Nonconservative stability problems for cantilever column subjected to subtangential forces

"CDM" discretization

N. M. Auciello
School of Engineering, University of Basilicata
Via dell'Ateneo Lucano 10,
85100 Potenza, Italy
nicola.auciello@unibas.it

Abstract—**In the present study, the dynamic behavior of cantilever column having a tip rigid body and subjected to the action sub-tangential forces. The solution of the problem is obtained through the Lagrange's approach and assuming as "CDM" discretization method [1,2]. The procedure, applied to cantilever column, is an alternative method to the usual FEM and Rayleigh-Ritz methodologies used in literature. The structures is reduced to a set of rigid bars linked together by means of elastic constraints. The system is reduced to a discrete problem to many parameters of freedom (MDOF) and you can study it using the usual theorem of classical mechanics. Evaluation schemes for flutter and divergence loads of non-conservative system are described and the static buckling loads and natural frequencies of beam-columns are compared through numerical examples. Finally, the influences of the tip rigid body on the dynamic behavior of the beam. The work ends with the analysis of a few numerical examples and results are compared with the ones obtained from authors mentioned in bibliography.**

Keywords- Cantilever beam, Lagrange's approach, "CDM" method, subtangential force

I. INTRODUCTION

There have been considerable number of papers on the dynamic stability of columns subjected to non-conservative forces. It is well known that if the type of instability is divergence, the critical loads of the system can be determined by the static approach while the critical loads for flutter should be determined using the dynamic criterion. The elastic instability of elastic non-uniform Euler-Bernoulli column (E-B) subjected to a tangential force has been examined by many instigators [3-8]. Chen [4] found that if a cantilever beam is subjected to a partially tangential force, then the instability mechanism for the beam is divergent if the tangency coefficient is less than or equal to values η_c. For uniform beam η_c=0.5. Recently, Auciello [8], study the instability behaviour of beams with variable cross section subjected to sub-tangential non-conservative follower forces, and the solution is numerically attained by using a Rayleigh-Ritz method. Instead, Marzani et al. [9], study the instability behaviour of beams with variable cross section subjected to sub-tangential non-conservative follower forces, and the solution is numerically attained by using a Differential Quadrature Method (DQM) procedure.

The dynamic stability of cantilevered columns having a rigid body at the tip, as a solid rocket motor to produce a tangential thrust, was first studied both theoretically and experimentally by Sugiyama et al [10]. Rao and Rao [11] examined the application of the static and dynamic stability criterion for obtaining the critical force of cantilevered columns subjected to a sub-tangential force.

Most of the studies so far made have been related to the column with large slenderness ratio. For tapered beam, short elastic columns and column with tip mass the parameters may be important in the stability analysis of the system. The aim of the present paper is to investigate the dynamic stability of a cantilever Euler-Bernoulli column having a tip rigid body an subjected to a sub-tangential force by using "CDM" discretization. The effect of non-conservativeness of the sub-tangential follower force and the tip mass parameters is fully discussed.

II. STATEMENT OF PROBLEM

Consider an elastic cantilever column having a tip rigid body and subjected to a sub-tangential force *P*. The direction of the force is specified by $\eta\theta$ as show in Fig.1, where θ is the angle of inclination of the tangent at tip end. When α=0, the direction of the acting force is vertical, i.e. the force is conservative. When α=1, it is tangential to the tip end, i.e. the force is purely non-conservative ; Beck type column [5].

The structure is reduced to a set of rigid bars linked together by means of elastic constraints, and consequently a stiffer structure than the real one is obtained. The beam is supposed to be divided into *t* rigid bars, linked together by means of elastic elements which allow relative rotation. Therefore, the structures is reduced to a finite-degree of freedom system (MDOF); the rotations of the *i*th rigid bar (φ_i). All the possible configurations are functions of the following vector:

$$\mathbf{c} = [\varphi_1, \varphi_2, \ \dots \ \varphi_t]^T \tag{1}$$

and the vertical components of the nodal displacements are given by the following expressions:

$$w_1 = 0, \quad w_2 = -\varphi_1 l_1 , \quad w_i = -\sum_1^{i-1} \varphi_i l_i , \quad w_{t+i} = -\sum_1^{i} \varphi_i l_i .$$

In matrix form, being **A** the displacements matrix, it is possible to write:

$$\mathbf{w} = \mathbf{A}\,\mathbf{c}\,. \tag{2}$$

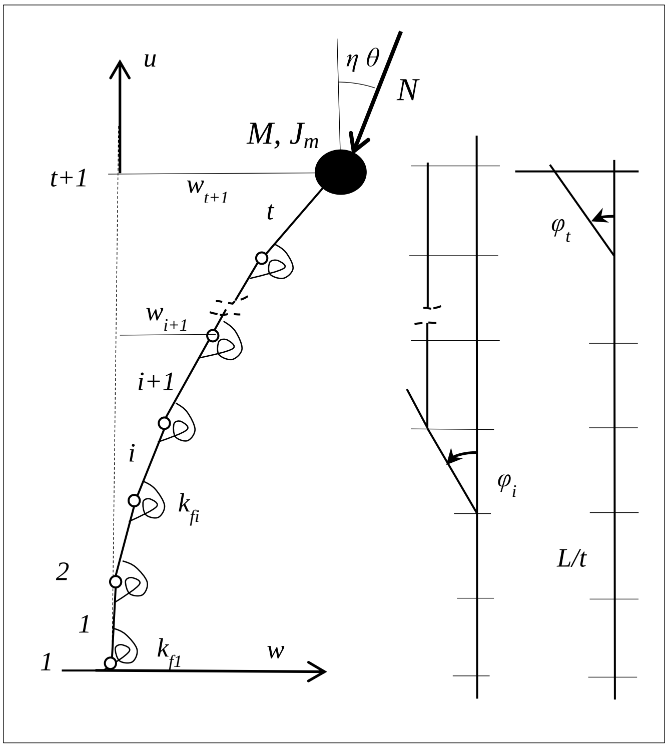

Figure 1. "CDM" discretization of a cantilever columns having a tip rigid body and subjectedto a subtangential force.

The axial components of the nodal displacements are given by the following expressions:

$$u_1 = 0, \quad u_2 = -\frac{\varphi_i^2}{2}\frac{L}{t}, \quad u_i = -\sum_{j=1}^{i-1}\frac{\varphi_j^2}{2}\frac{L}{t} \qquad i = 1.....t+1, \tag{3}$$

in matrix form:

$$u_M = u_{t+i} = -\frac{L}{t}\sum_{j=1}^{t}\frac{\varphi_j^2}{2} = -\frac{1}{2}\frac{L}{t}\mathbf{c}^T\,\mathbf{D}_l\,\mathbf{c}, \tag{4}$$

where $\boldsymbol{D}_l$ is the diagonal matrix of the terms *L/t*,

Similarly, the relative rotations between the two faces of the elastic cells are given by:

$$\Delta\varphi_1 = \varphi_1, \quad \Delta\varphi_2 = \varphi_2 - \varphi_1\,, \quad \Delta\varphi_i = \varphi_i - \varphi_{i-1}, \quad \Delta\varphi_{t+1} = 0$$

and in matrix form:

$$\boldsymbol{\Delta\varphi} = \mathbf{B}\mathbf{c}. \tag{5}$$

The rectangular matrices **A** and **B** have *t+1* rows and *t* columns, and each term can be calculated according to Fig.1; where:

$$\mathbf{A} = \begin{array}{c} \\ 1 \\ 2 \\ . \\ i \\ . \\ . \\ . \\ t+1 \end{array} \begin{array}{c} \begin{array}{cccccccc} 1 & 2 & & i & i+1 & & & t \end{array} \\ \left[\begin{array}{cccccccc} - & - & - & - & - & - & - & - \\ -1 & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot \\ -1 & -1 & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & - \\ -1 & -1 & -1 & -1 & \cdot\cdot & \cdot\cdot & \cdot\cdot & - \\ -1 & -1 & -1 & -1 & -1 & \cdot\cdot & \cdot\cdot & \cdot\cdot \\ - & - & - & - & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot \\ - & - & - & - & - & - & & \cdot\cdot \\ -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \end{array}\right] \end{array} \frac{L}{t},$$

and

$$\mathrm{B} = \begin{array}{c} \\ 1 \\ 2 \\ . \\ i \\ . \\ . \\ . \\ t+1 \end{array} \begin{array}{c} \begin{array}{cccccccc} 1 & 2 & & i & i+1 & & & t \end{array} \\ \left[\begin{array}{cccccccc} 1 & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot \\ -1 & 1 & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot \\ & -1 & 1 & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot \\ \cdot\cdot & \cdot\cdot & -1 & 1 & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot \\ \cdot\cdot & \cdot\cdot & \cdot\cdot & -1 & 1 & \cdot\cdot & \cdot\cdot & \cdot\cdot \\ \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot \\ \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & -1 & 1 & \cdot\cdot \\ \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & 0 & 0 \end{array}\right] \end{array},$$

Quite often it is possible to neglect both the axial and the shear deformation effects, limiting oneself to the bending deformations. In such hypothesis, at each "cell", the following relation between the relative rotation $\Delta\varphi_i$ and the moment M_i can be written, as follows:

$$M_i = L\frac{EI}{t}\Delta\varphi_i = k_{fi}\,\Delta\varphi_i, \tag{6}$$

where *E* is the Young modulus and *I* is the inertia cross sectional area.

III. Lagrange's Equations

The Lagrange's equations for free vibration of a distributed parameter are given by

$$\frac{d}{dt}\left(\frac{\partial T}{\partial\dot{\varphi}_i}\right) - \frac{\partial T}{\partial\varphi_i} + \frac{\partial U}{\partial\varphi_i} = Q_i, \qquad i = 1, 2....n \tag{7}$$

where *n* is the total number of modal coordinates and *T*, *U* are respectively the kinetic and the potential energy and Q_i the coefficients of the force non-conservative; Clough et al. [11].

The partial derivatives of *T* with respect to the generalized coordinates are needed.

A. Strain Energy

The strain energy is given by the bending strain energy, at *i-th* cells abscissa, the following linear relationship is supposed to hold:

$$L_i = \frac{1}{2}\left[M_i\,\Delta\varphi_i\right], \tag{8}$$

where M_i are the bending moment at the i-th cell abscissa. It is also:

10.18638/gv.2016.4.1.748
eISSN: 1339-9373, cdISSN: 1339-2778
ISBN: 978-80-554-1197-2

Put $\mathbf{K}_f$ the diagonal matrix of the terms k_{fi}, the strain energy of the whole structures is equal to:

$$U = \frac{1}{2}\sum_{1}^{2t} M_i \left(\Delta\varphi_i\right)^2 = \frac{1}{2}\mathbf{c}^T\mathbf{B}^T\mathbf{K}_f\mathbf{B}\mathbf{c} = \frac{1}{2}\mathbf{c}^T\mathbf{K}\,\mathbf{c} \quad (9)$$

B. *Kinetic Energy*

In order to deduce the kinetic energy of the structure, let us suppose that the distributed mass of the rigid bar will be concentrated at the nodal point of the rigid segments.

Therefore, the mass distribution becomes:

$$m_1 = \rho \frac{L}{2t} A_1,$$
$$m_i = \rho \frac{L}{t}\frac{A_i + A_{i+1}}{2} \qquad i = 2\ldots,t \qquad (10)$$
$$m_{t+1} = \rho \frac{L}{2t} A_{t+1} + M,$$

The kinetic energy also be expressed as a function of the Lagrangian coordinates; we have:

$$T = \frac{1}{2}\sum_{1}^{t+1} m_i\, \dot{w}_i^2 + \frac{1}{2} J_m\, \dot{\varphi}_t^2 =$$
$$= \frac{1}{2}\dot{\mathbf{c}}^T\,(\mathbf{A}^T\,\mathbf{m}\,\mathbf{A}\;\dot{\mathbf{c}} + \mathbf{J})\dot{\mathbf{c}} = \frac{1}{2}\dot{\mathbf{c}}^T\,\mathbf{M}\dot{\mathbf{c}}. \qquad (11)$$

where $\mathbf{J}(t,t)$ is the diagonal matrix of the terms;

$$J_i = 0 \qquad i = 1\ldots t-1$$

$$J_t = J_m \qquad i = 1\ldots t$$

C. *Axial force*

The elastic energy and the work by conservative forces of the system can be written as:

$$W_c = \frac{1}{2} N\, u_M = \frac{1}{2} N\,\mathbf{c}^T\,\mathbf{D}_l\,\mathbf{c}\,. \qquad (12)$$

The other, non-conservative part of the applied loads gives rise to the following virtual work;[12]:

$$\delta W_{nc} = N\sin\eta[u']_L \cong N\,\eta[\varphi]_L\,\delta w\big|_L = \qquad (13)$$
$$= N\,\eta\left[\varphi\right]_L\,\delta w\big|_L$$

Writing the previous expressions in terms of Lagrangian coordinates we have:

$$\delta W_{nc} = -N\,\eta\frac{L}{t}\mathbf{c}^T\mathbf{W}\,\delta\mathbf{c}\,, \qquad (14)$$

where $\boldsymbol{W}$ is the diagonal matrix

$$W_i = 0 \qquad i = 1,\ldots.t-1; \qquad W_t = 1. \qquad (15)$$

Most different cases can be faced, according to the η value. When η=0, the direction of the force is vertical, i.e. the force is conservative. When η=1, it is tangential to the tip end, i.e. the force is purely non-conservative. Thus the parameter η can also be referred to as the non-conservativeness parameter.

Put $\mathbf{K}_\text{n}(t,t) = (\mathbf{D}_l + \eta\,\mathbf{W})$, square matrix*:*

$$\mathbf{K}_\text{n} = \eta\frac{L}{t}\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ .. & .. & .. & .. & .. & .. & .. & .. \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ -1 & -1 & -1 & -1 & -1 & --1 & -1 & \frac{1-\eta}{\eta} \end{bmatrix},$$

the equation of motion (7) can be written as

$$\mathbf{M}\ddot{\mathbf{c}} + \left[\mathbf{K} - P\mathbf{K}_n\right]\mathbf{c} = \mathbf{0} \qquad (16)$$

and the free vibration frequencies are calculated as the eigenvalue problem imposing

$$\det\left[(\mathbf{K} - N\,\mathbf{K}_n) - \omega^2\mathbf{M}\right] = \mathbf{0}. \qquad (17)$$

From a computational point of view, the presence of unsymmetrical matrices leads to complex conjugate solutions, and an iterative approach seems to be the simplest solution algorithm. The stability of the system under consideration is determined by the sign of real part σ of the complex eigenvalue $\bar{\Omega} = \sigma \pm i\,\omega \;\; (i = \sqrt{-1})$. If $\sigma < 0$,the system is stable; if $\sigma > 0$ and $\omega = 0$, the system is statically unstable, i.e., divergence type instability; if $\sigma > 0$ and $\omega \neq 0$, the system is dynamically unstable, i.e., flutter type instability; if $\sigma = 0$, it has the critical follower force p_c arises.

Two different cases can be faced, according to the η value. If $\eta<\eta_c$ the normalized critical load p_c corresponds to $\Omega_1=0$, and it can be deduced using the static criterion.

The condition:

$$\det[\mathbf{K} + N\,\mathbf{K}_n] = 0, \qquad (18)$$

gives the solutions p_i and the critical load is $p_c=p_1$.
As η increases, a threshold value η_c is reached, beyond which the structure loses stability by flutter, and the static criterion is no longer applicable. At $\eta>\eta_c$ the solutions p_i of equation (12) turn out to be complex, and the critical load must be calculated using eqn. (12), corresponding to the coalescence of the first two free vibration frequencies.

eISSN: 1339-9373, cdISSN: 1339-2778
ISBN: 978-80-554-1197-2

IV. Numerical Results and Discussion

- *Nondimensional analysis*

In order to compare the results with those reported in the literature it is useful to introduce the functions *G(x)* and *H(x)* that define, in general terms, the geometric characteristics of the structure

$$A(x) = A_0\, G(x), \qquad I(x) = I_0\, H(x), \qquad (18,19)$$

where $A_0=\pi$ and $I_0=\pi/4$, are respectively the area and moment of inertia of the section at $x = 0$.

Introducing the following parameters

$$\xi = \frac{x}{L}, \quad m_t = \rho A_0 L, \quad \mu = \frac{M_m}{m_t}, \quad k_m^2 = \frac{J_m}{M_m}$$
$$\Omega_i^2 = \omega_i^2 \frac{\rho A_0 L^4}{\pi^2 E I_0}, \quad P = \frac{N L^2}{\pi^2 E I_0}. \qquad (20)$$

- *Uniform beam*

As a first example, for t=100, the column with constant cross section has been studied, subjected to a sub-tangential load, $\alpha=0$: G(x)=H(x)=1.

Using eqn. (18) the critical load is given, for different parameter value $\eta< \eta_c$. At $\eta=0$ we have two different value P_1 and P_2, whereas the difference P_1-P_2 diminishes with increasing η, and at $\eta =\eta_c=0.5$ the two values coalesce. The columns have two divergence instability force for $\eta<\eta_1$, and divergence ad flutter instability force for $\eta_1<\eta<\eta c$, and only one flutter instability force for $\eta>\eta_c$. In the case of E-B uniform column ($\alpha=0$), the boundary values non-conservativeness parameters, $\eta_1=0{,}3543$ and $\eta_c=0{,}5$. To verify the accuracy of the numerical calculation applied, the critical force obtained for a specific parameter η of the present paper was compared with the values reported by reference.

In Table I, the critical loads Pc for $\eta< \eta c$ are presented and compared with the results by Chen [4] and Auciello [8].

TABLE I. Comparison of critical force (η<0,5).

α=0	Chen [4]	Auciello [8]		Present (t=100)	
η	P_1	P_1	P_2	P_1	2
0	0,240	0,2499	2,2499	0,2499	2,2495
0,20	0,337	0,3369	2,0151	0,3369	2,0148
0,30	------	0,4109	1,8469	0,4108	1,8466
0,3543	------	0,4649	1,7236	1,4690	1,7293
0,40	------	0,5362	1,6071	0,5362	1,6069
0,45	------	0,6519	1,4279	0,6480	1,4278
0,48	------	0,7644	1,2671	0,7644	1,2670
0,49	0,829	0,8291	1,1868	0,8291	1,1867
0,50	1	1	1	1	1

In order to check the results of the η>0,5, in Table II are given the critical load and the instability is the type flutter. The dynamic criterion is followed, and the axial load is found, such that the first two frequencies coalesce, $\Omega_1=\Omega_2$. For $\eta > 0.5$ the critical load must be calculated using eqn. (12), corresponding to the coalescence of the first two free vibration frequencies. For the values of non-conservative parameter η=1, Beck column, the critical force are Pc=2.0315 approach (R-R) and Pc=2.0306 (CDM); Fig. 2.

TABLE II. Dependence of critical force (η>0,5).

α=0	Chen [4]		Auciello [8]		Present "CDM"	
η	P_C	$\Omega_1=\Omega_2$	P_C	$\Omega_1=\Omega_2$	P_C	$\Omega_1=\Omega_2$
0,51	**1,627**	**0,732**	**1,6267**	**0,7315**	**1,6262**	**0,7303**
0,52	-----	------	1,6274	0,7456	1,6267	0,7453
0,55	1,632	0,788	1,6321	0,7876	1,6315	0,7862
0,60	-----	-----	1,6473	0,8445	1,6466	0,8441
0,70	-----	-----	1,7009	0,9359	1,7000	0,9357
0,80	1,782	1,009	1,7815	1,0085	1,7812	1,0075
1,00	2,032	1,118	2,0315	1,1161	2,0306	1,1157

It is seen that the CDM approach lead to lower bound that Rayleigh-Ritz method; Auciello [13].The comparison of the results has shown a good agreement.

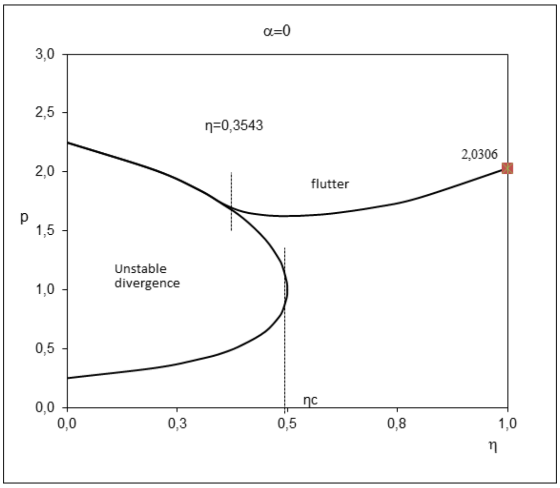

Figure 2. Effect of the non-conservative parameter for η=0,0-1.0.

A. *Effect of tip rigid body*

In order to demonstrate the effect of the tip rigid body of the column on the critical force, in Table II are reported the critical force for four cases of parameters μ and various non-conservative η.

TABLE III. Effect of tip rigid body on the critical force (η>0,5).

μ	η=1	η=0,8	η=0,7	η=0,6
0	2,0315	1,7813	1,7000	1,6468
0,01	1,9916	1,7629	1,6894	1,6420
0,1	1,7815	1,6701	1,6390	1,6260
1	1,6421	1,6674	1,6864	1,7115
10	1,8399	1,8635	1,8772	1,8928
100	1,9706	1,9800	1,9853	1,9913

For the values η<0,5, the increase of the mass of tip rigid body does not affect the change of the critical force, and the type of instability is divergence for all the values of the mass ratio μ. This is obvious because the divergence limit is not affect by mass. For η>0,5 as shown in Fig.3, however, the type of instability of the column is flutter.

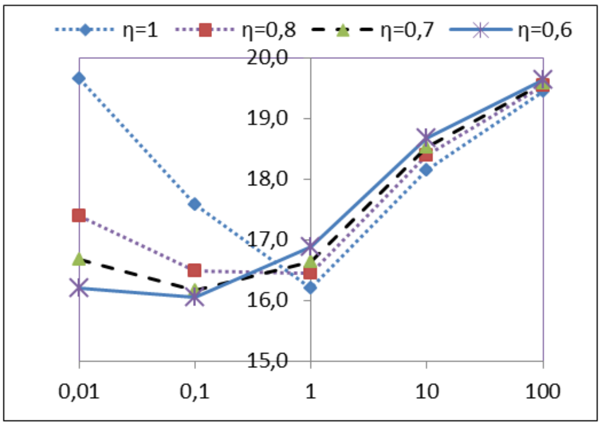

Figure 3. Effect of the mass of the tip rigid body on flutter force *P*c for η=0,6-1.0.

- *Tapered beam*

Let us consider here a linear circular tapered cantilever beam of length L clamped at ξ=0. The cross-section of the beam varies with geometrical tapered parameter α:

$$G(\xi)=(1+\alpha\,\xi)^2, \qquad H(\xi)=(1+\alpha\,\xi)^4 \qquad (21)$$

In the divergence region η<ηc, the static criterion is followed, and the axial force pc is found. The corresponding data for α=-0,5, ..0,5, are given in the Fig. 4.

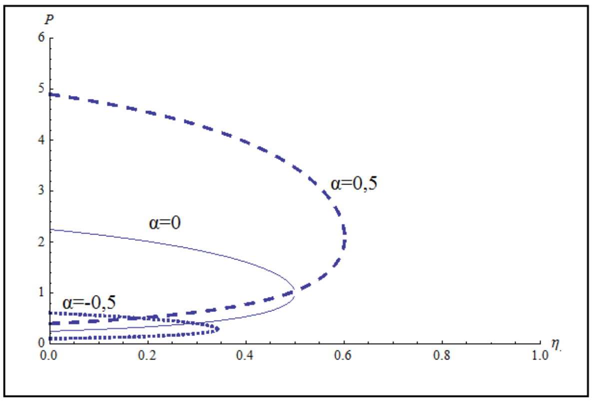

Figure 4. The nondimensional *P* vs the parameter η for various coefficients α.

The first two critical load coefficients *P*c are given for various values of the sub-tangential coefficient η, and the limiting value η_c defines the passage from divergence to flutter;

TABLE IV. VALUE η_C FOR VARIOUS PARAMETERS α.

	α=-0,5	α=0	α=0.25	α=0,5
η_c	0,3425	0,5	0,5560	0,6014
P_c	0,2937	1	1,4970	2,0896

V. CONCLUSIONS

In this paper the stability and dynamic analysis of cantilever beam in presence of a sub-tangential force. It has been shown that the proposed approach (CDM), when used together, can be of great interest in the study of the dynamic behaviour of beams subjected to sub-tangential force, at least in all the cases in which closed-form solution are not available. From the numerical results obtained, the following conclusions can be drawn;

- As to standard uniform column (α=0) the instability-type of the critical force is divergence for the non-conservative parameter η<0,5, regardless of the existence of the tip rigid body. The columns have two divergence forces for $\eta<\eta_1$, and divergence and flutter forces for $\eta_1<\eta<\eta_c$, and only one flutter force for $\eta>\eta_c$.
- It is confirmed that the effect of the tip rigid body parameter is negligibly for $\eta<\eta_c$. For the values of the non-conservativeness parameter $\eta>\eta_c$, however, the type of instability of the column is flutter.

REFERENCES

[1] V. Franciosi and N.M. Auciello, “Analisi modale degli archi su imposte elastiche comunque cedevoli”. Industria Italiana del Cemento, 1988, n.627, pp. 706-711.
[2] N.M. Auciello and A. Ercolano, “Nonlinear analysis of a beam subjected to axial and trasverse loads”. Comp. & Struct. 1992, n. 4, pp. 863-868.
[3] S.Y. Lee and Y.H. Kuo, “Exact solutions for the non-conservative elastic stability of non-uniform columns”. J. of Sound and Vibration, 1992, 155, pp. 291-301.
[4] Y.Z. Chen, “Interaction between compressive force and vibration frequency for a varying cross-section cantilever under action on generalized follower force”. J. Sound Vib. 2003, 259, 991-999.
[5] M. Beck, Die Knicklast des Einseitig Eingespannten, Tangential Gedrückten Stabes, ZAMP 3 (3) 1952, pp. 225-228. (in german)
[6] W. Glabisz, “Stability of non-prismatic rods subjected to non-conservative loads. Comput. & Struct. 1993, 46, pp. 479-486.
[7] N.M. Auciello and M.A. De Rosa, “Two approaches to the dynamic analysis of foundation beams subjected to subtangential forces”. Comput. Struct. , 2004, 82, pp. 519-524.
[8] N.M. Auciello, “Effect of subtangential parameter on the stability and dynamic of a cantilever tapered beams subjected to followed forces”, ARSA , Advanced Research in Scientific Areas 2012, December, 3. - 7. 2012, pp. 1721-1726.
[9] A. Marzani, F. Tornabene and E. Viola, “Nonconservative stability problems via generalized differential quadrature method”. J. Sound Vib. 2008, 315, pp. 176-196.
[10] Y. Sugiyama, J. Matsuike, B.J. Ryu, K. Katayama, S. Kinoi and N. Enomoto, “Effect of concentrated mass on stability of cantilevers under rosket thrust. AIAA Journal, 1993, 33, pp. 499-503.
[11] R.W. Cluogh and J. Penzien, “Dynamic of Structures”. McGraw-Hill Internatinal Edition, Civil Engineering Series, 1986.
[12] V.V. Bolotin, Nonconservative problems of the theorynof elastic stsability (English translation). New York:Permamon Press, 1963.
[13] N.M. Auciello, “On the transverse vibrations of non-uniform beams with axial loads and elastically reastrained ends”, Int. J. Mech. Scien., 2001, 43, pp. 193-208.