ON CANNON CONE TYPES AND VECTOR-VALUED MULTIPLICATIVE FUNCTIONS FOR GENUS-TWO-SURFACE-GROUP

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ABSTRACT. We consider Cannon cone types for a surface group of genus g, and we give algebraic criteria for establishing the cone type of a given cone and of all its sub-cones. We also re-prove that the number of cone types is exactly $8g(2g-1)+1$. In the genus 2 case, we explicitly provide the 48×48 matrix of cone types, M, and we prove that M is primitive, hence Perron-Frobenius. Finally we define vector-valued multiplicative functions and we show how to compute their values by means of M.

1. INTRODUCTION

Let Γ_g be a surface group of genus g. There are several definitions of cone types available for surface groups, in this paper we are interested on those called simply *Cannon cone types* [C], not to be confused to the *canonical Cannon cone types* [FP].

The aim of this paper is to give an overview on Cannon cone types, provide a matrix (for $g = 2$), called matrix of cone types, whose columns are cone types of successors of any element of Γ_2 with a given cone type, show that it is a Perron-Frobenius matrix, and to apply it in the computation of elementary multiplicative functions.

Even though some results on cone types are "folklore" by now, as far as we know there are no results available in the literature that can help us in constructing such a matrix. We try to fill this gap in the present work. Our approach to cone types is more of combinatorial/algebraic type than geometric.

We believe that there is a necessity to explicitly provide the matrix of cone types, for its potential use in numerical algorithms; this feeling, at the time of Cannon's manuscript, maybe was not so urgent. Recently,

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instead, in the work of Gouezel [G], the matrix of *canonical Cannon cone types* has been used to estimate numerically the lower bound of the spectral radius of a random walk on a genus 2 surface group, improving a previous result by Bartholdi [B].

Another possible application of the matrix of cone types is related to the construction of vector-valued elementary multiplicative functions, and the associated new class of representations on surface groups. The latter have been already defined for free groups by Kuhn and Steger in [KS] (see also [KSS]), further extended to virtual free groups by Iozzi, et al. in [IKS], and presently object of a work in progress on surface groups by Kuhn, Steger and their collaborators.

The structure of the paper is the following: after a review on cone types in Section 2, in Section 3 we establish an algebraic criterion to determine any element's cone type (which is equivalent to determine the cone type of each cone); this will lead us also to a proof for the well known fact that there are exactly $8g(2g-1)+1$ cone types in Γ_g. In Section 4 we provide cone types for each successors of the $48+1$ possible cone types in Γ_2. In Section 5 we construct the matrix of cone types and we show that it is a Perron-Frobenius matrix. Finally, in Section 6 we apply the matrix of cone types in the computation of elementary multiplicative functions.

In the meanwhile, we provide a drawing of (part of) the Cayley graph of the genus 2 surface group (octagons group) in a form that we believe new and hopefully useful, so to facilitate the reader to imagine a figure that is not easy to describe in a drawing.

2. Cannon cone types

To fix notations we recall some basic concepts on hyperbolic groups in the sense of Gromov, such as distance, length, geodesic, Cayley graph, etc... referring to Ohshika's book [O] for more details.

Definition 2.1. Given a group G, a subset $A \subset G$ is called a generator system of G if every element of G is expressed as a product of elements of A. G is said finitely generated if it has a finite generator system.

In this paper we assume that a generator system of G is symmetric, i.e. closed under inverses.

If G has a generator system A, then there is a canonical surjective group homomorphism $p : F(A) \to G$, whose kernel is called the set of relators. Here $F(A)$ is the free group on A, identified with the set of reduced words on A, i.e., words in which an element and its inverse are not juxtaposed. We consider the identity as the empty word.

If $R \subset G$ is a subset, then we denote by $<< R >>$ the normal closure of R in G, which is the intersection of all normal subgroups containing R. Intuitively, this is the smallest normal subgroup containing R. It is easy to see that the elements of $<< R >>$ are

$$g_1 x_1^{n_1} g_1^{-1} g_2 x_2^{n_2} g_2^{-1} \dots g_k x_k^{n_k} g_k^{-1},$$

for $n_1, \dots, n_k \in \mathbb{Z}$, $x_1 \dots x_k \in R$, and $g_1, \dots, g_k \in G$ (not necessarily distinct).

Definition 2.2. If G has a finite generator system A, we say that G is finitely presented if there is a finite set $R = \{w_1, \dots, w_n\} \subset \ker p \subset F(A)$ such that $<< R >>= \ker p$. Hence $G \cong F(A)/ << R >>$. In that case we write $G = \langle A | R \rangle$ and we call such a presentation of G a finite presentation. The words $w_1, ..., w_n$ are called relators.

The fundamental group, Γ_g, of a compact surface of genus $g \geq 2$, is a finitely presented group. Its usual presentation is

$$\Gamma_g = \langle a_1, \dots, a_g, b_1, \dots, b_g | [a_1, b_1] \cdots [a_g, b_g] \rangle.$$

where the bracket means the usual commutator $[a, b] = aba^{-1}b^{-1}$.

In this paper we shall deal manly with $g = 2$, and in this case, for simplicity, we shall write and fix the set of generators as follows

$$\Gamma_2 = \langle a, b, c, d | [a, b][c, d] = aba^{-1}b^{-1}cdc^{-1}d^{-1} \rangle. \tag{2.1}$$

Definition 2.3. Let G be a discrete, finitely generated group with a finite generator system A. The Cayley graph $\mathcal{G}$ of G with respect to A is a graph defined as follows.

(1) The vertices of $\mathcal{G}$ are the elements of G.
(2) The (unoriented) edges are (non-ordered) couples (x, xa) with $a \in A$.

We can introduce a metric, denoted by d, on a Cayley graph by letting the length of every edge be 1 and defining the distance between two vertices to be the minimum length of edges joining them.

The metric on $\mathcal{G}$ induces a metric on G when the latter is identified with the set of vertices of $\mathcal{G}$. We call this metric on G (still denoted by d) the word metric with respect to A. In particular, for $x \in G$, we call the distance from the identity "e", with respect to the word metric, the length of x, and we denote it by $|x|$.

We have also that, for all $x, y, z \in G$, $d(xy, xz) = d(y, z)$.

Definition 2.4. A geodesic segment joining two vertices x, y in $\mathcal{G}$ (or, more briefly, a geodesic from x to y) is a map f from a closed interval $[0, l] \subset \mathbb{R}$ to $\mathcal{G}$ such that $f(0) = x$, $f(l) = y$ and $d(f(t), f(s)) = |t - s|$

for all $t, s \in [0, l]$ (in particular $l = d(x, y)$. When there is no confusion, we also call the image of f a geodesic segment with endpoints x and y and we denote it by $\overline{xy}$. We should note that such a geodesic segment need not be unique.

If in the geodesic segment $\overline{xw}$ we have $x = e$ and $w = y \in G$, we say that w is a geodesic word (representing y). In this case $d(e, y) = |w|$.

Definition 2.5. Given three points $x, y, z \in \mathcal{G}$, a geodesic triangle $\Delta_{x,y,z}$ on $\mathcal{G}$ with vertices x, y, z, is formed by three geodesic segments $\overline{xy}, \overline{yz}, \overline{zx}$. The group G is said hyperbolic if there exists a constant $\delta > 0$, depending only on G, such that, for any geodesic triangle $\Delta_{x,y,z}$, one has that each $u \in \overline{xy}$ is at distance at most δ from $\overline{zx} \cup \overline{yz}$.

One should note that the constant δ in the previous definition has no much importance except in the case $\delta = 0$ ($\mathbb{R}$-trees).

Remark 2.6. Γ_g is hyperbolic.

Its Cayley graph, $\mathcal{G}$, is a planar graph, a tessellation of the hyperbolic space $\mathbb{H}^2$ with the following properties:

(1) Every vertex belongs to $4g$ polygons each of $4g$ edges and vertex angle $\frac{2\pi}{4g}$;
(2) Every two polygons share one (and only one) edge;
(3) $\mathcal{G}$ is bipartite and self-dual.

Definition 2.7. Given any two vertices $x, y \in \mathcal{G}$ we say that y is a successor of x if (x, y) is an edge and $|y| = |x| + 1$. In this case x is called a predecessor of y.

A simple realization for (part of) the Cayley graph of Γ_2 is given in Figure 1, available also at `https://www.geogebra.org/m/Jqayn5UZ`

Its center vertex is the identity "e", and, given any vertex, any successor is obtained by juxtaposing generators in counterclockwise verse in this order

$$a, \quad d, \quad c^{-1}, \quad d^{-1}, \quad c, \quad b, \quad a^{-1}, \quad b^{-1}. \tag{2.2}$$

We now give the definition of cone type for elements of G.

Definition 2.8 ([BH])**.** Let G be a group with finite generating set A and corresponding word metric d.

The cone type of an element $x \in G$, denoted by $\mathcal{C}(x)$, is the set of words $z \in F(A)$ such that

$$d(e, xz) = d(e, x) + |z|,$$

(hence $d(e, xz) = d(e, x) + |z| = d(e, x) + d(x, xz)$).

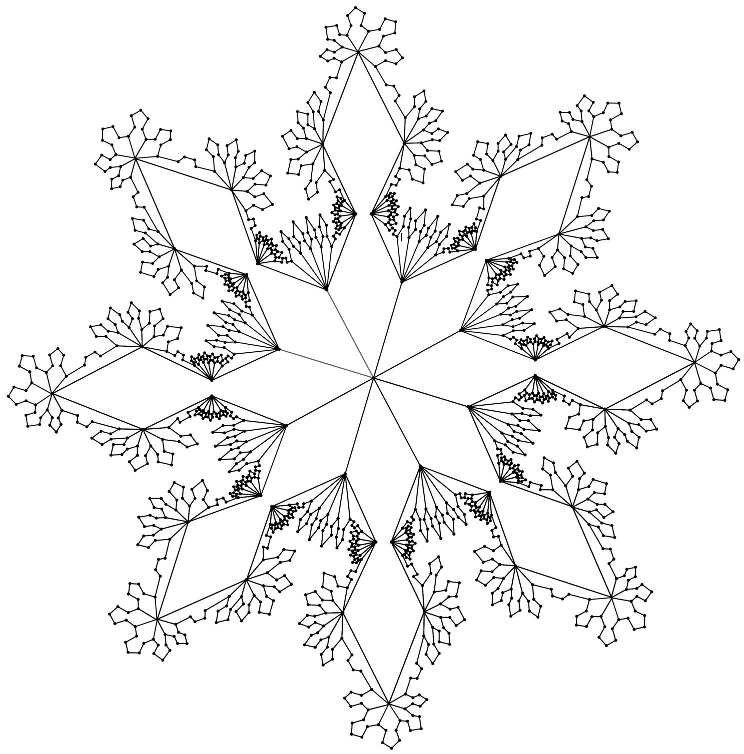

Figure 1. The Cayley graph of Γ_2 (part of).

In other words, if x is represented by a geodesic word u, then the cone type of x is the set of words z such that uz is also a geodesic.

An alternative definition of cone type involves the cone of a vertex.

Definition 2.9 ([G])**.** The cone of a vertex $x \in \mathcal{G}$, $\mathcal{C}(e,x)$, is the set of vertices $y \in \mathcal{G}$ for which there is a geodesic from e to y going through x.

The cone type of x is then defined as the set

$$\{x^{-1}y, \text{ for } y \in \mathcal{C}(e,x)\} = x^{-1}\mathcal{C}(e,x).$$

We see that the two definitions coincide once we identify G as the vertex set of $\mathcal{G}$, since if $z \in F(A)$ and $d(e,xz) = d(e,x) + |z|$, then $z = x^{-1}xz$ and $xz \in \mathcal{C}(e,x)$, since $|z| = d(x,xz)$. On the other hand, if $y \in \mathcal{C}(e,x)$, then

$$d(e,xx^{-1}y) = d(e,y) = d(e,x) + d(x,y) = d(e,x) + d(e,x^{-1}y).$$

Next we consider the definition of cone type for a cone, which relies on the action by isometries of the group G on its Cayley graph, this action is simply transitive on the vertices.

Definition 2.10. Given two vertices $x, y \in \mathcal{G}$ the cone at vertex y is

$$\mathcal{C}(x,y) = \{z \in G,\ d(x,z) = d(x,y) + d(y,z)\}.$$

The group G acts on the the collection of cones by (left) translation

$$z\mathcal{C}(x,y) = \mathcal{C}(zx, zy), \quad z \in G.$$

We say that two cones have the same type if they are in the same orbit. We may as well identify the set of cone-types with the set of cones $\mathcal{C}(x,e)$ whose vertex is $e \in G$.

From

$$\begin{aligned} \mathcal{C}(x) &= \{z \in G,\ d(e,xz) = d(e,x) + d(x,xz)\} = x^{-1}\mathcal{C}(e,x) \\ &= \{z \in G,\ d(x^{-1},z) = d(x^{-1},e) + d(e,z)\} \\ &= \mathcal{C}(x^{-1},e), \end{aligned}$$

we see that the cone type of x, $\mathcal{C}(x)$, is a representative of the cone type of $\mathcal{C}(x^{-1},e)$, and the latter cone has the same cone type of $\mathcal{C}(e,x)$ (same orbit).

A free group of rank m has $2m+1$ cone types with respect to any set of free generators. For more general hyperbolic groups we can refer to a result due to Cannon:

Theorem 2.11 (Theorem 2.18, [BH])**.** *If a group G is hyperbolic, then it has only finitely many cone types (with respect to any finite generating set).*

Proof. The proof is based on the following result:

Let $r \geq 1$ be an integer. Define the r-level of $g \in G$ as the set of elements h satisfying the following

$$|h| \leq r \text{ and } |gh| < |g|.$$

If the Cayley graph of G is δ-hyperbolic, the constant $r = 2\delta + 3$ is such that if two elements g_1 and g_2 have the same r-level, then the two cone types of g_1 and g_2 are the same. □

3. An algebraic criterion

In order to determine all the possible cone types in Γ_g, we shall establish an algebraic criterion to determine any element's cone type (which is equivalent, as shown in the previous section, to determine the cone type of each cone); this will lead us also to a proof for the well known fact that there are exactly $8g(2g-1)$ cone types in Γ_g, besides the cone type of the identity element. As far as we know, no proof of this fact is available in the literature.

It is important for us to determine the exact cone types of an element (and of each successor), in order to encode this informations in a matrix useful for algorithmic computations.

We focus on Γ_2, the general case being similar.

We need first some Lemmas.

Lemma 3.1. *If $w_1 w_2 \dots w_n$ is a geodesic word where each w_i is a generator, then*

$$\mathcal{C}(w_1 w_2 \dots w_n) \subset \mathcal{C}(w_2 \dots w_n) \subset \cdots \subset \mathcal{C}(w_n).$$

Proof. It is sufficient to show the first inclusion.

If $z \notin \mathcal{C}(w_2 \dots w_n)$, since $w_2 \dots w_n$ is a geodesic word,

$$d(e, w_2 \dots w_n z) < d(e, w_2 \dots w_n) + |z| = n - 1 + |z|$$

implies

$$\begin{aligned} d(e, w_1 w_2 \dots w_n z) &\leq d(e, w_1) + d(w_1, w_1 w_2 \dots w_n z) \\ &= d(e, w_1) + d(e, w_2 \dots w_n z) \\ &< 1 + n - 1 + |z| = d(e, w_1 w_2 \dots w_n) + |z|. \end{aligned}$$

Hence $z \notin \mathcal{C}(w_1 w_2 \dots w_n)$. $\square$

If we look for a proof of the opposite inclusion, we need to consider either any cyclic permutation of the relator

$$[a, b][c, d] = aba^{-1}b^{-1}cdc^{-1}d^{-1},$$

or of its inverse. To be short, we say that an element of Γ_2 belongs to $\mathcal{R}$ if it is represented by a sub-word of a cyclic permutation of the relator or of its inverse, or both (for example $ba^{-1}b^{-1}c \in \mathcal{R}$ while $ba^{-1}b^{-1}a \notin \mathcal{R}$). Note that elements in $\mathcal{R}$ have length at most 4. Also, from now on, when considering a geodesic word, say $u_1 u_2 \dots u_n$, we intend that each u_i is a generator.

Lemma 3.2. *Let $u_1, \dots, u_i$ be one, two or three generators, so that $i = 1, 2, 3$. Let $y \in \Gamma_2$, such that $u_1 \dots u_i \in \mathcal{C}(y)$. Assume that for* **any geodesic word** *$w_1 \dots w_J$, such that $y = w_1 \dots w_J$, the (geodesic) word $w_J u_1 \dots u_i$ does not belong to $\mathcal{R}$. Then:*

1. *If z is such that $u_1 \dots u_i z$ is a geodesic word, then for any geodesic $w_1 \dots w_J$ such that $y = w_1 \dots w_J$, the word $w_1 \dots w_J u_1 \dots u_i z$ is geodesic, too.*
2. *$\mathcal{C}(u_1 \dots u_i) \subset \mathcal{C}(y u_1 \dots u_i)$, and so $\mathcal{C}(u_1 \dots u_i) = \mathcal{C}(y u_1 \dots u_i)$.*

Proof. The second sentence follows from the first one and Lemma 3.1. Assume, on the contrary, that there exists a geodesic word $w_1 \dots w_J = y$ such that $w_1 \dots w_J u_1 \dots u_i z$ is not a geodesic word. Then it contains either couples like ss^{-1} (s being a generator) or a sequence of (at least) 5 generators in a cyclic permutation of the relator $[a,b][c,d]$ (or its inverse), or both.

The first case is excluded, since both $w_1 \dots w_J u_1 \dots u_i$ and $u_1 \dots u_i z$ are geodesic words.

In the second case, since both $w_1 \dots w_J u_1 \dots u_i$ and $u_1 \dots u_i z$ are geodesic words, and $i \leq 3$, we get that the sequence of 5 must contain $w_J u_1 \dots u_i$ against our assumption.

□

Example 3.3. Applying the Lemma 3.2 to points such as $x = bc$, we get $\mathcal{C}(c) = \mathcal{C}(bc)$.

For points such as $x = aba$, it yields $\mathcal{C}(aba) = \mathcal{C}(ba)$. Note that $ba \in \mathcal{R}$ and so the procedure stop here.

Consider now points such as $x = abcd$. In this case neither bcd nor $abcd$ belong to $\mathcal{R}$ so we get

$$\mathcal{C}(abcd) = \mathcal{C}(bcd) = \mathcal{C}(cd).$$

The above lemma does not apply to $x = dcd^{-1}c^{-1}a^{-1}dc$; in this case note that $c^{-1}a^{-1}dc \notin \mathcal{R}$ but since

$$x = dcd^{-1}c^{-1}a^{-1}dc = aba^{-1}b^{-1}a^{-1}dc,$$

and $b^{-1}a^{-1}dc \in \mathcal{R}$, the cone type of x is not the same of $a^{-1}dc$.

Finally we have

Lemma 3.4. *Let u_1, u_2, u_3, u_4 be four generators, such that $u_1u_2u_3u_4$ is a geodesic word in $\mathcal{R}$.*

Let $y \in \Gamma_2$, such that $u_1u_2u_3u_4 \in \mathcal{C}(y)$. We have:

(1) If z is such that $u_1u_2u_3u_4z$ is a geodesic word, then for any geodesic word $w_1 \dots w_J$ such that $y = w_1 \dots w_J$, we get that $w_1 \dots w_J u_1u_2u_3u_4z$ is a geodesic word, too.

(2) $\mathcal{C}(u_1u_2u_3u_4) = \mathcal{C}(yu_1u_2u_3u_4)$.

Proof. The second sentence follows from the first one and Lemma 3.1. Assume, on the contrary, that there exists a geodesic word $w_1 \dots w_J = y$ such that $w_1 \dots w_J u_1u_2u_3u_4z$ is not a geodesic word, then it contains either couples like ss^{-1} (s being a generator) or a sequence of (at least) 5 generators in a cyclic permutation of the relator $[a,b][c,d]$ (or its inverse), or both.
The first case is excluded, since both $w_1 \dots w_J u_1u_2u_3u_4$ and $u_1u_2u_3u_4z$ are geodesic words.

In the second case, since $u_1u_2u_3u_4z$ is a geodesic words, and since $u_1u_2u_3u_4 \in \mathcal{R}$, we get that the sequence of 5 must contain $w_J u_1u_2u_3u_4$ against the fact that $w_1 \dots w_J u_1u_2u_3u_4$ is a geodesic word. □

Example 3.5. Consider again $x = dcd^{-1}c^{-1}a^{-1}dc = aba^{-1}b^{-1}a^{-1}dc$. Since $b^{-1}a^{-1}dc \in \mathcal{R}$, the cone type of x is the same of $b^{-1}a^{-1}dc$.

The following lemma states the uniqueness of cone types for elements in $\mathcal{R}$.

Lemma 3.6. *Elements in $\mathcal{R}$ (of length at most 4) have distinct cone types.*

Hence, if $x, z \in \mathcal{R}$ and $\mathcal{C}(x) = \mathcal{C}(y)$, then $x = y$.

Proof. For any couple of words $x, z \in \mathcal{R}$ it is easy to explicitly provide an element $y \in \mathcal{C}(x)$ not in $\mathcal{C}(z)$ and vice versa. For example $b^{-1}a^{-1}d \in \mathcal{C}(a)$ while $b^{-1}a^{-1}d \notin \mathcal{C}(ba)$, since $bab^{-1}a^{-1}d = cdc^{-1}$. □

Lemma 3.7. *Let $x \in \Gamma_2$ and $u_1 \dots u_{|x|}$ be a geodesic word representing x. Assume the suffix $u_s = u_{|x|-2}u_{|x|-1}u_{|x|} \notin \mathcal{R}$. Then $\mathcal{C}(x) = \mathcal{C}(u_s)$.*

The same conclusion holds if $u_s = u_{|x|-1}u_{|x|}$.

Proof. The proof is an application of Lemma 3.2.

Let $u_p = u_1 \dots u_{|x|-3}$, and consider a geodesic word $y_1 \dots y_{|x|-3} = u_p$. Note that $u_s \in \mathcal{C}(u_p)$ since $u_1 \dots u_{|x|}$ is a geodesic word.

If $y_{|x|-3}u_s \in \mathcal{R}$, then also $u_s \in \mathcal{R}$, which is a contradiction. Hence $y_{|x|-3}u_s \notin \mathcal{R}$, and, by Lemma 3.2, we get $\mathcal{C}(x) = \mathcal{C}(u_p u_s) = \mathcal{C}(u_s)$. □

As a consequence of Lemmas 3.2, 3.4, and 3.6, we have

Proposition 3.8. *All the possible cone types of Γ_2 ($\mathcal{C}(e)$ excluded) are those determined by geodesic words obtained as sub-words, with length at most 4, of any cyclic permutation of the relator $[a, b][c, d]$ or of its inverse $[d, c][b, a]$, adding up to 48.*

Proof. The first sentence is a consequence of the previous Lemmas. The second one is established by looking for words of length at most 4 in any cyclic permutation of either $aba^{-1}b^{-1}cdc^{-1}d^{-1}$, or its inverse $dcd^{-1}c^{-1}bab^{-1}a^{-1}$, and adding up. Precisely we have:

- 8 words of length 1;
- $2(7 + 1)$ words of length 2, (7 in the relator, 1 in the following permutation, twice);
- $2(6+1+1)$ words of length 3, (6 in the relator, 1 in the following 2 permutations, twice);
- $(5 + 1 + 1 + 1)$ words of length 4, (5 in the relator, 1 in the following 3 permutation, just once, since for the inverse you get the same words).

Adding up we get $6 \times 8 = 48$. □

The above reasoning can be applied to a generic surface group Γ_g, leading to

Theorem 3.9. *All the possible cone types of Γ_g ($\mathcal{C}(e)$ excluded) are those determined by geodesic words obtained as sub-words, with length at most $2g$, of any cyclic permutation of either $[a_1, b_1] \dots [a_g, b_g]$, or its inverse $[b_g, a_g] \dots [b_1, a_1]$, adding up to $8g(2g-1)$.*

Proof. Looking for words of length at most $2g$ in any cyclic permutation of either the relator $[a_1, b_1] \dots [a_g, b_g]$ or its inverse $[b_g, a_g] \dots [b_1, a_1]$, we have:

- $4g$ words of length 1;
- $2((4g-1)+1)$ words of length 2, ($4g-1$ in the relator, 1 in the following permutation, twice);
- $2((4g-2)+1+1)$ words of length 3, ($4g-2$ in the relator, 1 in the following 2 permutations, twice);
- ......
- $2((4g-(2g-2))+(2g-2))$ words of length $2g-1$, ($4g-(2g-2)$ in the relator, 1 in the following $2g-1$ permutations, twice);
- $((4g-(2g-1))+(2g-1))$ words of length $2g$, ($4g-(2g-1)$ in the relator, 1 in the following $2g-1$ permutation, just once, since for the inverse you get the same words).

Adding up we get

$$
\begin{aligned}
&4g(1+\underbrace{2+2+\cdots+2}_{2g-2}+1) = 4g(2+2(2g-2)) \\
&= 8g(1+2g-2) = 8g(2g-1)
\end{aligned}
$$

□

4. Cone type of successors

We say, in short, that a cone type of an element of Γ_2 is a quadruple, triple, double, single, if it is one of the 48 cone types defined by elements in $\mathcal{R}$, of length, respectively $4, 3, 2, 1$.

It is convenient to organize the cone types in singles, doubles, triples, and quadruples. We follow the order shown in Table 1.

We now provide cone types for each successors of the 48 elements in Table 1. As it will be shown in Proposition 4.2, the list of cone types of successors depends only on the cone type.

- We start with the generator a. Successors of a are

  $$aa,\ ad,\ ac^{-1},\ ad^{-1},\ ac,\ ab,\ ab^{-1},$$

  and we need to find the cone type of the first 5 only, since they are not in $\mathcal{R}$.

Table 1. Cone types in Γ_2.

	singles		doubles		triples		quadruples
1	b^{-1}	9	$b^{-1}c$	25	$b^{-1}cd$	41	$b^{-1}cdc^{-1}$
2	a	10	$b^{-1}a^{-1}$	26	$b^{-1}a^{-1}d$	42	$aba^{-1}b^{-1}$
3	d	11	ab	27	aba^{-1}	43	$dc^{-1}d^{-1}a$
4	c^{-1}	12	ab^{-1}	28	$ab^{-1}a^{-1}$	44	$c^{-1}d^{-1}ab$
5	d^{-1}	13	dc^{-1}	29	$dc^{-1}d^{-1}$	45	$d^{-1}aba^{-1}$
6	c	14	dc	30	dcd^{-1}	46	$cdc^{-1}d^{-1}$
7	b	15	$c^{-1}d^{-1}$	31	$c^{-1}d^{-1}a$	47	$ba^{-1}b^{-1}c$
8	a^{-1}	16	$c^{-1}b$	32	$c^{-1}ba$	48	$a^{-1}b^{-1}cd$
		17	$d^{-1}a$	33	$d^{-1}ab$		
		18	$d^{-1}c^{-1}$	34	$d^{-1}c^{-1}b$		
		19	cd	35	cdc^{-1}		
		20	cd^{-1}	36	$cd^{-1}c^{-1}$		
		21	ba^{-1}	37	$ba^{-1}b^{-1}$		
		22	ba	38	bab^{-1}		
		23	$a^{-1}b^{-1}$	39	$a^{-1}b^{-1}c$		
		24	$a^{-1}d$	40	$a^{-1}dc$		

We note that for each $u = a, d, c^{-1}, d^{-1}, c$, the geodesic word $au \notin \mathcal{R}$. Hence by Lemma 3.2 we get that $\mathcal{C}(au) = \mathcal{C}(u)$. The same argument works for any of

$$b,\ c,\ d,\ a^{-1},\ b^{-1},\ c^{-1},\ d^{-1},$$

and we observe that successors of *singles* are either singles or doubles.

- We consider ab. Successors of ab are

$$aba,\ abd,\ abc^{-1},\ abd^{-1},\ abc,\ abb,\ aba^{-1},$$

and we have to find the cone type of the first 6 only, since the last one determines a cone type by itself.

Note that for each $u = d, c^{-1}, d^{-1}, c, b$, both the geodesic words abu, and bu, are not in $\mathcal{R}$. Hence by Lemma 3.2 applied twice, we get that $\mathcal{C}(abu) = \mathcal{C}(bu) = \mathcal{C}(u)$.

On the other hand, $aba \notin \mathcal{R}$, while $ba \in \mathcal{R}$. So we conclude that $\mathcal{C}(aba) = \mathcal{C}(ba)$. The same argument works for any double listed in Table 1, and we observe that successors of *doubles* can be singles, doubles or triples.

- Consider now aba^{-1}. Successors of aba^{-1} are

$$aba^{-1}d,\ aba^{-1}c^{-1},\ aba^{-1}d^{-1},\ aba^{-1}c,\ aba^{-1}b,\ aba^{-1}a^{-1},\ aba^{-1}b^{-1}$$

and we have to find the cone type of the first 6 only.

For each $u = c^{-1}, d^{-1}, c, b, a^{-1}$, both geodesic words $ba^{-1}u$, and $a^{-1}u$, are not in $\mathcal{R}$. Hence by Lemma 3.2 we get that

$$\mathcal{C}(aba^{-1}u) = \mathcal{C}(ba^{-1}u) = \mathcal{C}(a^{-1}u) = \mathcal{C}(u).$$

On the other hand, $ba^{-1}d \notin \mathcal{R}$, while $a^{-1}d \in \mathcal{R}$. So we conclude that

$$\mathcal{C}(aba^{-1}d) = \mathcal{C}(ba^{-1}d) = \mathcal{C}(a^{-1}d).$$

The same argument works for any of the triples listed in Table 1, and we observe that successors of *triples* are singles, doubles or quadruples.

- Finally let us consider $aba^{-1}b^{-1} = dcd^{-1}c^{-1}$. Successors of $aba^{-1}b^{-1}$ are

$$\begin{array}{lll} aba^{-1}b^{-1}a, & aba^{-1}b^{-1}d, & aba^{-1}b^{-1}c^{-1}, \\ aba^{-1}b^{-1}d^{-1}, & aba^{-1}b^{-1}a^{-1}, & aba^{-1}b^{-1}b^{-1}. \end{array}$$

Note that for $u = d, c^{-1}, a, b^{-1}$, all geodesic words $ba^{-1}b^{-1}u$, $a^{-1}b^{-1}u$, and $b^{-1}u$ are not in $\mathcal{R}$. Hence by Lemma 3.2, we get that $\mathcal{C}(aba^{-1}b^{-1}u) = \mathcal{C}(u)$.

On the other hand, $ba^{-1}b^{-1}a^{-1}$, and $a^{-1}b^{-1}a^{-1}$ are not in $\mathcal{R}$, while $b^{-1}a^{-1}$ is. So we conclude that

$$\mathcal{C}(ba^{-1}b^{-1}a^{-1}) = \mathcal{C}(a^{-1}b^{-1}a^{-1}) = \mathcal{C}(b^{-1}a^{-1}).$$

Similarly $ba^{-1}b^{-1}d^{-1} = cd^{-1}c^{-1}d^{-1}$, and $d^{-1}c^{-1}d^{-1}$ are not in $\mathcal{R}$, while $c^{-1}d^{-1}$ is. So we conclude that

$$\mathcal{C}(ba^{-1}b^{-1}d^{-1}) = \mathcal{C}(d^{-1}c^{-1}d^{-1}) = \mathcal{C}(c^{-1}d^{-1}).$$

The same argument above works for every quadruple listed in Table 1, and we observe that successors of *quadruple* are either singles or doubles.

Lemma 4.1. *If the cone type of $x \in \Gamma_2$ is $\mathcal{C}(z)$ where $z \in \mathcal{R}$, then there exists a geodesic word $u_1u_2 \dots u_{|x|-|z|+1} \dots u_{|x|}$ which represents x, ending with z (i.e. $u_{|x|-|z|+1} \dots u_{|x|} = z$), and such that $u_{|x|-|z|}z \notin \mathcal{R}$.*

Proof. We first show that there exists a geodesic word $u_1 \dots u_{|x|}$ which represents x and ends with z.

For any geodesic word $x_1 \dots x_J$ which represents x let us denote by $s_{x,k} = x_{J-k+1} \dots x_J$, $1 \leq k \leq 4$ the longest suffix which belongs to $\mathcal{R}$.

Let $n = |z|$, $1 \leq n \leq 4$, and, say, $z = z_1 \dots z_n$.

Now assume, on the contrary, that any geodesic word which represents x does not end with z. Then for any geodesic word $x_1 \dots x_J$ such that $x = x_1 \dots x_J$ we have $x_{J-n+1} \dots x_J \neq z_1 \dots z_n$.

If for some representation we have $k = 4$, then the cone type of x is $\mathcal{C}(s_{x,4})$ by Lemma 3.4, and by uniqueness of cone type we must have $z = s_{x,4}$ which is a contradiction.

Hence for all other representations of x we have $k < 4$. Let's take one with the longest suffix $s_{x,k} \in \mathcal{C}(x_1 \dots x_{J-k})$ (it exists since otherwise we find a shorter geodesic representing x).

If for any geodesic word $y_1 \dots y_{J-k}$ which represents $x_1 \dots x_{J-k}$ we have that $y_{J-k}s_{x,k} \notin \mathcal{R}$, then by Lemma 3.2,

$$\mathcal{C}(z) = \mathcal{C}(x) = \mathcal{C}(x_1 \dots x_{J-k}s_{x,k}) = \mathcal{C}(s_{x,k}),$$

and again, by uniqueness, $z = s_{x,k}$, a contradiction.

So there exists a geodesic word $y_1 \dots y_{J-k}$ representing $x_1 \dots x_{J-k}$ and $y_{J-k}s_{x,k} \in \mathcal{R}$. Thus we have found a representation of $x = y_1 \dots y_{J-k}s_{x,k}$, where the suffix $y_{J-k}s_{x,k} \in \mathcal{R}$ is longer then $s_{x,k}$, yielding again a contradiction. This complete the first part of the proof.

Next, given a geodesic word $u_1u_2 \dots u_{|x|-|z|}z$ which represents x we can exclude $u_{|x|-|z|}z \in \mathcal{R}$ if z is a quadruple, since the word is geodesic. In all other cases we have, by Lemma 3.1,

$$\mathcal{C}(z) = \mathcal{C}(x) \subset \mathcal{C}(u_{|x|-|z|}z) \subset \mathcal{C}(z),$$

and so, by Lemma 3.6, we get $u_{|x|-|z|}z \notin \mathcal{R}$.

□

Proposition 4.2. *If $x, z \in \Gamma_2$, $z \in \mathcal{R}$, and $\mathcal{C}(x) = \mathcal{C}(z)$, then the cone type of any of the successors of x is the same as the cone type of the successor of z corresponding to the same generator.*

Proof. By Lemma 4.1 there exists a geodesic word which represents x, say $u_1u_2 \dots u_{|x|-|z|+1} \dots u_{|x|}$, ending with z (i.e. $u_{|x|-|z|+1} \dots u_{|x|} = z$), and such that $u_{|x|-|z|}z \notin \mathcal{R}$.

Let us consider a successor y of x. We have $y = xa$, with $a \in A \cap \mathcal{C}(x)$ and $d(e, xa) = d(e, x) + 1$, yielding $a \in \mathcal{C}(z)$, and, by definition of cone type, $d(e, za) = d(e, z) + d(e, a) = d(e, z) + 1$. Hence za is a successor of z.

Also $a \in \mathcal{C}(x)$ implies that $u_1u_2 \dots u_{|x|-|z|}za$ is a geodesic word representing xa.

If $za \in \mathcal{R}$ is a quadruple, by Lemma 3.4 this yields $\mathcal{C}(xa) = \mathcal{C}(za)$.

If $za \in \mathcal{R}$ and $|za| \leq 3$, let $y = u_1u_2 \dots u_{|x|-|z|}$ and consider a geodesic word $v_1v_2 \dots v_{|x|-|z|} = y$.

Note that $za \in \mathcal{C}(y)$ so that $v_1v_2 \dots v_{|x|-|z|}za$ is geodesic.

If $v_{|x|-|z|}za \in \mathcal{R}$ then, also, $v_{|x|-|z|}z \in \mathcal{R}$ and, by Lemmas 3.2, and 3.1,

$$\mathcal{C}(v_{|x|-|z|}z) \subset \mathcal{C}(z) = \mathcal{C}(x) \subset \mathcal{C}(v_{|x|-|z|}z).$$

It follows $\mathcal{C}(v_{|x|-|z|}z) = \mathcal{C}(z)$, and by uniqueness of cone types we have $v_{|x|-|z|}z = z$ and so the contradiction $v_{|x|-|z|} = e$.

Hence $v_{|x|-|z|}za \notin \mathcal{R}$, and by Lemma 3.2

$$\mathcal{C}(xa) = \mathcal{C}(yza) = \mathcal{C}(za).$$

So it remain to consider the case $za \notin \mathcal{R}$.

If $za \notin \mathcal{R}$ and $|za| \leq 3$, we apply Lemma 3.7 to obtain $\mathcal{C}(xa) = \mathcal{C}(za)$.

If $za \notin \mathcal{R}$ and $|za| = 4$, then $|z| = 3$, and the cone type of $z = z_1z_2z_3$, is a triple. Also, by Lemma 3.2 applied to $y = z_1$, $\mathcal{C}(za)\mathcal{C}(z_2z_3a)$.

Since the cone type of a successor of a triple can only be a single, double, or quadruple, it follows $z_2z_3a \not in\mathcal{R}$.

Therefore we have by Lemma 3.7,

$$\mathcal{C}(xa) = \mathcal{C}(z_2z_3a) = \mathcal{C}(za).$$

The same reasoning apply for $za \notin \mathcal{R}$ and $|za| = 5$, recalling that quadruples do not have triples and quadruples as successors.

If $za \notin \mathcal{R}$ and $|za| = 5$, then $|z| = 4$, and the cone type of $z = z_1z_2z_3z_4$, is a quadruple. Since the cone type of a successor of a quadruple can only be either a single or a double, it follows $z_2z_3z_4a \not in\mathcal{R}$, (otherwise, by Lemma 3.4, $\mathcal{C}(za)\mathcal{C}(z_1z_3z_4a)$)).

Hence, by Lemma 3.2 applied to $y = z_1z_2$, $\mathcal{C}(za)\mathcal{C}(z_3z_4a)$.

Since the cone type of a successor of a quadruple can only be either a single or a double, it follows $z_3z_4a \not in\mathcal{R}$.

Therefore we have by Lemma 3.7,

$$\mathcal{C}(xa) = \mathcal{C}(z_3z_4a) = \mathcal{C}(za),$$

and the proof is complete. □

5. Matrix of cone types

Based on the discussion preceding Lemma 4.1, and Proposition 4.2 we can now construct a matrix, indexed by cone types, in which any column gives the cone type of any successor of the element whose cone type indexes the column. We work in the surface group of genus two, so we are speaking about a 48×48 matrix. We think it is better to provide the matrix as a block matrix. As we shall see, the matrix is sparse. It should be mentioned that any order of cone types indexing its columns (and corresponding rows) gives a similar matrix, hence we follow the order provided in Table 1.

Next we list the 16 blocks, $M_{i,j}$, $i, j = 1, \ldots, 4$, of the main matrix M. Note that indexes i, j refer to single, doubles, etc... for example, indexes $M_{1,4}$ means that rows are indexed by singles, and columns by quadruples. We set 0 whenever $M_{i,j}$ is the zero matrix, and I for the identity matrix.

$$M = \begin{pmatrix} M_{1,1} & M_{1,2} & M_{1,3} & M_{1,4} \\ M_{2,1} & M_{2,2} & M_{2,3} & M_{2,4} \\ 0 & I & 0 & 0 \\ 0 & 0 & M_{4,3} & 0 \end{pmatrix}. \tag{5.1}$$

The first block is an 8×8 matrix

$$M_{1,1} = \begin{pmatrix} 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 \end{pmatrix}.$$

In the same column it follows a 16×8 matrix

$$M_{2,1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

In the second column, first row, we have

$$M_{1,2} = \begin{pmatrix} 1&0&0&1&1&1&1&0&0&1&1&1&0&0&1&1 \\ 1&0&0&1&1&1&0&0&1&1&1&0&0&1&1&1 \\ 0&0&1&1&1&0&0&1&1&1&1&0&0&1&1&1 \\ 0&1&1&1&1&0&0&1&1&1&0&0&1&1&1&0 \\ 0&1&1&1&0&0&1&1&1&0&0&1&1&1&1&0 \\ 1&1&1&0&0&1&1&1&1&0&0&1&1&1&0&0 \\ 1&1&1&0&0&1&1&1&0&0&1&1&1&0&0&1 \\ 1&1&0&0&1&1&1&0&0&1&1&1&1&0&0&1 \end{pmatrix},$$

and then, in the same column,

$$M_{2,2} = \begin{pmatrix} 0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0 \\ 1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&1 \\ 0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0 \end{pmatrix}.$$

In the third column, we have

$$M_{1,3} = \begin{pmatrix} 1&1&0&0&1&1&0&0&0&0&1&1&1&1&1&1 \\ 1&1&0&0&0&0&1&1&0&0&1&1&1&1&1&1 \\ 1&1&0&0&0&0&1&1&1&1&1&1&1&1&0&0 \\ 0&0&1&1&0&0&1&1&1&1&1&1&1&1&0&0 \\ 0&0&1&1&1&1&1&1&1&1&0&0&1&1&0&0 \\ 0&0&1&1&1&1&1&1&1&1&0&0&0&0&1&1 \\ 1&1&1&1&1&1&0&0&1&1&0&0&0&0&1&1 \\ 1&1&1&1&1&1&0&0&0&0&1&1&0&0&1&1 \end{pmatrix}.$$

$$M_{2,3} = \begin{pmatrix}
0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0
\end{pmatrix}.$$

$$M_{4,3} = \begin{pmatrix}
1&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1&0&0&1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&1\\
0&1&0&0&0&0&0&0&0&0&0&0&0&0&1&0
\end{pmatrix}.$$

Finally in the last column we have

$$M_{1,4} = \begin{pmatrix}
1&1&0&0&0&0&1&1\\
1&1&1&0&0&0&0&1\\
1&1&1&1&0&0&0&0\\
0&1&1&1&1&0&0&0\\
0&0&1&1&1&1&0&0\\
0&0&0&1&1&1&1&0\\
0&0&0&0&1&1&1&1\\
1&0&0&0&0&1&1&1
\end{pmatrix}.$$

$$M_{2,4} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

Definition 5.1. A square non-negative matrix T is said to be primitive if there exists a positive integer k such that $T^k > 0$.

Proposition 5.2. *The matrix M is a primitive matrix.*

Proof. A direct computation shows that $M^5 > 0$.

More precisely, all elements in the first 8 rows of M^2 are strictly positive, and the same for the first 16 rows of M^3, the first 32 rows of M^4, and all rows in M^5. □

As a consequence, by Perron-Frobenius Theorem [S, Theorem 1.1], we obtain the following

Proposition 5.3. *There exists an eigenvalue r of M such that:*

1) *r is real, and $r > 0$;*
2) *r is associated to strictly positive left and right eigenvectors;*
3) *$r > |\lambda|$ for any eigenvalue $\lambda \neq r$;*
4) *The eigenvectors associated with r are unique up to constant multiples;*
5) *r is a simple root of the characteristic equation of T.*

6. Elementary multiplicative functions

In this section we recall the definition of vector-valued elementary multiplicative functions on Γ_2, by Kuhn and Steger, and we show how they can be easily computed by means of the matrix of cone types.

To set the vectorial context we need to define maps between finite dimensional vector spaces indexed by cone types. In this section we use letters as $a, b, \dots$ to denote arbitrary generators, not to be confused with the set of generators provided in (2.1).

We consider triples (a, c', c), where c, c' are cone types, and $a \in A$. We call a triple *admissible* if, given $c = \mathcal{C}(z)$ for some $z \in \Gamma_2$, we have $a \in A \cap \mathcal{C}(z)$, (so that if z is represented by a geodesic word u, then ua is also a geodesic) and $c' = \mathcal{C}(za)$, i.e. c' is the cone-type of the a-successor of z.

A matrix system (system in short) $(V_c, H_{a,c',c})$ consists of finite dimensional complex vector spaces V_c, for each cone-type c, and linear maps $H_{a,c',c} : V_c \to V_{c'}$ for each admissible triple (a, c', c). For non-admissible triples (a, c', c) we set $H_{a,c',c} = 0$.

Definition 6.1. For $x, y \in G$, consider the cone $\mathcal{C}(x, y)$ with cone-type $c = \mathcal{C}(x^{-1}y)$, $v_c \in V_c$.

The elementary multiplicative function $\mu[\mathcal{C}(x, y), v_c]$ is defined as

$$\mu[\mathcal{C}(x,y), v_c](z) = \begin{cases} 0, & z \notin \mathcal{C}(x,y), \\ v_c, & z = y, \\ \displaystyle\sum_{\substack{a \in A \\ ya \in \mathcal{C}(x,y) \\ c' = \mathcal{C}(x^{-1}ya)}} \mu[\mathcal{C}(x, ya), H_{a,c',c}(v_c)](z), & z \neq y. \end{cases}$$

Note that the action of Γ_2 on cones by translation (see Definition 2.10), $\gamma\mathcal{C}(x, y) = \mathcal{C}(\gamma x, \gamma y)$, $\gamma \in \Gamma_2$, implies

$$\mu[\mathcal{C}(x, y), v_c](\gamma^{-1}z) = \mu[\mathcal{C}(\gamma x, \gamma y), v_c](z). \tag{6.1}$$

The recursive definition yields an equivalent definition of elementary multiplicative function in terms of geodesics between two vertices.

Note that any geodesic between two vertices has the same length. Also, since Γ_2 is hyperbolic, the number of geodesics between two fixed vertices is finite.

We recall that elements of $\mathcal{R}$ are geodesic sub-word of a cyclic permutation of the fundamental relation $[a, b][c, d] = e$ or of its inverse $[d, c][b, a] = e$.

Consider any geodesic word representing $y \in G$, say $y = w_1 \dots w_n$, $w_i \in A$, and the set of all geodesic quadruples in $w_1 w_2 \dots w_n$ which belong to $\mathcal{R}$.

Definition 6.2. For $y \in G$, $\mathcal{R}_y$ is defined as the set of all geodesic quadruples, in any geodesic word representing y, which belong to $\mathcal{R}$, i.e.

$$\begin{aligned}\mathcal{R}_y &= \{w_i w_{i+1} w_{i+2} w_{i+3} \in \mathcal{R}, \\ &\quad \text{for all geodesic words } w_1 w_2 \dots w_n = y,\ w_i \in A\}.\end{aligned}$$

If $q \in \mathcal{R}_y$, the quadruple q' is called the *twin* of q if $q = q'$ represents the same group element, but $q \neq q'$ as a geodesic path in the Caley graph (q' always exists).

Note that if $q \in \mathcal{R}_y$, and q' is the twin of q, then also $q' \in R_y$. Also $R_y = \emptyset$ means that, if $y \in \mathcal{C}(e, a)$, for a given $a \in A$, there is only one geodesic from e to y passing through a.

Remark 6.3. If a given geodesic word contains more then one quadruple in $\mathcal{R}_y$, then they can have at most one element in common, since any two octagons have at most one edge in common. One such example is:

$$y = (aba^{-1}b^{-1})a^{-1}dc = (dcd^{-1}c^{-1})a^{-1}dc = aba^{-1}(a^{-1}b^{-1}cd).$$

Also note that the replacement of a quadruple q by its twin $q' = q$ can change the number of quadruples in R_y.

In the example above, $aba^{-1}b^{-1}a^{-1}dc$ contains 2 quadruples: $q_1 = aba^{-1}b^{-1}$ and $q_2 = b^{-1}a^{-1}dc$, but if we replace q_1 with $q_1' = q_1$ then the number of quadruples drops to 1. This reflects the number of different geodesic paths from e to y.

Hence it is necessary to introduce a notation which takes into account the several occurrences of quadruples.

Notation 6.4. The notation follows a hierarchical dyadic approach.

Since both the number of geodesic words representing y and the number of generators is finite, the *first quadruple* in R_y can be defined as follows. First let us fix an order on generators and on its inverses.

Let $i = 1, \dots, n$, be the smallest index such that $y = w_1 w_2 \dots w_n$, $w_i w_{i+1} w_{i+2} w_{i+3} \in R_y$, and w_i is the smallest in the given order on generators.

We set $w_{0,j} = w_j$, for $j = i, \dots, i+3$ and we define the *first quadruple* in R_y as $q_0 = w_{0,i} w_{0,i+1} w_{0,i+2} w_{0,i+3}$; its twin in R_y is denoted by $q_1 = w_{1,i} w_{1,i+1} w_{1,i+2} w_{1,i+3}$. Hence $q_0 = q_1$ as a geodesic word, but $q_0 \neq q_1$ as a geodesic path.

If we set $\delta_i = 0, 1$, at the nth stage $q_{(\delta_1,\dots,\delta_{n-1},0)}$ denotes the next quadruple in R_y after $q_{(\delta_1,\dots,\delta_{n-1})}$, while $q_{(\delta_1,\dots,\delta_{n-1},1)}$ denotes the twin of $q_{(\delta_1,\dots,\delta_{n-1},0)}$.

In the example above, ordering as in Table 1,

$$b^{-1} < a < d < c^{-1} < d^{-1} < c < b < a^{-1},$$

we have

$$\begin{aligned} y &= (aba^{-1}b^{-1})a^{-1}dc = (dcd^{-1}c^{-1})a^{-1}dc = aba^{-1}(a^{-1}b^{-1}cd). \\ q_0 &= aba^{-1}b^{-1}, \quad q_1 = dcd^{-1}c^{-1} \\ q_{(0,0)} &= b^{-1}a^{-1}dc, \quad q_{(0,1)} = a^{-1}b^{-1}cd. \end{aligned}$$

Finally, we shall use the notation $\delta = (\delta_1, \ldots, \delta_n)$, and q_δ for short; if $\delta = (\delta_1, \ldots, \delta_{n-1}, 0)$, and

$$q_\delta = w_{\delta_1,\ldots,\delta_{n-1},0,i} w_{\delta_1,\ldots,\delta_{n-1},0,i+1} w_{\delta_1,\ldots,\delta_{n-1},0,i+2} w_{\delta_1,\ldots,\delta_{n-1},0,i+3},$$

then we shall indicate its twin by setting $\delta' = (\delta_1, \ldots, \delta_{n-1}, 1)$, and

$$q_{\delta'} = w_{\delta_1,\ldots,\delta_{n-1},1,i} \ldots w_{\delta_1,\ldots,\delta_{n-1},1,i+3}.$$

If there is no confusion, we shall omit δ at all.

The following lemmas will be crucial for an alternative expression of elementary multiplicative functions.

Lemma 6.5. *Let $x, y \in G$ be such that $R_y \neq \emptyset$, and $y \in \mathcal{C}(x)$.*

Let $i = 1, \ldots, n$, be the smallest index such that $y = w_1 w_2 \ldots w_n$, and $w_i w_{i+1} w_{i+2} w_{i+3} \in R_y$. Denote it by $w_{0,i} w_{0,i+1} w_{0,i+2} w_{0,i+3}$, and its twin by $w_{1,i} w_{1,i+1} w_{1,i+2} w_{1,i+3}$.

Let q_0 be the first quadruple in R_y, say $y = w_1 w_2 \ldots w_n$ and $q_0 = w_{0,i} w_{0,i+1} w_{0,i+2} w_{0,i+3}$.

Let $a \in A \cap \mathcal{C}(x)$.

If $i = 1$, then

$$xy \in \mathcal{C}(e, xa) \Leftrightarrow \textit{either } w_{0,1} = w_1 = a, \textit{ or } w_{1,1} = a;$$

If $i > 1$, then

$$xy \in \mathcal{C}(e, xa) \Leftrightarrow w_1 = a.$$

Proof. First note that $xy \in \mathcal{C}(e, xa)$ means

$$d(e, xy) = d(e, xa) + d(xa, xy) = d(e, xa) + d(a, w_1 w_2 \ldots w_n).$$

Since $y \in \mathcal{C}(x)$, and $a \in \mathcal{C}(x)$, the latter is equivalent to

$$|x| + n = |x| + 1 + d(a, w_1 w_2 \ldots w_n).$$

Therefore $xy \in \mathcal{C}(e, xa)$ is equivalent to

$$n - 1 = d(a, w_1 w_2 \ldots w_n) = d(e, a^{-1} w_1 w_2 \ldots w_n). \tag{6.2}$$

If $i = 1$, and $a^{-1} w_1 \neq e$, then (6.2) implies $a^{-1} w_1 w_2 w_3 w_4 \in R$ and it equals a geodesic word of length 3. The latter implies also $w_1 w_2 w_3 w_4 \in R$. Being

$$w_1 w_2 w_3 w_4 = w_{0,1} w_{0,2} w_{0,3} w_{0,4} = w_{1,1} w_{1,2} w_{1,3} w_{1,4},$$

then

$$a^{-1}w_1w_2w_3w_4 = a^{-1}w_{1,1}w_{1,2}w_{1,3}w_{1,4},$$

and both are equal to a geodesic word of length 3, hence necessarily $a^{-1}w_{1,1} = e$.

If $i > 1$, then $w_1w_2w_3w_4 \notin R$. Hence (6.2) implies $a^{-1}w_1 = e$.

The reversed implications are immediate. □

A similar reasoning leads to the following lemma.

Lemma 6.6. *Let $x, y \in G$, such that $R_y = \emptyset$, and $y \in \mathcal{C}(x)$.*

Let $a \in A \cap \mathcal{C}(x)$. Then, if $w_1w_2\dots w_n$ is the (only) geodesic word representing y, we have

$$xy \in \mathcal{C}(e, xa) \Leftrightarrow w_1 = a.$$

In the following, the value of an elementary multiplicative function $\mu[\mathcal{C}(e,b), v]$ in $z \in \mathcal{C}(e,b)$ is obtained as a (finite) sum over all geodesic paths from b to z.

Proposition 6.7. *Let $b \in A$, $c_0 = \mathcal{C}(b)$, and $v_{c_0} \in V_{c_0}$. Then, for $z \in \mathcal{C}(e,b)$, $z \neq b$, we have*

$$\mu[\mathcal{C}(e,b), v_{c_0}](z) = \sum_{\substack{geodesic\ words\, w_1w_2\dots w_n \\ b^{-1}z = w_1w_2\dots w_n}} \left[\prod_{j=1}^{n} H_{w_j,c_j,c_{j-1}}\right](v_{c_0}),$$

with notation $c_0 = \mathcal{C}(b)$, $c_j = \mathcal{C}(bw_1\dots w_j)$, and

$$\prod_{j=1}^{n} H_{w_j,c_j,c_{j-1}} = H_{w_n,c_n,c_{n-1}} H_{w_{n-1},c_{n-1},c_{n-2}} \dots H_{w_1,c_1,c_0}.$$

Proof. Since $z \in \mathcal{C}(e,b)$, $z \neq b$, then $d(e,z) = d(e,b) + d(b,z)$ and we can write $b^{-1}z = w_1w_2\dots w_n$, where $w_1w_2\dots w_n$ is a geodesic word, $n = |z| - 1$. Set $y = b^{-1}z$.

We have either $R_y = \emptyset$, or $R_y \neq \emptyset$.

In the first case, $w_1w_2\dots w_n$ is the only geodesic word representing y, hence by Definition 6.1 and Lemma 6.6 applied to $x = b$,

$$\begin{aligned}
\mu[\mathcal{C}(e,b), v_{c_0}](z) &= \sum_{\substack{b' \in A \\ bb' \in \mathcal{C}(e,b) \\ c' = \mathcal{C}(bb')}} \mu[\mathcal{C}(e,bb'), H_{b',c',c_0}(v_{c_0})](z) \\
&= \mu[\mathcal{C}(e,bw_1), H_{w_1,c_1,c_0}(v_{c_0})](bw_1\dots w_n)
\end{aligned}$$

where $c_0 = \mathcal{C}(b)$, $c_1 = \mathcal{C}(bw_1)$, since all other instances are null. A repeated application of Lemma 6.6 yields

$$\begin{aligned} & \mu[\mathcal{C}(e,b), v_{c_0}](bw_1 \dots w_n) \\ = \;& \mu[\mathcal{C}(e, bw_1 \dots w_n), H_{w_n,c_n,c_{n-1}} \dots H_{w_1,c_1,c_0}(v_{c_0})](bw_1 \dots w_n) \\ = \;& H_{w_n,c_n,c_{n-1}} \dots H_{w_1,c_1,c_0}(v_{c_0}), \end{aligned}$$

as desired.

If $R_y \neq \emptyset$, instead, we consider the first quadruple $q_0 \in R_y$, and its twin q_1,

$$q_0 = w_{0,i_0} \dots w_{0,i_0+3} = w_{1,i_1} \dots w_{1,i_1+3} = q_1.$$

We consider the quadruple next to q_0, if any, and its twin, say

$$q_{(0,0)} = w_{(0,0),i_{(0,0)}} \dots w_{(0,0),i_{(0,0)}+3} = w_{(0,1),i_{(0,1)}} \dots w_{(0,1),i_{(0,1)}+3} = q_{(0,1)},$$

where $i_0 + 2 < i_{(0,0)}$ and $i_0 + 3 \leq i_{(0,0)}$.

Similarly for q_1, if any, say

$$q_{(1,0)} = w_{(1,0),i_{(1,0)}} \dots w_{(1,0),i_{(1,0)}+3} = w_{(1,1),i_{(1,1)}} \dots w_{(1,1),i_{(1,1)}+3} = q_{(1,1)},$$

with $i_1 + 3 \leq i_{(1,0)}$, and so on so forth...

Consider q_0. Lemma 6.5 implies

$$\mu[\mathcal{C}(e,b), v_{c_0}](z) = \sum_{\substack{b' \in A \\ bb' \in \mathcal{C}(e,b) \\ c' = \mathcal{C}(bb')}} \mu[\mathcal{C}(e,bb'), H_{b',c',c_0}(v_{c_0})](z)$$

$$= \begin{cases} \mu[\mathcal{C}(e,bw_{0,1}), H_{w_{0,1},c_{0,1},c_0}(v_{c_0})](bw_1 \dots w_4 \dots w_n) & \\ \quad & \text{if } i_0 = 1, \\ +\mu[\mathcal{C}(e,bw_{1,1}), H_{w_{1,1},c_{1,1},c_0}(v_{c_0})](bw_{1,1} \dots w_{1,4} \dots w_n), & \\ & \\ \mu[\mathcal{C}(e,bw_1), H_{w_1,c_1,c_0}(v_{c_0})](bw_1 \dots w_4 \dots w_n), & \text{if } i_0 > 1. \end{cases}$$

where, in the $i_0 = 1$ case, $w_{0,1} = w_1$, $c_1 = c_{0,1} = \mathcal{C}(bw_{0,1})$, and $c_{1,1} = \mathcal{C}(bw_{1,1})$.

Since the subsequent quadruple, if any, has in common with the previous one at most one element (either w_4 or $w_{1,4}$), see Remark 6.3, by Lemma 6.5, with obvious meaning of symbols, we get in the $i_0 = 1$

case (if there is no subsequent quadruple, we set $i_{(\delta_j,0)} = n+1$)

$$
\begin{aligned}
&\mu[\mathcal{C}(e,b), v_{c_0}](z) \\
= &\ \mu[\mathcal{C}(e, bw_{0,1}w_{0,2}w_{0,3}), H_{w_{0,3},c_{0,3},c_{0,2}} \dots H_{w_{0,1},c_{0,1},c_0}(v_{c_0})](bw_1 \dots w_n) \\
+ &\ \mu[\mathcal{C}(e, bw_{1,1}w_{1,2}w_{1,3}), H_{w_{1,3},c_{1,3},c_{1,2}} \dots H_{w_{1,1},c_{1,1},c_0}(v_{c_0})](bw_{1,1} \dots w_n) \\
= &\ \dots \\
= &\ \mu[\mathcal{C}(e, bw_{0,1} \dots w_{i_{(0,0)}-1}), H_{w_{i_{(0,0)}-1},c_{i_{(0,0)}-1},c_{i_{(0,0)}-2}} \dots \\
&\qquad \dots H_{w_{0,3},c_{0,3},c_{0,2}} \dots H_{w_{0,1},c_{0,1},c_0}(v_{c_0})](bw_1 \dots w_{i_{(0,0)}-1} \dots w_n) \\
+ &\ \mu[\mathcal{C}(e, bw_{1,1} \dots w_{i_{(1,0)}-1}), H_{w_{i_{(1,0)}-1},c_{i_{(1,0)}-1},c_{i_{(1,0)}-2}} \dots \\
&\qquad \dots H_{w_{1,3},c_{1,3},c_{1,2}} \dots H_{w_{1,1},c_{1,1},c_0}(v_{c_0})](bw_{1,1} \dots w_{i_{(1,0)}-1} \dots w_n),
\end{aligned}
$$

while, if $i_0 > 1$, we get

$$
\begin{aligned}
&\mu[\mathcal{C}(e,b), v_{c_0}](z) \\
= &\ \mu[\mathcal{C}(e, bw_1 \dots w_{i_0-1}), H_{w_{i_0-1},c_{i_0-1},c_{i_0-2}} \dots H_{w_1,c_1,c_0}(v_{c_0})](bw_1 \dots w_n).
\end{aligned}
$$

We apply Lemma 6.5 recursively to any summand so generated, for any subsequent quadruple and corresponding twin which contribute to a different way of writing y as a geodesic word.

After a finite number of steps,we get

$$
\begin{aligned}
&\mu[\mathcal{C}(e,b), v_{c_0}](z) \\
= &\sum_{\substack{\text{geodesic words } bw_1w_2\dots w_n \\ \text{such that } z=bw_1w_2\dots w_n}} \mu[\mathcal{C}(e, bw_1 \dots w_n), H_{w_n,c_n,c_{n-1}} \dots \\
&\qquad\qquad \dots H_{w_1,c_1,c_0}(v_{c_0})](bw_1 \dots w_n) \\
= &\sum_{\substack{\text{geodesic words } w_1w_2\dots w_n \\ \text{such that } b^{-1}z=w_1w_2\dots w_n}} \left[\prod_{j=1}^{n} H_{w_j,c_j,c_{j-1}}\right](v_{c_0}).
\end{aligned}
$$

□

We provide now a realization of elementary multiplicative functions in terms of the matrix of cone types M. The key point is that any admissible triple (a, c', c) is independent from a.

Proposition 6.8. *Let $x, y \in \Gamma_2$ such that $\mathcal{C}(x) = \mathcal{C}(y)$. Let $a \in A \cap \mathcal{C}(x)$ and $b \in A \cap \mathcal{C}(y)$. If $\mathcal{C}(xa) = \mathcal{C}(yb)$ then $a = b$.*

As a consequence, if triples (a, c', c), (b, c', c) are admissible, then $a = b$ (and we can simply write (c', c)).

Proof. There exists $z \in \mathcal{R}$ such that $\mathcal{C}(x) = \mathcal{C}(y) = \mathcal{C}(z)$. By Proposition 4.2, we get

$$\mathcal{C}(za) = \mathcal{C}(xa) = \mathcal{C}(yb) = \mathcal{C}(zb),$$

where $a, b \in A \cap \mathcal{C}(z)$. By results of Section 4 we know that different successors of $z \in \mathcal{R}$ have different cone types, yielding $za = zb$ and so $a = b$. $\square$

Let us consider the scalar case, first, where any matrix system is as follows: $V_c = \mathbb{C}$, for each cone type c, and, for any admissible triple (a, c', c), $H_{a,c',c}$ is multiplication by a non-zero complex number, while $H_{a,c',c} = 0$ otherwise. Let $M = (m_{c',c})$ be the matrix (5.1), indexed by cone types.

We note that, for any couple of cone types c', c and for any $a \in A$, $m_{c',c} H_{a,c',c} = H_{a,c',c}$, and the latter is non-zero only for admissible triples. Hence, by Proposition 6.8, we can set $H_{c',c} = m_{c',c} H_{a,c',c}$ and consider a new matrix $N = (H_{c',c})$, with non-zero entries in the same positions as M.

Also, for any cone type c, let $V_c = (0 \dots 1 \dots 0)$ be the vector with 1 at the c-position, and $E_c = V_c^\top V_c$ be the 48×48 matrix whose entries are all null except the (c, c) diagonal element (equal to 1).

Corollary 6.9 (Scalar case)**.** *Let $b \in A$, $c_0 = \mathcal{C}(b)$, and $v \in \mathbb{C}$. Then, for $z \in \mathcal{C}(e, b)$, $z \neq b$, we have*

$$\mu[\mathcal{C}(e,b), v](z) = \sum_{\substack{\text{geodesic words } w_1 w_2 \dots w_n \\ b^{-1} z = w_1 w_2 \dots w_n}} \left[\prod_{j=1}^{n} H_{c_j, c_{j-1}} \right] v$$

$$= V_{c_n} N \left[\sum_{\substack{\text{geodesic words } w_1 w_2 \dots w_n \\ b^{-1} z = w_1 w_2 \dots w_n}} E_{c_{n-1}} N E_{c_{n-2}} \dots N E_{c_1} \right] N V_{c_0}^\top v,$$

with notation $c_0 = \mathcal{C}(b)$, $c_j = \mathcal{C}(b w_1 \dots w_j)$, $c_n = \mathcal{C}(z)$.

Proof. It is easy to see that, for any triple (a, c', c),

$$V_{c'} N V_c^\top = (0 \dots 1 \dots 0) N \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} = H_{c',c},$$

and that the latter is $H_{a,c',c}$ if the triple is admissible, and 0 otherwise. $\square$

Also, by (6.1) we get a similar result for cones like $\mathcal{C}(b,e)$.

Proposition 6.10. *Let $b \in A$, $c_0 = \mathcal{C}(b^{-1})$, and $v_{c_0} \in V_{c_0}$. Then, for $z \in \mathcal{C}(b,e)$, $z \neq e$, we have*

$$\mu[\mathcal{C}(b,e), v_{c_0}](z) = \sum_{\substack{\text{geodesic words } w_1 w_2 \dots w_n \\ z = w_1 w_2 \dots w_n}} \left[\prod_{j=1}^{n} H_{w_j, c_j, c_{j-1}} \right] (v_{c_0}),$$

where $c_0 = \mathcal{C}(b^{-1})$, $c_j = \mathcal{C}(b^{-1} w_1 \dots w_j)$, and

$$\prod_{j=1}^{n} H_{w_j, c_j, c_{j-1}} = H_{w_n, c_n, c_{n-1}} H_{w_{n-1}, c_{n-1}, c_{n-2}} \dots H_{w_1, c_1, c_0}.$$

In the scalar case, with notation as in Corollary 6.9

$$\mu[\mathcal{C}(b,e), v](z) = \sum_{\substack{\text{geodesic words } w_1 w_2 \dots w_n \\ z = w_1 w_2 \dots w_n}} \left[\prod_{j=1}^{n} H_{c_j, c_{j-1}} \right] v$$

$$= V_{c_n} N \left[\sum_{\substack{\text{geodesic words } w_1 w_2 \dots w_n \\ z = w_1 w_2 \dots w_n}} E_{c_{n-1}} N E_{c_{n-2}} \dots N E_{c_1} \right] N V_{c_0}^{\top} v,$$

with notation $c_0 = \mathcal{C}(b^{-1})$, $c_j = \mathcal{C}(b^{-1} w_1 \dots w_j)$, $c_n = \mathcal{C}(b^{-1} z)$.

Proof. Since $b^{-1} z \neq b^{-1}$, and $b^{-1} z \in \mathcal{C}(e, b^{-1})$, by (6.1) and Proposition 6.7

$$\mu[\mathcal{C}(b,e), v_{c_0}](z) = \mu[\mathcal{C}(e,b^{-1}), v_{c_0}](b^{-1} z)$$

$$= \sum_{\substack{\text{geodesic words } w_1 w_2 \dots w_n \\ z = w_1 w_2 \dots w_n}} \left[\prod_{j=1}^{n} H_{w_j, c_j, c_{j-1}} \right] (v_{c_0}),$$

where $c_0 = \mathcal{C}(b^{-1})$, $c_j = \mathcal{C}(b^{-1} w_1 \dots w_j)$, $c_n = \mathcal{C}(b^{-1} z)$ and

$$\prod_{j=1}^{n} H_{w_j, c_j, c_{j-1}} = H_{w_n, c_n, c_{n-1}} H_{w_{n-1}, c_{n-1}, c_{n-2}} \dots H_{w_1, c_1, c_0}.$$

$\square$

The vector case follows naturally. Let d_c be the dimension of the (finite dimension) vector space V_c. $H_{a,c',c}$ can be identified with a $d_{c'} \times d_c$ matrix, and $H_{a,c',c} = 0$ for non-admissible triples. Let $d = \sum_c d_c$.

As before, for any couple of cone types c', c and for any $a \in A$, multiplication by the scalar $m_{c',c}$ yields $m_{c',c}H_{a,c',c} = H_{a,c',c}$, and the latter is zero for non-admissible triples. Hence, by Proposition 6.8, we can set $H_{c',c} = m_{c',c}H_{a,c',c}$ and define a (block) $d \times d$ matrix $\mathcal{N} = (H_{c',c})$.

Also, let $\mathcal{V}_c = (0 \dots I \dots 0)$ be the block matrix with the identity matrix I at the d_c-position, and $\mathcal{E}_c = \mathcal{V}_c^\top \mathcal{V}_c$ be the $d \times d$ block matrix whose entries are all null except the (d_c, d_c) diagonal element (equal to I).

Corollary 6.11 (Vector case)**.** *Let $b \in A$, $c_0 = \mathcal{C}(b)$, and $v \in V_{c_0}$. Then, for $z \in \mathcal{C}(e,b)$, $z \neq b$, we have*

$$\mu[\mathcal{C}(e,b),v](z) = \mathcal{V}_{c_n}\mathcal{N}\left[\sum_{\substack{\text{geodesic words } w_1w_2\dots w_n \\ b^{-1}z=w_1w_2\dots w_n}} \mathcal{E}_{c_{n-1}}\mathcal{N}\mathcal{E}_{c_{n-2}}\dots\mathcal{N}\mathcal{E}_{c_1}\right]\mathcal{N}\mathcal{V}_{c_0}^\top v,$$

with notation $c_0 = \mathcal{C}(b)$, $c_j = \mathcal{C}(bw_1 \dots w_j)$, $c_n = \mathcal{C}(z)$.

Proposition 6.10 extends similarly.

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