A mixed scheme of product integration rules in $(-1, 1)$

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Abstract

The paper deals with the numerical approximation of integrals of the type

$$I(f,y) := \int_{-1}^{1} f(x)k(x,y)dx, \quad y \in \mathrm{S} \subset \mathbb{R}$$

where f is a smooth function and the kernel $k(x,y)$ involves some kinds of "pathologies" (for instance, weak singularities, high oscillations and/or endpoint algebraic singularities). We introduce and study a product integration rule obtained by interpolating f by an extended Lagrange polynomial based on Jacobi zeros. We prove that the rule is stable and convergent with the order of the best polynomial approximation of f in suitable function spaces. Moreover, we derive a general recurrence relation for the new modified moments appearing in the coefficients of the rule, just using the knowledge of the usual modified moments. The new quadrature sequence, suitable combined with the ordinary product rule, allows to obtain a "mixed " quadrature scheme, significantly reducing the number of involved samples of f. Numerical examples are provided in order to support the theoretical results and to show the efficiency of the procedure.

Keywords: Lagrange Interpolation, Jacobi Polynomials, Quadrature rules

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1. Introduction

Let $w(x) = v^{\alpha,\beta}(x) := (1-x)^\alpha(1+x)^\beta$, $\alpha, \beta > -1$ and let $\{p_m(w)\}_m$ be the corresponding sequence of orthonormal polynomials. Since the zeros of $\mathcal{Q}_{2m+1} := p_m(w)p_{m+1}(w)$ are simple, the Lagrange polynomial $L_{2m+1}(w, f)$, interpolating f at the zeros of $\mathcal{Q}_{2m+1}$, can be considered. A peculiar aspect of the so called *extended Lagrange interpolation* is the approximation of functions by $2m$-degree polynomials by using the zeros of orthogonal polynomials of degree m and $m+1$. This allows to construct "high" degree interpolation processes, delaying in some sense the well known difficulties for computing the zeros of high degree orthogonal polynomials. Another interesting feature descends from the simple representation

$$L_{2m+1}(w, f) = p_{m+1}(w)\mathcal{L}_m\left(w, \frac{f}{p_{m+1}(w)}\right) + p_m(w)\mathcal{L}_{m+1}\left(w, \frac{f}{p_m(w)}\right), \quad (1)$$

where $\mathcal{L}_m(w, h)$ is the Lagrange polynomial interpolating h at the zeros of $p_m(w)$. Indeed, (1) highlights that, in the construction of the $2m$-degree interpolating polynomial $L_{2m+1}(w, f)$, one can reuse the previously computed m samples of f. It is possible to obtain other extended interpolation processes by using sequences of polynomials orthogonal w.r.t. different weight functions, provided that the interpolation zeros are sufficiently "far" among them. Extended interpolation processes have been extensively studied from several authors, in bounded and unbounded intervals, and with estimates of the error in different norms (see for instance [1], [2], [3], [4], [5], [6], [7]). Recently an application in quadrature has been proposed in [8], [9].

Here we consider integrals of the type

$$I(f, y) := \int_{-1}^{1} f(x)k(x, y)dx, \quad y \in \mathrm{S} \subset \mathbb{R}, \quad (2)$$

where f is a function with possible algebraic singularities in ± 1 and k is a kernel with some "pathologies" which are not reducible to any standard weight function. For instance k can be weakly singular or "nearly singular" or highly oscillating and, in addition, present an algebraic singular behavior at the endpoints ± 1.

Here we will employ extended Lagrange processes in constructing an *extended product quadrature rule*. Denoting by $\Sigma_{2m+1}(f)$ the quadrature sum obtained by replacing f with $L_{2m+1}(w, f)$ in (2), we state sufficient and necessary conditions under which the operator

$$\Sigma_{2m+1} : f \to \int_{-1}^{1} L_{2m+1}(w, f; x) k(x, y) dx$$

is bounded in a suitable subspace of L^1_u, where $u = v^{\gamma,\delta}$ with $\gamma, \delta \geq 0$. This result assures the stability of the rule, with a rate of convergence behaving as the error of the best polynomial approximation of f in weighted uniform norm.

From the computational point of view we will gather a general recurrence relation formula for the generalized modified moments arising in the extended rule coefficients, starting from the so-called modified moments.

Finally, we introduce an efficient quadrature scheme obtained by "mixing" the extended rule with the analogous "one-weight" product rule based on the Lagrange polynomial $\mathcal{L}_{m+1}(w, f)$. As we will show, the mixed sequence allows to reduce of one third the number of samples of f needed if we use the only one-weight product sequence and if we want to get the same speed of convergence.

The underlying idea of this apparently tortuous path, is the possible application to numerical methods for functional equations. For instance using the mixed quadrature scheme for solving Fredholm second kind integral equations it is possible to construct approximate solutions of degrees m and $2m+1$ respectively, by solving two systems of linear equations of orders m and $m+1$ (instead of m and $2m+1$).

The outline of this paper is as follows. Section 2 contains some auxiliary results.

In Section 3 we state a Theorem on the extended interpolating process that can be useful also in other contexts. Hence we introduce and study the extended quadrature rule, the compound quadrature sequence and some results about the stability and the convergence of the mixed scheme.

Section 4 contains some details about the computation of the coefficients of the extended rule and in Section 5 we produce some numerical tests, to confirm the theoretical estimates and for underlining the efficiency of the mixed scheme.

Finally, Section 6 contains the proofs of the main results.

2. Notation and preliminary results

In the sequel $\mathcal{C}$ will denote any positive constant which can be different in different formulas. Moreover $\mathcal{C} \neq \mathcal{C}(a, b, ..)$ will be used to note that the constant $\mathcal{C}$ is independent of $a, b, ...$ The notation $A \sim B$, where A and B are positive quantities depending on some parameters, will be used if and only if $(A/B)^{\pm 1} \leq \mathcal{C}$, with $\mathcal{C}$ a positive constant independent of the above parameters.

Denote by $\mathbb{P}_m$ the space of all algebraic polynomials of degree at most m and for any bivariate function $g(x, y)$, denote by g_y (g_x) the function in the only variable x (y). Finally, with a consolidate notation, we will set $v^{\rho,\sigma}(x) = (1-x)^\rho (1+x)^\sigma,\ \rho, \sigma \in \mathbb{R}$ and by $\varphi(x) = \sqrt{1-x^2}$.

For $1 \leq p < \infty$, and $u(x) = v^{\gamma,\delta}(x)$, $\gamma, \delta > -\dfrac{1}{p}$, let $L^p_u = L^p_u([-1, 1])$ be the space of measurable functions f s.t. $fu \in L^p([-1, 1])$, equipped with the norm

$$\|f\|_{L^p_u} = \left(\int_{-1}^{1} |f(x)u(x)|^p dx \right)^{\frac{1}{p}}.$$

Moreover with $\gamma, \delta \geq 0$, let $L^\infty_u([-1, 1]) =: C_u$ be the space of functions $f \in C^0((-1, 1))$ s.t. $fu \in C^0([-1, 1])$ and $\lim_{x \to 1^-} f(x)u(x) = 0$ if $\gamma > 0$ as well as $\lim_{x \to -1^+} f(x)u(x) = 0$ if $\delta > 0$, equipped with the norm $\|f\|_{C_u} = \sup_{|x| \leq 1} |f(x)|u(x)$. For smoother functions, with $r \in \mathbb{N}$, consider the Sobolev-type space

$$W^\infty_r(u) = \left\{ f \in C_u : f^{(r-1)} \in AC[-1, 1], \quad \|f^{(r)} \varphi^r u\|_\infty < \infty \right\},$$

where AC is the space of absolutely continuous functions, and equip it with the norm $\|f\|_{W^\infty_r(u)} = \|fu\|_\infty + \|f^{(r)} \varphi^r u\|_\infty$. Denoting by

$$E_m(f)_u = \inf_{P \in \mathbb{P}_m} \|(f - P)u\|_\infty,$$

the error of the best polynomial approximation of $f \in C_u$, we recall that (see for instance [10, p. 172]), $\forall f \in W_r^\infty(u),\ r \geq 1$,

$$E_m(f)_u \leq \mathcal{C} \frac{\|f\|_{W_r^\infty(u)}}{m^r}, \quad \mathcal{C} \neq \mathcal{C}(m,f), \quad . \tag{3}$$

Finally, setting $\log^+ f(x) = \log(\max(1, f(x)))$, by $L \log^+ L$ we denote the space of functions f defined in $[-1,1]$ s.t. $\|f(1+\log^+ f)\|_1 < +\infty$.

Consider the weight $w = v^{\alpha,\beta}, \quad \alpha, \beta > -1$, and let $\{p_m(w)\}_m$ be the corresponding sequence of orthonormal polynomials with positive leading coefficients, i.e.

$$p_m(w,x) = \gamma_m(w)x^m + \quad \text{terms of lower degree},\ \gamma_m(w) > 0.$$

Denoting by $\{x_k\}_{k=1}^m$ the zeros of $p_m(w)$ with $x_1 < x_2 < \cdots < x_m$, let $\mathcal{L}_m(w,f)$ be the Lagrange polynomial interpolating a given function f at the zeros of $p_m(w)$, i.e.

$$\mathcal{L}_m(w,f,x) = \sum_{k=1}^m \ell_{m,k}(w,x) f(x_k), \tag{4}$$

$$\ell_{m,k}(w,x) = \frac{p_m(w,x)}{p_m'(w,x_k)(x-x_k)} = \lambda_{m,k}(w) \sum_{j=0}^{m-1} p_j(w,x) p_j(w,x_k), \tag{5}$$

being $\{\lambda_{m,k}(w)\}_{k=1}^m$ the Christoffel numbers w.r.t. w (see, for instance, [10, (4.1.3), p.236]. About the weighted mean convergence of this interpolation process, we recall the following result [10, p. 348], which will be useful in the sequel.

Theorem 2.1. *Let $w = v^{\alpha,\beta}$, $u = v^{\gamma,\delta}$, $\gamma, \delta \geq 0$ and assume $\sup_{y\in S} \frac{k_y}{u} \in L\log^+ L$. Then for any $f \in C_u$ and with $\mathcal{C} \neq \mathcal{C}(m,f)$,*

$$\sup_{y \in S} \|\mathcal{L}_m(w,f)k(\cdot,y)\|_1 \leq \mathcal{C}\|fu\|_\infty \Leftrightarrow \sup_{y\in S} \frac{k_y}{\sqrt{w\varphi}} \in L^1([-1,1]), \frac{\sqrt{w\varphi}}{u} \in L^1([-1,1]).$$

Now, denoting by $\{y_k\}_{k=1}^{m+1}$ with $y_1 < y_2 < \cdots < y_{m+1}$ the zeros of $p_{m+1}(w)$ and recalling that $y_k < x_k < y_{k+1},\ k = 1, 2, \ldots, m$, we will set

$\{z_{2i-1} := y_i\}_{i=1}^{m+1},\ \{z_{2i} := x_i\}_{i=1}^m$ and $Q_{2m+1} := p_m(w)p_{m+1}(w)$.

With $\mathcal{L}_m$ defined as in (4), the *extended Lagrange polynomial* $L_{2m+1}(w,f)$ interpolating f at the zeros of Q_{2m+1} can be represented in the following form

$$\begin{aligned} L_{2m+1}(w,f;x) &= \sum_{k=1}^{2m+1} f(z_k)\frac{Q_{2m+1}(x)}{Q'_{m+1}(z_k)(x-z_k)} \\ &= p_{m+1}(w,x)\mathcal{L}_m\left(w,\frac{f}{p_{m+1}(w)},x\right)+p_m(w,x)\mathcal{L}_{m+1}\left(w,\frac{f}{p_m(w)},x\right). \quad (6) \end{aligned}$$

About the convergence of the extended interpolation polynomial, we recall

Theorem 2.2. *[3, Th. 3.4, p.79] Let u and w be generalized Jacobi weights. For any $f \in C^q([-1,1]), q \geq 1$, $\frac{w}{\varphi^q} \in L^1([-1,1])$, $\frac{u}{\varphi} \in L^1([-1,1])$, $\frac{u}{w\varphi} \in L^1([-1,1])$, we have*

$$\|(f-L_{2m+1}(w,f))u\|_1 \leq \frac{\mathcal{C}}{m^q}\omega\left(f^{(q)};\frac{1}{m}\right), \; m \geq 4q+5, \quad \mathcal{C} \neq \mathcal{C}(m,f),$$

where $\omega(f,t)$ denotes the ordinary modulus of continuity of f.

Remark 2.1. *Theorem 2.2 holds in a more general case since regards also the case of the simultaneous approximation of f and its derivatives; however we state here just what deals with our aims.*

For $I(f,y)$ in (2), consider the following "one-weight" product rule

$$I(f,y) \quad = \quad \int_{-1}^{1}\mathcal{L}_m(w,f,x)k(x,y)\,dx + e_m^{\mathcal{I}}(f,y) = \mathcal{I}_m(f,y)+e_m^{\mathcal{I}}(f,y), \quad (7)$$

where $\mathcal{I}_m(f,y)=\sum_{i=1}^{m}\mathcal{C}_i(y)f(x_i), \quad \mathcal{C}_i(y)=\int_{-1}^{1}\ell_{m,i}(w,x)k(x,y)dx.$

By Theorem 2.1 the stability and the convergence of the rule follows, i.e.

$$\sup_m \sup_{y\in \mathrm{S}} |\mathcal{I}_m(f,y)| \leq \mathcal{C}\|fu\|_\infty, \text{ and } \quad \sup_{y\in S}|e_m^{\mathcal{I}}(f,y)| \leq \mathcal{C}E_{m-1}(f)_u, \quad (8)$$

with $\mathcal{C} \neq \mathcal{C}(m,f)$.

Product rules were proposed by several authors (see for instance [11], [12]) and their strength is to compute problematic integrals with a rate of convergence depending on the smoothness of f and not on the kernel k. However, their main effort is due to the construction of the so called "modified moments" $\{M_j(y)\}_j$

needed to compute the coefficients $\{\mathcal{C}_i(y)\}_{i=1}^m$

$$\mathcal{C}_i(y) = \lambda_{m,i}(w) \sum_{j=0}^{m-1} p_j(w, x_i) M_j(y), \quad M_j(y) = \int_{-1}^{1} p_j(w, x) k(x, y) \, dx. \tag{9}$$

Usually the modified moments are obtained by recurrence relations, depending on the peculiarities of the kernel (see for instance [13], [14], [15] and the references therein). This means that there is no general procedure that allows us to build the modified moments regardless of the nature of the kernel.

3. Main results

First of all we want to give a result about the extended interpolation process which extend the results in Theorems 2.1 and 2.2.

Theorem 3.1. *Let $k(x, y)$ be defined in $[-1, 1] \times \mathrm{S}$, $u = v^{\gamma,\delta}$ with $\gamma, \delta \geq 0$ and $w = v^{\alpha,\beta}$. Under the assumptions*

$$\sup_{y \in \mathrm{S}} \left(\frac{k_y}{w\varphi} \right) \in L \log^+ L, \quad \frac{w}{u} \in L^\infty([-1, 1]), \tag{10}$$

for any $f \in C_u$ it is

$$\sup_m \sup_{y \in \mathrm{S}} \| L_{2m+1}(w, f) k_y \|_1 \leq \mathcal{C} \| f u \|_\infty, \quad \mathcal{C} \neq \mathcal{C}(f). \tag{11}$$

Moreover, if (11) holds true then

$$\sup_{y \in \mathrm{S}} \left(\frac{k_y}{w\varphi} \right) \in L^1([-1, 1]). \tag{12}$$

Remark 3.1. *We remark that (11) cannot be derived by Theorem 2.2, which holds for $f \in C^q([-1, 1]), q \geq 1$ and where the kernel k_y does not appear. We prove (11) in order to obtain a more general result which can be useful also in other contexts.*

Now we propose the following *extended quadrature rule* obtained by approximating f in (2) by the interpolating polynomial (6), i.e.

$$I(f, y) := \int_{-1}^{1} f(x) k(x, y) dx = \sum_{k=1}^{2m+1} f(z_k) \mathcal{D}_k(y) + e_{2m+1}^{\Sigma}(f, y) \tag{13}$$

$$:= \quad \Sigma_{2m+1}(f, y) + e_{2m+1}^{\Sigma}(f, y), \quad \mathcal{D}_k(y) = \int_{-1}^{1} \frac{Q_{2m+1}(x)}{Q'_{2m+1}(z_k)(x - z_k)} k(x, y) dx,$$

where $\Sigma_{2m+1}(f,y)$ is the quadrature sum and $e^{\Sigma}_{2m+1}(f,y)$ is the remainder term. The formula is exact for polynomials of degree not greater than $2m$ and first of all $\Sigma_{2m+1}(f,y)$ converges to $I(f,y)$, uniformly w.r.t. $y \in S$.

Theorem 3.2. *Let $k(x,y)$ be defined in $[-1,1]\times \mathrm{S}$, $u = v^{\gamma,\delta}$ with $\gamma,\delta \geq 0$ and $w = v^{\alpha,\beta}$. Under the assumptions (10) for any $f \in C_u$ it is*

$$\sup_m \sup_{y\in \mathrm{S}} |\Sigma_{2m+1}(f,y)| \leq \mathcal{C}\|fu\|_\infty, \quad \mathcal{C} \neq \mathcal{C}(f). \tag{14}$$

Moreover assuming in addition that

$$\sup_{y\in \mathrm{S}} \frac{k_y}{u} \in L^1([-1,1]), \tag{15}$$

the following error estimate holds true

$$\sup_{y\in \mathrm{S}} |e^{\Sigma}_{2m+1}(f,y)| \leq \mathcal{C}E_{2m}(f)_u, \quad \mathcal{C} \neq \mathcal{C}(m,f). \tag{16}$$

Remark 3.2. *We underline that for any $f \in W_r(u)$, $r \geq 1$, by (3) it follows*

$$\sup_{y\in \mathrm{S}} |e^{\Sigma}_{2m+1}(f,y)| \leq \mathcal{C}\frac{\|f\|_{W_r(u)}}{(2m)^r}, \quad \mathcal{C} \neq \mathcal{C}(m,f).$$

3.1. The computation of the quadrature coefficients

As already discussed, the main computational effort in constructing a product integration rule is the "exact" computation of the coefficients, since they strongly depend on the form of the kernel $k(x,y)$. Here, we determine a recurrence relation connecting the coefficients $\{\mathcal{C}_i(y)\}_{i=1}^m$ of the one-weight product rule (7) with the coefficients $\{\mathcal{D}_i(y)\}_{i=1}^{2m+1}$ of the extended quadrature rule (13). Hence by (6) and (5) we get

$$\begin{aligned} \Sigma_{2m+1}(f,y) &= \sum_{k=1}^{m} \mathcal{A}_k(y) f(x_k) + \sum_{k=1}^{m+1} \mathcal{B}_k(y) f(y_k) \\ &=: \sum_{k=1}^{m} \frac{f(x_k)}{p_{m+1}(w,x_k)} \lambda_{m,k}(w) \sum_{j=0}^{m-1} p_j(w,x_k) M_j^{m+1}(y) \\ &+ \sum_{k=1}^{m+1} \frac{f(y_k)}{p_m(w,y_k)} \lambda_{m+1,k}(w) \sum_{j=0}^{m} p_j(w,y_k) M_j^{m}(y) \end{aligned} \tag{17}$$

where

$$M_j^s(y) := \int_{-1}^{1} p_s(w,x)p_j(w,x)k(x,y)w(x)dx,$$

will be called *Generalized Modified Moments* (GMMs).

Now we show how to compute GMMs starting from the ordinary modified moments (MMs) $\{M_j(y)\}_j$ defined in (9). Just for the sake of simplicity, we treat the Gegenbauer case $w = v^{\alpha,\alpha}$, since the case $\alpha \neq \beta$ can be handled analogously. Following a trapezoidal scheme, (see for instance [16]), we propose an algorithm which maps the vector $\{M_j(y)\}_{j=0}^{2m}$ into the two arrays $\{M_j^m(y)\}_{j=0}^{m}$, $\{M_j^{m+1}(y)\}_{j=0}^{m-1}$. It is essentially based on the following *three terms recurrence relation* for the orthonormal Gegenbauer polynomials

$$p_0(w) \equiv \frac{1}{\sqrt{\int_{-1}^{1} w(x)dx}}, \quad p_1(w,x) = \frac{p_0(w)}{b_1}x, \quad b_1 = \frac{1}{\sqrt{3+2\alpha}}$$

$$xp_j(w,x) = b_j p_{j-1}(w,x) + b_{j+1}p_{j+1}(w,x) \quad b_j = \sqrt{\frac{j(j+2\alpha)}{4(j+\alpha)^2-1}}, \quad j = 2,3,\ldots.$$

Algorithm

Initialization: $\{M_j^0(y) = p_0(w)M_j(y)\}_{j=0}^{2m}$ and for $j = 1,\ldots,2m-1$

$$M_j^1(y) = \int_{-1}^{1} p_j(w,x)p_1(w,x)k_y(x)dx = \frac{1}{b_1}\left[b_{j+1}M_{j+1}^0(y) + b_j M_{j-1}^0(y)\right].$$

By the symmetry $M_j^k = M_k^j$, for $k \geq 2,\ k \leq j \leq 2m-k$

$$\begin{aligned} M_j^k(y) &= \int_{-1}^{1} p_j(w,x)p_k(w,x)k_y(x)dx \\ &= \frac{1}{b_k}\left[b_{j+1}M_{j+1}^{k-1}(y) + b_j M_{j-1}^{k-1}(y) - b_{k-1}M_j^{k-2}(y)\right]. \end{aligned}$$

We remark that the construction of GMMs requires $4m^2$ *flops* approximately.

We checked the sensitivity of the proposed recurrence relation by testing the algorithm for different kernels and comparing the values obtained by using double and quadruple machine precisions. To be more precise, denoting by $Md_j^{m+1}(y)$ and $Mq_j^{m+1}(y)$ the $j-th$ GMM in double and in quadruple precision respectively, we computed the quantities

$$\mathcal{M}_m = \max_{y\in T_q} \max_{0\le i\le m} \left| \frac{Md_j^{m+1}(y) - Mq_j^{m+1}(y)}{Mq_j^{m+1}(y)} \right|,$$

with $T_q = \left\{-1 + \frac{i}{10}\right\}_{i=0}^{20}$, for increasing values of m and by implementing the algorithm in MatLab (release R2018a). The largest value $m = 1000$ corresponds to the practical threshold value we can use, since for larger values of m the precision in computing the zeros of orthogonal polynomials and the Christoffel numbers, progressively reduces.

We report here, by way of example, the results obtained for the kernel $k(x, y) = |x - y|^\lambda$, with $\lambda = 0.5$, since for other λ the results are similar.

m	10	50	100	500	1000
$\mathcal{M}_m$	$4.2e-15$	$3.9e-14$	$4.0e-13$	$1.0e-12$	$6.3e-12$

As we can see, a moderate loss of accuracy occurs, consisting in losing 4-5 decimal digits at most, for $m = 1000$. This means that by using variable precision it should be sufficient to perform the computation by considering 4-5 additional digits. However, the implementation of the quadruple precision in the MatLab language is simpler than other ones and certainly its use is more than enough.

3.2. A mixed quadrature scheme

Now we propose to "compose" the sequences $\{\mathcal{I}_m(f)\}_m$ and $\{\Sigma_{2m+1}(f)\}_m$, in order to obtain a new sequence which approximates $I(f, y)$ with the same speed of convergence of $\{\mathcal{I}_{m+2}(f)\}_m$, but with a reduced number of function samples. In addition, in view of the relations between the MMs and GMMs, also the construction of the coefficients of the extended rules can be performed with an efficient procedure. To be more precise, for a fixed m, we consider the sequence $\mathcal{I}_m(f)$, $\Sigma_{2m+1}(f)$, $\mathcal{I}_{4m}(f), \Sigma_{8m+1}(f), \ldots$. Once we have constructed the $m-$th product rule (7), the samples of f are reused for computing the "extended" $2m + 1$ quadrature rule with only additional $m + 1$ samples. The scheme go on restarting from the $4m$−th product rule (7) and using the samples in the extended rule of order $8m+1$ etc.

In details, for any given $q \in \mathbb{N}$, consider the sequences $\{\mathcal{I}_{2^k m}(f)\}_{k=0}^{2q-1}$, and $\{\mathcal{I}_{2^{2k} m}(f), \Sigma_{2^{2k+1}m+1}(f)\}_{k=0}^{q-1}$. The first requires $m(2^{2q}-1)$ evaluations of the function f, while the mixed scheme only $q+\dfrac{2}{3}m(2^{2q}-1)$, i.e. by this way almost one third of the function evaluations is spared. Setting

$$\mathcal{T}_{2^n m}(f) = \begin{cases} \mathcal{I}_{2^n m}(f), & n = 0, 2, 4, 6, \dots \\ \Sigma_{2^n m+1}(f), & n \text{ odd} \end{cases}, \tag{18}$$

about the stability and the convergence of this mixed scheme we are able to prove the following:

Theorem 3.3. *Under the assumptions*

$$\begin{aligned} &\sup_{y \in S} \left(\frac{k_y}{w\varphi} \right) \in L \log^+ L, && \sup_{y \in S} \left(\frac{k_y}{u} \right) \in L \log^+ L, \\ &\frac{\sqrt{w\varphi}}{u} \in L^1([-1,1]), && \frac{w}{u} \in L^\infty((-1,1)), \end{aligned} \tag{19}$$

we have for any $f \in C_u$ and any fixed $n \in \mathbb{N}$,

$$\sup_{y \in S} |\mathcal{T}_{2^n m}(f, y)| \le \mathcal{C} \|fu\|_\infty \tag{20}$$

and

$$\sup_{y \in S} |I(f, y) - \mathcal{T}_{2^n m}(f, y)| \le \mathcal{C} E_{2^n m}(f)_u, \tag{21}$$

where in both cases $\mathcal{C} \neq \mathcal{C}(m, f)$.

4. Numerical Tests

In this section we propose some examples to test the mixed sequence (18) and to perform a comparison with the behaviour of the one-weight sequence (7). In each test we report the approximate values of the integral by means of the one-weight rule (**OWR**) and by the corresponding mixed sequence (**MixSeq**), for increasing values of m. Moreover, we specify the number $\#$ *feval.* of function evaluations, corresponding to **OWR** and **MixSeq**. We point out that all the computations have been performed in double-machine precision ($eps \approx 2.22044e-16$), except the routine for GMMs, performed in quadruple arithmetic precision in view of the mild instability of the algorithm.

Example 4.1. *Consider the following integral with an oscillating kernel*

$$I_1(f,y)=\int_{-1}^{1}\exp(x)(1+x^2)\sin(10x)(1-x^2)^{0.1}dx,\ u=v^{0,0},\ w=v^{0.1,0.1}$$
$$f(x)=\exp(x)(1+x^2),\quad k(x,y)=\sin(yx)(1-x^2)^{0.1},\quad y=10.$$

Since the function f is very smooth, the convergence is very fast (Table 1).

Table 1: Evaluation of $I_1(f,y)$ with $y=10$.

# $feval.$	**MixSeq**	# $feval.$	**OWR**
5	$-4.4e-01$	5	$-4.4e-01$
6	$-4.4405e-01$	11	$-4.44057e-01$
20	$-4.440579949773e-01$	20	$-4.440579949773e-01$
21	$-4.44057994977389e-01$	41	$-4.44057994977389e-01$

Example 4.2.

$$I_2(f,y)=\int_{-1}^{1}e^{|x-0.25|^{\frac{7}{2}}}\sin(xy)dx,\ u=w=v^{0,0}$$
$$f(x)=e^{|x-0.25|^{\frac{7}{2}}},\quad k(x,y)=\sin(xy),\ y=25.$$

In this case $f\in W_3^\infty(u)$ and the error behaves like $\frac{\|f^{(3)}\varphi^3\|_\infty}{m^3}$. Evaluating the seminorm, we have for $m=256$, $\frac{\|f^{(3)}\varphi^3\|_\infty}{m^3}\sim 1.8e-06$, *while for $m=256$ according to Table 2,* 12 *exact digits are achieved.*

Example 4.3.

$$I_3(f,y)=\int_{-1}^{1}\frac{\sin(1-x)^{\frac{9}{2}}\sqrt[4]{1-x^2}}{|x-y|^{0.3}}dx,\quad u=v^{0.25,0.25}=w$$
$$f(x)=\sin(1-x)^{\frac{9}{2}},\ k(x,y)=\frac{\sqrt[4]{1-x^2}}{|x-y|^{0.3}},\ y=-0.2.$$

Since $f\in W_9^\infty(u)$ the error behaves like $\frac{\|f^{(9)}u\varphi^9\|_\infty}{m^9}$. By $\|f^{(9)}u\varphi^9\|_\infty\sim 2e{+}12$, for $m=256$ the theoretical estimate assures 11 *digits, while the effective digits are* 15 *(Table 3).*

Table 2: Evaluation of $I_2(f,y)$ with $y=25$.

$\#$ *feval.*	**MixSeq**	$\#$ *feval.*	**OWR**
16	$2.811e-01$	16	$2.811e-01$
17	$2.8115e-01$	32	$2.81158e-01$
64	$2.811586e-01$	64	$2.811586e-01$
65	$2.811586223e-01$	128	$2.81158622e-01$
256	$2.8115862232e-01$	256	$2.811586223e-01$
257	$2.81158622325e-01$	512	$2.8115862235e-01$

Table 3: Evaluation of $I_3(f,y)$ with $y=-0.2$.

$\#$ *feval.*	**MixSeq**	$\#$ *feval.*	**OWR**
16	$6.5e-01$	16	$6.5e-01$
17	$6.50e-01$	32	$6.50e-01$
64	$6.50512850059e-01$	64	$6.50512850059e-01$
65	$6.5051285005932e-01$	128	$6.5051285005932e-01$
256	$6.50512850059325e-01$	256	$6.50512850059325e-01$

5. The Proofs

For any weight $w=v^{\alpha,\beta}$ we recall the following estimates, useful in the proofs:

$$|p_m(w,x)| \le \frac{\mathcal{C}}{\left(\sqrt{1-x}+\frac{1}{m}\right)^{\alpha+\frac{1}{2}}\left(\sqrt{1+x}+\frac{1}{m}\right)^{\beta+\frac{1}{2}}}, \quad |x|\le 1, \tag{22}$$

(see [17, (15)]),

$$\frac{1}{p'_m(w,x_k)} = \frac{\gamma_{m-1}(w)}{\gamma_m(w)} p_{m-1}(w,x_k)\lambda_{m,k}(w) \tag{23}$$

obtained by comparing [17, (3)] with the first expression for $\ell_{m,k}(w,x)$ in (5), and

$$\Delta x_k = x_{k+1}-x_k \sim \frac{1}{m}\varphi(x_k), \quad \lambda_{m,k}(w) \sim \Delta x_k w(x_k), \tag{24}$$

(see [18, p. 120]).

Next Lemma can be found in [19] (see also [20, p. 101, Prop. II. 4. 6]).

Lemma 5.1. *Let* $\{\{x_k = x_{m,k}\}_{k=1}^m\}_{m=2,3,\dots}$ *be arc cosine distributed (i.e. setting* $x_{m,k} = \cos\theta_{m,k}, k = 1,2,\dots,m$ *it is* $|\theta_{m,i} - \theta_{m,i+1}| \sim m^{-1}, 0 \le i \le m, \theta_{m,0} = \pi, \theta_{m,m+1} = 0$*) let* $l > 0$ *be a fixed integer,* $u(x) = v^{\gamma,\delta}$ *with* γ, $\delta > -1$. *Then, for any polynomial* $P_{lm} \in I\!P_{lm}$, *it results*

$$\sum_{k=1}^m \Delta x_k |P_{lm}u|(x_k) \le \mathcal{C}\|P_{lm}u\|_1, \quad \mathcal{C} \ne \mathcal{C}(P_{lm}, m).$$

Lemma 5.2. *Let be* $Q_{2m+1} = p_{m+1}(w)p_m(w)$. *With* $\{x_k\}_{k=1}^m$ *the zeros of* $p_m(w)$ *and by* $\{y_k\}_{k=1}^{m+1}$ *the zeros of* $p_{m+1}(w)$, *it is*

$$\frac{1}{|Q'_{2m+1}(x_k)|} \sim \Delta x_k w(x_k), \quad \frac{1}{|Q'_{2m+1}(y_k)|} \sim \Delta y_k w(y_k), \tag{25}$$

Proof. We omit the proof, since it easily follows taking into account (23), estimate $\dfrac{\gamma_m(w)}{\gamma_{m+1}(w)} \le \mathcal{C}, \quad \mathcal{C} \ne \mathcal{C}(m)$ and (24). □

Denoting by $H_B(g,t) = \displaystyle\int_B \frac{g(x)}{x-t}dx$ the Hilbert transform of the function g on the compact set B, we recall that, if $G \in L^\infty(B), F \in L\log^+ L(B)$, then [21, p.361]

$$\int_B GH_B(F) = -\int_B FH_B(G), \quad \|FH_B(G)\|_1 \le \mathcal{C}\|G\|_\infty(\|F\log^+ F\|_1 + 1), \tag{26}$$

where $\mathcal{C} \ne \mathcal{C}(F)$.

Proof of Theorem 3.1 . First we prove that **(10)** $\Rightarrow$ **(11)**. Denoting by $g_m = \mathrm{sgn}(L_{2m+1}(w,f)k_y)$, we get

$$\begin{aligned} \|L_{2m+1}(w,f)k_y\|_1 &= \left|\int_{-1}^1 L_{2m+1}(w,f)k_y(x)g_m(x)dx\right| \\ \le &\quad \left|\int_{-1}^1 p_{m+1}(w,x)\mathcal{L}_m\left(w, \frac{f}{p_{m+1}(w)}, x\right)k_y(x)g_m(x)dx\right| \\ + &\quad \left|\int_{-1}^1 p_m(w,x)\mathcal{L}_{m+1}\left(w, \frac{f}{p_m(w)}, x\right)k_y(x)g_m(x)dx\right|. \end{aligned}$$

Setting $\Pi(t) = \displaystyle\int_{-1}^1 \frac{Q_{2m+1}(x)q(x) - Q_{2m+1}(t)q(t)}{(x-t)} \frac{g_m(x)k_y(x)}{q(x)}dx$, with q an arbitrary polynomial of degree ml, l a fixed integer, by (25) we have

$$\begin{aligned}\|L_{2m+1}(w,f)k_y\|_1 &\leq \left|\sum_{k=1}^{m}\frac{f(x_k)}{Q'_{2m+1}(x_k)}\Pi(x_k)\right| + \left|\sum_{k=1}^{m+1}\frac{f(y_k)}{Q'_{2m+1}(y_k)}\Pi(y_k)\right| \\ &\leq \mathcal{C}\|fu\|_\infty\left(\sum_{k=1}^{m+1}\Delta y_k\frac{w(y_k)}{u(y_k)}|\Pi(y_k)| + \sum_{k=1}^{m}\Delta x_k\frac{w(x_k)}{u(x_k)}|\Pi(x_k)|\right).\end{aligned}$$

Taking into account that $\Pi \in \mathbb{P}_{2m+ml}$, we use Lemma 5.1 in order to obtain

$$\|L_{2m+1}(w,f)k_y\|_1 \leq \mathcal{C}\|fu\|_\infty \int_{-1}^{1}\frac{w(t)}{u(t)}|\Pi(t)|dt$$

and by the Remez-type inequality [22, (8.1.4)], setting $A_m = \left[-1+\frac{\mathcal{C}}{m^2}, 1-\frac{\mathcal{C}}{m^2}\right]$ we have

$$\begin{aligned}\|L_{2m+1}(w,f)k_y\|_1 \quad &\leq \quad \mathcal{C}\|fu\|_\infty\left\{\int_{A_m}\frac{w(t)}{u(t)}\left|H(Q_{2m+1}g_mk_y,t)\right|dt\right. \\ &+ \left.\int_{A_m}\frac{w(t)}{u(t)}|Q_{2m+1}(t)q(t)|\left|H\left(\frac{g_mk_y}{q},t\right)\right|dt\right\} \qquad (27) \\ &=: \quad \mathcal{C}\|fu\|_\infty\left\{\int_{A_m}\frac{w(t)}{u(t)}\left|H(F_{m,y},t)\right|dt\right. \\ &+ \left.\int_{A_m}|G_{m,y}(t)|\left|H\left(\frac{g_mk_y}{q},t\right)\right|dt\right\} \\ &=: \quad \mathcal{C}\|fu\|_\infty\left(J_1(y)+J_2(y)\right) \qquad (28)\end{aligned}$$

where $F_{m,y}(t) = Q_{2m+1}(t)g_m(t)k_y(t)$, $\quad G_{m,y}(t) = \frac{w(t)}{u(t)}\frac{Q_{2m+1}(t)q(t)w_m(t)}{w_m(t)}$,
and $w_m(t) = (\sqrt{1-x}+m^{-1})^{2\alpha+1}(\sqrt{1+x}+m^{-1})^{2\beta+1}$. By (22) it follows

$$|F_{m,y}(t)| \leq \mathcal{C}\frac{|k_y(t)|}{w_m(t)}$$

and under the assumptions in (10), in view of (26)

$$J_1(y) \quad \leq \quad \mathcal{C}\int_{-1}^{1}\frac{|k_y(t)|}{w(t)\varphi(t)}\log\left(1+\frac{|k_y(t)|}{w(t)\varphi(t)}\right)dt \leq \mathcal{C}. \qquad (29)$$

In order to estimate J_2 by a result in [17, p.682], we can choose the polynomial $q \in \mathbb{P}_{ml}$, such that under the assumption $\alpha > -\frac{1}{2}, \beta > -\frac{1}{2}$, $q(t) \sim (w\varphi)(t),\ t \in A_m$ and by (22) again, we have

$$|G_{m,y}(t)| \leq \mathcal{C}\frac{w(t)}{u(t)} \leq \mathcal{C}, \quad t \in A_m.$$

Therefore, under the assumption $\frac{k_y}{w\varphi} \in L\log^+ L$, by (26), it follows

$$J_2(y) \leq \mathcal{C} + \mathcal{C}\int_{-1}^{1} \frac{|k_y(t)|}{w(t)\varphi(t)} \log\left(1 + \frac{|k_y(t)|}{w(t)\varphi(t)}\right) dt \leq \mathcal{C}. \tag{30}$$

Combining (29) and (30) with (28) it follows

$$\sup_{y\in \mathrm{S}} \|L_{2m+1}(w,f)k_y\|_1 \leq \mathcal{C}\|fu\|_\infty, \quad \mathcal{C} \neq \mathcal{C}(m,f),$$

i.e. (11) is proved.

Now we prove **(11) $\Rightarrow$ (12).** Consider a linear piecewise function $\bar{f}(x)$ s.t. $\bar{f}(z_k) = \mathrm{sgn}(Q'_{2m+1}(z_k)(x - z_k))$, for $z_k = x_k$ and $\bar{f}(z_k) = 0$, for $z_k = y_k$.

Therefore $\|\bar{f}\|_\infty \leq 1$ and $|L_{2m+1}(w,\bar{f},x)k_y(x)| = \sum_{k=1}^{m} \frac{|Q_{2m+1}(x)|}{|Q'_{2m+1}(x_k)(x-x_k)|}|k_y(x)|$.

By (25) $|L_{2m+1}(w,\bar{f},x)k_y(x)| \geq \mathcal{C}|Q_{2m+1}(x)||k_y(x)| \sum_{k=1}^{m} \frac{\Delta x_k}{|x - z_k|} w(z_k)$ and since $|x - x_k| \leq 2$ and using $\sum_{k=1}^{m} \Delta x_k w(z_k) \geq \mathcal{C}$, we get $|L_{2m+1}(w,\bar{f},x)k_y(x)| \geq \mathcal{C}|Q_{2m+1}(x)||k_y(x)|$. Set $x_0 = -1 = -x_{m+1}$ and let $\epsilon > 0$ be "small". Defined $\epsilon_k = \frac{\epsilon}{4}\Delta x_k$ let $\zeta_m = \left(\bigcup_{k=1}^{m}[x_k - \epsilon_k, x_k + \epsilon_k]\right) \cup [x_0, x_0 + \epsilon_0] \cup [x_{m+1} - \epsilon_{m+1}, x_{m+1}]$ and let $C_{\zeta_m} = [-1,1]\setminus\zeta_m$. Hence by (11),

$$\begin{aligned}\|\bar{f}u\|_\infty &\geq \mathcal{C}\int_{-1}^{1} |L_{2m+1}(w,\bar{f},x)k_y(x)|dx \\ &\geq \mathcal{C}\int_{C_{\zeta_m}} |Q_{2m+1}(x)||k_y(x)|dx \geq \mathcal{C}\int_{C_{\zeta_m}} \frac{|k_y(x)|}{w(x)\varphi(x)}dx,\end{aligned}$$

where last inequality follows by (22). Since $meas(\zeta_m) \leq \epsilon$, we can conclude

$$\|\bar{f}u\|_\infty \geq \mathcal{C}\int_{-1}^{1}|L_{2m+1}(w,\bar{f},x)k_y(x)|dx \geq \mathcal{C}\int_{-1}^{1}\frac{|k_y(x)|}{w(x)\varphi(x)}dx$$

and the thesis follows. □

Proof of Theorem 3.2 . First we note that (14) immediately follows by (11). Now we prove (16). With $P \in \mathcal{P}_{2m}$, under assumption (15), we get

$$\begin{aligned}|e^{\Sigma}_{2m+1}(f,y)| &\leq \int_{-1}^{1} |(f(x) - P(x))k(x,y)|dx + |\Sigma_{2m+1}(f - P, y)| \\ &\leq \mathcal{C}\|(f-P)u\|_\infty \left(\int_{-1}^{1} \frac{|k_y(x)|}{u(x)}dx\right) \leq \mathcal{C}E_{2m}(f)_u. \ \square\end{aligned}$$

Proof of Theorem 3.3 . First of all we remark that under the assumptions (19) both (8) and Theorem 3.2 are true. Therefore, taking into account that, in view of (18), the sequence $\mathcal{T}_{2^n m}(f)$ is obtained by alternating two subsequences of $\{\mathcal{I}_m(f)\}_m$ and $\{\Sigma_{2m+1}(f)\}_m$, then (20) follows by the first inequality in (8) and by (14), while (21) follows by the second inequality in (8) and by (16). □

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