Computational model of vibration of Timoshenko beam under axial loads

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Abstract—**The free vibration characteristics of a cantilever tapered Timoshenko beam is analyzed in this study. First, the strain displacement relationship for the Timoshenko beam is formulated and used to derive the kinetic and strain energies in explicit analytical form. Second, Lagrange's variational principle is used to derive the governing differential equation of motion and the associated boundary conditions. Third, the Rayleigh-Ritz Method (R-R) is applied to the equation of motion and the boundary conditions to form a set of algebraic equations from which the frequency equation is derived. Next, a numerical algorithm implemented in the software package Mathematica is used to compute the natural frequencies. Also, the variation of the natural frequencies of vibration with respect to variations in the taper ratio and also the slenderness ratio is studied. The results obtained from the Timoshenko theory are compared with results obtained in literature to demonstrate the accuracy and relevance of the their application.**

Keywords- Timoshenko beam, R-R method, vibration analysis

I. Introduction

Elements offshore structures having the form of a column, wind turbines, aircraft propellers and turbo machinery are usually modeled as flexible beams. The dynamic behavior of these structures change significantly from their for axial force. This effect is due to the position the axial force which tends to vary the bending rigidity and hence, influences their natural frequencies of vibration.

For non-uniform beam are the most typical examples that can be reduced to a Timoshenko beam variable cross-section. By using this theory, free vibration frequencies have been obtained by many authors, employing finite element techniques.

Sigh and other [1-5] developed the eingen-problem formulation and solution technique which allow determination of the free vibrations of transversely vibrating beam having variety cross sections. For example, To [2] examined a beam with varying cross-section, for various boundary conditions, by using a cubic-linear interpolating function. For a cantilever uniform beam with a tip mass at the free end Abramovich et al. [3] and Cleghorn et al. [4] extended the analysis to eccentric masses. If the cross-section is supposed to vary according to a continuous law, Laura et al. [5] proposed an FEM-like algorithm. Auciello [6] used cell discretization metod(CDM) for determinate of free vibration of non-uniform beam for tip masse. Whereas, Auciello et al. in [7] presented the approximate analysis general solution of the vibration Timoshenko beam.

Esmailzadeh et al. [8] presented the exact, analytical solution of the vibration of Timoshenko beam subjected a two position to axial force.

In this paper, real discretization method (Rayleigh-Ritz) analysed a developed a study the instability behaviour of beams with axial force. Two sets of governing equations for transverse vibration of non-uniform Timoshenko beam subjected to axial load. The first set, the axial force is tangential to the axis of the beam. In the other set, the axial load have been taken normal to the normal to the shearing force and thus normal to the cross-section. Axial force tangential to the axis of the beam and axial force normal to the shearing force; Figure 1.

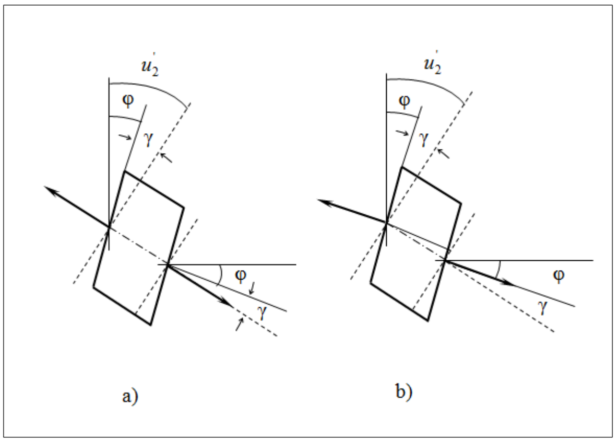

Figure 1. Timoshenko beam model; a) axial force tangential to the axis of the beam; b) axial force normal to the shearing force.

Referring to the Figure 1, it can be easily show the Hamilton's equation is used to derive the governing differential equation of motion.

II. Formulation of the Problem

If the Timoshenko model is assumed to be valid, then the displacements can be written as:

$$\mathbf{u} = \left[-x_2\,\varphi(x_1,t) \quad u_2(x_1,t) \quad 0 \right]^T, \tag{1}$$

where $\varphi(x_1,t)$ is the rotation of the cross section, which turn out to be different from the rotation θ of the neutral axis, so that the difference

$$\gamma_S = \frac{\partial u_2}{\partial x_1} - \varphi, \tag{2}$$

gives the additional rotation due to the shear deformation. According to (1) the strain components are given by:

$$\boldsymbol{\varepsilon} = \begin{bmatrix} -x_2 \dfrac{d\varphi}{dx_1} \\ \dfrac{du_2}{dx_1} - \varphi \end{bmatrix} \tag{3}$$

If the derivative with respect to x1 is written as an apex, the Hooke law for isotropic material gives the corresponding stress components;

$$\boldsymbol{\sigma} = \mathbf{D}\boldsymbol{\varepsilon}. \tag{4}$$

The equation of motion are derived via Hamilton's principle

$$\int_{t_1}^{t_2} (\delta U + \delta V - \delta T)\, dt = 0 \tag{5}$$

where δT, δU and δV are the variations of kinetic energy, strain energy and the potential energy of the beam associated with an initial axial tension load .

The kinetic energy of the beam is given as:

$$T = \frac{1}{2}\int_V \rho(\dot{u}_1^2 + \dot{u}_2^2 + \dot{u}_3^2)\, dV \tag{6}$$

and is rewritten using eqns (3) and integrating over the cross section, as

$$T = \frac{1}{2}\int_0^L \begin{bmatrix} \dot{u}_2 \\ \dot{\varphi} \end{bmatrix}^T \begin{bmatrix} \rho A & 0 \\ 0 & \rho I \end{bmatrix} \begin{bmatrix} \dot{u}_2 \\ \dot{\varphi} \end{bmatrix} dx_1 \tag{7}$$

where ρ is the mass density. For cantilever beam with tip mass, let us consider the case of the cantilever beam with a tip mass and rotatory inertia. Obviously we have to add in (6) the rate of kinetic energy due to the rotatory inertia:

$$T_M = \frac{\omega^2}{2}\left(M\,\dot{u}_{2L}^2 + J_M\,\varphi_L^2\right). \tag{7a}$$

The strain energy is given as

$$U = \frac{1}{2}\int_V \boldsymbol{\sigma}^T\,\boldsymbol{\varepsilon}\, dx_1 = \frac{1}{2}\int_V \boldsymbol{\varepsilon}^T \mathbf{D}\,\boldsymbol{\varepsilon}\, dx_1, \tag{8}$$

and the potential energy axial load (P):

$$V = \frac{P}{2}\int_0^L \left(\frac{\partial u_2}{\partial x_1}\right)^2 dx_1. \tag{9}$$

Case a) Axial force tangential to the axis of the beam.

In this case it is assumed the axial force is tangential to slope of the beam. For Bress-Timoshenko beam theory the eqn (4) can be written as, [8]:

$$\boldsymbol{\sigma} = \begin{bmatrix} EI & 0 \\ 0 & kGA \end{bmatrix} \begin{bmatrix} -x_2 \dfrac{d\varphi}{dx_1} \\ \dfrac{du_2}{dx_1} - \varphi \end{bmatrix} = \begin{bmatrix} -EI\, x_2\, \varphi' \\ kGA(u'_2 - \varphi) \end{bmatrix}, \tag{10}$$

where A is the cross sectional area, I is the moment of inertia, E is the Young modulus, G is the shear modulus, and k is the shear factor.

Eqn. (8) can be written as:

$$U = \frac{1}{2}\int_0^L \begin{bmatrix} -EI & \varphi' \\ kGA(u'_2 - \varphi) \end{bmatrix}^T \begin{bmatrix} -x_2 & \varphi' \\ u'_2 - \varphi \end{bmatrix} dx_1 \tag{11}$$

$$= \frac{1}{2}\int_0^L \left[EI(\varphi')^2 + kGA(u'_2 - \varphi)^2 \right] dx_1.$$

Case b) Axial force normal to the shearing force.
Eqn (4) can be written as

$$\boldsymbol{\sigma} = \begin{bmatrix} EI & 0 \\ 0 & (kGA - P) \end{bmatrix} \begin{bmatrix} -x_2 \dfrac{d\varphi}{dx_1} \\ \dfrac{du_2}{dx_1} - \varphi \end{bmatrix} = \begin{bmatrix} -EI\, x_2\, \varphi' \\ (kGA - P)(u'_2 - \varphi) \end{bmatrix}, \tag{12}$$

By substituting Eqs (12) into Eqs (8), The strain energy is given as

$$U = \frac{1}{2}\int_0^L \begin{bmatrix} -EI & \varphi' \\ kGA(u'_2 - \varphi) \end{bmatrix}^T \begin{bmatrix} -x_2 & \varphi' \\ u'_2 - \varphi \end{bmatrix} dx_1$$

$$= \frac{1}{2}\int_0^L \left[EI(\varphi')^2 + (kGA - P)(u'_2 - \varphi)^2 \right] dx_1. \tag{13}$$

In the hypothesis of separation of variables, $u_2(x,t)$ and $\varphi(x,t)$ an can be written as follows

$$u_2(x,t) = u_2(x)\cos\omega t, \quad \varphi(x,t) = \varphi(x)\cos\omega t \tag{14}$$

so, the kinetic energy is

$$T = \frac{\omega^2}{2}\int_0^L \rho(u_2^2 A + \varphi^2 I)\, dx_1. \tag{15}$$

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III. DISCRETIZATION MODEL

In the approximate formulation the transversal displacements are assumed to be linear combination of n independent functions which satisfy the boundary equations. If functions Φi and Ψi are chosen respecting the geometrical constraints the displacements can be written

$$u_2(x_1) = \mathbf{\Phi}^T \mathbf{q}_1, \qquad \varphi(x_1) = \mathbf{\Psi}^T \mathbf{q}_2 \quad , \tag{16}$$

where $\Phi_i(x_1)$ and $\Psi_i(x_1)$ are orthogonal functions and $\mathbf{q} = [\mathbf{q}_1 \quad \mathbf{q}_2]^T$ is the generalized displacements vector.

The approximate solution is developed by substituting the displacement distribution (eqn 16) into Hamilton's principle. Substituting (14) in eqs. (11,13) leads to the below bending strain energy. In both cases, the can be written as:

Case a) Strain energy eq. (11)

$$U_a = \frac{1}{2}\int_0^L EI(\mathbf{q}_2^T\, \mathbf{\Psi}'\mathbf{\Psi}'^T\, \mathbf{q}_2)\, dx_1 +$$

$$+\frac{1}{2}\int_0^L kGA\left[\mathbf{\Phi}'^T\, \mathbf{q}_1 - \mathbf{\Psi}^T \mathbf{q}_2\right]^T \left[\mathbf{\Phi}'^T\, \mathbf{q}_1 - \mathbf{\Psi}^T \mathbf{q}_2\right] dx_1. \tag{17}$$

In matrix form:

$$U^a = \frac{1}{2}\mathbf{q}^T(\mathbf{K}_f^a + \mathbf{K}_s^a)\mathbf{q} = \frac{1}{2}\mathbf{q}^T\mathbf{K}^a\, \mathbf{q}\,, \tag{17a}$$

where

$$\mathbf{K}_f^a = E\int_0^L I \begin{bmatrix} 0 & 0 \\ 0 & \mathbf{\Psi}'\mathbf{\Psi}'^T \end{bmatrix} dx_1$$

$$\mathbf{K}_s^a = kG\int_0^L A \begin{bmatrix} \mathbf{\Phi}'\mathbf{\Phi}'^T & -\mathbf{\Psi}\,\mathbf{\Phi}'^T \\ -\mathbf{\Psi}\,\mathbf{\Phi}'^T & \mathbf{\Psi}\mathbf{\Psi}^T \end{bmatrix} dx_1, \tag{17b}$$

Case b) Strain energy eq (13)

$$U_b = \frac{1}{2}\int_0^L EI(\mathbf{q}_2^T\, \mathbf{\Psi}'\mathbf{\Psi}'^T\, \mathbf{q}_2)\, dx_1 +$$

$$+\frac{1}{2}\int_0^L (kGA - N)\left[\mathbf{\Phi}'^T\, \mathbf{q}_1 - \mathbf{\Psi}^T \mathbf{q}_2\right]^T \left[\mathbf{\Phi}'^T\, \mathbf{q}_1 - \mathbf{\Psi}^T \mathbf{q}_2\right] dx_1, \tag{18}$$

and

$$U^b = \frac{1}{2}\mathbf{q}^T(\mathbf{K}_f^a + \mathbf{K}_s^b)\mathbf{q} = \frac{1}{2}\mathbf{q}^T\mathbf{K}^b\, \mathbf{q}\,, \tag{18a}$$

where:

(9)

$$\mathbf{K}_s^b = \mathbf{K}_s^a - N\int_0^L \begin{bmatrix} \mathbf{\Phi}'\mathbf{\Phi}'^T & -\mathbf{\Psi}\,\mathbf{\Phi}'^T \\ -\mathbf{\Psi}\,\mathbf{\Phi}'^T & \mathbf{\Psi}\mathbf{\Psi}^T \end{bmatrix} dx_1, \tag{18b}$$

or

$$\mathbf{K}_s^b = \mathbf{K}_s^a - N\,\mathbf{B}_N^b,$$

The potential energy axial load (P) can be rewriter as:

$$V = \frac{N}{2}\int_0^L \left[\mathbf{\Phi}'^T\, \mathbf{q}_1\right]^T \left[\mathbf{\Phi}'^T\, \mathbf{q}_1\right] dx_1 = \frac{1}{2}\mathbf{q}^T\mathbf{B}_N\, \mathbf{q}, \tag{19}$$

where,

$$\mathbf{B}_N = \int_0^L \begin{bmatrix} \mathbf{\Phi}'\mathbf{\Phi}'^T & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} dx_1\,, \tag{19a}$$

While, the kinetic energy:

$$T = \frac{\omega^2}{2}\int_0^L \rho\left[A\left(\mathbf{q}_1\mathbf{\Phi}\mathbf{\Phi}^T\mathbf{q}_1\right) + I\left(\mathbf{q}_2\mathbf{\Psi}\mathbf{\Psi}^T\mathbf{q}_2\right)\right] dx_1,$$

$$T = \frac{\omega^2}{2}\mathbf{q}^T\mathbf{M}_b\,\mathbf{q}. \tag{20}$$

where

$$\mathbf{M}_b = \int_0^L \rho \begin{bmatrix} A\mathbf{\Phi}\mathbf{\Phi}^T & 0 \\ 0 & I\,\mathbf{\Psi}\mathbf{\Psi}^T \end{bmatrix} dx_1\,, \tag{21}$$

The kinetic energy due to the tip mass can be written as:

$$T_M = \frac{\omega^2}{2}\left[\mathbf{q}_1\mathbf{\Phi}_L\mathbf{\Phi}_L^T\mathbf{q}_1 + \mathbf{q}_2\mathbf{\Psi}_L\mathbf{\Psi}_L^T\mathbf{q}_2\right]$$

$$= \frac{\omega^2}{2}\mathbf{q}^T\mathbf{M}_M\mathbf{q}, \tag{22}$$

where

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$$\mathbf{M}_M = \begin{bmatrix} M\,\mathbf{\Phi}_L\,\mathbf{\Phi}_L^T & 0 \\ 0 & J_M\,\mathbf{\Psi}_L\,\mathbf{\Psi}_L^T \end{bmatrix}. \tag{22a}$$

Obviously, the matrix Mb and Mm are symmetric and positive definite.

At last, in general the functional in (5) is written:

Case a)

$$\Pi_a = \frac{1}{2}\mathbf{q}^T(\mathbf{K}^a - N\,\mathbf{B}_N - \omega^2\,\mathbf{M})\mathbf{q}, \tag{23}$$

Case b)

$$\Pi_b = \frac{1}{2}\mathbf{q}^T(\mathbf{K}^b - N(\mathbf{B}_N + \mathbf{B}_N^b) - \omega^2\,\mathbf{M})\mathbf{q}, \tag{24}$$

In general, imposing the conditions (5) we get following homogeneous system in the unknown **q**:

$$(\mathbf{K} - \omega^2\mathbf{M})\,\mathbf{q} = \mathbf{0}, \tag{25}$$

which in turn leads to the frequency equation

$$\det(\mathbf{K} - \omega^2\,\mathbf{M}) = 0\,, \tag{26}$$

where $\mathbf{M}=\mathbf{M}_b+\mathbf{M}_m$.

As already said, the shape functions must obey the only geometric boundary conditions, so that it will be possible to write:

$$\Phi_1(x_1) = \sum_{j=0}^{n_u} a_j x_1^j\,, \qquad \Psi_1(x_1) = \sum_{j=0}^{n_\varphi} b_j x_1^j\,, \tag{27}$$

where n_u and n_φ are the geometric conditions which must be imposed on the vertical displacements and rotations, respectively. The coefficients ai and bi can be determined imposing the boundary conditions, whereas the higher order functions can be sought by means of the *Gram-Schmidt* [10-11] iterative method.

The geometric boundary conditions at the ends of the beam can be specified as follows;

$$x_1 = 0 \;\Rightarrow \begin{bmatrix} u_1 = 0 \\ u_{1,1} = 0 \end{bmatrix}, \qquad x_1 = L \;\Rightarrow \begin{bmatrix} u_1 \neq 0 \\ u_{1,1} \neq 0 \end{bmatrix}. \tag{28}$$

IV. Numerical Examples

In order to test the above suggested method, some numerical examples have been performed, for a beam with arbitrarily varying cross section, with area and moment of inertia given by the general relationships:

$$A(x_1) = A_0\;\left[1 - \alpha\, {}^{x_1}\!/_{L}\right], \quad I(x_1) = I_0\;\left[1 - \alpha\, {}^{x_1}\!/_{L}\right]^3, \tag{29}$$

where A_0 and I_0 are the cross sectional area and moment of at the abscissa x_1=0. It is also usual to introduce the following non-dimensional parameters:

$$P = \frac{N\,L^2}{E\,I_0},\;\; \mu = \frac{M_m}{M_t},\;\; \bar{k}^2 = \frac{J_M}{M_m\,L^2},\;\; M_t = \int_0^L \rho A_x\,dx_1. \tag{30}$$

whereas the free vibration frequencies are usually written as:

$$\Omega_i^2 = \omega_i^2\,L^4\,\frac{\rho A_0}{EI_0}, \qquad r^2 = \frac{I_0}{A_0}\;. \tag{31}$$

As a first comparison, let us consider the tapered beams with α=0,2 , subjected to axial forces as studies by Esmailzadeh and Ohadi [8]. The Poisson coefficient is equal to 0.25, E/G=2.6, the cross section is assumed to be rectangular, and consequently the shear factor is given by *k=5/6*. In the following we have used *6* polynomial trial function in order to approximate both displacements and rotations, so that the resulting problem has *12* degrees of freedom. The frequency coefficients Ω_i (i=1-4) have been calculated for an clamped carrying tip mass; μ=0,6. The results are given in Tab.I together with some results presented by Lee et al [9], which verifies the developed computer program for the non-uniform beams. The full agreement with the exact frequencies is quite evident, small discrepancies can be noticed only for the higher frequencies, but the error turns out to be than 0.1‰ .

In Tab.II are reported P_c critical buckling load parameter for cantilever beam with a tip mass; μ=0,6, α=0 and α=0.2.

TABLE I. First four frequency of a linearly thickness.

μ=0,6				
α=0,2	r=0,04	[8]	Present	Present r=0,1
P		Case b	Case b	Case b
0	Ω_1	1,85	1,8509	1,773
	Ω_2	14,44	14,4354	11,4183
	Ω_3	40,07	40,0733	26,7927
	Ω_4	74,24	74,2989	44,1279

The influence of the axial load on the natural frequencies for *r*=0.1 is shown in table III. The comparison of the results presented in Tab. II with the corresponding values taken from reference [8] shows good agreement and, therefore, verifies both the formulation and the developed computer program.

TABLE II. First four frequency vs (P_c)

μ=0,6					
α=0,2	r=0,04	[8]		Present	
P		Case 1	Case 2	Case 1	Case 2
0,006* k*G/(r^2)	Ω_1	1,18	1,18	1,1824	1,1822
P=1,2019	Ω_2	13,74	13,73	13,7395	13,7346
	Ω_3	39,36	39,33	39,3627	39,3346
	Ω_4	73,51	73,43	73,5704	73,4919
0,008*	Ω_1			0,8377	0,8374

k*G/(r^2)				
P=1,6025	Ω_2		13,4989	13,4924
	Ω_3		39,1224	39,0852
	Ω_4		73,3257	73,2208

A comparison of Tables I and 2 reveals that, as anticipate, the compressive axial force reduces all modes of natural frequencies of the beam. The reduction of the frequencies is small passing from the case a) in case b).

TABLE III. FIRST FOUR FREQUENCY VS (P_C); r=0,1.

μ=0,6			
α=0,2	r=0,1	Present	
P		Case 1	Case 2
0,006*k*G/(r^2)	Ω_1	1,6859	1,6852
P=1,2019	Ω_2	11,3101	11,2956
	Ω_3	26,6839	26,6353
	Ω_4	44,0199	43,9257
0,008*k*G/(r^2)	Ω_1	1,6551	1,6543
P=1,6025	Ω_2	11,2737	11,2543
	Ω_3	26,6476	26,5826
	Ω_4	43,9838	43,8581

In Tab. IV the non-dimensional critical loads are given, which have been used to obtained the previous pictures. The results obtained for case a) and case b).

TABLE IV. CRITICAL BUCKLING LOAD (P_C).

	P_c - case a		P_c - case b	
r	α=0,2	α=0	α=0,2	α=0
0,01	2,0218	2,4655	2,0218	2,4655
0,02	2,0130	2,4598	2,0132	2,4599
0,03	2,0099	2,4504	2,0100	2,4506
0,04	1,9999	2,4368	1,9994	2,4376
0,05	1,9866	2,4208	1,9869	2,4212
0,06	1,9708	2,4008	1,9713	2,4011
0,07	1,9524	2,3777	1,9530	2,3783
0,08	1,9318	2,3515	1,9326	2,3524
0,09	1,9089	2,3222	1,9099	2,3234
0,1	1,8839	2,2910	1,8852	2,2920

It can be seen that the critical bucking loads decrease for increase parameter r. The influence of the paper ratio α seems to be relevant for the cantilever beam.

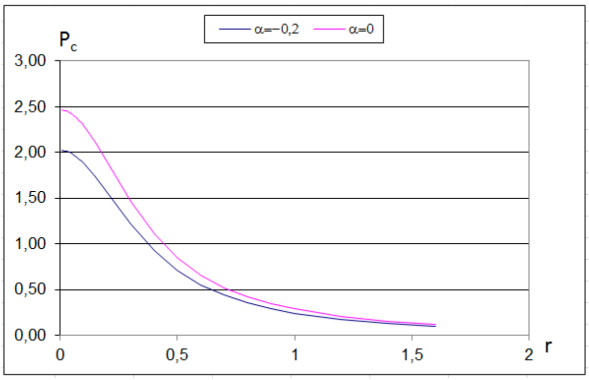

Figure 2. Critical buckling load (Pc) for various r; non-uniform beam.

It can be seen that the critical buckling loads of the uniform beam are greater than those of the non-uniform one, as expected, since the stiffness of the uniform beam is now higher. On the other hand, the mass of the uniform beam is also greater than that of a tapered beam. Variation of the buckling load (P_c) for various parameter r are presented in Figure 2.

V. CONCLUSION

A numerical technique has been presented for the analysis of non-uniform Timoshenko cantilever beam under axial force. The validity and accuracy of the technique have been verified through several numerical examples. The axial force effects considered in two different positions was analysed. Thus, this study demonstrates the reliability and convenience of the application of the Timoshenko theory. The natural frequencies are in excellent agreement with published results. Though for comparison purposes, the natural frequencies are kept accurate to the fourth decimal places, the precision of the natural frequencies can be increased and made as high as desired. The advantage of the procedure used is the generality of polynomial functions which only need to satisfy the essential conditions. The numerical examples have been completely carried through by means of the powerful symbolic software.

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