Chapter 2
Differential Operators and Approximation Processes Generated by Markov Operators

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2.1 Introduction

In recent years several investigations have been devoted to the study of large classes of (mainly degenerate) initial-boundary value evolution problems in connection with the possibility to obtain a constructive approximation of the associated positive C_0-semigroups by means of iterates of suitable positive linear operators which also constitute approximation processes in the underlying Banach function space. Usually, as a consequence of a careful analysis of the preservation properties of the approximating operators, such as monotonicity, convexity, Hölder continuity, and so on, it is possible to infer similar preservation properties for the relevant semigroups and, in turn, some spatial regularity properties of the solutions to the evolution problems (see, e.g., [CaEtAl99, AtCa00, Ma02, AlEtAl07, AlLe09], [AlCa94, Chapter 6] and the references therein).

More recently, by continuing along these directions, we started a research project in order to investigate the possibility of associating to a given Markov operator on the Banach space $C(K)$ of all real functions defined on a convex compact subset K of $\mathbb{R}^d$ ($d \geq 1$) some classes of differential operators as well as some suitable positive approximation processes. The main aim is to investigate whether these differential operators are generators of positive semigroups and whether the semigroups can

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be approximated by iterates of the approximation processes themselves. By means of a qualitative study of the approximation processes, the approximation formulas could guarantee a similar qualitative analysis of the positive semigroups and, consequently, of the solutions to the evolution equations governed by them.

In this survey paper we report some of the main ideas and results we have developed in this respect and which are documented in [AlEtAl14, AlEtAl14a, AlEtAl14b, AlEtAl16a, AlEtAl16b].

The differential operators considered within the framework of the theory fall into classes of operators of wide interest in the theory of evolution equations and in models of population dynamics and mathematical finance. The generation problems for these differential operators have been also studied with other methods which, however, do not allow to get spatial regularity properties of the solutions as well as information about their asymptotic behavior whereas these aspects are successfully obtained with the methods of approximation by positive operators.

Furthermore, the involved approximation processes are inspired by some classical ones and, among other things, they generalize the Bernstein operators and the Kantorovich operators in all one-dimensional and multi-dimensional convex domains on which the latter have been considered. These approximation processes seem to have an interest in their own also for the approximation of continuous functions and, in some cases, of p-th power integrable functions. For these reasons their study has been also deepened from several points of view of the approximation theory. The paper contains some noteworthy examples which offer a short outline of the possible application of the theory and show that diverse differential problems scattered in the literature can be encompassed in the present unifying approach (see, e.g., [CeCl01, MuRh11, Ta14]).

2.2 Canonical Elliptic Second-Order Differential Operators and Bernstein-Schnabl Operators

From now on we shall fix a convex compact subset K of the real Euclidean space $\mathbb{R}^d$ ($d \geq 1$) with non-empty interior $\mathrm{int}(K)$, and a *Markov operator* $T : C(K) \to C(K)$ on the space $C(K)$ of all real continuous functions on K, i.e. T is a positive linear operator on $C(K)$ such that $T(\mathbf{1}) = \mathbf{1}$, $\mathbf{1}$ being the constant function of value 1 on K.

By Riesz representation theorem it is known that, for every $x \in K$, there exists a (unique) probability Borel measure $\tilde{\mu}_x^T$ on K such that, for every $f \in C(K)$,

$$T(f)(x) = \int_K f \, d\tilde{\mu}_x^T .$$

Then, for every $n \geq 1$, we define *the n-th Bernstein-Schnabl operator B_n associated with T* by setting, for every $f \in C(K)$ and $x \in K$,

$$B_n(f)(x) := \int_K \cdots \int_K f\left(\frac{x_1 + \ldots + x_n}{n}\right) d\tilde{\mu}_x^T(x_1) \cdots d\tilde{\mu}_x^T(x_n).$$

For every $n \geq 1$, B_n is a positive linear operator from $C(K)$ into $C(K)$, $B_n(\mathbf{1}) = \mathbf{1}$ and hence, by positivity, $\|B_n\| = 1$. Moreover, $B_1 = T$.

If, in addition, the Markov operator T satisfies the assumption

$$T(h) = h \quad \text{for every } h \in \{pr_1, \ldots, pr_d\} \tag{2.1}$$

where, for each $i = 1, \ldots, d$, pr_i stands for the i^{th} coordinate function on K, i.e. $pr_i(x) = x_i$ for every $x = (x_1, \ldots, x_d) \in K$, then the sequence $(B_n)_{n \geq 1}$ is an approximation process on $C(K)$, i.e. for every $f \in C(K)$, $\lim_{n\to\infty} B_n(f) = f$ uniformly on K (see [AlEtAl14a, Theorem 3.1]; see also [AlEtAl14, Theorem 3.2.1]).

The class of Bernstein-Schnabl operators associated with T contains several examples of well-known sequences of operators as particular cases. In fact they generalize the classical Bernstein operators on the unit interval, on multi-dimensional simplices and hypercubes, and they share with them several preservation properties which are investigated in [AlEtAl14, Chapter 3] and [AlCa94, Chapter 6] (see also [AlEtAl14a] and the references therein).

Among the properties fulfilled by the B_n's, we recall here that, for every $m \geq 1$, all the Bernstein-Schnabl operators leave invariant the space $P_m(K)$ of (restriction to K of all) polynomials of degree at most m, under the additional hypothesis

$$T(P_m(K)) \subset P_m(K) \tag{2.2}$$

for every $m \geq 1$ (see [AlEtAl14a, Theorem 3.2]; see also [AlEtAl14, Lemma 4.1.1]).

Moreover, for every $u \in C^2(K)$, the following asymptotic formula holds:

$$\lim_{n\to\infty} n(B_n(u) - u) = W_T(u) \quad \text{uniformly on } K, \tag{2.3}$$

where W_T is the elliptic second-order differential operator defined as

$$W_T(u) := \frac{1}{2} \sum_{i,j=1}^{d} \alpha_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} \tag{2.4}$$

and the coefficients α_{ij}, for each $i, j = 1, \ldots, d$ and $x \in K$, are given by $\alpha_{ij}(x) := T(pr_i pr_j)(x) - (pr_i pr_j)(x)$ (see [AlEtAl14a, Theorem 4.2]; see also [AlEtAl14, Theorem 4.1.5]). W_T is referred to as the *canonical elliptic second-order differential operator associated with* T.

Operators (2.4) are of concern in the study of several diffusion problems arising in biology, financial mathematics, and other fields. As a matter of fact the study of such kind of differential operator presents some difficulties if tackled with the methods of the theory of PDEs. First of all the boundary ∂K of K is generally non-smooth due to the presence of possible sides and corners. Moreover, W_T degenerates on the set $\partial_T K$ of all interpolation points for T given by $\partial_T K := \{x \in K \mid T(f)(x) = f(x) \text{ for every } f \in C(K)\}$, which contains the extreme points of K by virtue of (2.1).

Now consider the following initial-boundary value problem associated with the couple $(W_T, C^2(K))$ and the initial datum u_0 belonging to a suitable subset of $C^2(K)$:

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) = \dfrac{1}{2}\displaystyle\sum_{i,j=1}^{d} \alpha_{ij}(x)\dfrac{\partial^2 u}{\partial x_i \partial x_j}(x,t) & x \in K, \quad t \geq 0; \\ u(x,0) = u_0(x) & x \in K. \end{cases} \tag{2.5}$$

The possibility to approximate (or to reconstruct) the solutions to (2.5) lies in the fact that, under hypotheses (2.1) and (2.2) on the Markov operator T, by applying a Trotter-Schnabl-type result (see [AlEtAl14, Corollary 2.2.3 and Remark 2.2.4]; see also [AlEtAl09, Theorem 2.1]), we get that

Theorem 1 *The operator $(W_T, C^2(K))$ is closable and its closure $(A_T, D(A_T))$ generates a Markov semigroup $(T(t))_{t\geq 0}$ on $C(K)$ such that, if $t \geq 0$ and $(k(n))_{n\geq 1}$ is a sequence of positive integers satisfying $\lim_{n\to\infty} k(n)/n = t$, then*

$$T(t)(f) = \lim_{n\to\infty} B_n^{k(n)}(f) \quad \textit{uniformly on } K \tag{2.6}$$

for every $f \in C(K)$ (here $B_n^{k(n)}$ denotes the iterate of B_n of order $k(n)$). Moreover, the subalgebra $P_\infty(K) := \bigcup_{m\geq 1} P_m(K)$ of $C(K)$, and hence $C^2(K)$ as well, is a core for $(A_T, D(A_T))$.

The approximation formula (2.6) will be briefly referred by saying that the sequence $(B_n)_{n\geq 1}$ is a *strongly admissible sequence* for the semigroup $(T(t))_{t\geq 0}$.

From the above theorem it follows that we can solve the abstract Cauchy problem associated with $(A_T, D(A_T))$ and the initial datum $u_0 \in D(A_T)$

$$\begin{cases} \dfrac{du}{dt}(x,t) = A_T(u(\cdot,t))(x) & x \in K, \quad t \geq 0; \\ u(x,0) = u_0(x) & x \in K, \end{cases} \tag{2.7}$$

which turns into the particular problem (2.5) when u_0 is in $C^2(K)$, since $A_T = W_T$ on $C^2(K)$, obtaining for it an approximation formula for the (unique) solution, i.e.

$$u(x,t) := T(t)(u_0)(x) = \lim_{n\to\infty} B_n^{k(n)}(u_0) \quad \text{uniformly on } K. \tag{2.8}$$

For more details on semigroup theory, we refer the reader to [EnNa00] or to [AlEtAl14, Chapter 2].

From (2.8) we can derive both numerical and qualitative information about the solutions of the Cauchy problems of kind (2.5) from the study of operators B_n.

Below we list some spatial regularity properties of the solutions to (2.7) which may be inferred by some preservation properties of the B_n's by means of (2.8).

Theorem 2 *Under the same assumptions of Theorem 1, the following statements hold true:*

(i) Given $M \geq 0$ and $0 < \alpha \leq 1$, let $Lip(M, \alpha)$ be the space of all Hölder continuous functions on K with exponent α and Hölder constant M. If, in addition, $T(\mathrm{Lip}(1,1)) \subset \mathrm{Lip}(1,1)$ and $u_0 \in \mathrm{Lip}(M,\alpha)$, then $u(\cdot,t) \in \mathrm{Lip}(M,\alpha)$ for every $t \geq 0$, $M \geq 0$ and $0 < \alpha \leq 1$.

(ii) Suppose that T maps continuous convex functions into (continuous) convex functions and that the quantity

$$\Delta(f;x,y) := B_2(f)(x) + B_2(f)(y) - 2\iint\limits_{K^2} f\left(\frac{s+t}{2}\right) d\tilde{\mu}_x^T(s) d\tilde{\mu}_x^T(t)$$

is positive for every $f \in C(K)$ and $x, y \in K$. If $u_0 \in D(A_T)$ is convex, then $u(\cdot,t)$ is convex for every $t \geq 0$.

We conclude this section by presenting several examples of Markov operators to which the previous theorems apply.

Example 1 Assume that $d \geq 2$ and that ∂K is an ellipsoid, i.e.

$$K = \left\{ x \in \mathbb{R}^d \mid Q(x-\overline{x}) := \sum_{i,j=1}^{d} r_{ij}(x_i - \overline{x}_i)(x_j - \overline{x}_j) \leq 1 \right\},$$

where $(r_{ij})_{i,j=1,\dots,d}$ is a real symmetric and positive-definite matrix and $\overline{x} \in \mathbb{R}^d$. Let L be a strictly elliptic differential operator of the form

$$L(u)(x) = \sum_{i,j=1}^{d} c_{ij} \frac{\partial^2 u(x)}{\partial x_i \partial x_j} \quad (u \in C^2(\mathrm{int}(K)), x \in \mathrm{int}(K))$$

associated with a real symmetric and positive matrix $(c_{ij})_{1 \leq i,j \leq d}$ and denote by $T_L : C(K) \to C(K)$ the Poisson operator associated with L. Thus, for every $f \in C(K)$, $T_L(f)$ denotes the unique solution to the Dirichlet problem

$$\begin{cases} Lu = 0 & \text{on } \mathrm{int}(K), \quad u \in C(K) \cap C^2(\mathrm{int}(K)); \\ u = f & \text{on } \partial K. \end{cases}$$

Note that T_L is a Markov operator (in particular a Markov projection) satisfying (2.1). Moreover, also (2.2) is verified.

The differential operator W_{T_L} associated with T_L, briefly denoted by W_L, is given by

$$W_L(u)(x) := \begin{cases} \dfrac{1 - Q(x-\overline{x})}{2\sum\limits_{i,j=1}^{d} r_{ij}c_{ij}} L(u)(x) & \text{if } x \in \mathrm{int}(K); \\ 0 & \text{if } x \in \partial K \end{cases}$$

$(u \in C^2(K), x \in K)$.

In particular, if K is the closed ball (with respect to $\|\cdot\|_2$) with center the origin of $\mathbb{R}^d$ and radius 1 and if L is the laplacian Δ, then

$$T_\Delta(f)(x) := \begin{cases} \dfrac{1-\|x\|_2^2}{\sigma_d} \displaystyle\int_{\partial K} \frac{f(z)}{\|z-x\|_2^d} d\sigma(z) & \text{if } \|x\|_2 < 1; \\ f(x) & \text{if } \|x\|_2 = 1 \end{cases}$$

($f \in C(K)$, $x \in K$), where σ_d and σ denote the surface area of the unit sphere in $\mathbb{R}^d$ and the surface measure on ∂K, resp., and, for every $u \in C^2(K)$ and $x \in K$,

$$W_\Delta(u)(x) := \begin{cases} \dfrac{1-\|x\|_2^2}{2d} \Delta(u)(x) & \text{if } \|x\|_2 < 1; \\ 0 & \text{if } \|x\|_2 = 1. \end{cases}$$

An explicit expression for the Bernstein-Schnabl operators associated with T_L can be found in [AlEtAl14, Subsection 3.1.4].

Example 2 Consider the d-dimensional simplex

$$K_d := \left\{ (x_1, \ldots, x_d) \in \mathbb{R}^d \;\middle|\; x_i \geq 0 \text{ for every } i = 1, \ldots, d \text{ and } \sum_{i=1}^{d} x_i \leq 1 \right\}$$

and the canonical projection T_d on K_d defined by

$$T_d(f)(x) := \left(1 - \sum_{i=1}^{d} x_i\right) f(v_0) + \sum_{i=1}^{d} x_i f(v_i)$$

($f \in C(K_d)$, $x = (x_1, \ldots, x_d) \in K_d$), where $v_0 := (0, \ldots, 0)$, $v_1 := (1, 0, \ldots, 0)$, …, $v_d := (0, \ldots, 0, 1)$ are the vertices of the simplex.

Then T_d is a Markov operator which satisfies (2.1) and (2.2) and the relevant differential operator W_{T_d} associated with T_d is given by

$$W_{T_d}(u)(x) = \frac{1}{2} \sum_{i=1}^{d} x_i(1-x_i) \frac{\partial^2 u}{\partial x_i^2}(x) - \sum_{1 \leq i < j \leq d} x_i x_j \frac{\partial^2 u}{\partial x_i \partial x_j}(x)$$

($u \in C^2(K_d)$, $x = (x_1, \ldots, x_d) \in K_d$). Note that the coefficients of W_{T_d} vanish on the vertices of the simplex.

The operator W_{T_d} falls into the class of the so-called Fleming-Viot operators which occur in the description of a stochastic process associated with a diffusion approximation of a gene frequency model in population genetics; more recently, they have been object of investigation by several authors.

Also in this case we have an explicit expression for the relevant operators B_n (see [AlEtAl14, formula (3.1.18)]).

Example 3 Let $S : C(K_d) \longrightarrow C(K_d)$ be the Markov operator defined by

$$S(f)(x) := \begin{cases} \left(1 - \frac{x_1}{1-\sum_{i=2}^{d} x_i}\right) f(0, x_2, \ldots, x_d) + \frac{x_1}{1-\sum_{i=2}^{d} x_i} & \\ \times f\left(1 - \sum_{i=2}^{d} x_i, x_2, \ldots, x_d\right) & \text{if } \sum_{i=2}^{d} x_i \neq 1; \\ f(0, x_2, \ldots, x_d) & \text{if } \sum_{i=2}^{d} x_i = 1 \end{cases}$$

$(f \in C(K_d),\ x = (x_1, \ldots, x_d) \in K_d)$. S verifies (2.1) and (2.2) and in this case the differential operator associated with S is defined as

$$W_S(u)(x) = \frac{1}{2} x_1 \left(1 - \sum_{i=1}^{d} x_i\right) \frac{\partial^2 u}{\partial x_1^2}(x)$$

$(u \in C^2(K_d),\ x = (x_1, \ldots, x_d) \in K_d)$. Note that W_S degenerates on the faces $\{x = (x_1, \ldots, x_d) \in K_d \mid x_1 = 0\}$ and $\left\{x = (x_1, \ldots, x_d) \in K_d \mid \sum_{i=1}^{d} x_i = 1\right\}$.

Example 4 Consider the convex combination of the above operators, that is the Markov operator $V := (T_d + S)/2$. Then V satisfies (2.1) and (2.2) and the differential operator associated with it becomes

$$W_V(u)(x) = \frac{1}{4}\left(2x_1(1 - x_1) - x_1 \sum_{i=2}^{d} x_i\right) \frac{\partial^2 u}{\partial x_1^2}(x)$$
$$+ \frac{1}{4} \sum_{i=2}^{d} x_i(1 - x_i) \frac{\partial^2 u}{\partial x_i^2}(x) - \frac{1}{4} \sum_{1 \leq i < j \leq d} x_i x_j \frac{\partial^2 u}{\partial x_i \partial x_j}(x)$$

$(u \in C^2(K_d),\ x = (x_1, \ldots, x_d) \in K_d)$. Observe that W_V degenerates on the vertices of K_d as well. In this case B_n's are defined as in the Subsection 3.1.6 of [AlEtAl14].

Example 5 Let $Q_d := [0, 1]^d$, $d \geq 1$, and for every $i = 1, \ldots, d$ consider a Markov operator U_i on $C([0, 1])$ satisfying (2.1) and (2.2). If $U := \bigotimes_{i=1}^{d} U_i$ is the tensor product of the family $(U_i)_{1 \leq i \leq d}$, then U is a Markov operator on $C(Q_d)$ satisfying (2.1) and (2.2). The differential operator in this case has the following form:

$$W_U(u)(x) = \frac{1}{2} \sum_{i=1}^{d} \alpha_i(x) \frac{\partial^2 u}{\partial x_i^2}(x),$$

($u \in C^2(Q_d)$, $x = (x_1, \ldots, x_d) \in Q_d$), where $\alpha_i(x) := U_i(e_2)(x_i) - x_i^2$ $(1 \le i \le d)$, e_2 being the function $e_2(t) = t^2$ for each $t \in [0, 1]$.

Bernstein-Schnabl operators on hypercubes are given by formula (3.1.30) of [AlEtAl14].

2.3 Other Classes of Differential Operators and Approximation Processes

In this section we discuss other classes of differential operators which are pre-generators of Markov semigroups along with the relevant strongly admissible sequences of positive operators. The differential operators are multiplicative and additive perturbations of the operator W_T and the relevant approximation processes can be obtained by means of suitable modification of Bernstein-Schnabl operators.

Multiplicative Perturbations and Lototsky-Schnabl Operators
Given a Markov operator T on $C(K)$ satisfying assumption (2.1), a new Markov operator U_λ can be constructed by setting $U_\lambda := \lambda T + (1-\lambda)I$, where $\lambda \in C(K)$, $0 \le \lambda \le 1$, I being the identity operator of $C(K)$.

The Bernstein-Schnabl operators B_{n,U_λ} associated with U_λ are referred to as the Lototsky-Schnabl operators associated with T and λ and they are simply denoted by $B_{n,\lambda}$, $n \ge 1$. Actually, if $f \in C(K)$, $x \in K$ and $n \ge 1$,

$$B_{n,\lambda}(f)(x) := \begin{cases} \displaystyle\sum_{k=0}^{n} \binom{n}{k} \lambda(x)^k (1-\lambda(x))^{n-k} B_k(f_{x,k/n})(x) & \text{if } x \notin \partial_T K, \\ f(x) & \text{if } x \in \partial_T K, \end{cases}$$

where $f_{x,k/n}(t) = f\left(\frac{k}{n}t + \left(1 - \frac{k}{n}\right)x\right)$ $(t \in K)$. It is possible to show that, on one hand, the multiplicative perturbation λW_T of the differential operator W_T is linked to the operators $B_{n,\lambda}$ by an asymptotic formula like (2.3), and on the other hand that the couple $(\lambda W_T, C^2(K))$ pre-generates a Markov semigroup for which a formula similar to (2.6) involving iterates of Lototsky-Schnabl operators holds under the same hypotheses (2.1) and (2.2) on T. For all details we refer the reader to [AlEtAl14, Section 5.1].

Special additive Perturbation and Generalized Kantorovich Operators
In the recent paper [AlEtAl16a] we introduce and study a new sequence of positive linear operators acting on $C(K)$ (and, in some particular cases, also in other function spaces). Namely, if T is a Markov operator on $C(K)$ satisfying (2.1), if a is a real positive number, and if $(\mu_n)_{n\ge1}$ is a weakly convergent sequence of probability Borel measures on K, we define the positive linear operator C_n by setting, for every $x \in K$ and $f \in C(K)$,

$$C_n(f)(x) = \int_K \cdots \int_K f\left(\frac{x_1 + \ldots + x_n + a x_{n+1}}{n+a}\right) d\tilde{\mu}_x^T(x_1) \cdots d\tilde{\mu}_x^T(x_n) d\mu_n(x_{n+1}).$$

Through an asymptotic formula like (2.3) such operators are connected with the following class of differential operators which turn into particular additive perturbations of W_T which are defined by setting, for every $u \in C^2(K)$ and $x \in K$,

$$V_T(u)(x) := W_T(u)(x) + \sum_{i=1}^{d} a(b_i - pr_i)\frac{\partial u}{\partial x_i},$$

where $b = (b_1, \ldots, b_d) \in K$ is the barycenter of the weak limit of $(\mu_n)_{n\geq 1}$ (see [AlEtAl16b]). Then $(V_T, C^2(K))$ is a pre-generator of a Markov semigroup for which the sequence $(C_n)_{n\geq 1}$ is a strongly admissible sequence. In some particular cases, the same result holds true also in L^p-spaces.

Additive Perturbation and Modified Bernstein-Schnabl Operators
Consider the complete differential operator

$$Z_T(u)(x) = W_T(u)(x) + \sum_{i=1}^{d} \beta_i(x)\frac{\partial u}{\partial x_i} + \gamma(x)u(x) \quad (u \in C^2(K), x \in K),$$

with $\beta_i, \gamma \in C(K)$ for which there exists $n_0 \geq 1$ such that $x + \dfrac{\beta(x)}{n} \in K$ and $1 + \dfrac{\gamma(x)}{n} \geq 0$ for each $x \in K$ and $n \geq n_0$.

By adapting an idea first developed in [AlAm05] (see also [AlRa99]), in order to study the solutions to the diffusion problem governed by Z_T we introduced and studied a further modification of the B_n's, in symbols M_n, defined by setting

$$M_n(f) := B_n\left((\mathbf{1} + \gamma/n) \times (f \circ (id + \beta/n))\right)$$

for every $f \in C(K)$, where $id(y) = y$ for every $y \in K$ (see [AlEtAl14, Section 5.2]). Also in this case we showed that then $(Z_T, C^2(K))$ is a pre-generator of a Markov semigroup for which the sequence $(M_n)_{n\geq 1}$ is a strongly admissible sequence.

In all the above-mentioned cases, by means of the relevant strongly admissible approximating sequences, we carried out a careful qualitative analysis of the preservation properties of the semigroups.

2.4 Final Remarks

1. Similar problems can be considered for convex compact subsets K of a (not necessarily finite dimensional) locally convex Hausdorff space X.
2. The crucial assumption (2.2) that the Markov operator T maps polynomials into polynomials of the same degree seems to have an independent interest on its own. We discuss several situations where it is satisfied in the final part of Section 4.5 of [AlEtAl14] as well as in [AlEtAl14b].

3. In some special cases we describe the asymptotic behavior of the Markov semigroup, i.e. we determine $\lim_{t \to +\infty} T(t)$, and we characterize the saturation class of $(B_n)_{n \geq 1}$ and the Favard class of $(T(t))_{t \geq 0}$ (see Subsection 4.5.4 and Appendix A.2 of [AlEtAl14]).
4. Fields of application of formula (2.6) include both probabilistic and numerical applications.

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