Hindawi
Journal of Function Spaces
Volume 2019, Article ID 2329060, 12 pages
https://doi.org/10.1155/2019/2329060

Review Article

On Sequences of J. P. King-Type Operators

Tuncer Acar,[1] Mirella Cappelletti Montano,[2] Pedro Garrancho,[3] and Vita Leonessa[4]

[1] *Department of Mathematics, Faculty of Science, Selcuk University, Selcuklu, Konya, Turkey*
[2] *Department of Mathematics, University of Bari, Bari, Italy*
[3] *Department of Mathematics, University of Jaén, Jaén, Spain*
[4] *Department of Mathematics, Computer Science and Economics, University of Basilicata, Potenza, Italy*

Correspondence should be addressed to Vita Leonessa; vita.leonessa@unibas.it

Received 28 February 2019; Accepted 2 May 2019; Published 16 May 2019

Academic Editor: Guozhen Lu

This survey is devoted to a series of investigations developed in the last fifteen years, starting from the introduction of a sequence of positive linear operators which modify the classical Bernstein operators in order to reproduce constant functions and x^2 on $[0,1]$. Nowadays, these operators are known as King operators, in honor of J. P. King who defined them, and they have been a source of inspiration for many scholars. In this paper we try to take stock of the situation and highlight the state of the art, hoping that this will be a useful tool for all people who intend to extend King's approach to some new contents within Approximation Theory. In particular, we recall the main results concerning certain King-type modifications of two well known sequences of positive linear operators, the Bernstein operators and the Szász-Mirakyan operators.

1. Introduction

The aim of this paper is to provide a survey on a series of recent investigations which are centered around the problem of obtaining better properties by modifying properly some well known sequences of positive linear operators in the underlying Banach function spaces.

Such results are principally inspired by the pioneering work [1]. In that paper the author, J. P. King, introduces a new sequence $(V_{n,r_n})_{n\geq 1}$ of positive linear Bernstein-type operators defined, for every $f \in C[0,1]$, $n \geq 1$ and $0 \leq x \leq 1$, by

$$V_{n,r_n}(f;x) = \sum_{k=1}^{n} \binom{n}{k} (r_n(x))^k (1-r_n(x))^{n-k} f\left(\frac{k}{n}\right), \quad (1)$$

$r_n : [0,1] \longrightarrow [0,1]$ being continuous functions for every $n \geq 1$. Such operators turn into the classical Bernstein operators B_n whenever, for any $n \geq 1$ and $0 \leq x \leq 1$, $r_n(x) = x$, but unlike the B_n's, they are not in general polynomial-type operators. In fact, for every $n \geq 1$ and $0 \leq x \leq 1$,

$$V_{n,r_n}(\mathbf{1}) = \mathbf{1},$$
$$V_{n,r_n}(e_1) = r_n,$$
$$V_{n,r_n}(e_2) = r_n^2 + \frac{r_n(1-r_n)}{n}, \quad (2)$$

where, for any $t \in [0,1]$, $\mathbf{1}(t) = 1$, and $e_i(t) = t^i$ for $i = 1, 2$. By applying Korovkin theorem to V_{n,r_n}, for every $f \in C[0,1]$, and $x \in [0,1]$, $\lim_{n\longrightarrow\infty} V_{n,r_n}(f;x) = f(x)$ if and only if $\lim_{n\longrightarrow\infty} r_n(x) = x$. Among all possible choices, King focuses his attention on the operators V_{n,r_n^*} that fix e_2, obtained by means of the generating functions

$$r_n^*(x) = \begin{cases} x^2 & \text{if } n = 1, \\ -\dfrac{1}{2(n-1)} + \sqrt{\dfrac{nx^2}{n-1} + \dfrac{1}{4(n-1)^2}} & \text{if } n \geq 2. \end{cases} \quad (3)$$

He shows that $(V_{n,r_n^*})_{n\geq 1}$ is a positive approximation process in $C[0,1]$. Moreover, the operator V_{n,r_n^*} interpolates f at the end points 0 and 1, and it is not a polynomial operator, because of (2) and (3). Through a quantitative estimate in terms of the classical first-order modulus of continuity, King also proves that the order of approximation of $V_{n,r_n^*}(f;x)$ to $f(x)$ is at least as good as the order of approximation of $B_n(f;x)$ to $f(x)$ for $0 \leq x < 1/3$.

A systematic study of the operators V_{n,r_n^*} is due to Gonska and Pițul [2], who determine new estimates for the rate of convergence in terms of the first and second moduli of continuity and, among the others, the behavior of the iterates $V_{n,r_n^*}^m$ as $m \longrightarrow +\infty$.

The A-statistical convergence of operators (1) is considered in [3].

King's idea inspires many other mathematicians to construct other modifications of well-known approximation processes fixing certain functions and to study their approximation and shape preserving properties.

In this review article we try to take stock of the situations and highlight the state of the art, hoping that this will be useful for all people that work in Approximation Theory and intend to apply King's approach in some new contexts.

The paper is organized as follows: after a brief history on what has been done in this research area up to now, in Sections 3 and 4 we illustrate certain King-type modifications of the well-known Bernstein and Szász-Mirakyan operators.

2. A Brief History

From King's work to nowadays, several investigations have been devoted to sequences of positive linear operators fixing certain (polynomial, exponential, or more general) functions. In this section we try to give some essential information about the construction of King-type operators. For all details we refer the readers to the references quoted in the text and we apologize in advance for any possible omission.

We begin to recall the contents of the first papers that generalize in some sense King's idea ([4–7]). In [5] Cárdenas-Morales, Garrancho, and Muñoz-Delgado present a family of sequences of linear Bernstein-type operators $B_{n,\alpha}$ $(n > 1)$, depending on a real parameter $\alpha \geq 0$, and fixing the polynomial function $e_2 + \alpha e_1$ (note that $B_{n,0} = V_{n,r_n^*}$). Among other things, the authors prove that if f is convex and increasing on $[0,1]$, then $f(x) \leq B_{n,\alpha}(f;x) \leq B_n(f;x)$ for every $x \in [0,1]$. Section 3.1 is indeed devoted to the operators $B_{n,\alpha}$. More general results can be found in [8].

On the other hand, in [6] Duman and Özarslan apply the King's original idea to Meyer-König and Zeller operators, and they obtain a better estimation error on the interval $[1/2, 1[$.

The generalizations in [4, 7] contain a different challenge: the authors propose King-type approximation processes in spaces of continuous functions on unbounded intervals.

In particular, in [7] (see also Examples 1) Duman and Özarslan consider the modified Szász-Mirakyan operators reproducing $\mathbf{1}$ and e_2 and obtain better error estimates on the whole interval $[0,\infty)$.

A study in full generality is undertaken in [4]. In fact, in that article, Agratini indicates how to construct sequences $(L_n^*)_{n\geq 1}$ of positive linear operators of discrete type that act on a suitable weighted subspace of $C[0,\infty)$ and preserve $\mathbf{1}$ and e_2. Besides the variant of Szász-Mirakjan operators, introduced independently in [7], he also constructs a variant of Baskakov and Bernstein-Chlodovsky operators.

In [9] Agratini investigates convergence and quantitative estimates for the bivariate version of the general operators previously considered in [4]. It is worthwhile noticing that the above results seem to be the only obtained in a multidimensional setting.

Subsequently, other articles appear. First, we recall the paper due to Duman, Özarslan, and Aktuğlu [10] in which Szász-Mirakyan-Beta type operators preserving e_2 are considered. Moreover, Duman and Özarslan, jointly with Della Vecchia ([11]), study a Kantorovich modification of Szász-Mirakjan type operators preserving linear functions, and they show their operators enable better error estimation on the interval $[1/2,\infty)$ than the classical Szász-Mirakjan-Kantorovich operators.

Post Widder and Stancu operators are instead object of a modification that preserves e_2 in polynomial weighted spaces, proposed by Rempulska and Skorupka in [12]. Also in this case better approximation properties than the original operators are achieved.

Another new general approach is considered by Agratini and Tarabie in [13] (see also [14]). The authors construct classes of discrete linear positive operators, acting on $[0,1]$ or on $[0,\infty)$, and preserving both the constants and the polynomial $e_2 + \alpha e_1$ $(\alpha \geq 0)$. Those classes of operators include the ones considered in [5] and a new modification of Szász-Mirakyan operators (see also [15]).

Modifications which fix constants and linear functions, or the function e_2, have been introduced in [16–20] (see also [21, Chapter 5]). In particular, such studies are concerned with modified Bernstein-Durrmeyer operators, Phillips operators, integrated Szász-Mirakjan operators, Beta operators of the second kind, and a Durrmeyer-Stancu type variant of Jain operators.

New King-type operators which reproduce e_1 and e_2 are studied in [22] by Braica, Pop and Indrea. Subsequently, Pop's school deals with modifications of Kantorovich type operators, Durrmeyer type operators, Schurer operators, Bernstein-type operators, and Baskakov operators, fixing exactly two test functions from the set $\{\mathbf{1}, e_1, e_2\}$, (see, e.g., [23, 24]).

Another general approach deserves to be mentioned. Coming back to the classical Bernstein operators B_n, in [25] Gonska, Pițul, and Rașa construct a sequence of King-type operators V_n^τ which preserve $\mathbf{1}$ and a strictly increasing function $\tau \in C[0,1]$, such that $\tau(0) = 0$ and $\tau(1) = 1$. Such operators are defined as $V_n^\tau(f) = B_n(f) \circ (B_n\tau)^{-1} \circ \tau$, and they are a positive approximation process in $C[0,1]$. Moreover, they preserve some global smoothness properties. The authors also discuss the monotonicity of the sequence $(V_n^\tau f)_{n\geq 1}$ when f is a convex and decreasing function. They establish a Voronovskaja-type theorem, and finally they prove a recursion formula generalizing a corresponding result valid for the classical Bernstein operators. Note that the class of operators presented in [25] recovers the cases previously studied in [1, 5].

Subsequently, the study of the operators V_n^τ has been deepened by Birou in [26], where he finds some conditions under which V_n^τ's provide a lower approximation error than the classical Bernstein operators for the class of decreasing and generalized convex functions (see, also [27]). Moreover, he analyzes some shape preserving properties in the case τ is a polynomial of degree at most 2, or $\tau(x) = (e^{bx} - 1)/(e^b - 1)$ ($x \in [0,1], b < 0$).

Very soon, the construction of the operators V_n^τ motivates other works.

In [27] the operators $B_n^\tau(f) = B_n(f \circ \tau^{-1}) \circ \tau$ which fix the function τ are studied and, among other things, they are compared with B_n's and V_n^τ's in the approximation of functions which are increasing and convex with respect to τ. The authors focus on the case for which B_n^τ and V_n^τ fix polynomials of degree m (see [28] for other generalizations of B_n's reproducing **1** and a strictly increasing polynomial). For more details about B_n^τ, see Section 3.2.

Subsequently, the above idea has been applied to other positive linear operators (see [29–33]).

In particular, in [32] the authors propose a generalization of the classical Szász-Mirakyan operators S_n by setting $S_n^\rho(f) = S_n(f \circ \rho^{-1}) \circ \rho$, where ρ is a continuously differentiable function on $[0,\infty)$ with $\rho(0) = 0$ and $\inf_{x\ge 0} \rho'(x) \ge 1$. We want to point out that this class of operators does not include the ones studied in [7]. However, very recently (see [34]; cf. Section 4.1), Aral, Ulusoy, and Deniz generalize the operators S_n^ρ, extending in this way the results contained in [7, 32]. See [35] for a modification of Baskakov-type operators in the spirit of what has been done for S_n^ρ.

We want to emphasize that the above constructions based on fixing suitable increasing functions do not recover the interesting case of linear operators fixing exponential functions, which has been a new and very popular direction in this research area in the last few years.

A sequence of Bernstein-type operators preserving $e^{\lambda_0 x}$ and $e^{\lambda_1 x}$, $\lambda_0, \lambda_1 \in \mathbb{R}$, $\lambda_0 \neq \lambda_1$, was already present in the literature (see [36, 37]).

In [38] a modification of Szász-Mirakyan operators preserving constants and e^{2ax}, $a > 0$, is considered, while in [39] another modification of Szász-Mirakyan operators reproducing e^{ax} and e^{2ax} ($a > 0$) is studied. For more details about these two different variants, see Section 4.2.

Later, the idea of preserving exponential functions of different type has been applied to some other well-known linear positive operators, for which approximation and shape preserving properties, as well as quantitative estimates and Voronovskaya-type theorems, are proven.

For papers inspired by [38, 39] we refer the readers to [40–42] and [43–46], respectively.

For modifications of linear operators preserving constants and e^{-x}, constants and e^{-2x}, or constants and e^{Ax}, $A \in \mathbb{R}$ cf. [47–51].

We end this section underlying that King's idea has been applied also to some q– or (p,q)– analogue operators (see, e.g., [52–56]) and to some sequences of operators involving orthogonal polynomials (see, e.g., [57]).

3. On Bernstein-Type Operators

In this section we review some results contained in [5, 27, 43], where the authors deal with different modifications of the Bernstein operators based on King's idea.

Let us start with some preliminaries. Throughout this section, $C[0,1]$ is the space of all continuous real valued functions on $[0,1]$, endowed with the sup norm $\|\cdot\|_\infty$ and the natural pointwise ordering. If $k \in \mathbb{N}$, the symbol $C^k[0,1]$ stands for the space of all continuously k-times differentiable functions on $[0,1]$.

We recall that the classical Bernstein operators are the positive linear operators $B_n : C[0,1] \longrightarrow C[0,1]$ defined by setting, for every $n \ge 1$, $f \in C[0,1]$, and $0 \le x \le 1$,

$$B_n(f;x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f\left(\frac{k}{n}\right). \quad (4)$$

It is very well known that the sequence $(B_n)_{n\ge 1}$ is an approximation process in $C[0,1]$; i.e., for every $f \in C[0,1]$, $\lim_{n\longrightarrow\infty} B_n(f) = f$ uniformly on $[0,1]$.

In what follows, it will be useful to recall the following inequality which is an estimate of the rate of the above approximation presented by Shisha and Mond: for any $C[0,1]$,

$$\left|B_n(f;x) - f(x)\right| \le \left(1 + \frac{x(1-x)/n}{\delta^2}\right)\omega_1(f,\delta), \quad (5)$$

where $\omega_1(f,\delta)$ is the first-order modulus of continuity.

Besides the usual notion of convexity, other notions of convexity will be considered (see [58]; see also [59]).

Let $\{u,v\}$ be an extended complete Tchebychev system on $[0,1]$.

A function $f : (0,1) \longrightarrow \mathbb{R}$ is said to be convex with respect to $\{u\}$ (in symbols $f \in \mathscr{C}(u)$), whenever

$$\begin{vmatrix} u(x_0) & u(x_1) \\ f(x_0) & f(x_1) \end{vmatrix} \ge 0, \quad 0 < x_0 < x_1 < 1. \quad (6)$$

Moreover, a function $f : (0,1) \longrightarrow \mathbb{R}$ is said to be convex with respect to $\{u,v\}$, in symbol $f \in \mathscr{C}(u,v)$, whenever

$$\begin{vmatrix} u(x_0) & u(x_1) & u(x_2) \\ v(x_0) & v(x_1) & v(x_2) \\ f(x_0) & f(x_1) & f(x_2) \end{vmatrix} \ge 0, \quad 0 < x_0 < x_1 < x_2 < 1. \quad (7)$$

If $f \in C[0,1]$, then (6) and (7) hold for $0 \le x_0 < x_1 < x_2 \le 1$.

For the convenience of the reader we split up the discussion into three subsections.

3.1. Bernstein-Type Operators Fixing Polynomials. In [5], the following Bernstein-type operators, depending on a real parameter $\alpha \ge 0$, are defined:

$$B_{n,\alpha}(f;x) := \sum_{k=0}^{n} \binom{n}{k} r_{n,\alpha}(x)^k (1 - r_{n,\alpha}(x))^{n-k} f\left(\frac{k}{n}\right) \quad (8)$$

($n \geq 1$, $f \in C[0,1]$, $x \in [0,1]$), where $\{r_{n,\alpha} : [0,1] \longrightarrow \mathbb{R}\}_{n>1}$ is the sequence of functions defined by

$$r_{n,\alpha}(x) := -\frac{n\alpha+1}{2(n-1)} + \sqrt{\frac{(n\alpha+1)^2}{4(n-1)^2} + \frac{n(\alpha x + x^2)}{n-1}} \quad (0 \leq x \leq 1). \tag{9}$$

It is easy to check that $B_{n,\alpha}f = (B_n f) \circ (r_{n,\alpha})$. Note that, if $= 0 B_{n,\alpha}$'s turn into the classical King operators (1), while if α goes to infinity they become the classical Bernstein operators.

The operators $B_{n,\alpha}$ are positive and map $C[0,1]$ into itself, and they fix the functions $\mathbf{1}$ and $e_2 + \alpha e_1$. Moreover, $B_{n,\alpha}(e_1) = r_{n,\alpha}$ and $B_{n,\alpha}(e_2) = (1/n) r_{n,\alpha} + ((n-1)/n) r_{n,\alpha}^2$.

Korovkin theorem can be applied in order to conclude that, for $f \in C[0,1]$, $\lim_{n\longrightarrow\infty} B_{n,\alpha}(f;x) = f(x)$ for $0 \leq x \leq 1$ since, for all $\alpha \geq 0$, $r_{n,\alpha}(x)$ converges to x.

Considering the first and second modulus of smoothness, the following quantitative estimates can be achieved:

$$\left|B_{n,\alpha}(f;x) - f(x)\right| \leq \left(1 + \frac{2x^2 + \alpha x - r_{n,\alpha}(x)(\alpha + 2x)}{\delta^2}\right) \omega_1(f,\delta), \tag{10}$$

$$\left|B_{n,\alpha}(f;x) - f(x)\right| \leq \frac{\left|r_{n,\alpha}(x) - x\right|}{\delta} \omega_1(f,\delta) + \left(1 + \frac{2x^2 + \alpha x - r_{n,\alpha}(x)(\alpha + 2x)}{2\delta^2}\right) \omega_2(f,\delta). \tag{11}$$

By comparing estimates (10) and (5), we have then the approximation error for the operators $B_{n,\alpha}$ is at least as good as the one for B_n's on the interval $[0, H_\alpha]$, where $H_\alpha = (1 - 2\alpha + \sqrt{1 + 8\alpha + 4\alpha^2})/6$. Indeed, we have that the inequality

$$2x^2 + \alpha x - r_{n,\alpha}(x)(\alpha + 2x) \leq \frac{x(1-x)}{n} \tag{12}$$

holds if and only if

$$0 \leq x \leq \frac{1 + n - 2n\alpha + \sqrt{1 + 2n + n^2 + 8n^2\alpha + 4n^2\alpha^2}}{2(1+3n)}. \tag{13}$$

Note that the right-end term in the above inequalities decreases to H_α as n goes to infinity. We point out that for $H_0 = 1/3$ we recover the result due to King, while for $\alpha \longrightarrow +\infty$ we get $H_\alpha \longrightarrow 1/2$; therefore King's result is improved.

The operators $B_{n,\alpha}$ share some shape preserving properties. We begin to recall that they map continuous and increasing functions into (continuous) increasing functions. Moreover, if f is convex and increasing, then $B_{n,\alpha}(f)$ is convex. Finally, if f is convex with respect to $\{1, e_2 + \alpha e_1\}$, then $B_{n,\alpha}(f) \geq f$ on $[0,1]$.

The operators $B_{n,\alpha}$ verify the following asymptotic formula:

$$\lim_{n\longrightarrow\infty} 2n\left(B_{n,\alpha}(f;x) - f(x)\right) = x(1-x)\left(f''(x) - \frac{2}{2x+\alpha} f'(x)\right), \tag{14}$$

for all functions $f \in C[0,1]$, which are two times differentiable at $x \in (0,1)$.

We end this subsection observing that if we impose additional conditions on f, we can get tangible improvements in the approximation error. In fact, if $f \in C[0,1]$ is increasing and if the divided difference $f[x_0, x_1, x_2]$ of f on the nodes $0 \leq x_0 < x_1 < x_2 \leq 1$ satisfy $f[x_0,x_1,x_2] \geq M$, M being a real strictly positive constant, there exists $\overline{\alpha} \geq 0$ such that

$$0 \leq B_{n,\alpha}(f;x) - f(x) < B_n(f;x) - f(x), \quad \text{for } \alpha \geq \overline{\alpha} \text{ and } 0 < x < 1. \tag{15}$$

In particular, $\overline{\alpha} := \min\{\alpha \geq 0 : (f(1) - f(0))/(1+\alpha) \leq M\}$. Note that, if $f \in C^2[0,1]$ is increasing and strictly convex and M is the lower bound of f'', then $\overline{\alpha} = 2f'(1)/M$.

3.2. Polynomial Operators Fixing Increasing Functions. The operators considered in the previous section fix $\mathbf{1}, e_2 + \alpha e_1$, but they are not polynomial-type operators. The construction of polynomial-type operators fixing the above functions is presented in [27]. In that paper operators of the form $B_n^\tau f = B_n(f \circ \tau^{-1}) \circ \tau$ are considered, where τ is any infinitely times continuously differentiable function on $[0,1]$, such that $\tau(0) = 0$, $\tau(1) = 1$ and $\tau'(x) > 0$. More precisely,

$$B_n^\tau(f;x) = \sum_{k=0}^{n} \binom{n}{k} \tau(x)^k (1 - \tau(x))^{n-k} \left(f \circ \tau^{-1}\right)\left(\frac{k}{n}\right), \quad f \in C[0,1], x \in [0,1]. \tag{16}$$

The Bernstein operators can be obtained as a particular case for $\tau = e_1$. On the other hand, if $\tau = (e_2 + \alpha e_1)/(1+\alpha)$, B_n^τ is a polynomial-type operator and $B_n^\tau(\tau) = \tau$. For a Durrmeyer variant of the operators B_n^τ we refer the readers to [29] (and for a genuine Durrmeyer variant see [33]).

We note that $B_n^\tau \tau^2 = \tau/n + ((n-1)/n)\tau^2$. From the positivity of B_n^τ, together with the fact that $\{1, \tau, \tau^2\}$ is an extended complete Tchebychev system on $[0,1]$, we easily get that $\lim_{n\longrightarrow\infty} B_n^\tau(f) = f$ uniformly on $[0,1]$.

Moreover, the operators B_n^τ map continuous and increasing functions into (continuous) and increasing functions. Finally, $B_n^\tau(f)$ is τ-convex of order k provided that f is so too (if $k \in \mathbb{N}$, we say that a function $f \in C^k[0,1]$ is τ-convex of order k whenever $D_\tau^m f = D^m(f \circ \tau^{-1}) \circ \tau$, D^k being the usual k-th differential operator).

For any function $f \in C[0,1]$, two times differentiable at $x \in (0,1)$, we have that

$$\lim_{n\longrightarrow\infty} 2n\left(B_n^\tau f(x) - f(x)\right) = \tau(x)(1 - \tau(x))\left(-\frac{\tau''(x) f'(x)}{\tau'^3} + \frac{f''(x)}{\tau'^2}\right). \tag{17}$$

We end this subsection by comparing the operators B_n^τ with B_n's.

First, if we take a positive constant K, whose existence is guaranteed by Freud [60], such that $K(t-x)^2 \leq \tau'^2$ for all

$t, x \in [0,1]$; we have the following estimate: for $f \in C[0,1]$, $\delta > 0$, and $x \in [0,1]$,

$$\begin{aligned} &\left|B_n^{\tau}(f;x) - f(x)\right| \\ &\quad \leq \omega_1(f,\delta)\left(1 + \frac{\tau'(x)\,\tau(x)\,(1-\tau(x))}{nK\delta^2}\right). \end{aligned} \tag{18}$$

Moreover, the following statement holds.

Theorem 1. *Let $f \in C^2[0,1]$. Suppose that there exists $n_0 \in \mathbb{N}$ such that*

$$f(x) \leq B_n^{\tau}(f;x) \leq B_n(f;x), \quad \forall n \geq n_0,\ x \in (0,1). \tag{19}$$

Then

$$\begin{aligned} f''(x) &\geq \frac{\tau''(x)}{\tau'(x)} f'(x) \\ &\geq \left(1 - \frac{x(1-x)\,\tau'^2}{\tau(x)(1-\tau(x))}\right) f''(x), \\ &\qquad\qquad x \in (0,1). \end{aligned} \tag{20}$$

In particular, $f''(x) \geq 0$.

Conversely, if (20) holds with strict inequalities at a given point $x_0 \in (0,1)$, then there exists $n_0 \in \mathbb{N}$ such that for $n \geq n_0$

$$f(x_0) < B_n^{\tau}(f;x_0) < B_n(f;x_0). \tag{21}$$

The proof is based on the comparison between the expression $x(1-x)$ and $\tau(x)(1-\tau(x))(-\tau''(x)f'(x)/\tau'^3 + f''(x)/\tau'^2)$ in the asymptotic formulae for B_n's and B_n^{τ}'s, respectively.

3.3. Fixing Increasing Exponential Functions. In this section we discuss the operators defined in [43]. From now on, set $a_n(x) := (e^{\mu x/n} - 1)/(e^{\mu/n} - 1)$ and recall that $\exp_\mu(x) := e^{\mu x}$ ($\mu > 0$). We define the sequence of positive linear operators $\mathscr{G}_n$ as

$$\mathscr{G}_n(f;x) = \exp_\mu(x)\, B_n\left(\frac{f}{\exp_\mu}; a_n(x)\right), \tag{22}$$

or, equivalently,

$$\begin{aligned} &\mathscr{G}_n(f;x) \\ &\quad = \sum_{k=0}^{n} \binom{n}{k} a_n(x)^k (1 - a_n(x))^{n-k} f\left(\frac{k}{n}\right) e^{-\mu k/n} e^{\mu x}, \end{aligned} \tag{23}$$

for $f : [0,1] \longrightarrow \mathbb{R}$, $n \geq 1$, and $0 \leq x \leq 1$. The functions fixed by these operators are $\exp_\mu$ and $\exp_\mu^2$ ($\mu > 0$). Moreover, for any $x \in [0,1]$ and $n \geq 1$, the following identities hold:

$$\begin{aligned} &\mathscr{G}_n(\mathbf{1};x) = e^{\mu(x-1)}\left(e^{\mu/n} + 1 - e^{\mu x/n}\right)^n, \\ &\mathscr{G}_n\left(\exp_\mu^3;x\right) = e^{\mu x}\left(e^{\mu(x+1)/n} + e^{\mu x/n} - e^{\mu/n}\right)^n, \\ &\mathscr{G}_n\left(\exp_\mu^4;x\right) \\ &\quad = e^{\mu x}\left(e^{\mu(x+2)/n} + e^{\mu(x+1)/n} + e^{\mu x/n} - e^{\mu/n} - e^{2\mu/n}\right)^n. \end{aligned} \tag{24}$$

Since $\{\mathbf{1}, \exp_\mu, \exp_\mu^2\}$ is an extended complete Tchebychev system, and the operators $\mathscr{G}_n$ are positive, they are an approximation process in $C[0,1]$ (i.e., for each $f \in C[0,1]$, $\lim_{n\longrightarrow\infty}\mathscr{G}_n(f;x) = f(x)$ uniformly w.r.t. $x \in [0,1]$).

Other (shape preserving) properties that this sequence verifies are

(i) if $f/\exp_\mu$ is increasing, then it is $\mathscr{G}_n(f)/\exp_\mu$;

(ii) if $f/\exp_\mu$ is increasing and convex, then $\mathscr{G}_n(f/\exp_\mu)$ is convex;

(iii) if $f \in \mathscr{C}(\exp_\mu)$, then $\mathscr{G}_n(f) \in \mathscr{C}(\exp_\mu)$ (see (6)).

Moreover,

$$\begin{aligned} &\left|\mathscr{G}_n(f;x) - f(x)\right| \leq \left|f(x)\right| \left(\mathscr{G}_n(\mathbf{1};x) - 1\right) \\ &\quad + \left(\mathscr{G}_n(\mathbf{1};x) + \frac{e^{2\mu x}\left(\mathscr{G}_n(\mathbf{1};x) - 1\right)}{\delta^2}\right) \\ &\quad \cdot \omega_1\left(f \circ \log_\mu;\delta\right), \end{aligned} \tag{25}$$

for $f \in C[0,1]$, $x \in (0,1)$, and $\delta > 0$. Here $\log_\mu$ denotes the inverse function of $\exp_\mu$. If $\mu \geq 1$, then $\omega_1(f \circ \log_\mu;\delta)$ can be replaced by $\omega_1(f;\delta)$.

For the operators $\mathscr{G}_n$, the following Voronovskaya-type result holds:

$$\begin{aligned} &\lim_{n\longrightarrow\infty} 2n\left(\mathscr{G}_n(f;x) - f(x)\right) \\ &\quad = x(1-x)\left(f''(x) - 3\mu f'^2 f(x)\right), \end{aligned} \tag{26}$$

if $f \in C[0,1]$ has second derivative at a point $x \in (0,1)$.

As in the previous subsection, by comparing the asymptotic formulae for B_n and $\mathscr{G}_n$, we are able to get an improvement in the approximation by means of operators $\mathscr{G}_n$ with respect to the operators B_n under certain conditions.

Theorem 2. *Let $f \in C^2[0,1]$. Suppose that there exists $n_0 \in \mathbb{N}$ such that*

$$f(x) \leq \mathscr{G}_n(f;x) \leq B_n(f;x), \quad \forall n \geq n_0,\ x \in (0,1). \tag{27}$$

Then

$$f''(x) \geq 3\mu f'^2 f(x) \geq 0, \quad x \in (0,1). \tag{28}$$

In particular, $f''(x) \geq 0$.

Conversely, if (28) holds with strict inequalities at a given point $x \in (0,1)$, then there exists $n_0 \in \mathbb{N}$ such that for $n \geq n_0$

$$f(x) < \mathscr{G}_n(f;x) < B_n(f;x). \tag{29}$$

We end this section by observing that if the following conjecture is true, we might obtain an even better improvement in the approximation error.

Conjecture 3. *If $f \in C[0,1]$ is such that $f \in \mathscr{C}(\exp_\mu)$ and $f \in \mathscr{C}(\exp_\mu, \exp_\mu^2)$, then for all $n \in \mathbb{N}$ and all $x \in [0,1]$, one has that $f(x) \leq \mathscr{G}_n(f;x) \leq B_n(f;x)$.*

4. On Szász-Mirakyan Type Operators

In the present section we pass to discuss sequences of positive linear operators acting on spaces of continuous functions on unbounded intervals. To this end, we need to fix preliminarily some notations and recall definition and main results concerning the classical Szász-Mirakyan operators.

First of all, we denote by $C[0,\infty)$ the space of all continuous real valued functions on $[0,\infty)$. We also indicate by $C_b[0,\infty)$ the subspace of all continuous bounded functions on $[0,\infty)$. The space $C_b[0,\infty)$, endowed with the sup-norm $\|\cdot\|_\infty$ and the natural pointwise ordering, is a Banach lattice. Moreover, the space of all continuous functions that converge at infinity will be denoted by $C^*[0,\infty)$.

In what follows, let φ be a weight function on $[0,\infty)$; we define

$$B_\varphi[0,\infty) = \{f : [0,\infty) \longrightarrow \mathbb{R} \mid \text{there exists } M_f \geq 0 \text{ such that } |f(x)| \leq M_f \varphi(x) \ \forall x \geq 0\}. \tag{30}$$

Clearly, $B_\varphi[0,\infty)$ is a normed space when endowed with the weighed norm

$$\|f\|_\varphi = \sup_{x\geq 0} \frac{|f(x)|}{\varphi(x)} \quad (f \in B_\varphi[0,\infty)). \tag{31}$$

Moreover, we denote by $C_\varphi[0,\infty)$ the space of all continuous functions in $B_\varphi[0,\infty)$, and by $C^*_\varphi[0,\infty)$ the space consisting of all functions in $C_\varphi[0,\infty)$ that converge at infinity. Finally, we say that $f \in U_\varphi[0,\infty)$ if f/φ is uniformly continuous.

It is well known that Szász-Mirakyan operators were introduced independently in the 1940s by J. Favard ([61]), G. M. Mirakjan ([62]), and O. Szász ([63]), and they are defined by setting

$$S_n(f;x) := \sum_{k=0}^{\infty} e^{-nx} \frac{(nx)^k}{k!} f\left(\frac{k}{n}\right) \quad (n \geq 1, x \geq 0), \tag{32}$$

for all functions $f : [0,\infty) \longrightarrow \mathbb{R}$ for which the series at the right-hand side is absolutely convergent. This space includes, in particular, all functions $f : [0,\infty) \longrightarrow \mathbb{R}$ such that $|f(x)| \leq M\exp(\alpha x)$ $(x \geq 0)$, for some $M \geq 0$ and $\alpha \in \mathbb{R}$.

In particular S_n's map $C_b[0,\infty)$ and $C^*[0,\infty)$ into themselves.

It might be useful for the following subsections to recall that (see [64, Lemma 3]), $S_n(\mathbf{1}) = \mathbf{1}$, $S_n(e_1) = e_1$, and $S_n(e_2) = e_2 + (1/n)e_1$.

Moreover, for every $x \geq 0$,

$$\begin{aligned} S_n(\psi_x(t);x) &= 0, \\ S_n(\psi_x^2(t);x) &= \frac{x}{n}, \end{aligned} \tag{33}$$

where, for every $y \geq 0$, $\psi_x(y) = y - x$.

It is well known that the sequence $(S_n)_{n\geq 1}$ is an approximation process in $C^*[0,\infty)$; more precisely, for every $f \in C^*[0,\infty)$, $\lim_{n\to\infty} S_n(f;x) = f(x)$ uniformly w.r.t. $x \in [0,\infty)$.

In particular, we recall that, taking (33) into account, for every $f \in C_b[0,\infty)$, $x \geq 0$ and $n \geq 1$ (see, for example, [65, Theorem 5.1.2]),

$$\begin{aligned} |S_n(f;x) - f(x)| &\leq 2\omega_1\left(f, \sqrt{S_n(\psi_x^2(t);x)}\right) \\ &= 2\omega_1\left(f, \sqrt{\frac{x}{n}}\right), \end{aligned} \tag{34}$$

where $\omega_1(f,\delta)$ denotes the classical first modulus of continuity.

This last result might be useful to compare the Szász-Myrakyan operators with suitable generalizations that fix different functions.

4.1. Generalized Szász-Mirakyan Operators. In this subsection, we examine the Szász-Mirakyan type operators studied in [34]. Let $\rho : [0,\infty) \longrightarrow \mathbb{R}$ be a function satisfying the following properties:

(a) ρ is continuously differentiable on $[0,\infty)$;

(b) $\rho(0) = 0$ and $\inf_{x\geq 0}\rho'(x) \geq 1$.

From now on, we set

$$\varphi(x) = 1 + \rho^2(x) \quad (x \geq 0), \tag{35}$$

and we consider the weighted spaces $B_\varphi[0,\infty)$, $C_\varphi[0,\infty)$, $C^*_\varphi[0,\infty)$, and $U_\varphi[0,\infty)$.

If $\rho(x) = x$ for each $x \geq 0$ the space $C_\varphi[0,\infty)$ (resp., $C^*_\varphi[0,\infty)$) becomes the classical weighed space

$$E_2 = \left\{f \in C[0,\infty) : \sup_{x\geq 0} \frac{f(x)}{1+x^2} \in \mathbb{R}\right\} \tag{36}$$

(resp.,

$$E_2^* = \left\{f \in C[0,\infty) : \lim_{x\to+\infty} \frac{f(x)}{1+x^2} \in \mathbb{R}\right\}). \tag{37}$$

The following result, proven in [66], shows that $\{\mathbf{1}, \rho, \rho^2\}$ is a Korovkin set in $C^*_\varphi[0,\infty)$.

Theorem 4. *Consider a sequence $(L_n)_{n\geq 1}$ of positive linear operators from $C_\varphi[0,\infty)$ into $B_\varphi[0,\infty)$. If*

$$\lim_{n\to\infty} \|L_n(\rho^\nu) - \rho^\nu\|_\varphi = 0 \quad \textit{for } \nu = 0,1,2, \tag{38}$$

and, then, for every $f \in C^*_\varphi[0,\infty)$,

$$\lim_{n\to\infty} \|L_n(f) - f\|_\varphi = 0. \tag{39}$$

After these preliminaries, set $\mathbb{N}_1 := \{n \in \mathbb{N} \mid n \geq n_0\}$, for a suitable $n_0 \in \mathbb{N}$. Given an interval $I \subset [0,\infty)$, consider two sequences $(\alpha_n)_{n\geq 1}$, $(\beta_n)_{n\geq 1}$ of functions on I such that, for any $n \in \mathbb{N}_1$,

(i) $\alpha_n, \beta_n : I \longrightarrow \mathbb{R}$ are positive functions on I;

(ii) $\beta_n(x) - \alpha_n(x) \geq 0$ for every $x \in I$.

In [34], the authors introduced and studied the sequence of the generalized Szász-Myrakjan operators, defined as

$$\widetilde{S}_n^{\rho}(f;x) = e^{-\alpha_n(x)} \sum_{k=0}^{\infty} \frac{(\beta_n(x))^k}{k!} \left(f \circ \rho^{-1}\right)\left(\frac{k}{n}\right) \tag{40}$$

for every $f \in C(I), n \in \mathbb{N}_1$ and $x \in I$.

Some conditions have to be imposed in order that the sequence $(\widetilde{S}_n^{\rho})_{n\geq n_0}$ is an approximation process in $C_{\varphi}^*[0,\infty)$, and, in particular, in order to verify (38).

More precisely, for any $n \geq n_0$, there exist $u_n, v_n : I \longrightarrow \mathbb{R}$ such that, for every $x \in I$,

$$\begin{gathered} |u_n(x)| \leq u_n^0, \\ |v_n(x)| \leq v_n^0, \\ \lim_{n\longrightarrow\infty} u_n^0 = \lim_{n\longrightarrow\infty} v_n^0 = 0, \end{gathered} \tag{41}$$

$$\widetilde{S}_n^{\rho}(\mathbf{1};x) = 1 + u_n(x), \tag{42}$$

$$\widetilde{S}_n^{\rho}(\rho;x) = \rho(x) + v_n(x). \tag{43}$$

Evaluating the operators $\widetilde{S}_n^{\rho}$ on $\mathbf{1}$ and ρ, it is easy to connect the sequences $(\alpha_n)_{n\geq n_0}$ and $(\beta_n)_{n\geq n_0}$ with $(u_n)_{n\geq n_0}$ and $(v_n)_{n\geq n_0}$, taking (40), (42), and (43) into account. More precisely, for every $x \in I$ and $n \geq n_0$,

$$\begin{gathered} \alpha_n(x) = n\frac{\rho(x) + v_n(x)}{1 + u_n(x)} - \log\left(1 + u_n(x)\right), \\ \beta_n(x) = n\frac{\rho(x) + v_n(x)}{1 + u_n(x)}. \end{gathered} \tag{44}$$

Accordingly, for any $n \geq n_0$, $f \in C(I)$ and $x \in I$,

$$\begin{aligned} \widetilde{S}_n^{\rho}(f;x) &= e^{-n((\rho(x)+v_n(x))/(1+u_n(x)))}\left(1 + u_n(x)\right) \\ &\cdot \sum_{k=0}^{\infty} \frac{1}{k!}\left(n\frac{\rho(x) + v_n(x)}{1 + u_n(x)}\right)^k \left(f \circ \rho^{-1}\right)\left(\frac{k}{n}\right). \end{aligned} \tag{45}$$

The operators $\widetilde{S}_n^{\rho}$ map $C_{\varphi}[0,\infty)$ into $B_{\varphi}[0,\infty)$. Moreover, since easy calculations show that, for every $x \in I$ and $n \geq n_0$,

$$\widetilde{S}_n^{\rho}(\rho^2;x) = \frac{(\rho(x) + v_n(x))^2}{1 + u_n(x)} + \frac{\rho(x) + v_n(x)}{n}, \tag{46}$$

by applying Theorem 4 to an extension of the operators $\widetilde{S}_n^{\rho}(f)$ to $[0,\infty)$,

$$\lim_{n\longrightarrow\infty} \sup_{x\in I} \frac{\left|\widetilde{S}_n^{\rho}(f;x) - |f(x)|\right|}{\varphi(x)} = 0. \tag{47}$$

Some estimates of the rate of convergence are available, by using a suitable modulus of continuity, introduced by Holhoş in [67]. More precisely, it is defined by setting, for every $f \in C_{\varphi}[0,\infty)$ and $\delta > 0$,

$$\omega_{\rho}(f;\delta) = \sup_{\substack{x,t\geq 0 \\ |\rho(t)-\rho(x)|\leq\delta}} \frac{|f(t) - f(x)|}{\varphi(t) + \varphi(x)}. \tag{48}$$

In particular, by using the results in [67], it can be proven that, for every $f \in C_{\varphi}[0,\infty)$ and $n \geq n_0$,

$$\begin{aligned} &\left\|\widetilde{S}_n^{\rho}(f) - f\right\|_{\varphi^{3/2}} \\ &\quad \leq \left(7 + 4u_n^0 + 2\left(2v_n^0 + \left(v_n^0\right)^2 + \frac{2}{n} + \frac{2v_n^0}{n}\right)\right) \\ &\quad \cdot \omega_{\rho}(f;\delta_n), \end{aligned} \tag{49}$$

where

$$\begin{aligned} \delta_n = {} & \frac{16}{n} + \frac{4}{n^2} + 3u_n^0 + 20v_n^0 + \frac{22v_n^0}{n} + \frac{4v_n^0}{n^2} + 8\left(v_n^0\right)^2 \\ & + \frac{6\left(v_n^0\right)^2}{n} + \left(v_n^0\right)^3 \\ & + 2\sqrt{\left(1 + u_n^0\right)\left(\frac{2}{n} + u_n^0 + 4v_n^0 + \frac{2v_n^0}{n} + \left(v_n^0\right)^2\right)}. \end{aligned} \tag{50}$$

Moreover, since $\lim_{\delta\longrightarrow 0}\omega_{\rho}(f;\delta) = 0$ if $f \in U_{\varphi}[0,\infty)$, from the latter formula and (41), we get that

$$\lim_{n\longrightarrow\infty} \left\|\widetilde{S}_n^{\rho}(f) - f\right\|_{\varphi^{3/2}} = 0 \tag{51}$$

for every $f \in U_{\varphi}[0,\infty)$.

Further, under suitable assumptions, it is possible to determine a Voronovskaya-type result involving $\widetilde{S}_n^{\rho}$'s. More precisely, assume that

$$\begin{gathered} \lim_{n\longrightarrow\infty} nu_n(x) = l_1, \\ \lim_{n\longrightarrow\infty} nv_n(x) = l_2. \end{gathered} \tag{52}$$

Moreover, consider a function $f \in C_{\varphi}[0,\infty)$ for which the function $f \circ \rho^{-1}$ is twice differentiable. If the second derivative of $f \circ \rho^{-1}$ is bounded on $[0,\infty)$, then, for every $x \in I$,

$$\begin{aligned} &\lim_{n\longrightarrow\infty} n\left(\widetilde{S}_n^{\rho}(f;x) - f(x)\right) \\ &\quad = f(x)\,l_1 + \left(l_2 - \rho(x)\,l_1\right)\left(f \circ \rho^{-1}\right)'(\rho(x)) \\ &\quad + \frac{1}{2}\rho(x)\left(f \circ \rho^{-1}\right)''(\rho(x)). \end{aligned} \tag{53}$$

The following examples show that, for suitable choices of the sequences $(u_n(x))_{n\geq n_0}$, $(v_n(x))_{n\geq n_0}$ and of the function ρ, operators (45) turn into well known Szász-Myrakjan type operators that fix certain functions and the results in [34] can be applied to those operators. For quantitative Voronovskaya theorems and the study of a Durrmeyer-type variant of the operators (40) see [68] and [69], respectively.

Examples 1. (1) If $I = [0,\infty)$, $u_n(x) = v_n(x) = 0$, and $\rho(x) = x$ for every $x \geq 0$, the operators $\widetilde{S}_n^{\rho}$ turn into the classical Szász-Myrakjan operators (32), which, as it is well known, preserve the function e_0 and $\rho = e_1$.

(2) If $I = [0,\infty)$, $\rho(x) = x$, $u_n(x) = 0$, and $v_n(x) = -1/2n + \sqrt{4n^2x^2+1}/2n - x$, then operators $\widetilde{S_n}^{\rho}$ turn into

$$D_n^*(f;x) = e^{(1-\sqrt{4n^2x^2+1})/2}\sum_{k=0}^{\infty}\frac{\left(\sqrt{4n^2x^2+1}-1\right)^k}{2^k k!}f\left(\frac{k}{n}\right) \tag{54}$$

($f \in E_2, n \geq 1, x \geq 0$), which were object of investigation in [7] and, in the spirit of King's work, preserve the function e_0 and $\rho^2 = e_2$.

In particular, when applied to D_n^*, (53) gives the following result. If $f \in E_2$ is a function which is twice differentiable and whose second derivative is bounded on $[0,\infty)$, then, for every $x \geq 0$,

$$\lim_{n\to\infty} n\left(D_n^*(f;x) - f(x)\right) = -\frac{1}{2}f'(x) + \frac{x}{2}f''(x). \tag{55}$$

Formula (55) holds true uniformly w.r.t. $x \geq 0$, if $f', f'' \in E_2^*$. An estimate of convergence in (55) can be found in [70, Corollary 4].

By means of [65, Theorem 5.1.2], we have that, for every $f \in C_b[0,\infty)$,

$$\left|D_n^*(f;x) - f(x)\right| \leq 2\omega_1\left(f, \sqrt{D_n^*(\psi_x^2(x))}\right). \tag{56}$$

We point out that, as shown in [7], for every $x \geq 0$

$$D_n^*\left(\psi_x^2(t);x\right) = 2x^2 + \frac{x}{n} - \frac{x\sqrt{4n^2x^2+1}}{n}. \tag{57}$$

Easy calculations prove that $D_n^*(\psi_x^2(t);x) \leq S_n(\psi_x^2(t);x)$ for every $x \geq 0$, so that, at least for $f \in C_b[0,\infty)$, the operators D_n^* provide a better approximation error than the classical Szász-Myrakjan operators S_n (see (34)).

(3) If $I = [1/(n_0-1),\infty)$, $u_n(x) = 1/(nx-1)$, $v_n(x) = 0$, and $\rho(x) = x$, then $\widetilde{S_n}^{\rho}$'s are exactly the operators studied in [71], given by

$$S_n^*(f;x) = \frac{nxe^{1-nx}}{nx-1}\sum_{k=0}^{\infty}\frac{(nx-1)^k}{k!}f\left(\frac{k}{n}\right) \tag{58}$$

($f \in E_2, n \geq n_0, x \in I$). Those operators fix the functions $\rho = e_1$ and $\rho^2 = e_2$. In this case, the Voronovskaya-type formula becomes

$$\lim_{n\to\infty} n\left(S_n^*(f;x) - f(x)\right) = \frac{f(x)}{x} - f'(x) + \frac{x}{2}f''(x), \tag{59}$$

for all $x \in I$ and all $f \in E_2$ which are twice differentiable and whose second derivative is bounded.

(4) For $I = [0,\infty)$, $u_n(x) = v_n(x) = 0$ for every $x \geq 0$ and considering an arbitrary function ρ satisfying (a) and (b), the operators $\widetilde{S_n}^{\rho}$ reduce to

$$S_n^{\rho}(f;x) = e^{-n\rho(x)}\sum_{k=0}^{\infty}\frac{(n\rho(x))^k}{k!}\left(f\circ\rho^{-1}\right)\left(\frac{k}{n}\right) \tag{60}$$

($f \in C_{\varphi}[0,\infty), x \in I, n \geq 1$) that were introduced and studied in [32] and preserve the functions e_0 and ρ.

In particular, for every $f \in C_{\varphi}[0,\infty)$,

$$\left\|S_n^{\rho}(f) - f\right\|_{\varphi^{3/2}} \leq \left(7 + \frac{4}{n}\right)\omega_{\rho}\left(f; \frac{20}{n} + \sqrt{\frac{8}{n}}\right). \tag{61}$$

Moreover, if $f \in C_{\varphi}[0,\infty)$ is a function such that $f\circ\rho^{-1}$ is twice differentiable and the second derivative of $f\circ\rho^{-1}$ is bounded on $[0,\infty)$, then, for every $x \geq 0$,

$$\lim_{n\to\infty} n\left(S_n^{\rho}(f;x) - f(x)\right) = \frac{1}{2}\rho(x)\left(f\circ\rho^{-1}\right)''(\rho(x)). \tag{62}$$

4.2. Fixing Increasing Exponential Functions. Another recent modification of the sequence of Szász-Mirakyan operators relies on the preservation of some exponential functions.

For functions $f \in C[0,\infty)$, such that the right-hand side below is absolutely convergent, Szász-Mirakyan operators reproducing the functions $\mathbf{1}$ and e^{2ax}, $a > 0$, are introduced in [38] and defined by

$$R_n^*(f;x) := e^{-n\alpha_n(x)}\sum_{k=0}^{\infty}\frac{(n\alpha_n(x))^k}{k!}f\left(\frac{k}{n}\right) \tag{63}$$

($x \geq 0, n \in \mathbb{N}$), in such a way that the conditions

$$R_n^*\left(e^{2at};x\right) = e^{2ax} \tag{64}$$

are satisfied for all x and all n. To provide condition (64), equality

$$\alpha_n(x) = \frac{2ax}{n\left(e^{2a/n}-1\right)} \tag{65}$$

must be held (for more details see [38]).

To investigate the approximation properties of the operators R_n^*, some preliminaries are needed. First, if $a \geq 0$, we get

$$\begin{aligned} R_n^*\left(e^{at};x\right) &= e^{n\alpha_n(x)(e^{a/n}-1)} = e^{2ax/(e^{a/n}+1)}, \\ R_n^*(\mathbf{1};x) &= 1, \\ R_n^*(e_1;x) &= \alpha_n(x) \\ R_n^*(e_2;x) &= \alpha_n^2(x) + \frac{\alpha_n(x)}{n}. \end{aligned} \tag{66}$$

Then, letting $\psi_x^k(t) := (t-x)^k$, $k = 0, 1, 2, \ldots$, we have

$$\begin{aligned} R_n^*\left(\psi_x^0(t);x\right) &= 1, \\ R_n^*\left(\psi_x^1(t);x\right) &= \alpha_n(x) - x, \\ R_n^*\left(\psi_x^2(t);x\right) &= \left(\alpha_n(x) - x\right)^2 + \frac{\alpha_n(x)}{n}. \end{aligned} \tag{67}$$

Moreover, considering equality (65), one can find

$$\lim_{n\longrightarrow\infty} n\left(\frac{2ax}{n\left(e^{2a/n}-1\right)}-x\right)=-ax,$$
$$\lim_{n\longrightarrow\infty} n\left(\left(\frac{2ax}{n\left(e^{2a/n}-1\right)}-x\right)^2+\frac{2ax}{n^2\left(e^{2a/n}-1\right)}\right) \tag{68}$$
$$=x.$$

In 1970, Boyanov and Veselinov [72] showed that uniform convergence of any sequence of positive linear operators acting on $C^*[0,\infty)$ can be checked as follows.

Theorem 5. *The sequence $A_n : C^*[0,\infty) \longrightarrow C^*[0,\infty)$ of positive linear operators satisfies the conditions*

$$\lim_{n\longrightarrow\infty} A_n\left(e^{-kt};x\right)=e^{-kt}, \quad k=0,1,2, \tag{69}$$

uniformly in $[0,\infty)$, if and only if

$$\lim_{n\longrightarrow\infty} A_n(f;x)=f(x) \tag{70}$$

uniformly in $[0,\infty)$, for all $f \in C^*[0,\infty)$.

A quantitative form for Theorem 5 can be given using the modulus of continuity on $C^*[0,\infty)$ introduced in [73, Corollary 3.2] and defined as

$$\omega^*(f,\delta)=\sup_{\substack{x,t\geq 0\\ |e^{-x}-e^{-t}|\leq\delta}}\left|f(x)-f(t)\right| \tag{71}$$

$(\delta>0, f\in C^*[0,\infty))$.

Theorem 6. *For $f \in C^*[0,\infty)$, we have*

$$\left\|R_n^*(f)-f\right\|_\infty \leq 2\omega^*\left(f;\sqrt{2\beta_n+\gamma_n}\right), \tag{72}$$

where

$$\beta_n=\left\|R_n^*\left(e^{-t};x\right)-e^{-x}\right\|_\infty,$$
$$\gamma_n=\left\|R_n^*\left(e^{-2t};x\right)-e^{-2x}\right\|_\infty. \tag{73}$$

Moreover, β_n and γ_n tend to zero as n goes to infinity so that $R_n^* f$ converges uniformly to f.

To investigate pointwise convergence of the operators R_n^* a quantitative Voronovskaya theorem is presented in [38] as well. Such a result allows establishing the rate of pointwise convergence and an upper bound for the error of approximation.

Theorem 7. *Let $f, f'' \in C^*[0,\infty)$. Then the inequality*

$$\left|n\left[R_n^*(f;x)-f(x)\right]+axf'(x)-\frac{x}{2}f''(x)\right|$$
$$\leq\left|p_n(x)\right|\left|f'(x)\right|+\left|q_n(x)\right|\left|f''(x)\right| \tag{74}$$
$$+2\left(2q_n(x)+x+r_n(x)\right)\omega^*\left(f'';\frac{1}{\sqrt{n}}\right)$$

holds for any $x \in [0,\infty)$, where

$$p_n(x):=nR_n^*\left(\psi_x(t);x\right)+ax,$$
$$q_n(x):=\frac{1}{2}\left(nR_n^*\left(\psi_x^2(t);x\right)-x\right), \tag{75}$$
$$r_n(x):=n^2\sqrt{R_n^*\left(\left(e^{-x}-e^{-t}\right)^4;x\right)}\sqrt{R_n^*\left(\psi_x^4(t);x\right)}.$$

As a uniform approximation result let us recall, as explained in [38], that the spaces $(C^*[0,\infty), \|\cdot\|_{[0,\infty)})$ and $(C[0,1], \|\cdot\|_{[0,1]})$ are isometrically isomorphic. Define $\psi(y):=e^{-y}$, $y\in[0,\infty)$, and let $T: C[0,1]\longrightarrow C^*[0,\infty)$ be given by

$$T(f)(y)=f^*(y)=f(\psi(y)), \qquad f\in C[0,1],\ y\in[0,\infty). \tag{76}$$

We remark that

$$\lim_{t\longrightarrow\infty} f^*(t)=\lim_{t\longrightarrow\infty} f(\psi(t))=f(0). \tag{77}$$

Clearly, T is linear and bijective. Moreover, for all $f\in C[0,1]$ one has

$$\|Tf\|_{[0,\infty)}=\sup_{t\in[0,\infty)}\left|f(\psi(t))\right|=\|f\|_{[0,1]}. \tag{78}$$

Hence T is an isometric isomorphism and

$$T^{-1}\left(f^*\right)=f^*\circ\psi^{-1}, \quad \text{for } f^*\in C^*[0,\infty). \tag{79}$$

Corollary 8. *For all $f^* \in C^*[0,\infty)$ $(f = f^*\circ\psi^{-1})$ and n large enough we have*

$$\left\|R_n^*f^*-f^*\right\|_{[0,\infty)}\leq\omega_1\left(f;\sqrt{\frac{1}{2}\left(\gamma_n+2\beta_n\right)}\right)_{[0,1]} + 2\omega_2\left(f;\sqrt{\frac{1}{2}\left(\gamma_n+2\beta_n\right)}\right)_{[0,1]}. \tag{80}$$

To see some of the advantages of new constructions of Szász-Mirakyan operators the following comparisons results were also presented in [38].

First, note that the definition of generalized convexity considered in $[0,1]$ (cf. (7)) can be given also in $[0,\infty)$ (see [59, 74]). More precisely, in this subsection we consider functions $f\in C[0,\infty)$ convex with respect to $\{1,\nu\}$, in short $\{1,\nu\}$-convex, where

$$\nu(x)=e^{2ax}, \quad a>0. \tag{81}$$

Observe that this is equivalent to $f\circ\nu^{-1}$ being convex in the classical sense. Moreover, if function $f\in C^2[0,\infty)$ (the space of twice continuously differentiable functions), then f is $\{1,\nu\}$-convex if and only if

$$f''(x)\geq 2af'(x), \quad x>0 \tag{82}$$

(see [26]).

Theorem 9. *Let $f \in C^2[0,\infty)$ be increasing and $\{1,v\}$-convex. Then*

$$f(x) \leq R_n^*(f;x) \leq S_n(f;x) \quad \text{for } x \geq 0. \tag{83}$$

The above-mentioned modified sequence of Szász-Mirakyan operators reproduces the functions $\mathbf{1}$ and e^{2ax}, $a > 0$. Another modification of Szász-Mirakyan operators reproducing the functions e^{ax} and e^{2ax}, $a > 0$, was introduced in [39] as

$$\mathcal{R}_n(f;x) = e^{-n\alpha_n(x)} \sum_{k=0}^{\infty} \frac{(n\beta_n(x))^k}{k!} f\left(\frac{k}{n}\right), \quad n \in \mathbb{N}, x \in [0,\infty), \tag{84}$$

where

$$\beta_n(x) = \frac{ax}{ne^{a/n}(e^{a/n}-1)}, \qquad \alpha_n(x) = \frac{ax\left(2-e^{a/n}\right)}{n(e^{a/n}-1)}. \tag{85}$$

This choice provides that

$$\mathcal{R}_n\left(e^{at};x\right) = e^{ax}, \qquad \mathcal{R}_n\left(e^{2at};x\right) = e^{2ax}. \tag{86}$$

For the operators $\mathcal{R}_n$, it can be shown that

(1) $\mathcal{R}_n(\mathbf{1};x) = e^{ax((e^{a/n}-1)/e^{a/n})}$,

(2) $\mathcal{R}_n(e_1;x) = (ax/ne^{a/n}(e^{a/n}-1))e^{ax((e^{a/n}-1)/e^{a/n})}$,

(3) $\mathcal{R}_n(e_2;x) = \{(ax/ne^{a/n}(e^{a/n}-1))^2 + ax/n^2e^{a/n}(e^{a/n}-1)\}e^{ax((e^{a/n}-1)/e^{a/n})}$,

and if one considers the central moment operator $\mu_n^s(x) = \mathcal{R}_n(\Psi_x^s;x)$ of order s ($s = 0,1,2\ldots$), the following formulae hold:

(1) $\mu_n^0(x) = e^{ax((e^{a/n}-1)/e^{a/n})}$,

(2) $\mu_n^1(x) = (ax/ne^{a/n}(e^{a/n}-1) - x)e^{ax((e^{a/n}-1)/e^{a/n})}$,

(3) $\mu_n^2(x) = \{(ax/ne^{a/n}(e^{a/n}-1) - x)^2 + ax/n^2e^{a/n}(e^{a/n}-1)\}e^{ax((e^{a/n}-1)/e^{a/n})}$.

Now set

$$\varphi(x) = 1 + e^{2ax} \quad (x \geq 0) \tag{87}$$

and consider the space $B_\varphi[0,\infty)$ (resp., $C_\varphi[0,\infty)$, $C_\varphi^*[0,\infty)$) defined by (30) and (31).

The first result on uniform convergence of sequence of the operators $\mathcal{R}_n$ was given in [39] by the following.

Theorem 10. *For each function $f \in C_\varphi^*[0,\infty)$*

$$\lim_{n\to\infty} \|\mathcal{R}_n(f) - f\|_\varphi = 0. \tag{88}$$

In order to approximate unbounded functions, the exponential weighed space $C_a[0,\infty)$ (with a fixed $a > 0$), consisting of $f \in C[0,\infty)$ satisfying the condition $|f(x)| \leq Me^{ax}$, where M is a positive constant, is considered and this space is a normed space with the norm

$$\|f\|_a = \sup_{x\in[0,\infty)} \frac{|f(x)|}{e^{ax}}. \tag{89}$$

Also let $C_a^k[0,\infty)$ be subspace of all functions $f \in C_a[0,\infty)$ such that $\lim_{x\to\infty}(|f(x)|/e^{ax}) = k$, where k is a positive constant. A weighted modulus of continuity is defined by

$$\widetilde{\omega}(f;\delta) = \sup_{|t-x|\leq\delta, x\geq 0} \frac{|f(t)-f(x)|}{e^{at}+e^{ax}}, \tag{90}$$

for $f \in C_a^k[0,\infty)$. We note that if $f \in C_a^k[0,\infty)$, then $\lim_{\delta\to 0}\widetilde{\omega}(f;\delta) = 0$ and $\widetilde{\omega}(f;m\delta) \leq 2m\widetilde{\omega}(f;\delta)$ for any $m \in \mathbb{N}$ (for more details we refer the readers to [39, Section 5]).

Theorem 11. *For $f \in C_a^k[0,\infty)$*

$$\|\mathcal{R}_n(f) - f\|_{5a/2} \leq \frac{a}{ne}\|f\|_a + C\widetilde{\omega}\left(f;\frac{1}{\sqrt{n}}\right), \tag{91}$$

where C is positive constant.

In [39, Section 6] a Voronovskaja-type result is also presented.

Theorem 12. *Let $f \in C_\varphi[0,\infty)$. If f is twice differentiable in $x \in [0,\infty)$ and f'' is continuous in x, and then*

$$\lim_{n\to\infty} n[\mathcal{R}_n(f,x) - f(x)] = a^2xf(x) - \frac{3}{2}axf'(x) + \frac{x}{2}f''(x). \tag{92}$$

Finally, the following saturation results for the sequence $(\mathcal{R}_n)_{n\geq 1}$ hold (see [39, Section 7]).

Theorem 13. *Let $f \in C_\varphi[0,\infty)$ and consider a bounded open interval $J \subset [0,\infty)$. Then, for each $x \in J$*

$$n(\mathcal{R}_nf(x) - f(x)) = o(1) \quad \text{if and only if } f \in \left\langle e^{ax}, e^{2ax}\right\rangle. \tag{93}$$

Theorem 14. *Let $f \in C_\varphi[0,\infty)$, $M \geq 0$ and let $J \subset [0,\infty)$ be a bounded open interval. Then, for each $x \in J$, one has that*

$$n|\mathcal{R}_nf(x) - f(x)| \leq M + o(1) \tag{94}$$

if and only if

$$\left|a^2xf(t) - \frac{3}{2}atf'(t) + \frac{t}{2}f''(t)\right| \leq M, \quad \text{for almost every } t \in J. \tag{95}$$

Conflicts of Interest

The authors declare that they have no conflicts of interest.

References

[1] J. P. King, "Positive linear operators which preserve x^2," *Acta Mathematica Hungarica*, vol. 99, no. 3, pp. 203–208, 2003.

[2] H. Gonska and P. Pitul, "Remarks on a article of J.P. King," *Commentationes Mathematicae Universitatis Carolinae*, vol. 46, no. 4, pp. 645–652, 2005.

[3] O. Duman and C. Orhan, "An abstract version of the Korovkin approximation theorem," *Publicationes Mathematicae Debrecen*, vol. 69, no. 1-2, pp. 33–46, 2006.

[4] O. Agratini, "Linear operators that preserve some test functions," *International Journal of Mathematics and Mathematical Sciences*, vol. 2006, Article ID 94136, 11 pages, 2006.

[5] D. Cardenas-Morales, P. Garrancho, and F. J. Munos-Delgado, "Shape preserving approximation by Bernstein-type operators which fix polinomials," *Applied Mathematics and Computation*, vol. 182, pp. 1615–1622, 2006.

[6] O. Duman and M. A. Özarslan, "MKZ type operators providing a better estimation on [1/2,1)," *Canadian Mathematical Society*, vol. 50, no. 3, pp. 434–439, 2007.

[7] O. Duman and M. A. Özarslan, "Szász-Mirakjan type operators providing a better error estimation," *Applied Mathematics Letters*, vol. 20, no. 12, pp. 1184–1188, 2007.

[8] X.-W. Xu, X.-M. Zeng, and R. Goldman, "Shape preserving properties of univariate Lototsky-Bernstein operators," *Journal of Approximation Theory*, vol. 224, pp. 13–42, 2017.

[9] O. Agratini, "On a class of linear positive bivariate operators of King type," *Studia Universitatis Babes-Bolyai*, vol. LI, no. 4, pp. 13–22, 2006.

[10] O. Duman, M. A. Özarslan, and H. Aktuglu, "Better error estimation for Szász-Mirakjan-Beta operators," *Journal of Computational Analysis and Applications*, vol. 10, no. 1, pp. 53–59, 2008.

[11] O. Duman, M. A. Özarslan, and B. D. Vecchia, "Modified Szász-Mirakjan-Kantorovich operators preserving linear functions," *Turkish Journal of Mathematics*, vol. 33, no. 2, pp. 151–158, 2009.

[12] L. Rempulska and K. Tomczak, "On approximation by Post-Widder and Stancu operators," *Kyungpook Mathematical Journal*, vol. 49, pp. 57–65, 2009.

[13] O. Agratini and S. Tarabie, "On approximating operators preserving certain polynomials," *Automation Computers Applied Mathematics*, vol. 17, no. 2, pp. 191–199, 2008.

[14] O. Agratini, "An asymptotic formula for a class of approximation processes of King's type," *Studia Scientiarum Mathematicarum Hungarica*, vol. 47, no. 4, pp. 435–444, 2010.

[15] Ö. G. Yilmaz, A. Aral, and F. Tasdelen Yesidal, "On Szász-Mirakyan type operators preserving polynomials," *Journal of Numerical Analysis and Approximation Theory*, vol. 46, no. 1, pp. 93–106, 2017.

[16] V. Gupta, "A note on modified Phillips operators," *Southeast Asian Bulletin of Mathematics*, vol. 34, pp. 847–851, 2010.

[17] V. Gupta and N. Deo, "A note on improved estimates for integrated Szász-Mirakyan operators," *Mathematica Slovaca*, vol. 61, no. 5, pp. 799–806, 2011.

[18] V. Gupta and O. Duman, "Bernstein Durrmeyer type operators preserving linear function," *Matematicki Vesnik*, vol. 62, no. 4, pp. 259–264, 2010.

[19] V. Gupta and R. Yadav, "Better approximation by Stancu Beta operators," *Revue d'Analyse Numérique et de Théorie de l'Approximation*, vol. 40, no. 2, pp. 149–155, 2011.

[20] V. N. Mishra, P. Patel, and L. N. Mishra, "The integral type modification of Jain operators and its approximation properties," *Numerical Functional Analysis and Optimization*, vol. 39, no. 12, pp. 1265–1277, 2018.

[21] V. Gupta and R. P. Agarwal, *Convergence Estimates in Approximation Theory*, Springer, New York, NY, USA, 2014.

[22] P. I. Braica, L. I. Piscoran, and A. Indrea, "About a King-type operator," *Applied Mathematics & Information Sciences*, vol. 6, no. 1, pp. 145–148, 2012.

[23] P. I. Braica and O. T. Pop, "Some general Kantorovich type operators," *Revue d'Analyse Numérique et de Théorie de l'Approximation*, vol. 41, no. 2, pp. 114–124, 2012.

[24] O. T. Pop, D. Barbosu, and P. I. Braica, "Bernstein-type operators which preserve exactly two test functions," *Studia Scientiarum Mathematicarum Hungarica*, vol. 50, no. 4, pp. 393–405, 2013.

[25] H. Gonska, P. Pitul, and I. Rasa, "General King-type operators," *Results in Mathematics*, vol. 53, no. 3-4, pp. 279–286, 2009.

[26] M. Birou, "A note about some general King-type operators," *Annals of the Tiberiu Popoviciu Seminar of Functional Equations, Approximation and Convexity*, vol. 12, pp. 3–16, 2014.

[27] D. Cárdenas-Morales, P. Garrancho, and I. Raşa, "Bernstein-type operators which preserve polynomials," *Computers & Mathematics with Applications*, vol. 62, no. 1, pp. 158–163, 2011.

[28] J. M. Aldaz and H. Render, "Generalized Bernstein operators on the classical polynomial spaces," *Mediterranean Journal of Mathematics*, vol. 15, no. 6, article 222, 22 pages, 2018.

[29] T. Acar, A. Aral, and I. Rasa, "Modified Bernstein-Durrmeyer operators," *Mathematics*, vol. 22, no. 1, pp. 27–41, 2014.

[30] A. M. Acu, P. N. Agrawal, and T. Neer, "Approximation properties of the modified Stancu operators," *Numerical Functional Analysis and Optimization*, vol. 38, no. 3, pp. 279–292, 2017.

[31] A.-M. Acu, N. Manav, and A. Ratiu, "Convergence properties of certain positive linear operators," *Results in Mathematics*, vol. 74, no. 1, article 8, p. 24, 2019.

[32] A. Aral, D. Inoan, and I. Raşa, "On the generalized Szász-Mirakyan operators," *Results in Mathematics*, vol. 65, no. 3-4, pp. 441–452, 2014.

[33] S. A. Mohiuddine, T. Acar, and M. Alghamdi, "Genuine modified Bernstein-Durrmeyer operators," *Journal of Inequalities and Applications*, vol. 104, 13 pages, 2018.

[34] A. Aral, G. Ulusoy, and E. Deniz, "A new construction of Szász-Mirakyan operators," *Numerical Algorithms*, vol. 77, no. 2, pp. 313–326, 2018.

[35] A. Erençin, A. Olgun, and F. Tasdelen, "Generalized Baskakov type operators," *Mathematica Slovaca*, vol. 67, no. 5, pp. 1269–1277, 2017.

[36] J. M. Aldaz, O. Kounchev, and H. Render, "Bernstein operators for exponential polynomials," *Constructive Approximation. An International Journal for Approximations and Expansions*, vol. 29, no. 3, pp. 345–367, 2009.

[37] J. M. Aldaz and H. Render, "Optimality of generalized Bernstein operators," *Journal of Approximation Theory*, vol. 162, no. 7, pp. 1407–1416, 2010.

[38] T. Acar, A. Aral, and H. Gonska, "On Szász-Mirakyan operators preserving e^{2ax}, $a > 0$," *Mediterranean Journal of Mathematics*, vol. 14, no. 1, article 6, p. 14, 2017.

[39] T. Acar, A. Aral, D. Cardenas-Morales, and P. Garrancho, "Szász-Mirakyan type operators which fix exponentials," *Results in Mathematics*, vol. 72, pp. 1393–1404, 2017.

[40] M. Bodur, Ö. Gürel Yılmaz, and A. Aral, "Approximation by baskakov-szász-stancu operators éreserving exponential functions," *Constructive Mathematical Analysis*, vol. 1, no. 1, pp. 1–8, 2018.

[41] S. N. Deveci, T. Acar, F. Usta, and O. Alagoz, "Approximation by Gamma type operators," submitted.

[42] Ö. G. Yilmaz, V. Gupta, and A. Aral, "On Baskakov operators preserving the exponential function," *Journal of Numerical Analysis and Approximation Theory*, vol. 46, no. 2, pp. 150–161, 2017.

[43] A. Aral, D. Cardenas-Morales, and P. Garrancho, "Bernstein-type operators that reproduce exponential functions," *Journal of Mathematical Inequalities*, vol. 12, no. 3, pp. 861–872, 2018.

[44] A. Aral, M. L. Limmam, and F. Ozsarac, "Approximation properties of Szász-Mirakyan-Kantorovich type operators," *Mathematical Methods in the Applied Sciences*, pp. 1–8, 2018.

[45] S. N. Deveci, F. Usta, and T. Acar, "Gamma operators reproducing exponential functions," submitted.

[46] F. Ozsarac and T. Acar, "Reconstruction of Baskakov operators preserving some exponential functions," *Mathematical Methods in the Applied Sciences*, pp. 1–9, 2018.

[47] T. Acar, M. Cappelletti Montano, P. Garrancho, and V. Leonessa, "On Bernstein-Chlodovsky operators preserving e^{2x}," In press.

[48] V. Gupta and A. M. Acu, "On Baskakov-Szász-Mirakyan type operators preserving exponential type functions," *Positivity*, vol. 22, pp. 919–929, 2018.

[49] V. Gupta and A. Aral, "A note on Szász-Mirakyan-Kantorovich type operators preserving e^{-x}," *Positivity*, vol. 22, pp. 415–423, 2018.

[50] V. Gupta and N. Malik, "Approximation with certains Szász-Mirakyan operators," *Khayyam Journal of Mathematics*, vol. 3, no. 2, pp. 90–97, 2017.

[51] V. Gupta and G. Tachev, "On approximation properties of Phillips operators preserving exponential functions," *Mediterranean Journal of Mathematics*, vol. 14, no. 4, article 177, 12 pages, 2017.

[52] T. Acar, P. N. Agrawal, and A. Sathish Kumar, "On a modification of (p,q)-Szász-Mirakyan operators," *Complex Analysis and Operator Theory*, vol. 12, no. 1, pp. 155–167, 2018.

[53] O. Agratini and O. Dogru, "Weighted approximation by q-Szász-King type operators," *Taiwanese Journal of Mathematics*, vol. 14, no. 4, pp. 1283–1296, 2010.

[54] Q.-B. Cai, "Approximation properties of the modification of q-Stancu-Beta operators which preserve x^2," *Journal of Inequalities and Applications*, vol. 2014, no. 505, 8 pages, 2014.

[55] M. Mursaleen and S. Rahman, "Dunkl generalization of q-Szász-Mirakjan operators which preserve x^2," *Filomat*, vol. 32, no. 3, pp. 733–747, 2018.

[56] N. I. Mahmudov, "q-Szász-Mirakjan operators which preserve x^2," *Journal of Computational and Applied Mathematics*, vol. 235, no. 16, pp. 4621–4628, 2011.

[57] N. Deo and M. Dhamija, "Charlier-Szász-Durrmeyer type positive linear operators," *Afrika Matematika*, vol. 29, no. 1-2, pp. 223–232, 2018.

[58] S. Karlin and W. Studden, *Tchebycheff Systems: with Applications in Analysis and Statistics*, Interscience, New York, NY, USA, 1966.

[59] Z. Ziegler, "Linear approximation and generalized convexity," *Journal of Approximation Theory*, vol. 1, pp. 420–443, 1968.

[60] G. Freud, "On approximation by positive linear methods I, II," *Studia Scientiarum Mathematicarum Hungarica*, vol. 3, pp. 365–370, 1968.

[61] J. Favard, "Sur les multiplicateurs d'interpolation," *Journal de Mathématiques Pures et Appliquées*, vol. 23, no. 9, pp. 219–247, 1944.

[62] G. M. Mirakjan, "Approximation of continuous functions with the aid of polynomials," *Doklady Akademii Nauk SSSR*, vol. 31, pp. 201–205, 1941 (Russian).

[63] O. Szász, "Generalization of Bernstein's polynomials to the infinite interval," *Journal of Research of the National Bureau of Standards*, vol. 45, pp. 239–245, 1950.

[64] M. Becker, "Global approximation theorems for Szász-Mirakjan and Baskakov operators in polynomial weight spaces," *Indiana University Mathematics Journal*, vol. 27, no. 1, pp. 127–142, 1978.

[65] F. Altomare and M. Campiti, *Korovkin-Type Approximation Theory and Its Applications, De Gruyter Studies in Mathematics*, vol. 17, Walter de Gruyter & Co., Berlin, Germany, 1994.

[66] A. D. Gadjiev, "A problem on the convergence of a sequence of positive linear operators on unbounded sets and theorems that are analogous to P.P. Korovkin's theorem," *Doklady Akademii Nauk SSSR*, vol. 218, pp. 1001–1004, 1974.

[67] A. Holhos, "Quantitative estimates for positive linear operators in weighted spaces," *General Mathematics*, vol. 16, no. 4, pp. 99–110, 2008.

[68] T. Acar, "Asymptotic formulas for generalized Szasz-Mirakyan operators," *Applied Mathematics and Computation*, vol. 263, pp. 233–239, 2015.

[69] T. Acar and G. Ulusoy, "Approximation properties of generalized Szasz-Durrmeyer Operators," *Periodica Mathematica Hungarica*, vol. 72, no. 1, pp. 64–75, 2016.

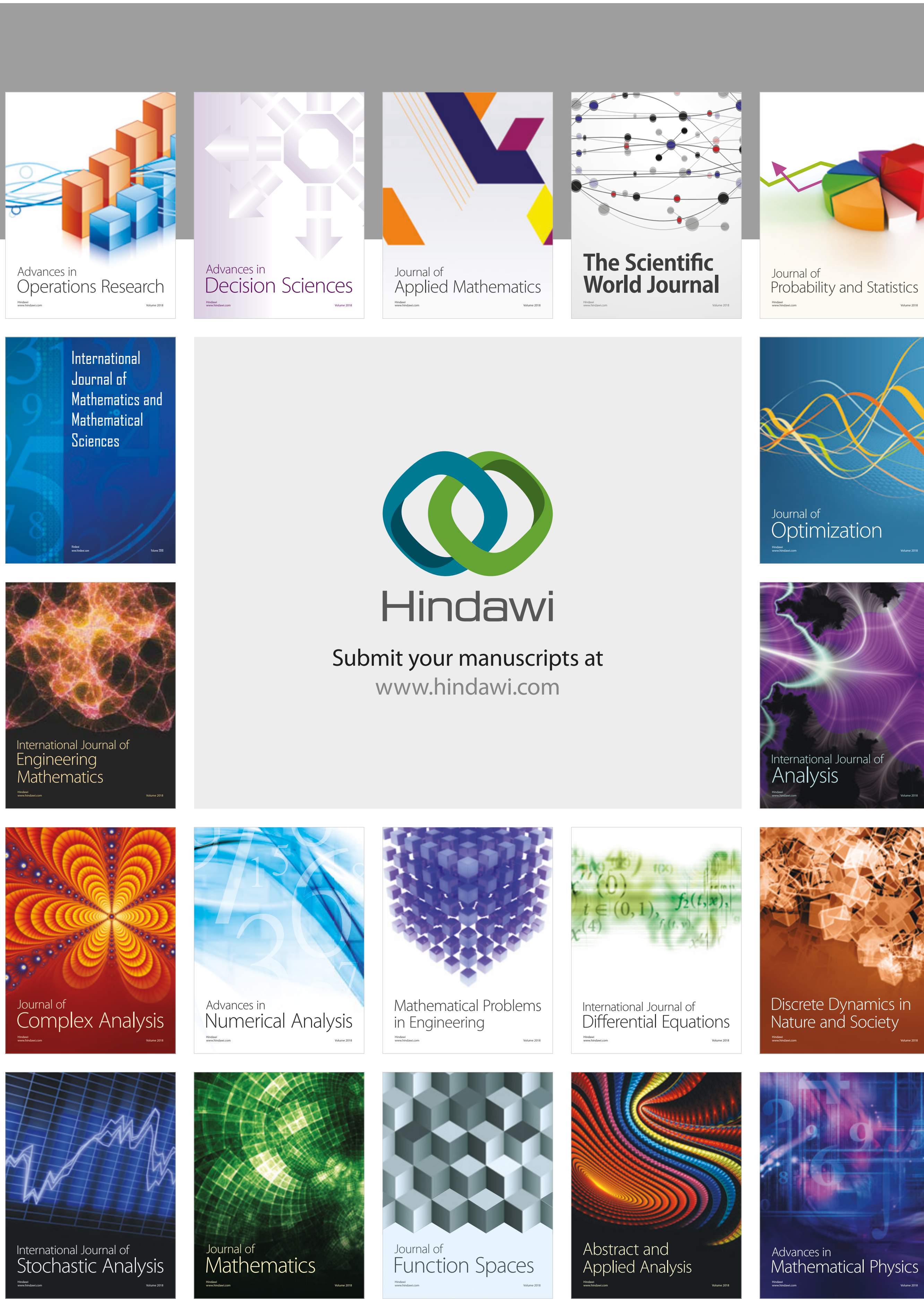
[70] C. Bardaro and I. Mantellini, "A quantitative asymptotic formula for a general class of discrete operators," *Computers & Mathematics with Applications. An International Journal*, vol. 60, no. 10, pp. 2859–2870, 2010.

[71] P. I. Braica, L. I. Piscoran, and A. Indrea, "Grafical structure of some King type operators," *Acta Universitatis Apulensis*, vol. 34, pp. 163–171, 2013.

[72] B. D. Boyanov and V. M. Veselinov, "A note on the approximation of functions in an infinite interval by linear positive operators," *Bulletin Mathematique de la Societe des Sciences Mathematiques de Roumanie*, vol. 14, no. 62, pp. 9–13, 1970.

[73] A. Holhos, "The rate of approximation of functions in an infinite interval by positive linear operators," *Studia Universitatis Babes-Bolyai Mathematica*, vol. LV, no. 2, pp. 133–142, 2010.

[74] M. Bessenyei, *Hermite-Hadamard-Type Inequalities for Generalized Convex Functions [Dissertation]*, University of Debrecen, 2004.


Advances in Operations Research
Advances in Decision Sciences
Journal of Applied Mathematics
The Scientific World Journal
Journal of Probability and Statistics
International Journal of Mathematics and Mathematical Sciences
Hindawi
Submit your manuscripts at
www.hindawi.com
Journal of Optimization
International Journal of Engineering Mathematics
International Journal of Analysis
Journal of Complex Analysis
Advances in Numerical Analysis
Mathematical Problems in Engineering
International Journal of Differential Equations
Discrete Dynamics in Nature and Society
International Journal of Stochastic Analysis
Journal of Mathematics
Journal of Function Spaces
Abstract and Applied Analysis
Advances in Mathematical Physics