_{n,h+1}(\varphi) + \omega_{n,h}(\varphi) \ge 0. \tag{3.28}$$

To this end, fix $h = 0, \dots, n-2$; then

$$\omega_{n,h+2}(\varphi) - 2\omega_{n,h+1}(\varphi) + \omega_{n,h}(\varphi) = \frac{\Gamma(n+a+b+2)}{\Gamma(h+a+3)\Gamma(n-h+b+1)}$$
$$\times \int_0^1 \Big((n-h+b)(n-h+b-1)t^{h+a+2}(1-t)^{n-h+b-2}$$
$$-2(h+a+2)(n-h+b)t^{h+a+1}(1-t)^{n-h+b-1}$$
$$+(h+a+2)(h+a+1)t^{h+a}(1-t)^{n-h+b}\Big)\,\varphi(t)\,dt\,.$$

Set

$$F(x) = x^{h+a+2}(1-x)^{n-h+b} \quad (0 \le x \le 1).$$

It is easy to prove that

$$\omega_{n,h+2}(\varphi) - 2\omega_{n,h+1}(\varphi) + \omega_{n,h}(\varphi) = \frac{\Gamma(n+a+b+2)}{\Gamma(h+a+3)\Gamma(n-h+b+1)} \int_0^1 F''(t)\varphi(t)\,dt\,.$$

Additionally, assume that $\varphi \in C^2([0,1])$; then, integrating by parts,

$$\int_0^1 F''(t)\varphi(t)\,dt = \int_0^1 F(t)\varphi''(t)\,dt \ge 0,$$

since φ is a convex function; this completes the proof of (3.28) under the additional hypothesis that $\varphi \in C^2([0,1])$.

On the other hand, if $\varphi \in C([0,1])$ is convex, then there exists a sequence $(\varphi_m)_{m\ge1}$, $\varphi_m \in C^2([0,1])$ and convex for every $m \ge 1$, such that $\lim_{m\to\infty}\varphi_m = \varphi$ uniformly on $[0,1]$; take, for example, for every $m \ge 1$, $\varphi_m = B_m(\varphi)$, B_m being the classical Bernstein polynomial operator of order m on $[0,1]$ ([4, Corollary 6.3.8]). Therefore,

$$\omega_{n,h+2}(\varphi) - 2\omega_{n,h+1}(\varphi) + \omega_{n,h}(\varphi) = \int_0^1 F''(t)\varphi(t)\,dt = \lim_{m\to+\infty}\int_0^1 F''(t)\varphi_m(t)\,dt \ge 0.$$

The proof is now complete. $\square$

If $d > 1$ it is no longer true that, if $f \in C(Q_d)$ is convex, then $M_n(f)$ is convex for every $n \ge 1$.

As a simple counterexample, it is enough to consider the function $f(x,y) = (x+y)^2$ $((x,y) \in Q_2)$. Then, for every $(x,y) \in Q_2$ and $n \ge 1$,

$$M_n(f)(x,y) = \frac{(a_1+1)(a_1+1) + n(2a_1+3)x + n(n-1)x^2}{(n+a_1+b_1+2)(n+a_1+b_1+3)}$$
$$+ 2\frac{(a_1+1+nx)(a_2+1+ny)}{(n+a_1+b_1+2)(n+a_2+b_2+2)} + \frac{(a_2+1)(a_2+1) + n(2a_2+3)y + n(n-1)y^2}{(n+a_2+b_2+2)(n+a_2+b_2+3)}$$

(see (3.15)), whose Hessian is not positive semi-definite.

Nonetheless, other weaker types of convexity are preserved under the M_n's, as the following result shows.

Theorem 3.7. *Let $f \in C(Q_d)$ be convex with respect to each variable. Then, for every $n \geq 1$, $M_n(f)$ is convex with respect to each variable.*

Proof. Fix $n \geq 1$ and $i = 1, \dots, d$. Then, for every $n \geq 1$ and $x = (x_1, \dots, x_d) \in Q_d$,

$$\frac{\partial^2 M_n(f)}{\partial x_i^2}(x) = n(n-1) \sum_{h_1=0}^{n} \sum_{h_2=0}^{n} \cdots \sum_{h_{i-1}=0}^{n} \sum_{h_{i+1}=0}^{n} \cdots \sum_{h_d=0}^{n} \prod_{\substack{j=1 \\ j \neq i}}^{d} \binom{n}{h_j} x_j^{h_j} (1-x_j)^{n-h_j}$$
$$\times \left\{ \sum_{h_i=0}^{n-2} (\omega_{n_d, h+2v_i}(f) - 2\omega_{n_d, h+v_i}(f) + \omega_{n_d, h}(f)) \binom{n-2}{h_i} x_i^{h_i} (1-x_i)^{n-h_i-2} \right\}. \tag{3.29}$$

The proof will be completed once we show that, for every $x \in Q_d$, $\frac{\partial^2 M_n(f)}{\partial x_i^2}(x) \geq 0$ or, equivalently, that, for every $h = (h_1, \dots, h_d)$, $0 \leq h_1, \dots, h_{i-1}, h_{i+1}, \dots, h_d \leq n$ and $h_i = 0, \dots, n-2$,

$$\omega_{n_d, h+2v_i}(f) - 2\omega_{n_d, h+v_i}(f) + \omega_{n_d, h}(f) \geq 0.$$

In fact, fix $h = (h_1, \dots, h_d)$, with $0 \leq h_1, \dots, h_{i-1}, h_{i+1}, \dots, h_d \leq n$ and $h_i = 0, \dots, n-2$. Then

$$\begin{aligned}
&\omega_{n_d, h+2v_i}(f) - 2\omega_{n_d, h+v_i}(f) + \omega_{n_d, h}(f) \\
&= \prod_{\substack{j=1 \\ j \neq i}}^{d} \frac{\Gamma(n + a_j + b_j + 2)}{\Gamma(h_j + a_j + 1)\Gamma(n - h_j + b_j + 1)} \int_{Q_{d-1}} dy_1 \dots dy_{i-1} dy_{i+1} \dots dy_d \\
&\times \prod_{\substack{j=1 \\ j \neq i}}^{d} y_j^{h_j + a_j} (1-y_j)^{n-h_j+b_j} \left(\omega_{n, h_i+2}(\varphi) - 2\omega_{n, h_i+1} + \omega_{n, h_i}(\varphi) \right),
\end{aligned}$$

where

$$\varphi(s) = f(y_1, \dots, y_{i-1}, s, y_{i+1}, \dots, y_d) \qquad 0 \leq s \leq 1,$$

with $(y_1, \dots y_{i-1}, y_{i+1}, \dots y_d) \in Q_{d-1}$ being fixed.

Since f is convex with respect to each variable, φ is convex in $[0,1]$ too and this, together with formula (3.28) in Proposition 3.6, completes the proof. □

4 The positive semigroups generated by Fleming-Viot type differential operators on the hypercube

After the necessary preliminaries of the previous sections, we finally proceed to look more closely at the degenerate second-order elliptic differential operator defined by

$$A(u)(x) = \sum_{i=1}^{d} x_i(1-x_i)\frac{\partial^2 u}{\partial x_i^2}(x) + (a_i + 1 - (a_i + b_i + 2)x_i)\frac{\partial u}{\partial x_i}(x) \quad (4.1)$$

for every $u \in C^2(Q_d)$ and $x = (x_1, \dots, x_d) \in Q_d$, where $a_1, \dots, a_d, b_1, \dots, b_d \in \mathbf{R}$, $a_i > -1$ and $b_i > -1$ for all $i = 1, \dots, d$.

Operators similar to (4.1) have been already studied in several papers (see, e.g., [5, Section 5.8], [6], [8], [13], [16] and the references therein).

The special case where $b_i = a_{i+1}$, with $a_1, \dots, a_{d+1} \in \mathbf{R}$ and $a_i > -1$ for all $i = 1, \dots, d+1$, has been investigated in [16].

The difficulties in studying operators (4.1) lie in the fact that they degenerate on the boundary of Q_d, which is not smooth because of the presence of sides and corners.

In this section, we will show that operator (4.1) is the pregenerator of a Markov semigroup on $C(Q_d)$ and of a positive contraction semigroup in $L^p(Q_d, \mu_{a,b})$; moreover, both these semigroups are obtained as a limits of suitable iterates of the operators M_n we studied in Section 3.

First of all we prove that operator (4.1) is related to operators M_n through an asymptotic formula.

For a given $x \in Q_d$, we denote by $\Psi_x \in C(Q_d)$ the function defined by

$$\Psi_x(y) := y - x \qquad (y \in Q_d), \quad (4.2)$$

and by $d_x \in C(Q_d)$ the function defined by

$$d_x(y) := \|y - x\|_2 \qquad (y \in Q_d), \quad (4.3)$$

where $\|\cdot\|_2$ is the Euclidian norm in $\mathbf{R}^d$.

Since, for every $y = (y_1, \dots, y_d) \in Q_d$, and $i = 1, \dots, d$

$$(pr_i \circ \Psi_x)(y) = pr_i(y - x) = y_i - x_i = pr_i(y) - x_i,$$

we have that

$$d_x^2 = \sum_{i=1}^{d} (pr_i \circ \Psi_x)^2 \quad (4.4)$$

and

$$d_x^4 = \sum_{i,j=1}^{d} (pr_i \circ \Psi_x)^2 (pr_j \circ \Psi_x)^2. \quad (4.5)$$

Theorem 4.1. *For every $u \in C^2(Q_d)$,*

$$\lim_{n\to\infty} n(M_n(u) - u) = A(u) \qquad \textit{uniformly on } Q_d. \quad (4.6)$$

Therefore, considering the measure $\mu_{a,b} \in M_1^+(Q_d)$ having as density the function defined by (3.1)*, then*

$$\int_{Q_d} A(u)\, d\mu_{a,b} = 0, \tag{4.7}$$

i.e., $\mu_{a,b}$ is an infinitesimally invariant measure for the operator A.

Proof. According to [5, Theorem 1.5.2], in order to prove (4.6), we have to show that, for every $i, j = 1, \ldots, d$, the following conditions hold true:

(a) $\lim_{n\to\infty} nM_n(pr_i \circ \Psi_x)(x) - \beta_i(x) = 0$ uniformly w.r.t. $x \in Q_d$,

(b) $\lim_{n\to\infty} nM_n((pr_i \circ \Psi_x)(pr_j \circ \Psi_x))(x) - 2\alpha_{ij}(x) = 0$ uniformly w.r.t. $x \in Q_d$,

(c) $\sup_{n\geq 1, x\in Q_d} nM_n(d_x^2)(x) < +\infty$,

and

(d) $\lim_{n\to\infty} nM_n(d_x^4)(x) = 0$ uniformly w.r.t. $x \in Q_d$,

where, for a fixed $x = (x_1, \ldots, x_d) \in Q_d$, d_x and Ψ_x are given by (4.3) and (4.2), respectively, and, for every $i, j = 1, \ldots, d$,

$$\beta_i(x) = a_i + 1 - (a_i + b_i + 2)x_i$$

and

$$\alpha_{ij}(x) = \begin{cases} 0 & \text{if } i \neq j, \\ x_i(1 - x_i) & \text{if } i = j. \end{cases}$$

We start by verifying condition (a). For any $i = 1, \ldots, d$ and $x = (x_1, \ldots, x_d) \in Q_d$, according to (3.10),

$$\begin{aligned} M_n(pr_i \circ \Psi_x)(x) &= M_n(e_1 \circ pr_i - x_i\mathbf{1})(x) \\ &= M_{n,a_i,b_i}(e_1)(x_i) - x_i = \frac{(a_i + 1) - (a_i + b_i + 2)x_i}{n + a_i + b_i + 2}, \end{aligned}$$

so that we get the required assertion.

To prove statement (b) we preliminary notice that, according to (3.15), for every $i, j = 1, \ldots, d$, $i \neq j$, and for every $x = (x_1, \ldots, x_d) \in Q_d$,

$$\begin{aligned} &M_n((pr_i \circ \Psi_x)(pr_j \circ \Psi_x))(x) = M_n((e_1 \circ pr_i - x_i\mathbf{1})(e_1 \circ pr_j - x_j\mathbf{1}))(x) \\ &= M_{n,a_i,b_i}(e_1 - x_i\mathbf{1})(x_i) M_{n,a_j,b_j}(e_1 - x_j\mathbf{1})(x_j); \end{aligned}$$

hence, taking (3.10) into account,

$$\lim_{n\to\infty} nM_n((pr_i \circ \Psi_x)(pr_j \circ \Psi_x))(x) = 0$$

uniformly w.r.t. $x \in Q_d$.

Let us now assume that $i = j$ and let us set $u := (e_1 - x_i\mathbf{1})^2$. Then

$$\begin{aligned}
&\lim_{n\to\infty} nM_n((pr_i \circ \Psi_x)^2)(x) = \lim_{n\to\infty} n(M_{n,a_i,b_i}(u)(x_i) - u(x_i)) \\
&= x_i(1 - x_i)u''(x_i) + (a_i + 1 - (a_i + b_i + 2)x_i)u'(x_i) = 2x_i(1 - x_i)
\end{aligned}$$

uniformly w.r.t. $x = (x_1, \dots, x_d) \in Q_d$ (see (3.13)).

Condition (c) easily follows from the previous calculations and from (4.4).

Finally, for every $x = (x_1, \dots, x_d) \in Q_d$, from (4.5) we get

$$M_n(d_x^4)(x) = \sum_{i,j=1}^{d} M_n((pr_i \circ \Psi_x)^2(pr_j \circ \Psi_x)^2)(x).$$

Let $i, j = 1, \dots, d$, $i \neq j$. Then, if we set $u = (e_1 - x_i\mathbf{1})^2$ and $v = (e_1 - x_j\mathbf{1})^2$, we have

$$\begin{aligned}
&nM_n((pr_i \circ \Psi_x)^2(pr_j \circ \Psi_x)^2)(x) = nM_{n,a_i,b_i}((e_1 - x_i\mathbf{1})^2)(x_i)M_{n,a_j,b_j}((e_1 - x_j\mathbf{1})^2)(x_j) \\
&= n(M_{n,a_i,b_i}(u)(x_i) - u(x_i))M_n(v)(x_j);
\end{aligned}$$

hence, taking (3.12) and (3.13) into account,

$$\lim_{n\to\infty} nM_n((pr_i \circ \Psi_x)^2(pr_j \circ \Psi_x)^2)(x) = 0$$

uniformly w.r.t. $x = (x_1, \dots, x_d) \in Q_d$.

On the other hand, if $i = j$ and $w := (e_1 - x_i\mathbf{1})^4$, then

$$\begin{aligned}
&\lim_{n\to\infty} nM_n((pr_i \circ \Psi_x)^4)(x) = \lim_{n\to\infty} n(M_{n,a_i,b_i}(u)(x_i) - u(x_i)) \\
&= x_i(1 - x_i)u''(x_i) + (a_i + 1 - (a_i + b_i + 2)x_i)u'(x_i) = 0
\end{aligned}$$

uniformly w.r.t. $x = (x_1, \dots, x_d) \in Q_d$, and this completes the proof of (4.6).

Finally, formula (4.7) is a consequence of the invariance of the M_n's under the measure $\mu_{a,b}$. $\square$

The next result shows that the operator $(A, C^2(Q_d))$ pregenerates a Markov semigroup $(T(t))_{t\geq 0}$ on $C(Q_d)$; moreover, a representation formula for such semigroup, involving suitable iterates of the operators M_n, is also provided. By means of such a representation formula, we shall deduce some preservation properties of the semigroup itself and we shall describe its asymptotic behaviour.

For unexplained terminology concerning semigroup theory, we refer, e.g., to [5, Chapter 2].

Theorem 4.2. *The differential operator $(A, C^2(Q_d))$ defined by (4.1) is closable and its closure $(B, D(B))$ generates a Markov semigroup $(T(t))_{t\geq 0}$ on $C(Q_d)$ such that, if $f \in C(Q_d)$, $t \geq 0$ and $(k(n))_{n\geq 1}$ is a sequence of positive integers satisfying $\lim_{n\to\infty} k(n)/n = t$, then*

$$T(t)(f) = \lim_{n\to\infty} M_n^{k(n)}(f) \quad \textit{uniformly on } Q_d, \tag{4.8}$$

where each $M_n^{k(n)}$ denotes the iterate of M_n of order $k(n)$.

Moreover, $\mathbb{P}_\infty := \bigcup_{m=1}^{\infty} \mathbb{P}_m$, and hence $C^2(Q_d)$, is a core for $(B, D(B))$ and $T(t)(\mathbb{P}_m) \subset \mathbb{P}_m$ for every $t \geq 0$ and $m \geq 1$.

Considering the measure $\mu_{a,b} \in M_1^+(Q_d)$ with density the function $w_{a,b}(x)$ $(x \in Q_d)$ defined by (3.1), then, for every $f \in C(Q_d)$ and $n \geq 1$,

$$\lim_{t\to+\infty} T(t)(f) = \lim_{m\to\infty} M_n^m(f) = \int_{Q_d} f d\mu_{a,b} \tag{4.9}$$

uniformly on Q_d, and the measure $\mu_{a,b}$ is the unique invariant measure on Q_d for both the sequence $M_{n\geq 1}$ and the semigroup $(T(t))_{t\geq 0}$.

Finally, if $f \in Lip(Q_d)$, then, for every $n, m \geq 1$ and $t \geq 0$,

$$\left\|M_n^m(f) - \int_{Q_d} f d\mu_{a,b}\right\|_\infty \leq 2\left(1 + \frac{\omega}{n}\right)^m |f|_{Lip} \tag{4.10}$$

and

$$\left\|T(t)(f) - \int_{Q_d} f d\mu_{a,b}\right\|_\infty \leq 2\exp(\omega t)|f|_{Lip}, \tag{4.11}$$

where ω is defined by (3.21).

Proof. First of all we remark that each subspace $\mathbb{P}_m$, $m \geq 1$, of $C^2(Q_d)$ is finite dimensional, it is invariant under the M_n's $(n \geq 1)$ by virtue of (3.16), and $\mathbb{P}_\infty$ is dense in $C(Q_d)$.

Moreover, from Theorem 4.1, we get

$$\lim_{n\to\infty} n(M_n(u) - u) = A(u) \quad \text{uniformly on } Q_d,$$

for every $u \in C^2(Q_d)$, and hence for every $u \in \mathbb{P}_\infty$.

From [5, Corollary 2.2.11] it follows that $(A, C^2(Q_d))$ is closable and its closure $(B, D(B))$ is the generator of a contraction C_0-semigroup $(T(t))_{t\geq 0}$ on $C(Q_d)$ such that, for every $t \geq 0$ and $f \in C(Q_d)$,

$$T(t)(f) = \lim_{n\to\infty} M_n^{k(n)}(f) \quad \text{uniformly on } Q_d$$

for every sequence $(k(n))_{n\geq 1}$ of positive integers such that $\lim_{n\to\infty} k(n)/n = t$. Moreover, $\mathbb{P}_\infty$ is a core for $(B, D(B))$.

Formula (4.8) implies that each $T(t)$ $(t \geq 0)$ is a Markov operator.

From (3.16) it also follows that, if $f \in \mathbb{P}_m$ for some $m \geq 1$, then $M_n^k \in \mathbb{P}_m$ for every $n, k \geq 1$; hence, for every $t \geq 0$ and every sequence $(k(n))_{n\geq 1}$ of positive integers such that $\lim_{n\to\infty} k(n)/n = t$, we get

$$T(t)(f) = \lim_{n\to\infty} M_n^{k(n)}(f) \in \mathbb{P}_m,$$

since $\mathbb{P}_m$ is closed.

On account of Corollary 2.5 of [8] (see, also, [5, Corollary 1.4.6]), in order to prove (4.9)-(4.11), it is enough to show the following two conditions:

(i) for any $n \geq 1$, $\mu_{a,b}$ is an invariant measure for M_n;

(ii) there exist $\omega < 0$ such that, for every $n \geq 1$ and $f \in Lip(Q_d)$, $M_n(f) \in Lip(Q_d)$, and $|M_n(f)|_{Lip} \leq \left(1 + \dfrac{\omega}{n}\right) |f|_{Lip}$.

This follows at once from what we observed in Section 3 and from Theorem 3.4.

Finally, since the measure $\mu_{a,b}$ is invariant for the restrictions of the operators M_n, $n \geq 1$, to $C(Q_d)$, clearly it is so for each iterate of M_n of order m, $m, n \geq 1$, and for the semigroup $T(t)$ $(t \geq 0)$, thanks to the approximation formula (4.8). On the other hand, if $\nu \in M_1^+(Q_d)$ is a further invariant measure for $T(t)$ $(t \geq 0)$, then, passing to the limit under the integral sign with respect to ν in (4.8), we get that

$$\int_X f\, d\nu = \int_X f\, d\mu_{a,b}$$

for every $f \in C(Q_d)$ and hence $\nu = \mu_{a,b}$. The same reasoning applies if ν is invariant for M_n, $n \geq 1$, because, in this case, ν is also invaniant of all the iterates of the M_n's. □

Remarks 4.3.

1. In [5, Theorem 5.6.3 and Theorem 5.8.4] the pregeneration property of differential operators more general than (4.1) has been also studied with approximation theory methods. In particular, in that monograph, the authors proved an approximation formula for $(T(t))_{t\geq 0}$, similar to (4.8), involving another sequence of positive linear operators on $C(Q_d)$, referred to as the modified Bernstein-Schnabl operators. Anyway, such operators cannot be defined in weighted L^p-spaces and that wouldn't allow us to investigate the extension of the semigroup $(T(t))_{t\geq 0}$ to spaces of weighted integrable functions, which is one of the aims of this paper. Moreover, the present approach allows to derive several qualitative properties of the semigroup including its asymptotic behaviour.

2. Theorem 4.2 extends similar results obtained in [8, Section 3.1] and [6, Section 4] in the special case where $a_i = b_i = 0$ for every $i = 1, \ldots, d$.

In this case the measure $\mu_{a,b}$ is, indeed, the Borel-Lebesgue measure on Q_d and the operators M_n turn into the Kantorovich operators.

3. According to [5, Remark 2.2.12], if $u, v \in C(Q_d)$ and $\lim_{n\to\infty} n(M_n(u) - u) = v$ uniformly on Q_d, then $u \in D(B)$ and $B(u) = v$.

In particular, if $\lim_{n\to\infty} n(M_n(u) - u) = 0$ uniformly on Q_d, then $u \in D(B)$ and $B(u) = 0$ (a saturation result for the operators M_n, $n \geq 1$).

Consider the abstract Cauchy problem associated with $(B, D(B))$ (see Theorem 4.2)

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) = B(u(\cdot,t))(x) & x \in Q_d, \quad t \geq 0, \\ \\ u(x,0) = u_0(x) & u_0 \in D(B), \quad x \in Q_d. \end{cases} \tag{4.12}$$

Since $(B, D(B))$ generates a Markov semigroup $(T(t))_{t\geq 0}$, (4.12) admits a unique solution $u : Q_d \times [0, +\infty[\longrightarrow \mathbf{R}$ given by $u(x,t) = T(t)(u_0)(x)$ for every $x \in Q_d$ and $t \geq 0$ (see, e.g., [17, Chapter A-II]). Hence, taking (4.8) into account, we may approximate such a solution in terms of iterates of the M_n's, namely

$$u(x,t) = T(t)(u_0)(x) = \lim_{n\to\infty} M_n^{k(n)}(u_0)(x), \tag{4.13}$$

where $(k(n))_{n\geq 1}$ is a sequence of positive integers satisfying $\lim_{n\to\infty} k(n)/n = t$, and the limit is uniform with respect to $x \in Q_d$.

Note that B coincides with A on $C^2(Q_d)$; therefore, if $u_0 \in \mathbb{P}_m$ $(m \geq 1)$ then $u(x,t)$ is the unique solution to the Cauchy problem

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) = \displaystyle\sum_{i=1}^{d} x_i(1-x_i)\frac{\partial^2 u}{\partial x_i^2}(x,t) + (a_i+1-(a_i+b_i+2)x_i)\frac{\partial u}{\partial x_i}(x,t) & x \in Q_d, t \geq 0, \\ \\ u(x,0) = u_0(x) & x \in Q_d, \end{cases}$$

and

$$u(\cdot,t) \in \mathbb{P}_m \quad \text{for every } t \geq 0. \tag{4.14}$$

Moreover, each $u(\cdot,t), t \geq 0$, and u_0 have the same integrals with respect to the measure $\mu_{a,b}$ and, thanks to formula (4.9),

$$\lim_{t\to+\infty} u(x,t) = \int_{Q_d} u_0 \, d\mu_{a,b} \tag{4.15}$$

uniformly w.r.t. $x \in Q_d$.

Next, we enlighten other spatial regularity properties of the solution $u(\cdot,t)$ of (4.12), which, however, we state in terms of the semigroup $(T(t))_{t\geq 0}$.

Theorem 4.4. *The following statements hold true:*

(a) *$T(t)(Lip(Q_d)) \subset Lip(Q_d)$ for every $t \geq 0$; moreover, for every $f \in Lip(Q_d)$ and $t \geq 0$,*

$$|T(t)(f)|_{Lip} \leq \exp(\omega t)|f|_{Lip}; \tag{4.16}$$

in particular, if $f \in Lip(M, 1)$, then, for every $t \geq 0$,

$$T(t)(f) \in Lip(M \exp(\omega t), 1),$$

where ω is defined by(3.21).

(b) *For every $f \in C(Q_d)$, $t \geq 0$, $\delta > 0$,*

$$\Omega(T(t)(f), \delta) \leq (1 + \exp(\omega t))\Omega(f, \delta). \tag{4.17}$$

Moreover, if $M > 0$ and $0 < \alpha \leq 1$,

$$T(t)(Lip(M, \alpha)) \subset Lip(M \exp(\alpha\omega t), \alpha) \subset Lip(M, \alpha) \tag{4.18}$$

(see (2.7)*).*

(c) *If $f \in C(Q_d)$ is convex with respect to each variable, then so is $T(t)(f)$ for every $t \geq 0$. In particular, if $d = 1$ and if $f \in C([0, 1])$ is convex, then $T(t)(f)$ is convex for every $t \geq 0$.*

Proof. To prove statement (a) we first note that, in Theorem 3.4 we established that $M_n(Lip(Q_d)) \subset Lip(Q_d)$ for every $n \geq 1$. Since $Lip(Q_d)$ is closed under the uniform norm, by iterating this inclusion and taking (4.8) into account, we get that $T(t)((Lip(Q_d)) \subset Lip(Q_d)$.

Inequality (4.16) along with the last part of the statement, follow from Theorem 3.4 and formula (4.8), as, given $t \geq 0$ and considering a sequence $(k(n))_{n\geq 1}$ of positive integers such that $k(n)/n \to t$ as $n \to \infty$, then $|M_n^{k(n)}(f)|_{Lip} \leq \left(1 + \frac{\omega}{n}\right)^{k(n)} |f|_{Lip}$ and $\left(1 + \frac{\omega}{n}\right)^{k(n)} \to \exp(\omega t)$ as $n \to \infty$.

Statement (b) is a direct consequence of (4.16) and [4, Corollary 6.1.20].

Finally, statement (c) holds true because of Proposition 3.6, Proposition 3.7 and formula (4.8). □

The next result shows the semigroup $(T(t))_{t\geq 0}$ can be extended to $L^p(Q_d, \mu_{a,b})$, ($p \in [1, +\infty[$), where $\mu_{a,b}$ is the Borel probability measure introduced in Theorem 4.2. Moreover, a representation formula similar to (4.8), as well as the asymptotic behaviour of the extended semigroup (see (4.9)), can be established.

Theorem 4.5. *For every $1 \leq p < +\infty$, the semigroup $(T(t))_{t\geq 0}$ on $C(Q_d)$ (see Theorem 4.2) extends to a unique positive contraction semigroup $(T_p(t))_{t\geq 0}$ on $L^p(Q_d, \mu_{a,b})$, whose generator is an extension of $(B, D(B))$ to $L^p(Q_d, \mu_{a,b})$ and $\mathbb{P}_\infty$ is a core for it.*

Moreover, if $f \in L^p(Q_d, \mu_{a,b})$ *and* $(k(n))_{n\geq 1}$ *is a sequence of positive integers satisfying* $\lim_{n\to\infty} k(n)/n = t$*, then*

$$T_p(t)(f) = \lim_{n\to\infty} M_n^{k(n)}(f) \quad \text{in } L^p(Q_d, \mu_{a,b}). \tag{4.19}$$

Finally, if $f \in L^p(Q_d, \mu_{a,b})$ *and* $n \geq 1$*,*

$$\lim_{t\to+\infty} T_p(t)(f) = \lim_{m\to\infty} M_n^m(f) = \int_{Q_d} f d\mu_{a,b} \tag{4.20}$$

in $L^p(Q_d, \mu_{a,b})$*.*

Proof. For every $t \geq 0$, denote by $T_p(t)$ the unique extension of $T(t)$ to $L^p(Q_d, \mu_{a,b})$ as explained in Section 2. The operator $T_p(t)$ is a positive linear contraction on $L^p(Q_d, \mu_{a,b})$. Since $C(Q_d)$ is dense in $L^p(Q_d, \mu_{a,b})$ and $\|\cdot\|_p \leq \|\cdot\|_\infty$ on $C(Q_d)$, it is easily seen that $(T_p(t))_{t\geq 0}$ is a strongly continuous semigroup on $L^p(Q_d, \mu_{a,b})$, its generator is an extension of $(B, D(B))$ to $L^p(Q_d, \mu_{a,b})$ and $\mathbb{P}_\infty$ is a core for it, by virtue of Theorem 4.2.

Finally, formulas (4.19) and (4.20) can be obtained from the corresponding ones (4.8) and (4.9) taking again into account that $C(Q_d)$ is dense in $L^p(Q_d, \mu_{a,b})$ and that, as remarked before Theorem 3.2, each restriction $M_n \mid_{L^p(Q_d,\mu_{a,b})}$ coincides with the continuous extension of $M_n \mid_{C(Q_d)}$ to $L^p(Q_d, \mu_{a,b})$. $\square$

References

[1] A. A. Albanese, M. Campiti and E. M. Mangino, *Regularity properties of semigroups generated by some Fleming-Viot type operators*, J. Math. Anal. Appl. **335** (2007), 1259-1273.

[2] A. A. Albanese and E. M. Mangino, *Analyticity of a class of degenerate evolution equations on the canonical simplex of* $\mathbf{R}^d$ *arising from Fleming-Viot processes*, J. Math. Anal. Appl. **379** (2011), no. 1, 401-424.

[3] F. Altomare, *Korovkin-type Theorems and Approximation by Positive Linear Operators*, Surveys in Approximation Theory, Vol. 5, 2010, 92-164, free available on line at http://www.math.technion.ac.il/sat/papers/13/, ISSN 1555-578X.

[4] F. Altomare and M. Campiti, Korovkin-Type Approximation Theory and its Applications, *de Gruyter Studies in Mathematics* **17**, *Walter de Gruyter, Berlin-New York*, 1994.

[5] F. Altomare, M. Cappelletti Montano, V. Leonessa and I. Raşa, Markov Operators, Positive Semigroups and Approximation Processes, *de Gruyter Studies in Mathematics* **61**, *Walter de Gruyter GmbH, Berlin/Boston*, 2014.

[6] F. Altomare, M. Cappelletti Montano, V. Leonessa and I. Raşa, *Elliptic differential operators and positive semigroups associated with generalized Kantorovich operators*, J. Math. Anal. Appl. **458** (1) (2018), 153-173.

[7] F. Altomare and I. Raşa, *On some classes of diffusion equations and relates approximation problems*, in: M- G- de Bruin, H. D. Mache and J. Szabados (Eds), Trends and Applications in Constructive Approximation, ISNM, **151** (Birkhäuser Velag, Basel, 2005), 13-26.

[8] R. Altomare and I. Raşa, *Lipschitz contractions, unique ergodicity and asymptotics of Markov semigroups*, Bollettino U.M.I. (9) **V** (2012), 1-17.

[9] A. Attalienti and M. Campiti, *Degenerate evolution problems and Beta-type operators*, Studia Math. **140** (2) (2000), 117-139.

[10] H. Bauer, Measure and Integration Theory, *de Gruyter Studies in Mathematics* **26**, *Walter de Gruyter GmbH, Berlin/Boston*, 2011.

[11] E. E. Berdysheva and K. Jetter, *Multivariate Bernstein Durrmeyer operators with arbitrary weight functions*, J. Approx. Theory **162** (2010), 576-598.

[12] H. Berens and Y. Xu, *On Bernstein-Durrmeyer polynomials with Jacobi weights*, in: C. K. Chui (Ed.), Approximation Theory and Functional Analysis, Academic Press, Boston, 1991, 25-46.

[13] S. Cerrai and Ph. Clément, *Schauder estimates for a degenerate second order elliptic operator on a cube*, J. Diff. Eq. **242** (2007), 287-321.

[14] M. Campiti and I. Raşa, *Qualitative properties of a class of Fleming-Viot operators*, Acta Math. Hungar. **103** (2004), no. 1-2, 55-69.

[15] U. Krengel, Ergodic Theorems, *de Gruyter Studies in Mathematics* **6**, *W. de Guyter, Berlin/New York*, 1985.

[16] D. Mugnolo and A. Rhandi, *On the domain of a Fleming-Viot type operator on a* L^p*-space with invariant measure*, Note Mat. **31** (2011), no. 1, 139-148.

[17] R. Nagel (Ed.), One-parameter semigroups of positive operators, *Lecture Notes in Math.* **1184**, *Springer-Verlag, Berlin*, 1986.

[18] R. Paltanea, *Sur un opérateur polynomial défini sur l'ensemble des fonctions intégrables*, Univ. Babeş-Bolyai, Cluj-Napoca, 83-2 (1983), 101-106.

[19] T. Vladislav and I. Raşa, Analiza Numerica, Aproximare, problema lui Cauchy abstracta, proiectori Altomare, *Ed. Tehnica, Bucuresti*, 1999.

[20] Sh. Waldron, *A generalization of beta integral and the limit of Bernstein-Durrmeyer operator with Jacobi weights*, J. Approx. Theory **122** (2003), 141-150.

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