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On differential operators associated with Markov operators

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A B S T R A C T

In this paper we introduce and study a new class of elliptic second-order differential operators on a convex compact subset K of $\mathbf{R}^d$, $d \geqslant 1$, which are associated with a Markov operator T on $\mathscr{C}(K)$ and which degenerate on a suitable subset of K containing its extreme points. Among other things, we show that the closures of these operators generate Markov semigroups. Moreover, we prove that these semigroups can be approximated by means of iterates of suitable positive linear operators, which are referred to as the Bernstein–Schnabl operators associated with T. As a consequence we show that the semigroups preserve polynomials of a given degree as well as Hölder continuity, which gives rise to some spatial regularity properties of the solutions of the relevant evolution equations.

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1. Introduction

This paper is mainly concerned with the study of a new class of second-order differential operators on compact subsets of $\mathbf{R}^d$, $d \geqslant 1$, which can be associated with a Markov operator.

More precisely, given a convex compact subset K of $\mathbf{R}^d$ with non-empty interior and a Markov operator T on $\mathscr{C}(K)$ (i.e., a positive linear operator T on $\mathscr{C}(K)$ such that $T(\mathbf{1}) = \mathbf{1}$), we shall consider and study the following elliptic second-order differential operator W_T, defined by setting, for every $u \in \mathscr{C}^2(K)$,

$$W_T(u) := \frac{1}{2}\sum_{i,j=1}^{d} \alpha_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j}, \tag{1}$$

where, for each $i, j = 1, \ldots, d$, $\alpha_{ij} := T(pr_i pr_j) - pr_i pr_j$ and pr_i stands for the i-th coordinate function.

One of the difficulties in studying operators (1) lies in the fact that the boundary ∂K of K is generally non-smooth, due to the presence of possible sides and corners; moreover, since we assume that T preserves the coordinate functions, the operator W_T degenerates on a subset of K containing the set $\partial_e K$ of all the extreme points of K.

On the other hand, operators (1) are of concern in the study of several diffusion problems arising in biology, financial mathematics and other fields, so that they seem to be worthy of a comprehensive and thorough study.

In this paper we are mainly interested in proving that, under suitable assumptions on T, the operator $(W_T, \mathscr{C}^2(K))$ is closable and its closure is the generator of a Markov semigroup $(T(t))_{t\geqslant 0}$ on $\mathscr{C}(K)$.

Our approach is based on a Trotter-type result due to Schnabl [25]; in fact, in order to prove our generation result, we consider a suitable sequence $(B_n)_{n\geqslant 1}$ of positive linear operators, the so-called *Bernstein–Schnabl operators* associated with T, which are strictly connected with operator (1) via an asymptotic formula, and then, by means of them, we obtain the generation result by also showing that the generated semigroup $(T(t))_{t\geqslant 0}$ can be represented in terms of suitable iterates of the B_n's. As a consequence we show that the semigroups preserve polynomials of a given degree as well as Hölder continuity, which, in turn, allows to highlight some spatial regularity properties of the solutions of the relevant evolution equations.

The sequence of Bernstein–Schnabl operators is an approximation process in $\mathscr{C}(K)$, so that, from the point of view of Approximation Theory, it seems to have an interest on its own. For this reason, even if in this work we investigate several properties of $(B_n)_{n\geqslant 1}$, we shall undertake a deeper and more accurate study of them in a forthcoming monograph [14]. In the same monograph we shall also deepen the investigations of other additional preserving properties of the semigroups by relating them with the relevant ones held by the operators B_n.

Our results generalize those of [2,4] (see also [6, Chapter 6]), in which the author is concerned with operator (1), in the special case where T is a positive projection satisfying suitable assumptions. The more general framework of the present paper guarantees to notably enlarge the class of differential operators by including, in particular, those which can be obtained by usual operations with Markov operators such as convex combinations, compositions, tensor products and so on.

Finally we point out that in [13] the authors have undertaken a similar study of operator (1) in the context of the unit interval $[0,1]$. This paper represents a full extension to higher dimensions which discloses new and more difficult problems.

2. Preliminaries

In this section we collect the main notation of the paper and we recall some results that will be useful in the sequel.

From now on, let us consider a convex compact subset K of $\mathbf{R}^d$, $d \geqslant 1$, whose interior $\operatorname{int}(K)$ is assumed to be non-empty. We shall denote by $\|\cdot\|_2$ the Euclidean norm on $\mathbf{R}^d$, i.e., $\|x\|_2 := \sqrt{\sum_{i=1}^d x_i^2}$ $(x = (x_1, \dots, x_d) \in \mathbf{R}^d)$.

As usual we shall denote by $\mathscr{F}(K)$ the space of all real-valued functions on K and by $\mathscr{C}(K)$ the space of all real-valued continuous functions on K; the space $\mathscr{C}(K)$, endowed with the natural (pointwise) ordering and the sup-norm $\|\cdot\|_\infty$, is a Banach lattice.

Moreover, we denote by $\mathscr{C}^2(K)$ the space of all real-valued (continuous) functions on K that are twice continuously differentiable on $\operatorname{int}(K)$ and whose partial derivatives of order $\leqslant 2$ can be continuously extended to K. For $u \in \mathscr{C}^2(K)$ and $i, j = 1, \dots, d$, we shall continue to denote by $\frac{\partial u}{\partial x_i}$ and $\frac{\partial^2 u}{\partial x_i \partial x_j}$ the continuous extensions to K of the partial derivatives $\frac{\partial u}{\partial x_i}$ and $\frac{\partial^2 u}{\partial x_i \partial x_j}$.

For every $i = 1, \dots, d$, we shall denote by pr_i the i-th coordinate function on K (i.e., $pr_i(x) = x_i$ for every $x = (x_1, \dots, x_d) \in K$) and by $\mathbf{1}$ the constant function of constant value 1 on K.

Moreover, we denote by B_K the σ-algebra of all Borel subsets of K. The symbol $M^+(K)$ (resp., $M_1^+(K)$) stands for the space of all regular Borel measures (resp., probability Borel measures) on K; in particular, for every $x \in K$, ε_x denotes the unit mass concentrated at x, i.e., for every $B \in B_K$,

$$\varepsilon_x(B) := \begin{cases} 1 & \text{if } x \in B; \\ 0 & \text{if } x \notin B. \end{cases}$$

Consider a Markov operator $T : \mathscr{C}(K) \longrightarrow \mathscr{C}(K)$, i.e., a positive linear operator on $\mathscr{C}(K)$ such that $T(\mathbf{1}) = \mathbf{1}$.

It is well known that for every $x \in K$ there exists a (unique) $\tilde{\mu}_x^T \in M_1^+(K)$ such that, for every $f \in \mathscr{C}(K)$,

$$T(f)(x) = \int_K f \, d\tilde{\mu}_x^T. \tag{2.1}$$

The family $(\tilde{\mu}_x^T)_{x \in K}$ is also referred to as the *continuous selection of probability Borel measures on K associated with T*.

Given a Markov operator T on $\mathscr{C}(K)$, first of all we are interested in describing a particular subset of K, the so-called *set of interpolation points for T*, that is denoted by $\partial_T K$ and defined as

$$\partial_T K := \{x \in K \mid T(f)(x) = f(x) \text{ for every } f \in \mathscr{C}(K)\}. \tag{2.2}$$

To this end we recall that, given a linear subspace L of $\mathscr{C}(K)$, the *Choquet boundary* $\partial_L K$ of L is the subset of all points $x \in K$ such that, if $\tilde{\mu} \in M^+(K)$ and $\int_K h \, d\tilde{\mu} = h(x)$ for every $h \in L$, then $\int_K f \, d\tilde{\mu} = f(x)$ for every $f \in \mathscr{C}(K)$, i.e., $\tilde{\mu} = \varepsilon_x$.

If L contains the constants and separates the points of K, then $\partial_L K$ is non-empty and every $h \in L$ attains its minimum and maximum on $\partial_L K$ (see, e.g., [6, Corollary 2.6.5]).

We shall also set

$$M := \{h \in \mathscr{C}(K) \mid T(h) = h\}. \tag{2.3}$$

Clearly, M is contained in the range of T that will be also denoted by

$$H := T(\mathscr{C}(K)) = \{T(f) \mid f \in \mathscr{C}(K)\}. \tag{2.4}$$

The subspace M contains the constants and, if it separates the points of K, then its Choquet boundary $\partial_M K$ is non-empty. Moreover, since each $g \in M$ attains its minimum and its maximum on $\partial_M K$, if $g = 0$ on $\partial_M K$, then necessarily $g = 0$.

The next result has been obtained in [5, Theorem 2.1].

Theorem 2.1. *Consider a Markov operator $T : \mathscr{C}(K) \longrightarrow \mathscr{C}(K)$ such that the subspace M defined by (2.3) separates the points of K. Then*

$$\emptyset \neq \partial_M K \subset \partial_T K \subset \partial_H K \tag{2.5}$$

(*see* (2.2) *and* (2.4)).

Moreover, if V is an arbitrary subset of M separating the points of K, then

$$\partial_T K = \{x \in K \mid T(h^2)(x) = h^2(x) \text{ for every } h \in V\}. \tag{2.6}$$

Finally, if $(h_n)_{n\geqslant 1}$ *is a finite or countable family in* M *separating the points of* K *and such that the series* $\Phi := \sum_{n=1}^{\infty} h_n^2$ *is uniformly convergent, then*

$$\partial_T K = \{x \in K \mid T(\Phi)(x) = \Phi(x)\}. \tag{2.7}$$

Remarks 2.2. 1. There always exists a sequence $(h_n)_{n\geqslant 1}$ in M such that the series $\sum_{n=1}^{\infty} h_n^2$ converges uniformly on K.

Indeed, M is separable (because $\mathscr{C}(K)$ is separable) and hence there exists a countable dense family $(\varphi_n)_{n\geqslant 1}$ of M. Then, it is sufficient to set $h_n := \frac{\varphi_n}{2^n(\|\varphi_n\|_\infty+1)}$ for every $n \geqslant 1$.

2. Assume that T is a Markov operator such that $T(h) = h$ for all $h \in \{\mathbf{1}, pr_1, \ldots, pr_d\}$ i.e., $P_1(K) \subset M$, where $P_1(K)$ is the space of (the restriction to K of) all polynomials of degree at most 1.

Then, by combining [6, Proposition 2.6.3] and (2.5), we have that

$$\partial_e K \subset \partial_M K \subset \partial_T K \subset \partial_H K, \tag{2.8}$$

where the symbol $\partial_e K$ denotes the set of the *extreme* points of K; more precisely, $\partial_e K$ is the set of those points $x_0 \in K$ such that $K \setminus \{x_0\}$ is convex, i.e., if $x_1, x_2 \in K$ and $\lambda \in \mathbf{R}$, $0 < \lambda < 1$, and if $x_0 = \lambda x_1 + (1-\lambda)x_2$, then $x_0 = x_1 = x_2$.

We pass now to recall an asymptotic formula for an arbitrary sequence of positive linear operators on $\mathscr{C}(K)$.

Given $x \in K$, we denote by $\Psi_x : K \longrightarrow \mathbf{R}^d$ the mapping defined by

$$\Psi_x(y) := y - x \quad (y \in K) \tag{2.9}$$

and by $\Phi_x : K \longrightarrow \mathbf{R}$ the function defined by

$$\Phi_x(y) := \|y - x\|_2 \quad (y \in K). \tag{2.10}$$

Furthermore, we consider a second-order differential operator of the form

$$A(u)(x) := \frac{1}{2}\sum_{i,j=1}^{d} \alpha_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j}(x) + \sum_{i=1}^{d} \beta_i(x)\frac{\partial u}{\partial x_i}(x) + \gamma(x)u(x) \tag{2.11}$$

($u \in \mathscr{C}^2(K)$, $x \in K$), where $\alpha_{ij}, \beta_i, \gamma \in \mathscr{F}(K)$, $i, j = 1, \ldots, d$.

The following result is a special case of a general asymptotic formula proved in [7, Theorem 3.5].

Theorem 2.3. *Let* $\alpha_{ij}, \beta_i, \gamma \in \mathscr{F}(K)$, $i, j = 1, \ldots, d$, *and consider the differential operator* A *defined by* (2.11). *Furthermore, consider a divergent sequence* $(\varphi(n))_{n\geqslant 1}$ *of positive integers and a sequence* $(L_n)_{n\geqslant 1}$ *of positive linear operators from* $\mathscr{C}(K)$ *into* $\mathscr{F}(K)$. *Assume that there exists a subset* G *of* K *such that*

(i) $\lim_{n\to\infty} \varphi(n)(L_n(\mathbf{1})(x) - 1) - \gamma(x) = 0$ *uniformly w.r.t.* $x \in G$;
(ii) $\lim_{n\to\infty} \varphi(n)L_n(pr_i \circ \Psi_x)(x) - \beta_i(x) = 0$ *uniformly w.r.t.* $x \in G$ *for every* $i = 1, \dots, d$;
(iii) $\lim_{n\to\infty} \varphi(n)L_n((pr_i \circ \Psi_x)(pr_j \circ \Psi_x))(x) - \alpha_{ij}(x) = 0$ *uniformly w.r.t.* $x \in G$, *for every* $i, j = 1, \dots, d$;
(iv) $\sup_{\substack{x\in G\\ n\geqslant 1}} \varphi(n)L_n(\Phi_x^2)(x) < +\infty$;
(v) *there exists* $q \in \mathbf{R}$, $q > 2$, *such that* $\lim_{n\to\infty} \varphi(n)L_n(\Phi_x^q)(x) = 0$ *uniformly w.r.t.* $x \in G$.

Then, for every $u \in \mathscr{C}^2(K)$,

$$\lim_{n\to\infty} \varphi(n)\big(L_n(u) - u\big) = A(u) \tag{2.12}$$

uniformly on G.

3. Bernstein–Schnabl operators associated with Markov operators

In this section we shall consider a sequence of positive linear operators, the so-called *Bernstein–Schnabl operators*, that will be the main tool to study the differential operators we shall present in Section 4.

Bernstein–Schnabl operators, introduced by Schnabl in [25] and successively by Grossmann [20] and Nishishiraho [22,23] in different contexts, were intensively studied by Altomare (see [2], [6, Section 6.1] and the references therein) in association with a positive projection on $\mathscr{C}(K)$ and through the last twenty years they have been the subject of several researches and generalizations. For more details, we refer the interested reader to the survey [12] and its numerous references, but also to [9–11,24], among many others.

From now on, fix a Markov operator $T : \mathscr{C}(K) \longrightarrow \mathscr{C}(K)$, K being a convex compact subset of $\mathbf{R}^d$, $d \geqslant 1$, and the continuous selection $(\tilde{\mu}_x^T)_{x\in K}$ of probability Borel measures associated with T as in (2.1).

Moreover, we assume that

$$T(h) = h \quad \text{for each } h \in \{\mathbf{1}, pr_1, \dots, pr_d\}. \tag{3.1}$$

Then, for any $n \geqslant 1$, we consider the positive linear operator $B_n : \mathscr{C}(K) \longrightarrow \mathscr{C}(K)$ defined by setting, for every $f \in \mathscr{C}(K)$ and $x \in K$,

$$B_n(f)(x) = \int_K \cdots \int_K f\left(\frac{x_1 + \cdots + x_n}{n}\right) d\tilde{\mu}_x^T(x_1) \cdots d\tilde{\mu}_x^T(x_n). \tag{3.2}$$

B_n will be referred to as the *n-th Bernstein–Schnabl operator associated with* T.
Clearly, $B_n(\mathbf{1}) = \mathbf{1}$ and hence $\|B_n\| = 1$. Moreover, $B_1 = T$.

We point out that, if $K = [0,1]$ and we consider the canonical projection $T_1 : \mathscr{C}([0,1]) \longrightarrow \mathscr{C}([0,1])$ defined by setting, for every $f \in \mathscr{C}([0,1])$ and $x \in [0,1]$,

$$T_1(f)(x) := xf(1) + (1-x)f(0), \tag{3.3}$$

then the B_n's turn into the classical Bernstein operators on $[0,1]$.

For some further examples of Bernstein–Schnabl operators we refer the interested reader to [6,8,13] and the references quoted therein.

In defining Bernstein–Schnabl operators, assumption (3.1) is not essential. In fact, it will be needed in order to prove that the sequence $(B_n)_{n\geqslant 1}$ is an approximation process in $\mathscr{C}(K)$, as the following result shows.

Theorem 3.1. *Let $(B_n)_{n\geqslant 1}$ be the sequence of Bernstein–Schnabl operators associated with a Markov operator T on $\mathscr{C}(K)$ satisfying (3.1). Then, for every $i, j = 1, \dots, d$ and $n \geqslant 1$,*

$$B_n(pr_i) = pr_i \tag{3.4}$$

and

$$B_n(pr_i pr_j) = \frac{1}{n}T(pr_i pr_j) + \frac{n-1}{n} pr_i pr_j. \tag{3.5}$$

In particular,

$$B_n\left(pr_i^2\right) = \frac{1}{n}T\left(pr_i^2\right) + \frac{n-1}{n} pr_i^2. \tag{3.6}$$

Moreover, for every $f \in \mathscr{C}(K)$,

$$\lim_{n\to\infty} B_n(f) = f \quad \text{uniformly on } K. \tag{3.7}$$

Finally, for every $n \geqslant 1$ and $f \in \mathscr{C}(K)$,

$$B_n(f) = f \quad \text{on } \partial_T K. \tag{3.8}$$

Proof. Fix $n \geqslant 1$; then (3.4) easily follows taking (3.1) and (3.2) into account.

Moreover, for every $i, j = 1, \dots, d$, $n \geqslant 1$ and $x_1, \dots, x_n \in K$, with $x_l = (x_l^1, \dots, x_l^d)$ for every $l = 1, \dots, n$,

$$\begin{aligned} pr_i\left(\frac{x_1 + \dots + x_n}{n}\right) pr_j\left(\frac{x_1 + \dots + x_n}{n}\right) &= \frac{1}{n^2}\left(\sum_{l=1}^n x_l^i \sum_{m=1}^n x_m^j\right) \\ &= \frac{1}{n^2}\left(\sum_{l=1}^n x_l^i x_l^j + \sum_{\substack{l,m\in\{1,\dots,n\}\\ l\neq m}} x_l^i x_m^j\right); \end{aligned}$$

hence, for every $x \in K$, we get

$$B_n(pr_i pr_j)(x) = \int_K \cdots \int_K (pr_i pr_j)\left(\frac{x_1 + \cdots + x_n}{n}\right) d\tilde{\mu}_x^T(x_1) \cdots d\tilde{\mu}_x^T(x_n)$$
$$= \frac{1}{n} \int_K (pr_i pr_j)(x_l)\, d\tilde{\mu}_x^T(x_l) + \frac{n-1}{n}(pr_i pr_j)(x),$$

and this completes the proof of (3.5) and (3.6).

Finally, (3.7) immediately follows from (3.4) and (3.6), because of the Korovkin-type Theorem 4.4.6 of [6] (see also [26]), and (3.8) is a consequence of (2.2). $\square$

In order to determine some shape preserving properties of the sequence $(B_n)_{n \geqslant 1}$, for every $m \geqslant 1$ we denote by $P_m(K)$ the (restriction to K of all) polynomials of degree at most m. Moreover, we set

$$P_\infty(K) := \bigcup_{m \geqslant 1} P_m(K); \tag{3.9}$$

clearly, $P_\infty(K)$ a subalgebra of $\mathscr{C}(K)$ and, by the Stone–Weierstrass theorem, it is dense in $\mathscr{C}(K)$.

Finally, for every $m \geqslant 1$ and $n \geqslant 1$ we introduce the symbol $F(m,n)$ to denote the set of all mappings $\sigma : \{1, \dots, m\} \longrightarrow \{1, \dots, n\}$.

If $\sigma \in F(m,n)$, consider the equivalence relation R_σ on $\{1, \dots, m\}$ defined by

$$R_\sigma := \{(i,j) \mid i, j = 1, \dots, m \text{ and } \sigma(i) = \sigma(j)\} \tag{3.10}$$

and the corresponding subdivisions $R_1^\sigma, \dots, R_{s_\sigma}^\sigma$ of $\{1, \dots, m\}$ in equivalence classes. Clearly, $\sum_{j=1}^{s_\sigma} \operatorname{card}(R_j^\sigma) = m$.

After these preliminaries, we can state the following result.

Proposition 3.2. *If $h_1, \dots, h_m \in P_1(K)$ $(m \geqslant 1)$, then, for every $n \geqslant 1$,*

$$B_n\left(\prod_{j=1}^m h_j\right) = \frac{1}{n^m} \sum_{\sigma \in F(m,n)} \prod_{j=1}^{s_\sigma} T\left(\prod_{i \in R_j^\sigma} h_i\right). \tag{3.11}$$

Therefore, if $T(P_m(K)) \subset P_m(K)$ for every $m \geqslant 1$, then

$$B_n(P_m(K)) \subset P_m(K) \tag{3.12}$$

for every $n, m \geqslant 1$.

Proof. Fix $n, m \geqslant 1$; first we recall that, if $(\alpha_{ij})_{\substack{1\leqslant i\leqslant n\\ 1\leqslant j\leqslant m}}$ is a matrix of real numbers, then

$$\prod_{j=1}^{m}\sum_{i=1}^{n}\alpha_{ij} = \sum_{\sigma\in F(m,n)}\prod_{j=1}^{m}\alpha_{\sigma(j)j}.$$

Accordingly, if $x_1, \dots, x_n \in K$, we get

$$\begin{aligned}\left(\prod_{j=1}^{m}h_j\right)\left(\frac{x_1+\dots+x_n}{n}\right) &= \frac{1}{n^m}\prod_{j=1}^{m}\sum_{i=1}^{n}h_j(x_i) = \frac{1}{n^m}\sum_{\sigma\in F(m,n)}\prod_{j=1}^{m}h_j(x_{\sigma(j)})\\ &= \frac{1}{n^m}\sum_{\sigma\in F(m,n)}\prod_{j=1}^{s_\sigma}\prod_{i\in R_j^\sigma}h_i(x_{\sigma(j)}).\end{aligned}$$

On account of (3.2) and (3.4) the result follows. □

As a consequence of Proposition 3.2, the following result easily follows.

Corollary 3.3. *Let $x \in K$ and $h_1, h_2, h_3, h_4 \in P_1(K)$ satisfying $h_i(x) = 0$ for $i = 1, \dots, 4$. Then, for every $n \geqslant 1$,*

$$\begin{aligned}B_n(h_1h_2h_3h_4)(x) = \frac{1}{n^3}\big[&T(h_1h_2h_3h_4)(x) + (n-1)T(h_1h_2)(x)T(h_3h_4)(x)\\ &+ (n-1)T(h_1h_3)(x)T(h_2h_4)(x)\\ &+ (n-1)T(h_1h_4)(x)T(h_2h_3)(x)\big].\end{aligned}\tag{3.13}$$

In particular, if $h, k \in P_1(K)$ and $h(x) = k(x) = 0$, then

$$\begin{aligned}B_n\big(h^2k^2\big)(x) = \frac{1}{n^3}\big[&T\big(h^2k^2\big)(x) + (n-1)T\big(h^2\big)(x)T\big(k^2\big)(x)\\ &+ 2(n-1)T(hk)^2(x)\big].\end{aligned}\tag{3.14}$$

4. Differential operators associated with Markov operators

This article is mainly concerned with the study of particular second-order differential operators associated with Markov operators. More precisely, given a Markov operator T on $\mathscr{C}(K)$ satisfying (3.1), it is possible to construct a suitable differential operator $W_T : \mathscr{C}^2(K) \longrightarrow \mathscr{C}(K)$ defined by setting, for every $u \in \mathscr{C}^2(K)$,

$$W_T(u) := \frac{1}{2}\sum_{i,j=1}^{d}\alpha_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j}\tag{4.1}$$

where, for each $i, j = 1, \dots, d$ and $x = (x_1, \dots, x_d) \in K$,

$$\alpha_{ij}(x) := T(pr_ipr_j)(x) - (pr_ipr_j)(x) = T\big((pr_i - x_i)(pr_j - x_j)\big)(x).\tag{4.2}$$

Accordingly, if $\xi_1, \dots, \xi_d \in \mathbf{R}$, then

$$\sum_{i,j=1}^{d} \alpha_{ij}(x)\xi_i\xi_j = T\left(\left(\sum_{i=1}^{d} \xi_i(pr_i - x_i)\right)^2\right)(x) \geqslant 0,$$

which implies that W_T is elliptic. Moreover, it degenerates on $\partial_T K$ (see (2.2)) and, in particular, on $\partial_e K$ (see Remark 2.2, 2)), because $\alpha_{ij} = 0$ on $\partial_T K$ for every $i, j = 1, \dots, d$.

The operator W_T will be referred to as the *elliptic second-order differential operator associated with the Markov operator T*.

Differential operators of the form (4.1) are of concern in the study of diffusion problems arising from different areas such as biology, mathematical finance, physics. As far as we know, these differential operators have been studied in the special case when T is a positive projection. For a rather complete overview we refer to Chapter 6 of [6], where several examples are presented. Below we discuss some further ones.

Examples 4.1. 1. Consider a Markov operator T on $\mathscr{C}([0,1])$ satisfying (3.1), i.e.,

$$T(e_1) = e_1, \tag{4.3}$$

where $e_1(x) := x$ $(0 \leqslant x \leqslant 1)$.

Then, for every $u \in \mathscr{C}^2([0,1])$ and $x \in [0,1]$,

$$W_T(u)(x) = \frac{\alpha(x)}{2} u''(x), \tag{4.4}$$

with

$$\alpha(x) := T(e_2)(x) - x^2 \tag{4.5}$$

and $e_2(x) := x^2$ $(0 \leqslant x \leqslant 1)$.

In particular, if T_1 is the positive projection on $\mathscr{C}([0,1])$ defined by (3.3), then

$$W_{T_1}(u)(x) = \frac{x(1-x)}{2} u''(x) \quad \bigl(u \in \mathscr{C}^2\bigl([0,1]\bigr),\ 0 \leqslant x \leqslant 1\bigr). \tag{4.6}$$

The differential operator W_{T_1} was intensively studied by Altomare (see [3,6]) and typically occurs in a one-dimensional diffusion model arising from population dynamics (see [6, Section 6.3.4]).

Coming back to the general case, since $e_2 \leqslant e_1$ on $[0,1]$, from (4.3) and the Hölder inequality, it follows that

$$0 \leqslant \alpha(x) \leqslant x(1-x) \quad (0 \leqslant x \leqslant 1). \tag{4.7}$$

By Theorem 2.1, $\partial_T([0,1]) = \{x \in [0,1] \mid T(e_2)(x) = x^2\}$ and hence, if $\partial_T([0,1]) = \{0,1\}$, then

$$0 < \alpha(x) \quad (0 < x < 1). \tag{4.8}$$

Moreover, $\alpha \in \mathscr{C}([0,1])$ and $\alpha(0) = \alpha(1) = 0$.

Conversely, if $\alpha \in \mathscr{C}([0,1])$ satisfies (4.7), then there always exists a Markov operator T on $\mathscr{C}([0,1])$ satisfying (3.1) such that $\alpha = T(e_2) - e_2$.

Indeed, it is enough to consider the function $\lambda :]0,1[\longrightarrow \mathbf{R}$ defined by

$$\lambda(x) := \frac{\alpha(x)}{x(1-x)} \quad (0 < x < 1) \tag{4.9}$$

and the Markov operator $T : \mathscr{C}([0,1]) \longrightarrow \mathscr{C}([0,1])$ defined by setting, for every $f \in \mathscr{C}([0,1])$,

$$T(f)(x) := \begin{cases} \lambda(x)T_1(f)(x) + (1-\lambda(x))f(x) & \text{if } 0 < x < 1; \\ f(x) & \text{if } x = 0,1 \end{cases} \tag{4.10}$$

(see (3.3)).

2. Consider the d-dimensional simplex

$$K_d := \left\{ (x_1,\dots,x_d) \in \mathbf{R}^d \;\middle|\; x_i \geqslant 0 \text{ for every } i = 1,\dots,d \text{ and } \sum_{i=1}^d x_i \leqslant 1 \right\} \tag{4.11}$$

and the projection T_d on the K_d defined by

$$T_d(f)(x) := \left(1 - \sum_{i=1}^d x_i\right) f(v_0) + \sum_{i=1}^d x_i f(v_i) \tag{4.12}$$

$(f \in \mathscr{C}(K_d),\ x = (x_1,\dots,x_d) \in K_d)$, where

$$v_0 := (0,\dots,0), \qquad v_1 := (1,0,\dots,0), \quad \dots, \quad v_d := (0,\dots,0,1)$$

are the vertices of the simplex. Then the differential operator W_{T_d} associated with T_d is given by

$$\begin{aligned} W_{T_d}(u)(x) &= \frac{1}{2}\sum_{i,j=1}^d x_i(\delta_{ij} - x_j)\frac{\partial^2 u}{\partial x_i \partial x_j}(x) \\ &= \frac{1}{2}\sum_{i=1}^d x_i(1-x_i)\frac{\partial^2 u}{\partial x_i^2}(x) - 2\sum_{1 \leqslant i < j \leqslant d} x_i x_j \frac{\partial^2 u}{\partial x_i \partial x_j}(x) \end{aligned} \tag{4.13}$$

$(u \in \mathscr{C}^2(K_d),\ x = (x_1,\dots,x_d) \in K_d)$; δ_{ij} stands for the Kronecker symbol.

The operator (4.13) falls into the class of the so-called Fleming–Viot operators, that were introduced by Feller in [19] in the description of a stochastic process associated with a diffusion approximation of a gene frequency model in population genetics; more recently, they have been object of investigation by several authors (see, e.g., [1,16,18] and the references quoted therein).

The coefficients of W_{T_d} vanish on the vertices of the simplex. Below we describe other differential operators with different degeneracies.

Let $S : \mathscr{C}(K_d) \longrightarrow \mathscr{C}(K_d)$ be the Markov operator defined by

$$S(f)(x) := \begin{cases} (1 - \frac{x_1}{1-\sum_{i=2}^d x_i}) f(0, x_2, \dots, x_d) & \\ \quad + \frac{x_1}{1-\sum_{i=2}^d x_i} f(1 - \sum_{i=2}^d x_i, x_2, \dots, x_d) & \text{if } \sum_{i=2}^d x_i \neq 1; \\ f(0, x_2, \dots, x_d) & \text{if } \sum_{i=2}^d x_i = 1 \end{cases} \tag{4.14}$$

$(f \in \mathscr{C}(K_d),\ x = (x_1, \dots, x_d) \in K_d)$.

Then

$$S(pr_1 pr_j) = \begin{cases} (1 - \sum_{i=2}^d pr_i) pr_1 & \text{if } j = 1; \\ pr_1 pr_j & \text{if } 1 < j \leqslant d \end{cases} \tag{4.15}$$

and $S(pr_i pr_j) = pr_i pr_j$ for every $1 < i \leqslant j \leqslant d$.

Therefore the differential operator associated with S is given by

$$W_S(u)(x) = \frac{1}{2} x_1 \left(1 - \sum_{i=1}^d x_i\right) \frac{\partial^2 u}{\partial x_1^2}(x) \tag{4.16}$$

$(u \in \mathscr{C}^2(K_d),\ x = (x_1, \dots, x_d) \in K_d)$.

Thus, W_S degenerates on the faces $\{x = (x_1, \dots, x_d) \in K_d \mid x_1 = 0\}$ and $\{x = (x_1, \dots, x_d) \in K_d \mid \sum_{i=1}^d x_i = 1\}$.

Finally, note that the differential operator associated with the Markov operator $V := \frac{T_d + S}{2}$ is given by

$$\begin{aligned} W_V(u)(x) = \frac{1}{4} \Bigg(& \left(2x_1(1 - x_1) - x_1 \sum_{i=2}^d x_i\right) \frac{\partial^2 u}{\partial x_1^2}(x) \\ & + \sum_{i=2}^d x_i(1 - x_i) \frac{\partial^2 u}{\partial x_i^2}(x) - \sum_{1 \leqslant i < j \leqslant d} x_i x_j \frac{\partial^2 u}{\partial x_i \partial x_j}(x) \Bigg) \end{aligned} \tag{4.17}$$

$(u \in \mathscr{C}^2(K_d),\ x = (x_1, \dots, x_d) \in K_d)$.

Therefore, W_V degenerates on the vertices of K_d as well.

3. Let $Q_d := [0, 1]^d$, $d \geqslant 1$, and for every $i = 1, \dots, d$ consider a Markov operator U_i on $\mathscr{C}([0, 1])$ satisfying (4.3).

If $U := \bigotimes_{i=1}^d U_i$ is the tensor product of the family $(U_i)_{1\leqslant i\leqslant d}$ (see [6, pp. 32–36]), then U is a Markov operator on $\mathscr{C}(Q_d)$ satisfying (3.1) and

$$W_U(u)(x) = \frac{1}{2}\sum_{i=1}^{d} \alpha_i(x)\frac{\partial^2 u}{\partial x_i^2}(x), \tag{4.18}$$

$(u \in \mathscr{C}^2(Q_d),\ x = (x_1,\dots,x_d) \in Q_d)$, where

$$\alpha_i(x) := U_i(e_2)(x_i) - x_i^2 \quad (1 \leqslant i \leqslant d). \tag{4.19}$$

Moreover,

$$0 \leqslant \alpha_i(x) \leqslant x_i(1-x_i) \quad \bigl(1 \leqslant i \leqslant d,\ x = (x_1\dots,x_d) \in Q_d\bigr). \tag{4.20}$$

Conversely, on account of Example 4.1, 1), if $(\alpha_i)_{1\leqslant i\leqslant d}$ is a family of continuous functions on $[0,1]$ satisfying (4.20), then there always exists a Markov operator U on $\mathscr{C}(Q_d)$ satisfying (3.1) whose differential operator satisfies (4.18).

Finally note that, if $U_i = T_1$ for every $i = 1,\dots,d$ (see (3.3)), then

$$W_U(u)(x) = \frac{1}{2}\sum_{i=1}^{d} x_i(1-x_i)\frac{\partial^2 u}{\partial x_i^2}(x) \tag{4.21}$$

$(u \in \mathscr{C}^2(Q_d),\ x = (x_1,\dots,x_d) \in Q_d)$.

Coming back to the general definition (4.1), in the next result we prove that the differential operator W_T is strictly connected with the Bernstein–Schnabl operators B_n (see (3.2)), since it occurs in an asymptotic formula involving the sequence $(B_n)_{n\geqslant 1}$.

Theorem 4.2. *For every $u \in \mathscr{C}^2(K)$,*

$$\lim_{n\to\infty} n\bigl(B_n(u) - u\bigr) = W_T(u) \quad \textit{uniformly on } K. \tag{4.22}$$

Proof. The proof consists in applying Theorem 2.3; in particular, we keep the same notation as there.

For any $n \geqslant 1$, $i = 1,\dots,d$ and $x = (x_1,\dots,x_d) \in K$, we have that $B_n(\mathbf{1}) = \mathbf{1}$ and, by this last equality and (3.4), $B_n(pr_i \circ \Psi_x)(x) = 0$, which shows conditions (i) and (ii) of Theorem 2.3 with $\gamma = 0$ and $\beta_i = 0$.

On the other hand, by applying (3.5) and (3.1), for $i,j = 1,\dots,d$,

$$\begin{aligned} B_n\bigl((pr_i \circ \Psi_x)(pr_j \circ \Psi_x)\bigr)(x) &= B_n\bigl((pr_i - x_i)(pr_j - x_j)\bigr)(x) \\ &= \frac{1}{n}T\bigl((pr_i - x_i)(pr_j - x_j)\bigr)(x) = \frac{1}{n}\alpha_{ij}(x) \end{aligned}$$

and, from this, condition (iii) of Theorem 2.3 holds true.

Similar reasoning shows condition (iv) in Theorem 2.3, since

$$B_n\big(\Phi_x^2\big)(x) = B_n\left(\sum_{i=1}^{d}(pr_i - x_i)^2\right)(x)$$
$$= \frac{1}{n}T\left(\sum_{i=1}^{d}(pr_i - x_i)^2\right)(x) = \frac{1}{n}\big[T(e)(x) - e(x)\big];$$

here Φ_x is defined by (2.10) and $e(x) := \sum_{i=1}^{d} x_i^2$.

Finally, by (3.14), we get

$$nB_n\big(\Phi_x^4\big)(x) = \sum_{i,j=1}^{d} nB_n\big((pr_i - x_i)^2(pr_j - x_j)^2\big)(x)$$
$$= \frac{1}{n^2}\sum_{i,j=1}^{d}\big[T\big((pr_i - x_i)^2(pr_j - x_j)^2\big)(x)$$
$$+ (n-1)T\big((pr_i - x_i)^2\big)(x)T\big((pr_j - x_j)^2\big)(x)$$
$$+ 2(n-1)T\big((pr_i - x_i)(pr_j - x_j)\big)^2(x)\big].$$

Therefore, there exists a suitable positive constant M, depending on K only, such that

$$nB_n\big(\Phi_x^4\big)(x) \leqslant \frac{3n-2}{n^2}M \quad \text{for every } x \in K,$$

which implies condition (v) of Theorem 2.3, for $q = 4$. This completes the proof. □

Finally, the next result allows us to show, not only that the operator $(W_T, \mathscr{C}^2(K))$ (pre)-generates a Markov semigroup on $\mathscr{C}(K)$, but also to determine an approximation formula for such a semigroup by means of suitable iterates of the Bernstein–Schnabl operators associated with T.

Before stating it, we recall that a *core* for a linear operator $A : D(A) \longrightarrow \mathscr{C}(K)$ is a linear subspace D_0 of $D(A)$ which is dense in $D(A)$ with respect to the graph norm $\|u\|_A := \|A(u)\|_\infty + \|u\|_\infty$ $(u \in D(A))$.

Theorem 4.3. *Let K be a convex compact subset of $\mathbf{R}^d$, $d \geqslant 1$, having non-empty interior and consider a Markov operator T on $\mathscr{C}(K)$ satisfying (3.1). Furthermore, assume that*

$$T\big(P_m(K)\big) \subset P_m(K) \quad \textit{for every } m \geqslant 2. \tag{4.23}$$

Then the operator $(W_T, \mathscr{C}^2(K))$ is closable and its closure $(A_T, D(A_T))$ generates a Markov semigroup $(T(t))_{t\geqslant 0}$ on $\mathscr{C}(K)$ such that, if $t \geqslant 0$ and $(k(n))_{n\geqslant 1}$ is a sequence of *positive integers satisfying* $\lim_{n\to\infty} \frac{k(n)}{n} = t$,

$$T(t)(f) = \lim_{n\to\infty} B_n^{k(n)}(f) \quad \textit{uniformly on } K \tag{4.24}$$

for every $f \in \mathscr{C}(K)$.

Moreover, $P_\infty(K)$, *and hence* $\mathscr{C}^2(K)$ *as well, is a core for* $(A_T, D(A_T))$.

Finally, for every $t \geqslant 0$ *and* $m \geqslant 1$,

$$T(t)\big(P_m(K)\big) \subset P_m(K) \tag{4.25}$$

and, if $t \geqslant 0$ *and* $f \in \mathscr{C}(K)$,

$$T(t)(f) = f \quad \textit{on } \partial_T K. \tag{4.26}$$

Proof. Let $(B, D(B))$ be the linear operator defined by

$$B(u) := \lim_{n\to\infty} n\big(B_n(u) - u\big)$$

for every $u \in D(B)$, where

$$D(B) := \Big\{v \in \mathscr{C}(K) \;\Big|\; \text{there exists } \lim_{n\to\infty} n\big(B_n(v) - v\big) \text{ uniformly on } K\Big\}.$$

From Theorem 4.2 we get that $\mathscr{C}^2(K)$ (and, in particular, $P_\infty(K)$) is contained in $D(B)$ and $B = W_T$ on $\mathscr{C}^2(K)$. Moreover, each $P_m(K)$, $m \geqslant 1$, is closed and invariant under every operator B_n by virtue of Proposition 3.2 and assumptions (3.1) and (4.23). Because of [6, Theorem 1.6.8], $(B, D(B))$ is closable and its closure $(\overline{B}, D(\overline{B}))$ generates a contraction C_0-semigroup $(T(t))_{t\geqslant 0}$ on $\mathscr{C}(K)$ satisfying (4.24). Since $(B, D(B))$ is a closable extension of $(W_T, \mathscr{C}^2(K))$, then $(W_T, \mathscr{C}^2(K))$ is closable as well and, denoting its closure by $(A_T, D(A_T))$, we get $D(A_T) \subset D(\overline{B})$ and $\overline{B} = A_T$ on $D(A_T)$.

From (3.12), (3.1) and (4.24) it also follows that $T(t)(P_m(K)) \subset P_m(K)$ for every $t \geqslant 0$ and hence $P_\infty(K)$ is invariant under $(T(t))_{t\geqslant 0}$. By [17, Chapter II, Proposition 1.7], we infer that $P_\infty(K)$ is a core for $(\overline{B}, D(\overline{B}))$. In particular, it turns out that $\mathscr{C}^2(K)$ is a core for $(\overline{B}, D(\overline{B}))$. On the other hand, $\mathscr{C}^2(K)$ is a core for $(A_T, D(A_T))$, so that $(A_T, D(A_T)) = (\overline{B}, D(\overline{B}))$.

Finally, (4.25) is a consequence of (4.23) and Proposition 3.2, and (4.26) follows at once from (3.8). □

An inspection of the proof of Theorem 4.3 shows that if $u, v \in \mathscr{C}(K)$ and $\lim_{n\to\infty} n(B_n(u) - u) = v$ uniformly on K, then $u \in D(A_T)$ and $A_T(u) = v$.

As a consequence, we get the following “small” saturation result for Bernstein–Schnabl operators.

Corollary 4.4. *Under the same assumptions of Theorem 4.3, if* $u \in \mathscr{C}(K)$ *and* $\lim_{n\to\infty} n(B_n(u) - u) = 0$ *uniformly on* K, *then* $u \in D(A_T)$ *and* $A_T(u) = 0$.

Below we show some examples of differential operators (4.1) satisfying the assumptions of Theorem 4.3.

Examples 4.5. 1. Consider the Markov operators T_d and S on the d-dimensional simplex K_d and their convex combination $V := \frac{T_d+S}{2}$ as in Example 4.1, 2). Clearly, the operator T_d satisfies (3.1) and (4.23) (note that $T_d(\mathscr{C}(K_d)) \subset P_1(K_d)$). Similarly, S (and hence V) verifies the same properties since, if $m_1, \dots, m_d$ are positive integers, then

$$S\big(pr_1^{m_1} \cdots pr_d^{m_d}\big) = \begin{cases} pr_2^{m_2} \cdots pr_d^{m_d} & \text{if } m_1 = 0; \\ (\mathbf{1} - \sum_{i=2}^{d} pr_i)^{m_1-1} pr_1 pr_2^{m_2} \cdots pr_d^{m_d} & \text{if } m_1 \geqslant 1. \end{cases}$$

Therefore, Theorem 4.3 and Corollary 4.4 apply to the differential operators (4.13), (4.16) and (4.17).

2. Consider a family $(U_i)_{1 \leqslant i \leqslant d}$ of Markov operators on $\mathscr{C}([0,1])$ satisfying (4.3) and (4.23). Then the tensor product $U := \bigotimes_{i=1}^{d} U_i$ on $\mathscr{C}(Q_d)$ (see Example 4.1, 3)) verifies (4.23) (and (3.1)) as well because

$$U\big(pr_1^{m_1} \cdots pr_d^{m_d}\big) = \big(U_1\big(e_1^{m_1}\big) \circ pr_1\big) \cdots \big(U_d\big(e_1^{m_d}\big) \circ pr_d\big)$$

for every positive integers $m_1, \dots, m_d$.

Therefore, Theorem 4.3 applies to the differential operator (4.18).

Remark 4.6. Consider the abstract Cauchy problem

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) = A_T\big(u(\cdot,t)\big)(x) & x \in K,\ t \geqslant 0, \\ u(x,0) = u_0(x) & u_0 \in D(A_T),\ x \in K. \end{cases} \tag{4.27}$$

It is well-known that, since $(A_T, D(A_T))$ generates a Markov semigroup, (4.27) admits a unique solution $u : K \times [0, +\infty[\longrightarrow \mathbf{R}$ given by $u(x,t) = T(t)(u_0)(x)$ for every $x \in K$ and $t \geqslant 0$ (see, e.g., [21, Chapter A-II]). Hence, by Theorem 4.3, we can approximate such a solution in terms of iterates of Bernstein–Schnabl operators, namely

$$u(x,t) = T(t)(u_0)(x) = \lim_{n \to \infty} B_n^{k(n)}(u_0)(x), \tag{4.28}$$

where $(k(n))_{n \geqslant 1}$ is a sequence of positive integers satisfying $\lim_{n\to\infty} \frac{k(n)}{n} = t$, and the limit is uniform with respect to $x \in K$.

We recall that A_T coincides with W_T on $\mathscr{C}^2(K)$; therefore, if $u_0 \in P_m(K)$ $(m \geqslant 1)$ then $u(x,t)$ is the unique solution to the Cauchy problem

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) = \dfrac{1}{2} \displaystyle\sum_{i,j=1}^{d} \alpha_{ij}(x) \dfrac{\partial^2 u}{\partial x_i \partial x_j}(x,t) & x \in K,\ t \geqslant 0, \\ u(x,0) = u_0(x) & x \in K \end{cases}$$

(see (4.2)), and

$$u(\cdot,t) \in P_m(K) \quad \text{for every } t \geqslant 0. \tag{4.29}$$

Formula (4.28) can be fruitfully employed in order to obtain other "spatial regularity properties" of the solutions $u(\cdot,t)$ of (4.27) (as in (4.29)), by deducing them from the corresponding ones held by the B_n's.

As an example, we preliminarily prove that Bernstein–Schnabl operators preserve Hölder-continuity and, to this end, we need to fix some further notation.

First of all, given $M \geqslant 0$ and $0 < \alpha \leqslant 1$, we denote by

$$\operatorname{Lip}(M,\alpha) := \bigl\{ f \in \mathscr{C}(K) \bigm| \bigl|f(x) - f(y)\bigr| \leqslant M\|x-y\|^{\alpha} \text{ for every } x, y \in K \bigr\}$$

the space of all *Hölder continuous functions with exponent α and Hölder constant M on K*; here $\|\cdot\|$ stands for an arbitrary norm on $\mathbf{R}^d$. Clearly, $\operatorname{Lip}(M,1)$ is the space of all *Lipschitz-continuous functions with Lipschitz constant M on K*.

Moreover, we consider a Markov operator T on $\mathscr{C}(K)$ (not necessarily satisfying (3.1)) and we assume that

$$T\bigl(\operatorname{Lip}(1,1)\bigr) \subset \operatorname{Lip}(1,1), \tag{4.30}$$

or, equivalently,

$$T\bigl(\operatorname{Lip}(M,1)\bigr) \subset \operatorname{Lip}(M,1) \tag{4.31}$$

for every $M \geqslant 0$.

For any $n \geqslant 2$, $f \in \mathscr{C}(K)$ and $x_1, \dots, x_{n-1} \in K$, consider the function $f_{x_1,\dots,x_{n-1}} : K \longrightarrow \mathbf{R}$ defined by

$$f_{x_1,\dots,x_{n-1}}(t) := f\left(\frac{x_1 + \dots + x_{n-1} + t}{n}\right) \quad (t \in K). \tag{4.32}$$

Furthermore, for every $x \in K$, consider the function

$$f^x_{x_1,\dots,x_{n-2}}(t) := T(f_{x_1,\dots,x_{n-2},t})(x) \quad (t \in K) \tag{4.33}$$

and, for every $k = 3, \dots, n-1$, define recursively the functions $f^x_{x_1,\dots,x_{n-k}} : K \longrightarrow \mathbf{R}$ by setting

$$f^x_{x_1,\dots,x_{n-k}}(t) := T\bigl(f^x_{x_1,\dots,x_{n-k},t}\bigr)(x) \quad (t \in K). \tag{4.34}$$

Finally, set

$$f^x(t) = T\bigl(f^x_t\bigr)(x) \quad (t \in K). \tag{4.35}$$

After these preliminaries, we prove that the sequence $(B_n)_{n\geqslant 1}$ preserves Hölder-continuous functions.

Theorem 4.7. *Under assumption (4.30), for any $n \geqslant 1$,*

$$B_n\bigl(\mathrm{Lip}(M,\alpha)\bigr) \subset \mathrm{Lip}(M,\alpha) \tag{4.36}$$

for every $n \geqslant 1$, $M \geqslant 0$ and $\alpha \in]0,1]$.

Proof. To prove (4.7), by virtue of [15, Corollary 7], it is enough to show that, for every $M \geqslant 0$,

$$B_n\bigl(\mathrm{Lip}(M,1)\bigr) \subset \mathrm{Lip}(M,1). \tag{1}$$

For $n = 1$, we have that $B_1 = T$, so that (1) is a consequence of (4.31).

Fix $n \geqslant 2$, $f \in \mathrm{Lip}(M,1)$ for some $M \geqslant 0$, $x \in K$ and consider the functions defined by (4.32)–(4.35). By finite induction, it is easy to prove that

$$\bigl\| f^x_{x_1,\dots,x_{k-1},u} - f^x_{x_1,\dots,x_{k-1},v}\bigr\|_\infty \leqslant \frac{M}{n}\|u-v\|$$

for any $u, v \in K$ and $k = 1,\dots,n-1$; hence, $f^x_{x_1,\dots,x_k} \in \mathrm{Lip}(M/n,1)$ and $f^x \in \mathrm{Lip}(M/n,1)$.

Moreover, for every $y \in K$ and $k = 1,\dots,n-1$, from (4.31) it follows that

$$T\bigl(f^y_{x_1,\dots,x_k}\bigr)(x) \leqslant T\bigl(f^y_{x_1,\dots,x_k}\bigr)(y) + \frac{M}{n}\|x-y\|$$

and

$$T\bigl(f^y\bigr)(x) \leqslant T\bigl(f^y\bigr)(y) + \frac{M}{n}\|x-y\|.$$

Since $f^y_{x_1,\dots,x_{n-1}} = f^x_{x_1,\dots,x_{n-1}}$, we obtain

$$\begin{aligned}
B_n(f)(x) &= \int_K \cdots \int_K f\left(\frac{x_1+\cdots+x_n}{n}\right) d\tilde{\mu}^T_x(x_1)\cdots d\tilde{\mu}^T_x(x_n) \\
&= \int_K \cdots \int_K T\bigl(f^x_{x_1,\dots,x_{n-1}}\bigr)(x)\, d\tilde{\mu}^T_x(x_1)\cdots d\tilde{\mu}^T_x(x_{n-1}) \\
&= \int_K \cdots \int_K T\bigl(f^y_{x_1,\dots,x_{n-1}}\bigr)(x)\, d\tilde{\mu}^T_x(x_1)\cdots d\tilde{\mu}^T_x(x_{n-1}) \\
&\leqslant \int_K \cdots \int_K T\bigl(f^y_{x_1,\dots,x_{n-1}}\bigr)(y)\, d\tilde{\mu}^T_x(x_1)\cdots d\tilde{\mu}^T_x(x_{n-1}) + \frac{M}{n}\|x-y\|
\end{aligned}$$

$$
\begin{aligned}
&\leqslant \cdots \leqslant \int_K T\big(f^y_{x_1}\big)(y)\, d\tilde{\mu}^T_x(x_1) + \frac{(n-1)M}{n}\|x-y\| \\
&= \int_K f^y(x_1)\, d\tilde{\mu}^T_x(x_1) + \frac{(n-1)M}{n}\|x-y\| \\
&= T\big(f^y\big)(x) + \frac{(n-1)M}{n}\|x-y\| \leqslant T\big(f^y\big)(y) + M\|x-y\|.
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
B_n(f)(y) &= \int_K \cdots \int_K f\left(\frac{x_1+\cdots+x_n}{n}\right) d\tilde{\mu}^T_y(x_1)\cdots d\tilde{\mu}^T_y(x_n) \\
&= \int_K \cdots \int_K T\big(f^y_{x_1,\ldots,x_{n-1}}\big)(y)\, d\tilde{\mu}^T_y(x_1)\cdots d\tilde{\mu}^T_y(x_{n-1}) \\
&= \cdots = \int_K T\big(f^y_{x_1}\big)(y)\, d\tilde{\mu}^T_y(x_1) = T\big(f^y\big)(y).
\end{aligned}
$$

Accordingly,

$$
\big|B_n(f)(x) - B_n(f)(y)\big| \leqslant M\|x-y\|
$$

and this completes the proof. □

From the previous result we infer that if, in addition, T satisfies (3.1) and (4.23), then the Markov semigroup $(T(t))_{t\geqslant 0}$ (see Theorem 4.3), on account of (4.24), preserves Hölder continuous functions, as the following corollary shows.

Corollary 4.8. *If T is a Markov operator on $\mathscr{C}(K)$ satisfying (3.1), (4.23) and (4.30), then*

$$
T(t)\big(\mathrm{Lip}(M,\alpha)\big) \subset \mathrm{Lip}(M,\alpha) \tag{4.37}
$$

for every $t \geqslant 0$, $M \geqslant 0$ and $\alpha \in\,]0,1]$.

Hence, if $u_0 \in D(A_T) \cap \mathrm{Lip}(M,\alpha)$, for some $M \geqslant 0$ and $\alpha \in\,]0,1]$, and if we consider the abstract Cauchy problem (4.27) with initial datum u_0, then, because of (4.28), we have that the solution $u(\cdot,t)$ of (4.27) belong to $\mathrm{Lip}(M,\alpha)$ for every $t \geqslant 0$.

In the same spirit of the preceding remarks further properties of the semigroup $(T(t))_{t\geqslant 0}$ will be investigated in [14].

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