On the Hermite-Fejér interpolation based at the zeros of generalized Freud polynomials

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Abstract. This paper deals with a special Hermite-Fejér interpolation process based at the zeros of generalized Freud polynomials which are orthogonal with respect to the weight $w(x) = |x|^\alpha e^{-|x|^\beta}$, $x \neq 0$, $\alpha > -1$, $\beta > 1$. We prove its uniform convergence for functions belonging to a suitable space of functions equipped with a weighted uniform norm.

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1. Introduction

Let $w(x) = |x|^\alpha e^{-|x|^\beta}$, $x \neq 0$, $\alpha > -1$, $\beta > 1$, be a generalized Freud weight and let $\{p_m(w)\}_m$ be the corresponding sequence of polynomials with positive leading coefficients and orthonormal in $\mathbb{R}$ w.r.t. the weight w. The zeros of $p_m(w)$ are symmetrical with respect to the origin and lie in the interval $\left(-a_m\left(1-\frac{c}{m^{2/3}}\right),\ a_m\left(1-\frac{c}{m^{2/3}}\right)\right)$, where $c > 0$ and $a_m := a_m(w) \sim m^{\frac{1}{\beta}}$ is the so-called Maskar-Rakmanov-Saff number (M-R-S number). We denote by x_k, $k = 1, 2, \ldots, \left\lfloor\frac{m}{2}\right\rfloor$, the positive zeros of $p_m(w)$ and by x_{-k}, $k = 1, 2, \ldots, \left\lfloor\frac{m}{2}\right\rfloor$, the corresponding negative ones ($\lfloor a\rfloor$ stands for the largest integer smaller than or equal to $a \in \mathbb{R}^+$). We assume that m is even, then

$$-a_m\left(1-\frac{c}{m^{2/3}}\right) < x_{-\frac{m}{2}} < \ldots < x_{-1} < x_1 < \ldots < x_{\frac{m}{2}} < a_m\left(1-\frac{c}{m^{2/3}}\right).$$

The Hermite-Fejér polynomial based at these zeros has the form

$$F_m(w, f, x) = \sum_{k=-\frac{m}{2}}^{\frac{m}{2}} l_k^2(x) v_k(x) f(x_k),$$

where f is a continuous function on the real axis $\mathbb{R}$ ($f \in C^0(\mathbb{R})$),

$$l_k(x) = \frac{p_m(w,x)}{p'_m(w,x_k)(x-x_k)}$$

are the fundamental Lagrange polynomials and

$$v_k(x) = 1 - 2\, l'_k(x_k)(x-x_k).$$

The operator $F_m(w)$ has been considered in [7] and [9] when the parameter α of the weight w is null. The error estimates proved in such papers were improved by J. Szabados in [15] and by V.E.S. Szabó in [16]. Recently in [5, 6, 3, 4] the case $\alpha \neq 0$ has been studied assuming that the interpolated functions are uniformly continuous in $\mathbb{R}$. However this represents a strong limitation.

In this paper, following a procedure adopted in the last years for several approximation processes with exponential weights (see, for example, [10, 11, 13, 12, 1, 2], we introduce a simpler interpolant of Hermite-Fejér-type that we will denote by $F^*_{m+2}(w,f)$. We will prove the uniform convergence of the sequence $\{F^*_{m+2}(w,f)\}_m$ to the function f in suitable spaces of functions equipped with a weighted uniform norm. The results are new and include those ones existing in the literature.

2. Main results

Also using an idea of J. Szabados, we consider the Hermite-Fejér interpolation polynomial based on the zeros $\{x_k\}_{k=-\frac{m}{2},\ldots,\frac{m}{2}}$ and the two extra points $\pm a_m$. Letting $x_{\frac{m}{2}+1} = a_m$ and $x_{-\frac{m}{2}-1} = -a_m$, such a polynomial has the form

$$F_{m+2}(w,f,x) = \sum_{|k|\le \frac{m}{2}+1} \ell_k^2(x)\nu_k(x)f(x_k),$$

where $f \in C^0(\mathbb{R})$,

$$\ell_k(x) = l_k(x)\frac{a_m^2 - x^2}{a_m^2 - x_k^2}, \quad k = -\frac{m}{2},\ldots,\frac{m}{2}, \tag{2.1}$$

$$\ell_{\pm(\frac{m}{2}+1)} = \frac{a_m \pm x}{2a_m}\,\frac{p_m(w,x)}{p_m(w,\pm a_m)},$$

and

$$\nu_k(x) = 1 - 2\,\ell'_k(x_k)(x-x_k). \tag{2.2}$$

We have $F_{m+2}(w,f) \in \mathbb{P}_{2m+2}$ and $F_{m+2}(w,f,x_k) = f(x_k)$, $|k| \le \frac{m}{2}+1$. Now we recall an important property of the exponential weights considered here.

Letting $u(x) = |x|^\gamma e^{-|x|^\beta}$, $\gamma > 0$ not integer, $\beta > 1$, we introduce the space of functions

$$C_u = \left\{ f \in C^0(\mathbb{R}),\ \lim_{\substack{|x|\to\infty \\ x\to 0}} f(x)u(x) = 0 \right\}, \tag{2.3}$$

with the norm

$$\|f\|_{C_u} = \sup_{x\in\mathbb{R}} |f(x)|u(x) = \|fu\|, \tag{2.4}$$

and we will write $\|f\|_A := \sup_{x\in A} |f(x)|$, $A \subset \mathbb{R}$. In C_u the Jackson theorem holds true [14] and the M-R-S number $a_m(u)$ related to u is equivalent to the one related to the weight w, then $a_m(u) \sim m^{\frac{1}{\beta}}$.

Now, for every polynomial $Q_m \in \mathbb{P}_m$ ($\mathbb{P}_m$ denotes the set of all polynomials of degree at most m) the following relations hold true [14]:

$$\|Q_m u\| \le \mathcal{C}\|Q_m u\|_{\mathcal{I}_m}, \quad \mathcal{I}_m = \left[-a_m, -c\frac{a_m}{m}\right] \cup \left[c\frac{a_m}{m}, a_m\right], \tag{2.5}$$

and

$$\|Q_m u\|_{\{x \,:\, |x|>(1+\delta)a_m\}} \le \mathcal{C}e^{-\mathcal{A}m}\|Q_m u\|, \tag{2.6}$$

where $\delta > 0$ and $c, \mathcal{C}, \mathcal{A}$ are independent of m and Q_m.

As a consequence of (2.6), for all $f \in C_u$ and $0 < \theta < 1$ fixed, the weight function u satisfies the following property:

$$\|fu\| \le \mathcal{C}\left[\|fu\|_{[-\theta a_m, \theta a_m]} + E_M(f)_u\right],$$

where $\mathcal{C}$ is independent of f, $E_M(f)_u$ is the error of best approximation of f in C_u and $M = \lfloor \frac{\theta m}{1+\theta} \rfloor \sim m$. Then $\|fu\|_{\mathbb{R}\setminus[-\theta a_m, \theta a_m]}$ is equivalent to the best approximation and $\|fu\|_{[-\theta a_m, \theta a_m]}$ is the dominant part of $\|fu\|$. This fact suggests us to apply the operator $F_{m+2}(w)$ to a finite section of the function f. Then, with $j = j(m)$ defined as

$$x_j = \min\{x_k \;:\; x_k \ge \theta a_m\}$$

and χ_j the characteristic function of the interval $[-x_j, x_j]$, we define

$$F^*_{m+2}(w, f, x) = F_{m+2}(w, \chi_j f, x) = \sum_{|k|\le j} \ell_k^2(x)\nu_k(x)f(x_k). \tag{2.7}$$

By definition

$$F^*_{m+2}(w, f, x_i) = f(x_i), \quad |i| \le j,$$

and

$$F^*_{m+2}(w, f, x_i) = 0, \quad |i| > j.$$

We introduce a second space of functions, $C_{\bar{u}}$, defined as in (2.3)-(2.4) with $u(x)$ replaced by $\bar{u}(x) = u(x)\log(2 + |x|)$. Obviously $C_{\bar{u}} \subset C_u$. Now we show that $F^*_{m+2}(w) : C_{\bar{u}} \to C_u$ is a bounded map. We can establish the following lemma.

Lemma 2.1. *Assume that the parameters α and γ of the weights w and u satisfy the condition*

$$\alpha > -1, \quad \gamma > 0, \quad 0 < \gamma - \alpha < 1, \tag{2.8}$$

then

$$\|F^*_{m+2}(w, f)u\| \le \mathcal{C}\|f\bar{u}\|_{[-x_j, x_j]},$$

where $\mathcal{C}$ is independent of m and $f \in C_{\bar{u}}$.

In order to evaluate the error $(f - F^*_{m+2}(w, f))u$, we point out that the weight u has a zero at the origin. Thus the more suitable modulus of smoothness in $C_{\bar{u}}$ is [14]

$$\omega(f,t)^*_{\bar{u}} = \Omega(f,t)_{\bar{u}} + \inf_{P \in \mathbb{P}_0} \|(f-P)\bar{u}\|_{[-c\frac{a_m}{m}, c\frac{a_m}{m}]},$$

being

$$\Omega(f,t)_{\bar{u}} = \sup_{0<h\le t} \|(\Delta_h f)\bar{u}\|_{\mathbb{R}\setminus[-c\frac{a_m}{m}, c\frac{a_m}{m}]},$$

with

$$\Delta_h f(x) = f\left(x + \frac{h}{2}\right) - f\left(x - \frac{h}{2}\right).$$

Using [14], it is easy to verify that

$$E_m(f)_{\bar{u}} = \inf_{P \in \mathbb{P}_m} \|(f-P)\bar{u}\| \le \mathcal{C}\omega\left(f, \frac{a_m}{m}\right)^*_{\bar{u}} \tag{2.9}$$

and

$$\lim_m \omega\left(f, \frac{a_m}{m}\right)^*_{\bar{u}} = 0.$$

Now, we can establish the convergence theorem

Theorem 2.2. *Under the assumption of Lemma* 2.1 *on the parameters* α *and* γ*, we get*

$$\|(f - F^*_{m+2}(w,f))u\| \le \mathcal{C}\left[\omega\left(f, \frac{a_m}{m}\log m\right)^*_{\bar{u}} + \frac{a_m}{m}\log m \|f\bar{u}\|\right],$$

where $\mathcal{C}$ *is independent of* m *and* f.

Next remark completes the previous result.

Remark 2.3. In this paper, employing the two additional points $\pm a_m$ together with the "truncation", we construct the operator $F^*_{m+2}(w)$ that is simpler than $F_{m+2}(w)$, since a smaller number of evaluations of the function is used, and that, in addition, satisfies Theorem 2.2. The operator $F_{m+2}(w)$ with or without the additional points $\pm a_m$ and with or without the "truncation", by contrast, can converge for classes of functions more restricted than those considered here.

For the sake of simplicity we chose the weight $|x|^\alpha e^{-|x|^\beta}$. However it is possible to replace $e^{-|x|^\beta}$ with $e^{-Q(x)}$, provided that the latter is a Freud weight (see [8]) under the condition that $\frac{Q'(x)}{\log(2+|x|)}$ increases in $[0, +\infty)$.

3. Proofs

We recall some inequalities that will be useful in the sequel:

$$\left| p_m^2(w,x)w(x)\sqrt{a_m^2 - x^2 + \frac{a_m^2}{m^{2/3}}} \right| \le \mathcal{C}, \quad \frac{a_m}{m} \le |x| \le a_m, \tag{3.1}$$

$$\frac{1}{|p_m'^2(w,x_k)|w(x_k)} \sim \Delta^2 x_k \sqrt{a_m^2 - x_k^2}, \quad |k| \le \frac{m}{2}, \tag{3.2}$$

$$\Delta x_k \sim \frac{a_m}{m} \sim \frac{1}{m^{1-\frac{1}{\beta}}}, \quad |k| \le j, \tag{3.3}$$

and

$$|l_k'(x_k)| \le \mathcal{C}\left[\frac{|x_k|}{a_m^2} + \beta|x_k|^{\beta-1} + \frac{1}{|x_k|}\right]. \tag{3.4}$$

These inequalities can be deduced from [5, Theorem 3.6, p. 42].

Proof of Lemma 2.1. Using (3.1) and (3.2), we get

$$\begin{aligned} \frac{\ell_k^2(x)u(x)}{\bar{u}(x_k)} &\le \mathcal{C}\left|\frac{x}{x_k}\right|^{\gamma-\alpha}\left(\frac{a_m^2-x^2}{a_m^2-x_k^2}\right)^{\frac{3}{2}}\frac{\Delta^2 x_k}{\log(2+|x_k|)(x-x_k)^2} && (3.5)\\ &\le \mathcal{C}\left|\frac{x}{x_k}\right|^{\gamma-\alpha}\frac{\Delta^2 x_k}{(x-x_k)^2}, \quad \frac{a_m}{m} \le |x| \le a_m, && (3.6)\end{aligned}$$

being $|k| \le j$. Moreover, by definition (2.1) and (3.4), we have

$$\begin{aligned} |\ell_k'(x_k)| &\le \frac{2|x_k|}{a_m^2-x_k^2} + |l_k'(x_k)| \\ &\le \mathcal{C}\left[\frac{|x_k|}{a_m^2-x_k^2} + \frac{|x_k|}{a_m^2} + \beta|x_k|^{\beta-1} + \frac{1}{|x_k|}\right]. \end{aligned}$$

Consequently, by definition (2.2),

$$\begin{aligned} |\nu_k(x)| &\le 1+\mathcal{C}\left[\frac{|x_k|}{a_m^2-x_k^2} + \frac{|x_k|}{a_m^2} + \beta|x_k|^{\beta-1} + \frac{1}{|x_k|}\right]|x-x_k| \\ &\le 1+\mathcal{C}\left[1+\frac{1}{|x_k|} + |x_k|^{\beta-1}\right]|x-x_k|. && (3.7)\end{aligned}$$

Moreover, also in virtue of (3.7) and (3.3), it is easy to verify that

$$\frac{\ell_d^2(x)u(x)}{\bar{u}(x_d)}|\nu_d(x)| \le \mathcal{C}, \tag{3.8}$$

being x_d a zero closest to x. Thus, recalling (2.7) and taking into account (3.6) and (3.8), for $\frac{a_m}{m} < |x| \le a_m$,

$u(x)|F_{m+2}^*(w,f,x)| \le$

$$\le \mathcal{C}\|f\bar{u}\|_{[-x_j,x_j]}\left[\sum_{\substack{|x_k|\le|x_j|\\ k\ne d-1,d,d+1}}\left|\frac{x}{x_k}\right|^{\gamma-\alpha}\frac{\Delta^2 x_k}{(x-x_k)^2}\frac{\nu_k(x)}{\log(2+|x_k|)}+1\right]. \tag{3.9}$$

Now we estimate the sum in (3.9). We first consider the case $|x| > 2$ and write

$$\begin{aligned} u(x)|F^*_{m+2}(w,f,x)| &\leq \mathcal{C}\|f\bar{u}\|_{[-x_j,x_j]}\left[\sum_{|x_1|\leq|x_k|\leq 1} + \sum_{1<|x_k|\leq\frac{|x|}{2}}\right. \\ &+ \left.\sum_{\frac{|x|}{2}<|x_k|<|x_{d-2}|} + \sum_{|x_{d+2}|\leq|x_k|<|x_j|}\right] \\ &=: \mathcal{C}\|f\bar{u}\|_{[-x_j,x_j]}\left[\sigma_1(x)+\sigma_2(x)+\sigma_3(x)+\sigma_4(x)\right]. \end{aligned}$$

For $|x_1| \leq |x_k| \leq 1$, (3.7) becomes

$$|\nu_k(x)| \leq \mathcal{C}\frac{|x|}{|x_k|}.$$

Then, taking also into account $x - x_k > \frac{x}{2}$, $\gamma - \alpha < 1$, (3.3) and $\frac{a_m}{m} < |x|$, we have

$$\begin{aligned} \sigma_1(x) &\leq \mathcal{C}\sum_{|x_1|\leq|x_k|\leq 1}\left|\frac{x}{x_k}\right|^{\gamma-\alpha+1}\frac{\Delta^2 x_k}{(x-x_k)^2} \\ &\leq \mathcal{C}|x|^{\gamma-\alpha-1}\frac{a_m}{m}\sum_{|x_1|\leq|x_k|\leq 1}\Delta x_k|x_k|^{\alpha-\gamma-1} \\ &\leq \mathcal{C}\sum_{|x_1|\leq|x_k|\leq 1}\Delta x_k|x_k|^{\alpha-\gamma} \leq \mathcal{C}\int_{-1}^{1}|t|^{\alpha-\gamma}dt \leq \mathcal{C}. \end{aligned}$$

We note that for $1 < |x_k| \leq |x_j|$, (3.7) becomes

$$|\nu_k(x)| \leq 1 + \mathcal{C}|x_k|^{\beta-1}|x-x_k|, \quad \beta > 1, \tag{3.10}$$

and, then,

$$\sigma_2(x) \leq \sum_{1<|x_k|\leq\frac{|x|}{2}}\left|\frac{x}{x_k}\right|^{\gamma-\alpha}\frac{\Delta^2 x_k}{(x-x_k)^2}\left[1+\mathcal{C}\frac{|x_k|^{\beta-1}}{\log(2+|x_k|)}|x-x_k|\right].$$

Thus, using (3.3) and $x - x_k > \frac{x}{2}$, we get

$$\begin{aligned} \sigma_2(x) &\leq \mathcal{C}\frac{a_m}{m}|x|^{\gamma-\alpha-2}\sum_{1<|x_k|\leq\frac{|x|}{2}}|x_k|^{\alpha-\gamma}\,\Delta x_k \\ &+ \frac{\mathcal{C}}{\log m}\sum_{1<|x_k|\leq\frac{|x|}{2}}\left|\frac{x}{x_k}\right|^{\gamma-\alpha}\frac{\Delta x_k}{(x-x_k)} \\ &\leq \mathcal{C} + \frac{\mathcal{C}}{\log m}\int_1^{\frac{x}{2}}\left|\frac{x}{t}\right|^{\gamma-\alpha}\frac{dt}{(x-t)} \\ &\leq \mathcal{C} + \frac{\mathcal{C}}{\log m}\int_0^{\frac{1}{2}}y^{\alpha-\gamma}\frac{dy}{(1-y)} \leq \mathcal{C} + \frac{\mathcal{C}}{\log m}, \end{aligned}$$

being $\alpha-\gamma>-1$. Moreover, taking into account (3.10), (3.3) and $|x|\sim|x_k|$, we obtain

$$\begin{aligned}\sigma_3(x) &\le \sum_{\frac{|x|}{2}<|x_k|<|x_{d-2}|}\frac{\Delta^2 x_k}{(x-x_k)^2}\left[1+\mathcal{C}\frac{|x_k|^{\beta-1}}{\log(2+|x_k|)}|x-x_k|\right]\\ &\le \mathcal{C}+\mathcal{C}\sum_{\frac{|x|}{2}<|x_k|<|x_{d-2}|}\frac{\Delta x_k}{|x-x_k|}\frac{|x_k|^{\beta-1}\Delta x_k}{\log(2+|x_k|)}\\ &\le \mathcal{C}+\frac{\mathcal{C}}{\log m}\int_{\frac{x}{2}}^{x-\Delta x_d}\frac{dt}{x-t}\sim\frac{1}{\log m}\log\frac{x}{2\Delta x_d}\sim 1.\end{aligned}$$

Finally, proceeding as done for the estimate of $\sigma_3(x)$, it results

$$\sigma_4(x)\le\mathcal{C}.$$

Summing up, taking into account (2.5),

$$\|F^*_{m+2}(w,f)u\|\le\mathcal{C}\|F^*_{m+2}(w,f)u\|_{\mathcal{I}_m}\le\mathcal{C}\|f\bar{u}\|_{[-x_j,x_j]},\quad |x|\ge 2,$$

The case $|x|<2$ is simpler and we omit the details. □

In order prove Theorem 2.2, we need some further considerations. We recall that if g is continuous with its first derivative, the Hermite polynomial interpolating g and g', based on the zeros x_k, $|k|\le\frac{m}{2}+1$, is given by

$$\begin{aligned}H_{m+2}(w,g,x) &= \sum_{|k|\le\frac{m}{2}+1}\ell_k^2(x)\nu_k(x)g(x_k)+\sum_{|k|\le\frac{m}{2}+1}\ell_k^2(x)(x-x_k)g'(x_k)\\ &=: F_{m+2}(w,g,x)+G_{m+2}(w,g,x).\end{aligned}$$

Letting $G^*_{m+2}(w,g,x)=G_{m+2}(w,\chi_j g,x)$, the proposition that follows will be useful to our aims.

Proposition 3.1. *Assuming that the parameters α and γ satisfy* (2.8)*, then, for every $g,\ g'\in C_u$, we have*

$$\|G^*_{m+2}(w,g)u\|\le\mathcal{C}\frac{a_m}{m}\log m\|g'u\|_{[-x_j,x_j]},\tag{3.11}$$

where $\mathcal{C}$ is independent of m and f. Moreover, for every polynomial $Q_M\in I\!P_M$, with $M\le\lfloor\frac{\theta m}{1+\theta}\rfloor$, $0<\theta<1$, we get

$$\|H_{m+2}(w,(1-\chi_j)Q_M)u\|\le\mathcal{C}e^{-\mathcal{A}m}\|Q_M u\|,\tag{3.12}$$

where $\mathcal{C}$ and $\mathcal{A}$ are independent of m and Q_M.

Proof. In order to prove the inequality (3.11) we recall (3.6). Then, using (3.3), we get

$$|G^*_{m+2}(w,g,x)|u(x)\quad\le\quad\mathcal{C}\frac{a_m}{m}\|g'u\|_{[-x_j,x_j]}\sum_{|k|\le j}\left|\frac{x}{x_k}\right|^{\gamma-\alpha}\frac{\Delta x_k}{|x-x_k|}.$$

Since, by similar arguments to those used for the proof of Lemma 2.1, we obtain

$$\sum_{|k|\le j} \left|\frac{x}{x_k}\right|^{\gamma-\alpha} \frac{\Delta x_k}{|x-x_k|} \le \mathcal{C}\log m,$$

(3.11) easily follows. To prove (3.12) we limit ourself to show that

$$\|G_{m+2}(w,(1-\chi_j)Q_M)u\| \le \mathcal{C}e^{-\mathcal{A}m}\|Q_M u\|, \tag{3.13}$$

being the estimate of $\|F_{m+2}(w,(1-\chi_j)Q_M)u\|$ similar.

Now, using (3.5), we get

$$\begin{aligned} &|G_{m+2}(w,(1-\chi_j)Q_M,x)|u(x) \le \\ &\quad\le \mathcal{C}\|Q_M'u\|_{[x_j,+\infty)} \sum_{|k|>j} \left|\frac{x}{x_k}\right|^{\gamma-\alpha} \left(\frac{a_m^2-x^2}{a_m^2-x_k^2}\right)^{\frac{3}{2}} \frac{\Delta^2 x_k}{|x-x_k|} \\ &\quad\le \mathcal{C}m^\tau \|Q_M'u\|_{[x_j,+\infty)}, \end{aligned}$$

for some $\tau>0$. Finally, by (2.6) and the Bernstein theorem [8], we obtain

$$\begin{aligned} m^\tau\|Q_M'u\|_{[\theta a_m,+\infty)} &\le \mathcal{C}m^\tau e^{-\bar{\mathcal{A}}m}\|Q_M'u\| \le \mathcal{C}\frac{m^{1+\tau}}{a_m}e^{-\bar{\mathcal{A}}m}\|Q_M u\| \\ &\le \mathcal{C}e^{-\mathcal{A}m}\|Q_M u\| \end{aligned}$$

and (3.13) easily follows. □

Now we can prove Theorem 2.2.

Proof of Theorem 2.2. Let $P_N \in \mathbb{P}_N$, $N = \left\lfloor \frac{M}{\log M} \right\rfloor$, $M = \left\lfloor \frac{\theta m}{1+\theta} \right\rfloor$, be the polinomial of best approximation of $f \in C_{\bar{u}}$. We have

$$\begin{aligned} f - F_{m+2}^*(w,f) &= f - P_N + H_{m+2}(w,P_N) - F_{m+2}^*(w,f) \\ &= f - P_N + F_{m+2}^*(w,f-P_N) \\ &+ G_{m+2}^*(w,P_N) + H_{m+2}(w,(1-\chi_j)P_N), \end{aligned}$$

from which, using Lemma 2.1 and Proposition 3.1, we get

$$\|(f-F_{m+2}^*(w,f))u\| \le \mathcal{C}\left[\|(f-P_N)\bar{u}\| + \frac{a_N}{N}\|P_N'\bar{u}\| + e^{-\mathcal{A}m}\|P_N\bar{u}\|\right].$$

Since

$$\|P_N\bar{u}\| \le 2\|f\bar{u}\|,$$

recalling (2.9) we have

$$\|(f-P_N)\bar{u}\| \le \mathcal{C}\omega\left(f,\frac{a_N}{N}\right)_{\bar{u}}^* \sim \omega\left(f,\frac{a_m}{m}\log m\right)_{\bar{u}}^*,$$

and by [14, th 2.3, p. 291] we get

$$\begin{aligned} \frac{a_N}{N}\|P_N'\bar{u}\| &\le \mathcal{C}\left[\omega\left(f,\frac{a_N}{N}\right)_{\bar{u}}^* + \frac{1}{N}\|f\bar{u}\|\right] \\ &\sim \omega\left(f,\frac{a_m}{m}\log m\right)_{\bar{u}}^* + \frac{a_m}{m}\log m\|f\bar{u}\|, \end{aligned}$$

the theorem follows. □

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