Journal of Algebraic Combinatorics
https://doi.org/10.1007/s10801-020-00997-6

Eulerian polynomials via the Weyl algebra action

Jose Agapito[2] · Pasquale Petrullo[1] · Domenico Senato[1] · Maria M. Torres[2]

Received: 17 March 2020 / Accepted: 16 November 2020

Abstract
Through the action of the Weyl algebra on the geometric series, we establish a generalization of the Worpitzky identity and new recursive formulae for a family of polynomials including the classical Eulerian polynomials. We obtain an extension of the Dobiński formula for the sum of rook numbers of a Young diagram by replacing the geometric series with the exponential series. Also, by replacing the derivative operator with the q-derivative operator, we extend these results to the q-analogue setting including the q-hit numbers. Finally, a combinatorial description and a proof of the symmetry of a family of polynomials introduced by one of the authors are provided.

Keywords Eulerian polynomials · Weyl algebra · Rook numbers · Permutation statistics · Formal power series

1 Introduction

This paper is mainly motivated by the idea of developing a theory for Eulerian polynomials and their generalizations through the formalism of the Weyl algebra. Our starting point is a family of polynomials, occasionally called hit polynomials [4,5], already covered in Riordan's book [16] in the late 1950s, and introduced by Kaplansky

✉ Pasquale Petrullo
pasquale.petrullo@unibas.it

Jose Agapito
jaruiz@ciencias.ulisboa.pt

Domenico Senato
domenico.senato@unibas.it

Maria M. Torres
mmtorres@ciencias.ulisboa.pt

1 Dipartimento di Scienze Umane, Università degli Studi della Basilicata, Via Nazario Sauro 85, Potenza, Italy

2 Departamento de Matemática, Faculdade de Ciências, Universidade de Lisboa, 1749-016 Campo Grande, Lisboa, Portugal

Published online: 02 December 2020

and Riordan [14]. Among other reasons, hit polynomials are interesting because of their combinatorial properties linked to rook numbers. Let us recall some notions and briefly describe the context. A non-attacking rook placement on a board D is a set P of boxes of D with no two boxes in the same row or column. The number $r_k(D)$ of non-attacking rook placements P on D with $|P| = k$ is said to be the k-th rook number of D. If $D = D_\lambda$ is the Young diagram of a partition λ, then we write $r_k(\lambda)$ for the k-th rook number of D_λ. In particular, for the staircase partition $\delta_n := (n, n-1, \dots, 1)$, it is well-known that the rook numbers $r_k(\delta_{n-1})$ are the Stirling numbers of the second kind $S(n, n-k)$. In this sense, the sum $R_\lambda = \sum_k r_k(\lambda)$ can be regarded as a generalized Bell number. By identifying the permutations in the symmetric group $\mathfrak{S}_n$ with the placements on the square diagram D_n consisting of n rows of length n, for any partition λ such that $D_\lambda \subseteq D_n$, we set

$$\mathcal{A}_{n,\lambda}(x) := \sum_{\sigma \in \mathfrak{S}_n} x^{|\sigma \cap D_\lambda|}.$$

The polynomials $\mathcal{A}_{n,\lambda}(x)$ often occur within the well developed literature on rook theory [4,6,9–14]. It is well-known that the classical Eulerian polynomials $A_n(x)$ arise as $\mathcal{A}_{n,\delta_{n-1}}(x)$. In Sect. 3, we will show that $\mathcal{A}_{n,\delta_{n-r}}(x)$ agrees with the polynomial ${}^rA_n(x)$ introduced by Foata and Schützenberger [7]. This connection motivates a generalized notion of the excedance statistic that allows another combinatorial description of the polynomial $\mathcal{A}_{n,\lambda}(x)$. A classical formula of Frobenius, relating the Stirling numbers of the second kind and the Eulerian polynomials, extends in a straightforward manner to the following identity [4]

$$\mathcal{A}_{n,\lambda}(x) = \sum_{k \geq 0} r_k(\lambda)\, (n-k)!\, (x-1)^k. \tag{1}$$

Based on a q-analogue of rook numbers, Garsia and Remmel [8] provided a q-analogue for the polynomials $\mathcal{A}_{n,\lambda}(x)$ that generalizes identity (1). Dworkin [5] further studied the recursive properties of such polynomials and also gave a direct combinatorial interpretation of their coefficients, the q-hit numbers.

In the seventies, Navon [15] showed that rook placements also provide a natural combinatorial framework for the algebras generated by annihilation and creation operators, and in particular for the so-called normal ordering problem [2,3,17]. Recall that, if $\mathbf{X}$ denotes the operator of multiplication by x, and $\mathbf{D} = \frac{d}{dx}$ denotes the usual derivative operator, then $\mathbf{DX} - \mathbf{XD} = 1$ and the algebra generated by $\mathbf{X}$ and $\mathbf{D}$ is referred to as the Weyl algebra. The normal ordering of any product $\mathbf{\Pi}$ involving a occurrences of the operator $\mathbf{X}$ and b occurrences of the operator $\mathbf{D}$ is given by

$$\mathbf{\Pi} = \sum_{k \geq 0} r_k(\lambda)\, \mathbf{X}^{a-k}\mathbf{D}^{b-k},$$

where λ is a suitable partition associated with $\mathbf{\Pi}$. In this setting, the Stirling numbers of the second kind arise as the normal ordering coefficients of $\mathbf{\Pi} = (\mathbf{XD})^n$.

We show that the polynomials $\mathcal{A}_{n,\lambda}(x)$ naturally describe the action of any product of the operators $\mathbf{D}$ and $\mathbf{X}$ on the geometric series $1/(1-x)$. More precisely, given a partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_l)$, we define an operator $\mathbf{\Pi}_\lambda$ such that for any square diagram D_n containing D_λ,

$$\mathbf{\Pi}_\lambda \mathbf{D}^{n-\lambda_1} \frac{1}{1-x} = \frac{\mathcal{A}_{n,\lambda^{(n)}}(x)}{(1-x)^{n+1}},$$

where $\lambda^{(n)}$ is a partition that we call the reduced complement of λ in D_n (Theorem 5). A first consequence of this point of view is that the polynomials of Garsia and Remmel arise when the operator $\mathbf{\Pi}_{\lambda,q} \mathbf{D}_q^{n-\lambda_1}$, obtained from $\mathbf{\Pi}_\lambda \mathbf{D}^{n-\lambda_1}$ by replacing $\mathbf{D}$ with the q-derivative $\mathbf{D}_q$, acts on $1/(1-x)$. More precisely, they are the polynomials $\mathcal{A}_{n,\lambda}(x, q)$ such that

$$\mathbf{\Pi}_{\lambda,q} \mathbf{D}_q^{n-\lambda_1} \frac{1}{1-x} = \frac{\mathcal{A}_{n,\lambda^{(n)}}(x, q)}{(1-x)(1-xq)\cdots(1-xq^n)}.$$

In addition, straightforward manipulations of derivatives and formal power series allow us to establish a generalization of the classical Worpitzky identity (Corollary 6), a remarkably and seemingly new property of the polynomials $\mathcal{A}_{n,\lambda}(x)$ with respect to derivation (Corollary 7), and a recursion formula to compute $\mathcal{A}_{n,\lambda}(x)$ (Corollary 8). When $\lambda = \delta_{n-r}$ a new recursive formula relating the polynomials ${}^rA_n(x)$ and the classical Eulerian polynomials is obtained. In turn, each of these results provide a corresponding q-analogue simply by replacing $\mathbf{D}$ with $\mathbf{D}_q$ (Corollaries 9,10,11). Furthermore, by letting $\mathbf{\Pi}_\lambda \mathbf{D}^{n-\lambda_1}$ act on the formal power series expansion of e^x, we recover an extension of the classical Dobiński formula for the Bell numbers (identity (27)), and its q-analogue (identity (28)). Finally, we provide a combinatorial description and a proof of the symmetry property of the polynomials $A_{r,s,n}(x)$ (Proposition 13), defined by

$$(\mathbf{X}^r \mathbf{D}^s)^n \frac{1}{1-x} = \frac{A_{r,s,n}(x)}{(1-x)^{sn+1}},$$

and introduced by one of the authors of the present paper [1].

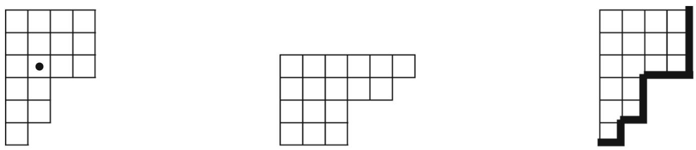
2 Partitions and rook numbers

By a *partition*, we mean a finite non-increasing vector $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_l)$ of positive integers called *parts* of λ. The number of parts of λ is called the *length* of λ, and denoted by $\ell(\lambda)$. The Young diagram (or Ferrers board) of λ is a left-aligned array of boxes, displayed in $\ell(\lambda)$ rows consisting of $\lambda_1, \lambda_2, \dots, \lambda_l$ boxes, from top to bottom. In analogy with matrix notation, given a Young diagram D, we let $D_{i,j}$ denote the box of D occurring at the i-th row (counting from top to bottom) and at the j-th column (counting from left to right). For instance, the Young diagram of $\lambda = (4, 4, 4, 2, 2, 1)$ is shown in Fig. 1A, with a bullet drawn in the box $D_{3,2}$. The *conjugate* of λ is the partition λ' whose diagram $D_{\lambda'}$ is obtained by reflecting D_λ with respect to its main diagonal. For example, the conjugate of $\lambda = (4, 4, 4, 2, 2, 1)$ is $\lambda' = (6, 5, 3, 3)$ and its Young diagram is shown in Fig. 1B. The *border* of a Young diagram D is by definition the subset of those sides lying at the rightmost position in a row, or at a lowest position in a column. The border of $D_{(4,4,4,2,2,1)}$ is highlighted in Fig. 1c.

Given any vectors $\boldsymbol{r} = (r_1, r_2, \ldots, r_k)$ and $\boldsymbol{u} = (u_1, u_2, \ldots, u_k)$ of positive integers, we let $\lambda_{\boldsymbol{r},\boldsymbol{u}}$ denote the unique partition whose Young diagram has border with horizontal strips of lengths $r_1, r_2, \ldots, r_k$ (from left to right), and vertical strips of lengths $u_1, u_2, \ldots, u_k$ (from bottom to top). For instance, we have $\lambda_{(1,1,2),(1,2,3)} = (4, 4, 4, 2, 2, 1)$ as one may check from the horizontal and vertical strips in Fig. 2.

Given two partitions λ and μ, we write $\lambda \subseteq \mu$ to mean that $D_\lambda \subseteq D_\mu$. Moreover, we let D_n denote the square Young diagram of n rows, and for any partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ such that $D_\lambda \subseteq D_n$, we call *reduced complement* of λ in D_n the partition $\lambda^{(n)} := (n - \lambda_l, n - \lambda_{l-1}, \ldots, n - \lambda_1)$. In terms of Young diagrams, $D_{\lambda^{(n)}}$ is obtained from D_n by removing the boxes of D_λ, deleting all the rows of D_n lying below D_λ, then rotating by 180°. For instance, the reduced complement of $(2, 2, 1)$ in D_4 is $(3, 2, 2)$ and of $(6, 6, 3, 3)$ in D_9 is $(6, 6, 3, 3)$. They are obtained by rotating the white diagrams in Fig. 3.

A *non attacking rook placement* on a Young diagram D, simply *placement* from now on, is a set P of blocks of D with no two boxes occurring in the same row or column. The number of placements on D_λ consisting of k boxes, usually called the *k-th rook number of* λ, will be denoted by $r_k(\lambda)$. For instance, we have $r_3(4, 3, 1) = 4$ and indeed the four placements of three boxes on $D_{(4,3,1)}$ are depicted in Fig. 4.



(a) D_λ **(b)** $D_{\lambda'}$ **(c)** The border of D_λ

Fig. 1 Young diagrams and their border

Fig. 2 Horizontal and vertical strips of a border

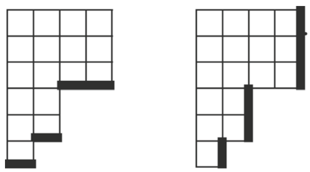
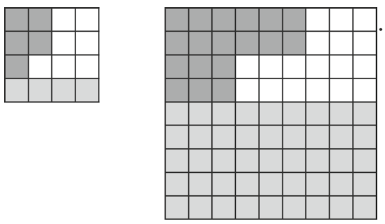


Fig. 3 The reduced complement (white boxes) of a partition (dark gray boxes)



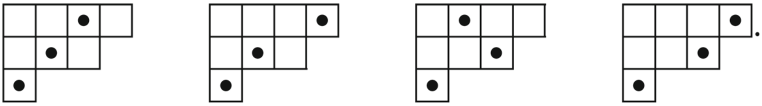


Fig. 4 A bullet is marked in each box of the placement

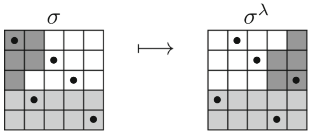
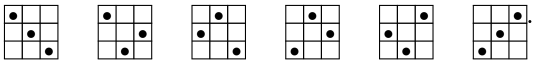


Fig. 5 D_λ in dark gray, B_λ in light gray

A placement of n boxes on D_n can be identified with a permutation matrix of order n. Thus, denoting the symmetric group of degree n by $\mathfrak{S}_n$, we will consider the permutation $\sigma = \sigma_1\sigma_2\ldots\sigma_n$ and the placement $\{D_{1,\sigma(1)}, D_{2,\sigma(2)}, \ldots, D_{n,\sigma(n)}\}$ on $D = D_n$ as the same object. For instance, we identify the permutations 123, 132, 213, 231, 312, 321 in $\mathfrak{S}_3$ with the following placements on D_3:



Note that σ^{-1} is obtained by reflecting σ in the main diagonal of D_n. Hence, for all $\sigma \in \mathfrak{S}_n$ and for all λ such that $D_\lambda \subseteq D_n$ we have

$$\left|\sigma \cap D_\lambda\right| = \left|\sigma^{-1} \cap D_{\lambda'}\right|. \tag{2}$$

Moreover, given $\sigma \in \mathfrak{S}_n$, let $\sigma^\lambda = \sigma_1^\lambda\sigma_2^\lambda\ldots\sigma_n^\lambda$ be defined by

$$\sigma_i^\lambda := \begin{cases} n+1-\sigma_{\ell(\lambda)+1-i} & \text{if } 1 \le i \le \ell(\lambda); \\ n+1-\sigma_{n+1+\ell(\lambda)-i} & \text{if } \ell(\lambda)+1 \le i \le n. \end{cases} \tag{3}$$

It is easy to deduce that $\sigma \mapsto \sigma^\lambda$ is a bijective map. Now, set

$$A_\lambda := \{D_{i,j} \mid 1 \le i \le \ell(\lambda),\ 1 \le j \le n\} \text{ and } B_\lambda := D \setminus A_\lambda.$$

Observe that σ^λ is obtained by separately rotating by 180° the rectangles A_λ and B_λ (with respect to their center). For instance, let $\lambda = (2, 2, 1)$, $n = 5$ and $\sigma = 13425$, then we have $\sigma^\lambda = 23514$ as depicted in Fig. 5.

As $\left|\sigma \cap A_\lambda\right| = \ell(\lambda)$, we obtain

$$\left|\sigma \cap D_\lambda\right| = \ell(\lambda) - \left|\sigma^\lambda \cap D_{\lambda^{(n)}}\right|. \tag{4}$$

3 Generalized Eulerian polynomials

Given a partition λ, and a positive integer n such that $D_\lambda \subseteq D_n$, we define the polynomial $\mathcal{A}_{n,\lambda}(x)$ as follows:

$$\mathcal{A}_{n,\lambda}(x) := \sum_{\sigma \in \mathfrak{S}_n} x^{|\sigma \cap D_\lambda|}. \tag{5}$$

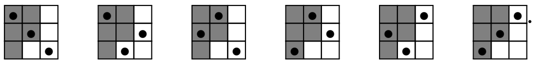
Moreover, we set

$$\mathcal{A}_{n,k,\lambda} := \left|\{\sigma \in \mathfrak{S}_n \ : \ |\sigma \cap D_\lambda| = k\}\right|, \text{ for } k = 0, 1, \ldots, n, \tag{6}$$

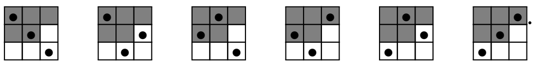
and obtain

$$\mathcal{A}_{n,\lambda}(x) := \sum_{k \geq 0} \mathcal{A}_{n,k,\lambda}\, x^k.$$

Example 1 Let $\lambda = (2, 2, 1)$ and $n = 3$. In order to obtain $\mathcal{A}_{3,(2,2,1)}(x)$, we compute the cardinality of $\sigma \cap D_\lambda$, for each $\sigma \in \mathfrak{S}_3$.



We get $\mathcal{A}_{3,(2,2,1)}(x) = 4x^2 + 2x$. Note that by reflecting with respect to the main diagonal of D_3 (i.e., taking images under the bijection $\sigma \mapsto \sigma^{-1}$) one obtains $\mathcal{A}_{3,(3,2)}(x) = 4x^2 + 2x = \mathcal{A}_{3,\lambda'}(x)$,



Proposition 1 *Given a partition λ and a positive integer n such that $D_\lambda \subseteq D_n$, we have*

(i) $\mathcal{A}_{n,\lambda}(1) = n!$;
(ii) $\mathcal{A}_{n,\lambda'}(x) = \mathcal{A}_{n,\lambda}(x)$;
(iii) $\mathcal{A}_{n,\lambda^{(n)}}(x) = x^{\ell(\lambda)}\mathcal{A}_{n,\lambda}(1/x)$.

Proof From (5) and (2), we have (i) and (ii), respectively. Moreover, by means of $\sigma \mapsto \sigma^\lambda$ and (4) we have

$$x^{\ell(\lambda)}\mathcal{A}_{n,\lambda}(1/x) = \sum_{\sigma \in \mathfrak{S}_n} x^{\ell(\lambda) - |\sigma \cap D_\lambda|} = \sum_{\sigma \in \mathfrak{S}_n} x^{|\sigma^\lambda \cap D_{\lambda^{(n)}}|} = \mathcal{A}_{n,\lambda^{(n)}}(x),$$

which gives (i). □

Note that (iii) means that the coefficients of $\mathcal{A}_{n,\lambda}(x)$, read in decreasing order of degree, agree with the coefficients of $\mathcal{A}_{n,\lambda^{(n)}}(x)$, read in increasing order of degree. For instance, if $\lambda = (3, 3, 2, 1)$ then $\lambda^{(7)} = (6, 5, 4, 4)$ and in fact we have

$$\mathcal{A}_{7,(3,3,2,1)}(x) = 192x^3 + 1704x^2 + 2496x + 648$$

and

$$\mathcal{A}_{7,(6,5,4,4)}(x) = 648x^4 + 2496x^3 + 1704x^2 + 192x.$$

In particular, the following symmetry property holds.

Corollary 2 *Let n be a positive integer and λ a partition such that $D_\lambda \subseteq D_n$. If $\lambda^{(n)} = \lambda$ then*

$$\mathcal{A}_{n,\lambda}(x) = x^{\ell(\lambda)} \mathcal{A}_{n,\lambda}(1/x). \tag{7}$$

Moreover, if $(\lambda')^{(n)} = \lambda'$ then

$$\mathcal{A}_{n,\lambda}(x) = x^{\lambda_1} \mathcal{A}_{n,\lambda}(1/x). \tag{8}$$

Proof Identity (7) follows from $\lambda = \lambda^{(n)}$ and (iii). Identity (8) follows from (iii) taking into account that $\ell(\lambda') = \lambda_1$. □

An explicit expansion of $\mathcal{A}_{n,\lambda}(x)$ in terms of the basis $\{(x-1)^i \mid i \geq 0\}$ has been known since [14], where it is proved by using the inclusion–exclusion principle. Here, we provide an alternative and explicit proof.

Theorem 3 *Given a partition λ and a positive integer n such that $D_\lambda \subseteq D_n$, we have*

$$\mathcal{A}_{n,\lambda}(x) = \sum_{i\geq 0} r_i(\lambda)\,(n-i)!\,(x-1)^i. \tag{9}$$

Proof By (5) we have

$$\mathcal{A}_{n,\lambda}(x+1) = \sum_{\sigma\in\mathfrak{S}_n} (x+1)^{|\sigma\cap D_\lambda|} = \sum_{(\sigma,B)\in\text{Pairs}} x^{|B|},$$

where Pairs denotes the set of all (σ, B) such that $\sigma \in \mathfrak{S}_n$ and $B \subseteq (\sigma \cap D_\lambda)$. Note that for all $(\sigma, B) \in$ Pairs, B is a placement on D_λ. Now, for any given placement B_0 on D_λ, let us count the pairs (σ, B) such that $B = B_0$. Assume $|B_0| = i$ and consider the permutation σ^{B_0} obtained by adding to B_0 the $n-i$ available boxes on the main diagonal of $D := D_n$, that is

$$\sigma^{B_0} := B_0 \cup \{D_{i,i} \mid D_{i,j} \notin B_0 \text{ for all } j = 1, 2, \ldots, n\}.$$

Clearly $(\sigma^{B_0}, B_0) \in$ Pairs. Moreover, we obtain all the pairs of type (σ, B_0) by permuting the $n-i$ columns of D with no boxes in $\sigma^{B_0} \setminus B_0$. As there are $r_i(\lambda)$ placements

B on D_λ with $|B| = i$, the number of pairs (σ, B) such that $|B| = i$ is $r_i(\lambda)\,(n-i)!$. We recover

$$\mathcal{A}_{n,\lambda}(x+1) = \sum_{i \geq 0} r_i(\lambda)\,(n-i)!\,x^i,$$

which gives (9) when x is replaced by $x - 1$. □

Example 2 Let r be a nonnegative integer. Following Foata and Schützenberger [7], we consider the polynomial

$${}^{r}A_n(x) := \sum_{\sigma \in \mathfrak{S}_n} x^{\mathrm{exc}_r(\sigma)},$$

where

$$\mathrm{exc}_r(\sigma) := \Big|\{i \mid 1 \leq i \leq n,\ \sigma_i \geq i + r\}\Big|.$$

Clearly, ${}^{1}A_n(x)$ is the classical Eulerian polynomial. Now, let $\sigma \mapsto \sigma'$ denote the bijection defined on $\mathfrak{S}_{n+r}$ by $\sigma'_i := n + r + 1 - \sigma_i$, for $i = 1, 2, \ldots n + r$. Observe that $\sigma_i \leq n + 1 - i$ if and only if $\sigma'_i \geq r + i$. As a consequence, we obtain

$$\mathcal{A}_{n+r,\delta_n}(x) = \sum_{\sigma \in \mathfrak{S}_{n+r}} x^{|\sigma \cap D_{\delta_n}|} = \sum_{\sigma \in \mathfrak{S}_{n+r}} x^{\mathrm{exc}_r(\sigma')} = {}^{r}A_{n+r}(x), \tag{10}$$

or equivalently ${}^{r}A_n(x) = \mathcal{A}_{n,\delta_{n-r}}(x)$. From (9), we recover the following Frobenius identity for the polynomials ${}^{r}A_n(x)$ [7]:

$${}^{r}A_n(x) = \sum_{k \geq 0} S(n+1-r, n+1-r-k)\,(n-k)!\,(x-1)^k.$$

The following generalization of the notion of excedance is motivated by Example 2. Given a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$, a positive integer n such that $D_\lambda \subseteq D_n$, and a permutation $\sigma = \sigma_1\sigma_2\ldots\sigma_n \in \mathfrak{S}_n$, we set

$$\mathrm{exc}_\lambda(\sigma) := \Big|\{i \mid 1 \leq i \leq n,\ \sigma_i > n + 1 - \lambda_i\}\Big|, \tag{11}$$

where $\lambda_i = 0$ is assumed for $\ell(\lambda) < i \leq n$. As before, the complement bijection $\sigma \mapsto \sigma'$ provides

$$\Big|\sigma \cap D_\lambda\Big| = \mathrm{exc}_\lambda(\sigma'),$$

so that we get

$$\mathcal{A}_{n,\lambda}(x) = \sum_{\sigma \in \mathfrak{S}_n} x^{\mathrm{exc}_\lambda(\sigma)}. \tag{12}$$

4 The Weyl algebra action

Let $\mathbf{D}, \mathbf{X} \colon \mathbb{Z}[x] \to \mathbb{Z}[x]$ denote the derivative operator and the operator of multiplication by x, respectively. As $\mathbf{DX} - \mathbf{XD} = 1$ the following normal ordering problem may be posed: given any product $\mathbf{\Pi}$ involving a occurrences of the operator $\mathbf{D}$ and b occurrences of the operator $\mathbf{X}$, find the coefficients $c_i(\mathbf{\Pi})$ satisfying

$$\mathbf{\Pi} = \sum_{i \geq 0} c_i(\mathbf{\Pi})\, \mathbf{X}^{b-i} \mathbf{D}^{a-i}.$$

A beautiful answer to this problem was given by Navon [15] in terms of placements on Young diagrams. Here, we recast Navon's result following the work of Varvak [17]. For any partition λ, we set

$$\mathbf{\Pi}_\lambda := \mathbf{D}^{r_1}\mathbf{X}^{u_1}\mathbf{D}^{r_2}\mathbf{X}^{u_2} \cdots \mathbf{D}^{r_k}\mathbf{X}^{u_k}, \tag{13}$$

where $\boldsymbol{r} = (r_1, r_2, \ldots, r_k)$ and $\boldsymbol{u} = (u_1, u_2, \ldots, u_k)$ are the unique vectors satisfying $\lambda = \lambda_{\boldsymbol{r},\boldsymbol{u}}$. Note that $\lambda_1 = r_1 + r_2 + \cdots + r_k$ and $\ell(\lambda) = u_1 + u_2 + \cdots + u_k$.

Theorem 4 *For any partition λ, we have*

$$\mathbf{\Pi}_\lambda = \sum_{i \geq 0} r_i(\lambda)\, \mathbf{X}^{\ell(\lambda)-i} \mathbf{D}^{\lambda_1 - i}. \tag{14}$$

Proof Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$. A straightforward computation shows that $\mathbf{\Pi}_\lambda\, 1 = r_{\lambda_1}(\lambda)\, x^{\ell(\lambda)-\lambda_1}$. Set

$$\mu := (\underbrace{\lambda_1, \lambda_1, \ldots, \lambda_1}_{m+1}, \lambda_2, \ldots, \lambda_l) \text{ and } \mu \setminus \lambda := (\underbrace{\lambda_1, \lambda_1, \ldots, \lambda_1}_{m}).$$

It follows that $\mathbf{\Pi}_\lambda\, x^m = \mathbf{\Pi}_\lambda \mathbf{X}^m\, 1 = \mathbf{\Pi}_\mu\, 1 = r_{\lambda_1}(\mu)\, x^{m+\ell(\lambda)-\lambda_1}$. On the other hand, we may compute $r_{\lambda_1}(\mu)$ in the following alternative way,

$$r_{\lambda_1}(\mu) = \sum_{k \geq 0} r_k(\lambda)\, r_{\lambda_1 - i}(\mu \setminus \lambda) = \sum_{i \geq 0} r_i(\lambda)\, \frac{m!}{(m - \lambda_1 - i)!}.$$

Then, we conclude

$$\sum_{i \geq 0} r_i(\lambda)\, \mathbf{X}^{\ell(\lambda)-i} \mathbf{D}^{\lambda_1 - i} x^m = r_{\lambda_1}(\mu)\, x^{m+\ell(\lambda)-\lambda_1} = \mathbf{\Pi}_\lambda\, x^m.$$

□

The following theorem makes explicit the connection between the Weyl algebra and the polynomials $\mathcal{A}_{n,\lambda}(x)$.

Theorem 5 *For any partition* λ *and any positive integer* n *such that* $D_\lambda \subseteq D_n$, *we have*

$$\mathbf{\Pi}_\lambda \mathbf{D}^{n-\lambda_1} \frac{1}{1-x} = \frac{\mathcal{A}_{n,\lambda^{(n)}}(x)}{(1-x)^{n+1}}. \tag{15}$$

Proof By (14) we obtain

$$\begin{aligned}\mathbf{\Pi}_\lambda \mathbf{D}^{n-\lambda_1} \frac{1}{1-x} &= \sum_{i\geq 0} r_i(\lambda)\, \mathbf{X}^{\ell(\lambda)-i} \mathbf{D}^{n-i} \frac{1}{1-x} \\ &= \sum_{i\geq 0} r_i(\lambda)\, \big(n-i\big)!\, \frac{x^{\ell(\lambda)-i}}{(1-x)^{n-i+1}},\end{aligned}$$

hence

$$(1-x)^{n+1} \mathbf{\Pi}_\lambda \mathbf{D}^{n-\lambda_1} \frac{1}{1-x} = \sum_{i\geq 0} r_i(\lambda)\, \big(n-i\big)!\, x^{\ell(\lambda)-i}\, (1-x)^i. \tag{16}$$

Moreover, by (9) we have

$$x^{\ell(\lambda)}\, \mathcal{A}_{n,\lambda}(1/x) = \sum_{i\geq 0} r_i(\lambda)\, (n-i)!\, x^{\ell(\lambda)-i} (1-x)^i. \tag{17}$$

Finally, by comparing (17), (16) and Proposition 1 (iii), we have

$$\frac{\mathcal{A}_{n,\lambda^{(n)}}(x)}{(1-x)^{n+1}} = \frac{x^{\ell(\lambda)} \mathcal{A}_{n,\lambda}(1/x)}{(1-x)^{n+1}} = \mathbf{\Pi}_\lambda \mathbf{D}^{n-\lambda_1} \frac{1}{1-x}.$$

□

A first consequence of (15) is the following extension of the Worpitzky identity for Eulerian polynomials.

Corollary 6 *Let* m *be a positive integer. For any partition* $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ *and any positive integer* n *such that* $D_\lambda \subseteq D_n$, *we have*

$$\prod_{i=0}^{n-1} (m + \lambda'_{n-i} - i) = \sum_{k\geq 0} \binom{m+\ell(\lambda)-k}{n} \mathcal{A}_{n,k,\lambda^{(n)}}, \tag{18}$$

where $\lambda' = (\lambda'_1, \lambda'_2, \ldots, \lambda'_{l'})$ *is the conjugate of* λ *and we assume that* $\lambda'_i = 0$ *for* $i > l' = \lambda_1$.

Proof Set

$$\mu := (\underbrace{n, n, \ldots, n}_{m}, \lambda_1, \lambda_2, \ldots, \lambda_l)$$

and observe that

$$r_n(\mu) = \prod_{i=0}^{n-1} (m + \lambda'_{n-i} - i).$$

Moreover, we have $\mathbf{\Pi}_\lambda \mathbf{D}^{n-\lambda_1}\, x^m = \mathbf{\Pi}_\mu\, 1 = r_n(\mu)\, x^{m+\ell(\lambda)-n}$ and then the left-hand side of (15) is given by

$$\mathbf{\Pi}_\lambda \mathbf{D}^{n-\lambda_1}\, \frac{1}{1-x} = \sum_{m\geq 0} \prod_{i=0}^{n-1} (m + \lambda'_{n-i} - i)\, x^{m+\ell(\lambda)-n}.$$

From (6), the right-hand side of (15) may be rewritten as

$$\sum_{i\geq 0} \left(\sum_{k\geq 0} \binom{n+i-k}{n} \mathcal{A}_{n,k,\lambda^{(n)}} \right) x^i.$$

Hence, (18) follows by extracting the coefficient of $x^{m-n+\ell(\lambda)}$ from both sides in (15). □

Example 3 Setting $\lambda = (n-1, n-2, \ldots, r)$ in (18), and observing that $\lambda^{(n)} = \delta_{n-r}$, we obtain the following Worpitzky identity [7],

$$m^{n-r}\, \frac{m!}{(m-r)!} = \sum_{k\geq 0} \binom{m+r-k}{n}\, {}^r\!A_{n,k}.$$

Of course, $r = 1$ leads to the Worpitzky identity for Eulerian numbers:

$$m^n = \sum_{k\geq 0} \binom{m+1-k}{n} A_{n,k}.$$

A further consequence of (15) is a remarkable property of the polynomials $\mathcal{A}_{n,\lambda}(x)$ with respect to derivation. In terms of the underlined Young diagrams, this property encodes the evolution of the polynomials $\mathcal{A}_{n,\lambda}(x)$, for a fixed partition λ, with respect to square diagrams D_n of increasing size.

Corollary 7 *For any partition λ and any positive integer n such that $D_\lambda \subseteq D_n$, we have*

$$\mathbf{D}\, \frac{\mathcal{A}_{n,\lambda}(x)}{(1-x)^{n+1}} = \frac{\mathcal{A}_{n+1,\lambda}(x)}{(1-x)^{n+2}}. \tag{19}$$

Proof If $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ then we set $\lambda + 1 := (\lambda_1 + 1, \lambda_2 + 1, \ldots, \lambda_l + 1)$. Note that the reduced complements of λ in D_n and of $\lambda + 1$ in D_{n+1} agree, hence from (15) we have

$$\mathbf{D}\,\frac{\mathcal{A}_{n,\lambda^{(n)}}(x)}{(1-x)^{n+1}} = \mathbf{D}\mathbf{\Pi}_\lambda \mathbf{D}^{n-\lambda_1}\,\frac{1}{1-x} = \mathbf{\Pi}_{\lambda+1}\mathbf{D}^{(n+1)-(\lambda_1+1)}\,\frac{1}{1-x} = \frac{\mathcal{A}_{n+1,\lambda^{(n)}}(x)}{(1-x)^{n+2}}.$$

□

Identity (19) suggests that the polynomials $\mathcal{A}_{n,\lambda}(x)$ indexed by the smallest n such that $D_\lambda \subseteq D_n$, play a special role. Indeed, for any partition λ, we set

$$n(\lambda) := \max\{\lambda_1, \ell(\lambda)\} \tag{20}$$

and define

$$\mathcal{A}_\lambda(x) := \mathcal{A}_{n(\lambda),\lambda}(x). \tag{21}$$

Hence, we obtain the following recursive rule.

Corollary 8 *For any partition λ and any positive integer n such that $D_\lambda \subseteq D_n$, we have*

$$\mathcal{A}_{n,\lambda}(x) = (1-x)^{n+1}\,\mathbf{D}^{n-n(\lambda)}\,\frac{\mathcal{A}_\lambda(x)}{(1-x)^{n(\lambda)+1}}. \tag{22}$$

Proof Identity (22) follows by iterating (19). □

Remark 1 Note that, by Proposition 1 (iii) and (7) we have $\mathcal{A}_{\delta_n}(x) = x\,A_n(x)$. Therefore, by setting $\lambda = \delta_{n-r}$ in (22), the polynomials ${}^rA_n(x)$ are obtained via suitable derivatives involving the classical Eulerian polynomials,

$${}^rA_n(x) = (1-x)^{n+1}\,\mathbf{D}^r\,\frac{x\,A_{n-r}(x)}{(1-x)^{n-r+1}}.$$

5 *q*-analogues arising from the *q*-Weyl algebra

Let $\mathbf{D}_q$ denote the q-derivative operator acting on the polynomial $p(x)$ according to the following rule,

$$\mathbf{D}_q\,p(x) = \frac{p(qx)-p(x)}{qx-x}.$$

We have $\mathbf{D}_q\mathbf{X} - q\mathbf{X}\mathbf{D}_q = 1$ and the algebra generated by $\mathbf{X}, \mathbf{D}_q$ is a q-analogue of the Weyl algebra. Now, let $[i] := 1+q+\cdots+q^{i-1}$ denote the q-integer, and for all partitions λ, let $\mathbf{\Pi}_{\lambda,q}$ be obtained from (13) by replacing $\mathbf{D}$ with $\mathbf{D}_q$. As $\mathbf{D}_q^i\,x^m = [m][m-1]\cdots[m-i+1]\,x^{m-i}$, straightforward computations show that

$$\mathbf{\Pi}_{\lambda,q}\mathbf{D}_q^{n-\lambda_1}\,\frac{1}{1-x} = \sum_{m\geq 0}\prod_{i=0}^{n-1}[m+\lambda'_{n-i}-i]\,x^{m-n+\ell(\lambda)}. \tag{23}$$

Note that the right-hand side of (23) agrees with the right-hand side of identity (I.11) in the paper of Garsia and Remmel [8] , as can be seen by setting $a_{i+1} = n - \ell(\lambda) + \lambda'_{n-i}$ for $0 \le i \le n-1$, that is by setting $\lambda = \mu^{(n)}$ for $\mu := (a_n, a_{n-1}, \dots, a_1)$. Now, we let $\mathcal{A}_{n,\lambda^{(n)}}(x,q)$ denote the polynomial defined by

$$\mathbf{\Pi}_{\lambda,q}\mathbf{D}_q^{n-\lambda_1}\frac{1}{1-x} = \frac{\mathcal{A}_{n,\lambda^{(n)}}(x,q)}{(1-x)(1-xq)\cdots(1-xq^n)}, \tag{24}$$

and the right-hand side of (I.12) in [8] ensures that $Q_A(x,q) = \mathcal{A}_{n,\lambda^{(n)}}(x,q)$ when the partition λ is chosen such that $a_{i+1} = n - \ell(\lambda) + \lambda'_{n-i}$ for $0 \le i \le n-1$. First, we recall that

$$\frac{1}{(1-x)(1-xq)\cdots(1-xq^n)} = \sum_{k\ge 0}\begin{bmatrix} n+k \\ n \end{bmatrix} x^k.$$

Moreover, we define $\mathcal{A}_{n,k,\lambda^{(n)}}(q)$ by

$$\mathcal{A}_{n,\lambda^{(n)}}(x,q) = \sum_{k\ge 0}\mathcal{A}_{n,k,\lambda^{(n)}}(q)\,x^k,$$

and compare the coefficients of (23) and (24) to obtain the following q-analogue of Corollary 6.

Corollary 9 *Let m be a positive integer. For any partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_l)$ and any positive integer n such that $D_\lambda \subseteq D_n$, we have*

$$\prod_{i=0}^{n-1}[m + \lambda'_{n-i} - i] = \sum_{k\ge 0}\begin{bmatrix} m + \ell(\lambda) - k \\ n \end{bmatrix}\mathcal{A}_{n,k,\lambda^{(n)}}(q),$$

where $\lambda' = (\lambda'_1, \lambda'_2, \dots, \lambda'_{l'})$ is the conjugate of λ and we assume that $\lambda'_i = 0$ for $i > l' = \lambda_1$.

Moreover, simply by replacing $\mathbf{D}$ with $\mathbf{D}_q$ in the proof of Corollary 7, we obtain the following q-analogue of (19).

Corollary 10 *For any partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_l)$ and any positive integer n such that $D_\lambda \subseteq D_n$, we have*

$$\mathbf{D}_q\frac{\mathcal{A}_{n,\lambda}(x,q)}{(1-x)(1-xq)\cdots(1-xq^n)} = \frac{\mathcal{A}_{n+1,\lambda}(x,q)}{(1-x)(1-xq)\cdots(1-xq^{n+1})}.$$

We let $\mathcal{A}_\lambda(x,q) := \mathcal{A}_{n(\lambda),\lambda}(x,q)$ and easily obtain the q-analogue of the recursive property (22).

Corollary 11 *For any partition* $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ *and any positive integer* n *such that* $D_\lambda \subseteq D_n$, *we have*

$$\mathcal{A}_{n+1,\lambda}(x,q) = (1-x)(1-xq)\cdots(1-xq^{n+1})\mathbf{D}_q^{n-n(\lambda)}\,\frac{\mathcal{A}_\lambda(x,q)}{(1-x)(1-xq)\cdots(1-xq^{n(\lambda)})}.$$

We explicitly remark that the polynomials $\mathcal{A}_{n,k,\lambda}(q)$ are the so-called q-hit numbers [5].

6 Further generalizations and applications

6.1 An application to the operator $(\mathbf{X}^r\mathbf{D}^s)^n$

We now consider the polynomials $A_{r,s,n}(x)$ introduced in [1] and defined by

$$(\mathbf{X}^r\mathbf{D}^s)^n\,\frac{1}{1-x} = \frac{A_{r,s,n}(x)}{(1-x)^{sn+1}},$$

for all positive integers $r \leq s$ and $n \geq 1$. Let $\boldsymbol{r} = (r_1, r_2, \ldots, r_n)$ and $\boldsymbol{u} = (u_1, u_2, \ldots, u_n)$ satisfy $r_1 = r_2 = \ldots = r_n = s$ and $u_1 = u_2 = \ldots = u_n = r$, set $\delta_{r,s,n} := \lambda_{\boldsymbol{r},\boldsymbol{u}}$. The Young diagram of $\delta_{r,s,n}$ is obtained from D_{δ_n} by replacing each box in D_{δ_n} with a rectangular diagram of s columns and r rows. For example, the Young diagram of $\delta_{2,3,2}$ is $D_{(6,6,3,3)}$, as shown in Fig. 3 (dark gray) as a subset of D_9. We denote by $\mathrm{exc}_{r,s,n}$ the deformation of the excedance statistic induced by $\lambda = \delta_{r,s,n}$ via (11). In particular, for all $\sigma \in \mathfrak{S}_{sn}$, we have

$$\mathrm{exc}_{r,s,n-1}(\sigma) = \big|\{i = (i_1-1)r + i_2 \mid 1 \leq i_1 \leq n-1,\ 1 \leq i_2 \leq r,\ \sigma_i > si_1)\}\big|. \tag{25}$$

Note that, as $\delta_{1,1,n-1} = \delta_{n-1}$ (by convention $\delta_0 = (1)$), we have $\mathrm{exc}_{1,1,n-1}(\sigma) = \mathrm{exc}(\sigma)$ for all $\sigma \in \mathfrak{S}_n$. The following result gives a combinatorial explanation for the identity $A_{r,s,n}(1) = (sn)!$ [1].

Proposition 12 *For all positive integers* $r \leq s$ *and* $n \geq 1$, *we have*

$$A_{r,s,n}(x) = x^r \mathcal{A}_{sn,\delta_{r,s,n-1}}(x) = x^r \sum_{\sigma\in\mathfrak{S}_{sn}} x^{\mathrm{exc}_{r,s,n-1}(\sigma)}. \tag{26}$$

Proof Let $\lambda := \delta_{s,r,n-1}$. From

$$(\mathbf{X}^r\mathbf{D}^s)^n = \mathbf{X}^r(\mathbf{D}^s\mathbf{X}^r)^{n-1}\mathbf{D}^s = \mathbf{X}^r\,\mathbf{\Pi}_\lambda\mathbf{D}^{sn-s(n-1)},$$

by virtue of Theorem 5 we obtain

$$\frac{A_{r,s,n}(x)}{(1-x)^{sn+1}} = \mathbf{X}^r\,\mathbf{\Pi}_\lambda\mathbf{D}^{sn-s(n-1)}\,\frac{1}{1-x} = \frac{x^r\,\mathcal{A}_{sn,\lambda^{(sn)}}(x)}{(1-x)^{sn+1}}.$$

As $\delta_{r,s,n-1} = \delta^{(sn)}_{r,s,n-1}$,

$$A_{r,s,n}(x) = x^r \mathcal{A}_{sn,\delta_{r,s,n-1}}(x),$$

and via (12) we deduce (26). □

Now, we prove the following result originally conjectured in [1].

Proposition 13 *For all positive integers $r \leq s$ and $n \geq 1$, we have*

$$A_{r,s,n}(x) = x^{r(n-1)} A_{r,s,n}(1/x).$$

Proof By taking into account Proposition 1(iii), as $\delta_{r,s,n-1} = \delta^{(sn)}_{r,s,n-1}$, and since $\ell(\delta_{r,s,n-1}) = r(n-1)$, from (26), we have

$$x^{r(n-1)} A_{r,s,n}(1/x) = x^r x^{\ell(\delta_{r,s,n-1})} \mathcal{A}_{sn,\delta_{r,s,n-1}}(1/x) = A_{r,s,n}(x).$$

□

6.2 Generalizations of the Dobiński formula

One may think to replace the geometric series $1/(1-x)$ in (15) and let any product $\mathbf{\Pi}$ act on an arbitrary power series $f(x)$. More interestingly, one may look for those series $f(x)$ such that $\mathbf{\Pi}\, f(x)$ has some combinatorial interest. Let us discuss the case $f(x) = e^x$, which leads to an extension of the Dobiński formula. Indeed, by (14) one obtains

$$\mathbf{\Pi}_\lambda \mathbf{D}^{n-\lambda_1} e^x = e^x \sum_{k \geq 0} r_k(\lambda)\, x^{\ell(\lambda)-k} = e^x\, x^{\ell(\lambda)}\, R_\lambda(1/x),$$

where $R_\lambda(x) = \sum_k r_k(\lambda)\, x^k$ is the well-known rook polynomial associated with D_λ. On the other hand, by expanding e^x we also have

$$\mathbf{\Pi}_\lambda \mathbf{D}^{n-\lambda_1} e^x = \sum_{m \geq 0} \prod_{i=0}^{n-1} (m + \lambda'_{n-i} - i)\, \frac{x^{m-n+\ell(\lambda)}}{m!},$$

and then

$$\sum_{m \geq 0} \prod_{i=0}^{n-1} (m + \lambda'_{n-i} - i)\, \frac{x^{m-n+\ell(\lambda)}}{m!} = e^x\, x^{\ell(\lambda)}\, R_\lambda(1/x).$$

Setting $x = 1$ and $R_\lambda := R_\lambda(1)$ we obtain the following generalization of the Dobiński formula

$$\sum_{m \geq 0} \frac{\prod_{i=0}^{n-1} (m + \lambda'_{n-i} - i)}{m!} = e\, R_\lambda. \tag{27}$$

The classical case arises when $\lambda = \delta_{n-1}$, and then $R_{\delta_{n-1}} = B_n$ is the n-th Bell number,

$$\sum_{m\geq 0} \frac{m^n}{m!} = e\, B_n.$$

Moreover, replacing n with sn, setting $\lambda = \delta_{r,s,n-1}$ and $B_{r,s,n} := R_{\delta_{r,s,n-1}}$, we get a Dobiński formula for the sum of all generalized Stirling numbers $S_{r,s}(n,k) := r_{sn-k}(\delta_{r,s,n-1})$ [2],

$$\sum_{m\geq 0} \frac{1}{\big(m-(s-r)n\big)!} \prod_{i=1}^{n} \frac{\big(m-(s-r)i\big)!}{\big(m-(s-r)i-r\big)!} = e\, B_{r,s,n}.$$

In particular, when $r = s$, we recover

$$\sum_{m\geq 0} \frac{1}{m!} \frac{m!^n}{(m-r)!^n} = e\, B_{r,r,n}.$$

In closing, to recover a q-analogue of (27), set

$$\varepsilon(x) := \sum_{k\geq 0} \frac{x^k}{[k]!},$$

where $[k]! := [1][2]\cdots[k]$, and observe that $\mathbf{D}_q\, \varepsilon(x) = \varepsilon(x)$. We deduce

$$\mathbf{\Pi}_{\lambda,q} \mathbf{D}_q^{n-\lambda_1}\, \varepsilon(x) = \varepsilon(x) \sum_{k\geq 0} r_k(\lambda,q)\, x^{\ell(\lambda)-k} = \varepsilon(x)\, x^{\ell(\lambda)}\, R_\lambda(1/x, q),$$

where $R_\lambda(x,q) = \sum_k r_k(\lambda,q)\, x^{\ell(\lambda)-k}$, and the $r_k(\lambda,q)$ are the q-rook numbers arising here as the normal ordering coefficients of $\mathbf{\Pi}_{\lambda,q}\mathbf{D}_q^{n-\lambda_1}$ (Theorem 6.1 in [17]). Finally, we set $\varepsilon := \varepsilon(1)$ and $R_\lambda(q) := R_\lambda(1,q)$ and obtain the following result.

Proposition 14 *For any partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ such that $D_\lambda \subseteq D_n$, we have*

$$\sum_{m\geq 0} \frac{\prod_{i=0}^{n-1} [m + \lambda'_{n-i} - i]}{[m]!} = \varepsilon\, R_\lambda(q), \tag{28}$$

where $\lambda' = (\lambda'_1, \lambda'_2, \ldots, \lambda'_{l'})$ is the conjugate of λ and we assume that $\lambda'_i = 0$ for $i > l' = \lambda_1$.

Acknowledgements The first and the fourth authors were partially supported by FCT-Fundação para a Ciência e a Tecnologia (Portugal), through the project UIDB/04721/2020.

Funding Open access funding provided by University degli Studi della Basilicata within the CRUI-CARE Agreement.

References

1. Agapito, J.: On symmetric polynomials with only real zeros and nonnegative γ-vectors. Linear Algebra Appl. **451**, 260–289 (2014)
2. Blasiak, P.: Combinatorics of boson normal ordering and some applications. Concepts Phys. **1**, 177–278 (2004)
3. Blasiak, P., Flajolet, P.: Combinatorial models of creation-annihilation. Sém. Lothar. Combin. **65**, Article B65c (2011)
4. Butler, F., Can, M., Haglund, J., Remmel, J.B.: Rook Theory Notes, preprint, 2020. http://www.math.ucsd.edu/~remmel/
5. Dworkin, M.: An interpretation for Garsia and Remmel q-hit numbers. J. Combin. Theory Ser. A **81**, 149–175 (1998)
6. Foata, D., Schützenberger, M.-P.: On the rook polynomials of Ferrers relations. In: Combinatorial Theory and its Applications, II (Proc. Colloq., Balatonfüred, 1969), vol. 2, pp. 413–436 (1970)
7. Foata, D., Schützenberger, M.-P.: Théorie Géométrique des Polynômes Eulériens. Lectures Notes in Mathematics, vol. 138. Springer, Berlin (1970)
8. Garsia, A.M., Remmel, J.B.: Q-Counting rook configurations and a formula of Frobenius. J. Combin. Theory Ser. A **41**, 246–275 (1986)
9. Goldman, J.R., Joichi, J.T., Reiner, D.L., White, D.E.: Rook theory II: Boards of binomial type. SIAM J. Appl. Math. **31**(4), 618–633 (1976)
10. Goldman, J.R., Joichi, J.T., White, D.E.: Rook theory I. Rook equivalence of Ferrers boards. Proc. Amer. Math. Soc. **52**, 485–492 (1975)
11. Goldman, J.R., Joichi, J.T., White, D.E.: Rook theory III. Rook polynomials and the chormatic structure of graphs. J. Combin. Theory Ser. B **25**, 135–142 (1978)
12. Goldman, J.R., Joichi, J.T., White, D.E.: Rook Theory IV. Orthogonal sequences of rook polynomials. Stud. Appl. Math. **56**(3), 267–272 (1977)
13. Goldman, J.R., Joichi, J.T., White, D.E.: Rook Theory V. Rook polynomials, Möbius inversion and the umbral calculus. J. Combin. Theory Ser. A **21**(2), 230–239 (1976)
14. Kaplansky, I., Riordan, J.: The problem of the rooks and its applications. Duke Math. J. **13**, 259–268 (1946)
15. Navon, A.M.: Combinatorics and fermion algebra. Nuovo Cimento Soc. Ital. Fis. B **16**, 324–330 (1973)
16. Riordan, J.: Introduction to Combinatorial Analysis. Wiley, New York (1958)
17. Varvak, A.: Rook numbers and the normal ordering problem. J. Combin. Theory Ser. A **112**, 292–307 (2005)

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.