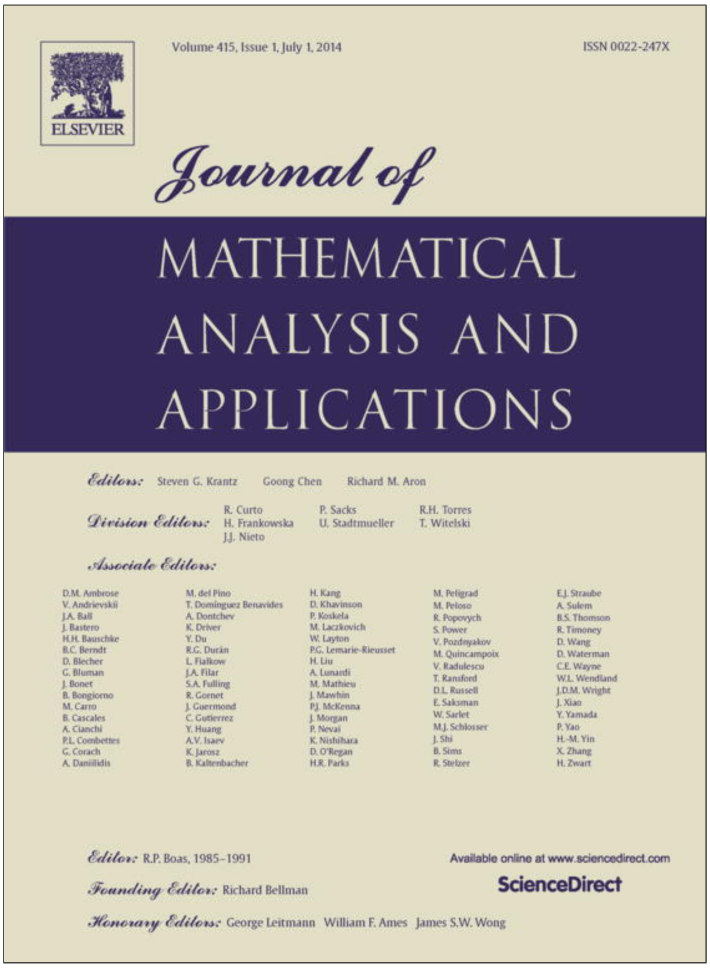
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On Markov operators preserving polynomials

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ABSTRACT

The paper is concerned with a special class of positive linear operators acting on the space $C(K)$ of all continuous functions defined on a convex compact subset K of $\mathbf{R}^d$, $d \geqslant 1$, having non-empty interior. Actually, this class consists of all positive linear operators T on $C(K)$ which leave invariant the polynomials of degree at most 1 and which, in addition, map polynomials into polynomials of the same degree. Among other things, we discuss the existence of such operators in the special case where K is strictly convex by also characterizing them within the class of positive projections. In particular we show that such operators exist if and only if ∂K is an ellipsoid. Furthermore, a characterization of balls of $\mathbf{R}^d$ in terms of a special class of them is furnished. Additional results and illustrative examples are presented as well.

0. Introduction

The paper is concerned with a special class of positive linear operators acting on the space $C(K)$ of all continuous functions defined on a convex compact subset K of $\mathbf{R}^d$, $d \geqslant 1$, having non-empty interior. Actually, this class consists of all positive linear operators T on $C(K)$ which leave invariant the polynomials of degree at most 1 and which, in addition, map polynomials into polynomials of the same degree.

The interest for such operators comes from the study of a special differential operator $(W_T, C^2(K))$ which we carefully investigated in [6] and which is defined as

$$W_T(u) := \frac{1}{2}\sum_{i,j=1}^{d} \alpha_{ij}\frac{\partial^2 u}{\partial x_i\,\partial x_j}$$

$(u \in C^2(K))$, where $\alpha_{ij} := T(pr_i pr_j) - pr_i pr_j$ $(i, j = 1, \dots, d)$ and each pr_i denotes the i-th coordinate function on K.

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The differential operator W_T is elliptic and it degenerates on a subset of K which contains the set of the extreme points $\partial_e K$ of K. In [6] we showed that, if T maps polynomials into polynomials of the same degree, then $(W_T, C^2(K))$ is closable in $C(K)$ and its closure generates a Markov semigroup on $C(K)$ which can be represented as a limit of suitable iterates of particular positive linear operators associated with T, namely the Bernstein–Schnabl operators associated with T, which have been deeply investigated in [5] and, more recently, in [6] and in the forthcoming monograph [7].

The main aim of the paper is to look more closely at this preservation property which seems to have an independent own interest. Among other things, we discuss the existence of such operators in the special case where K is strictly convex, i.e., $\partial_e K = \partial K$, by also characterizing them within the class of positive projections on $C(K)$ (for the bi-dimensional case see [11]). In particular we show that such operators exist if and only if ∂K is an ellipsoid. Furthermore, a characterization of balls of $\mathbf{R}^d$ in terms of a special class of them is furnished. Illustrative examples and additional results involving the tensor products and the convex convolution products of positive linear operators are presented as well.

1. Preliminaries on positive linear operators

Throughout this paper K will be a convex compact subset of $\mathbf{R}^d$, $d \geqslant 1$, with non-empty interior $\mathrm{int}(K)$. As usual we denote by $C(K)$ the space of all real-valued continuous functions on K and by $C^2(K)$ the space of all real-valued continuous functions on K that are twice continuously differentiable on $\mathrm{int}(K)$ and whose partial derivatives up to the order two can be continuously extended to K. For $u \in C^2(K)$ and $i, j = 1, \ldots, d$, we shall continue to denote by $\frac{\partial u}{\partial x_i}$ and $\frac{\partial^2 u}{\partial x_i \partial x_j}$ the continuous extensions to K of the partial derivatives $\frac{\partial u}{\partial x_i}$ and $\frac{\partial^2 u}{\partial x_i\, \partial x_j}$. The space $C(K)$, endowed with the supremum norm $\|f\|_\infty := \sup_{x\in K} |f(x)|$ $(f \in C(K))$ and the natural (pointwise) ordering, is a Banach lattice.

We also denote by $\mathbf{1}$ the constant function of constant value 1 on K and, for every $i \in \{1, \ldots, d\}$, by pr_i the i-th coordinate function on K, i.e., $pr_i(x) = x_i$ for every $x = (x_i)_{1\leqslant i\leqslant d} \in K$.

Let B_K be the σ-algebra of all Borel subsets of K and denote by $M^+(K)$ (resp., $M_1^+(K)$) the subset of all Borel measures (resp., the subset of all probability Borel measures) on K. In particular, for every $x \in K$, the symbol ε_x stands for the unit mass concentrated at x, i.e., for every $B \in B_K$,

$$\varepsilon_x(B) := \begin{cases} 1 & \text{if } x \in B; \\ 0 & \text{if } x \notin B. \end{cases}$$

If $\tilde{\mu} \in M^+(K)$, then $Supp(\tilde{\mu})$ denotes the support of $\tilde{\mu}$, i.e., the complement of the largest open subset of K having measure zero with respect to $\tilde{\mu}$.

Given a *Markov operator* $T : C(K) \to C(K)$, i.e., a positive linear operator such that $T(\mathbf{1}) = \mathbf{1}$, by the Riesz representation theorem there exists a unique family $(\tilde{\mu}_x^T)_{x\in K}$ in $M_1^+(K)$ such that

$$T(f)(x) = \int_K f \, d\tilde{\mu}_x^T \quad \bigl(f \in C(K),\ x \in K\bigr). \tag{1.1}$$

Such a family is said to be the *continuous selection of probability Borel measures on K associated with T*. By means of $(\tilde{\mu}_x^T)_{x\in K}$ we can construct the so-called *Bernstein–Schnabl operators associated with T* which are defined by setting, for every $n \geqslant 1$, $x \in K$ and $f \in C(K)$,

$$B_n(f)(x) = \int_K \cdots \int_K f\left(\frac{x_1 + \cdots + x_n}{n}\right) d\tilde{\mu}_x^T(x_1) \cdots d\tilde{\mu}_x^T(x_n). \tag{1.2}$$

Note that by the continuity property of the product measure it follows that $B_n(f) \in C(K)$. Moreover, $B_1 = T$.

For a comprehensive survey on these operators (including noteworthy examples), we refer to [5, Chapter 6] and to the references contained in the relevant notes. More recent results can be also found in [3,6–10,12].

Here we only point out that, if in addition the Markov operator T satisfies

$$T(h) = h \quad \text{for every } h \in \{\mathbf{1}, pr_1, \dots, pr_d\}, \tag{1.3}$$

then the sequence $(B_n)_{n\geqslant 1}$ is a positive approximation process in $C(K)$, that is

$$\lim_{n\to\infty} B_n(f) = f \quad \text{uniformly on } K \text{ for every } f \in C(K). \tag{1.4}$$

Another useful tool that we shall use in the sequel is the notion of Choquet boundary.

Given a linear subspace H of $C(K)$, the *Choquet boundary* of H is the subset of all points $x \in K$ such that, if $\tilde{\mu} \in M^+(K)$ and $\int h\,d\tilde{\mu} = h(x)$ for every $h \in H$, then $\int f\,d\tilde{\mu} = f(x)$ for every $f \in C(K)$, i.e., $\tilde{\mu} = \varepsilon_x$. It will be denoted by $\partial_H K$. If H contains the constants and separates the points of K, then $\partial_H K$ is non-empty and every $h \in H$ attains its minimum and maximum on $\partial_H K$ (see, e.g., [5, Corollary 2.6.5]). Therefore, if $f, g \in H$ and if $f = g$ on $\partial_H K$, then $f = g$ on K.

An important example of Choquet boundary is the set $\partial_e K$ of the *extreme points* of K. They are defined as those points $x_0 \in K$ such that $K \setminus \{x_0\}$ is convex, i.e., if $x_1, x_2 \in K$ and $\lambda \in \mathbf{R}$, $0 < \lambda < 1$, and if $x_0 = \lambda x_1 + (1-\lambda)x_2$, then $x_0 = x_1 = x_2$.

Indeed, denote by $P_1(K)$ the space of (the restriction to K of) all polynomials of degree at most 1. Clearly, $P_1(K)$ contains the constants and separates the points of K. As a matter of fact, it turns out that

$$\partial_{P_1(K)} K = \partial_e K \tag{1.5}$$

(for a proof see, e.g., [5, Proposition 2.6.3]).

Now let us consider a Markov operator $T : C(K) \to C(K)$ and set

$$M := \bigl\{h \in C(K) \bigm| T(h) = h\bigr\}. \tag{1.6}$$

Clearly, M is contained in the range of T which will be denoted by

$$H := T\bigl(C(K)\bigr) = \bigl\{T(f) \bigm| f \in C(K)\bigr\}. \tag{1.7}$$

The subspace M contains the constants and hence, if it separates the points of K, its Choquet boundary $\partial_M K$ is non-empty.

In the sequel, the following subset

$$\partial_T K := \bigl\{x \in K \bigm| T(f)(x) = f(x) \text{ for every } f \in C(K)\bigr\} \tag{1.8}$$

will play an important role. Its elements are also called the *interpolation points* of the operator T.

The next result has been obtained in [4, Theorem 2.1].

Theorem 1.1. *Consider a Markov operator $T : C(K) \to C(K)$ such that the subspace M defined by (1.6) separates the points of K. Then*

$$\emptyset \neq \partial_M K \subset \partial_T K \subset \partial_H K \tag{1.9}$$

(*see* (1.7) *and* (1.8)).

Moreover, if V is an arbitrary subset of M separating the points of K, then

$$\partial_T K = \{x \in K \mid T(h^2)(x) = h^2(x) \text{ for every } h \in V\}. \tag{1.10}$$

Finally, if $(h_n)_{n\geqslant 1}$ is a finite or countable family of M, separating the points of K and such that the series $\Phi := \sum_{n=1}^{\infty} h_n^2$ is uniformly convergent, then $\Phi \leqslant T(\Phi)$ and

$$\partial_T K = \{x \in K \mid T(\Phi)(x) = \Phi(x)\}. \tag{1.11}$$

Remarks 1.2. 1. There always exists a sequence $(h_n)_{n\geqslant 1}$ in M separating the points of K and such that the series $\sum_{n=1}^{\infty} h_n^2$ converges uniformly on K. Indeed, M is separable (because $C(K)$ is so) and hence there exists a countable dense family $(\varphi_n)_{n\geqslant 1}$ of M which separates the points of K. Then, it is sufficient to set $h_n := 2^{-n}\varphi_n(\|\varphi_n\|_\infty + 1)^{-1}$ $(n \geqslant 1)$.

2. Whenever T satisfies condition (1.3), i.e., $P_1(K) \subset M$, then, from Theorem 1.1 and (1.5), it follows that

$$\partial_e K \subset \partial_M K \subset \partial_T K \subset \partial_H K. \tag{1.12}$$

Below we discuss some cases where the inclusions in (1.9) are equalities (see [4, Proposition 2.4 and Theorem 1.1] and [5, Remark 3 to Theorem 3.3.3]). To this end, we recall that a Markov operator T on $C(K)$ is said to be a *projection* if $T^2(f) = T(f)$ for every $f \in C(K)$.

Proposition 1.3. *Under the same assumptions of Theorem 1.1, the following statements are equivalent*:

(a) *There exists a subset V of M separating the points of K such that $T^2(h^2) = T(h^2)$ for every $h \in V$, i.e., $T(V^2) \subset M$.*
(b) *T is a projection.*
(c) *There exists a finite or countable family $(h_n)_{n\geqslant 1}$ in M separating the points of K such that the series $\Phi := \sum_{i=1}^{\infty} h_n^2$ is uniformly convergent and $T^2(\Phi) = T(\Phi)$.*

Finally, if (a), (b) *or* (c) *holds true, then $M = H$ (see* (1.7)*) and hence $\partial_M K = \partial_T K = \partial_H K$. Moreover, for every $x \in K$,*

$$Supp(\tilde{\mu}_x^T) \subset \partial_T K = \partial_H K, \tag{1.13}$$

$\tilde{\mu}_x^T$ being defined by (1.1), *and hence, for every $f, g \in C(K)$,*

$$T(f) = T(g) \quad \text{provided } f = g \text{ on } \partial_H K. \tag{1.14}$$

Finally we recall that a simplex of $\mathbf{R}^d$ is the convex hull of some $d + 1$ affinely independent points (we recall that p points $x_1, \ldots, x_p \in \mathbf{R}^d$ are said to be *affinely independent* if for every $\lambda_1, \ldots, \lambda_p \in \mathbf{R}$ satisfying $\sum_{i=1}^{p} \lambda_i x_i = 0$ and $\sum_{i=1}^{p} \lambda_i = 0$, it turns out that $\lambda_1 = \cdots = \lambda_p = 0$).

Therefore, the subset

$$K_d := \left\{(x_1, \ldots, x_d) \in \mathbf{R}^d \;\middle|\; x_i \geqslant 0 \text{ for every } i = 1, \ldots, d \text{ and } \sum_{i=1}^{d} x_i \leqslant 1\right\}, \tag{1.15}$$

being the convex hull of $\{v_0, \ldots, v_d\}$, where

$$v_0 := (0, \ldots, 0), \quad v_1 := (1, 0, \ldots, 0), \quad \ldots, \quad v_d := (0, \ldots, 0, 1), \tag{1.16}$$

is a simplex in $\mathbf{R}^d$ and it is called the *canonical simplex* of $\mathbf{R}^d$. Note that, if such is the case, $\partial_e K_d = \{v_0, \ldots, v_d\}$.

According to the next theorem [5, Corollary 1.5.9], when K is a simplex then on $C(K)$ there exists a (unique) natural positive projection T on $C(K)$ such that $T(C(K)) = P_1(K)$.

Theorem 1.4. *Given a convex compact subset K of $\mathbf{R}^d$, $d \geqslant 1$, the following statements are equivalent*:

(a) *K is a simplex.*
(b) *For every $x \in K$ there exists a unique $\tilde{\mu}_x \in M_1^+(K)$ such that $\tilde{\mu}_x(K \setminus \overline{\partial_e K}) = 0$ and*

$$\int_K h \, d\tilde{\mu}_x = h(x) \quad \textit{for every } h \in P_1(K).$$

(c) *Every continuous function $f : \partial_e K \to \mathbf{R}$ can be continuously extended to a (unique) function $\tilde{f} \in P_1(K)$.*
(d) *There exists a (unique) positive projection $T : C(K) \to C(K)$ such that $T(C(K)) = P_1(K)$.*

Moreover, if one of these statements holds true, then for every $f \in C(K)$ and $x \in K$,

$$T(f)(x) = \int_K f \, d\tilde{\mu}_x = \widetilde{f|_{\partial_e K}}(x). \tag{1.17}$$

Given a simplex K of $\mathbf{R}^d$, the positive projection $T : C(K) \to C(K)$ given by (1.17) is referred to as the *canonical positive projection associated with K*. Thus, for every $f \in C(K)$, $T(f)$ is the unique continuous affine function on K that coincides with f on $\partial_e K$.

In the case $K = K_d$, $d \geqslant 1$, the canonical projection is given by

$$T_d(f)(x) := \left(1 - \sum_{i=1}^{d} x_i\right) f(v_0) + \sum_{i=1}^{d} x_i f(v_i) \tag{1.18}$$

($f \in C(K_d)$, $x = (x_1, \ldots, x_d) \in K_d$, $v_0, \ldots, v_d$ as in (1.16)).

In particular, for $d = 1$,

$$T_1(f)(x) := (1 - x) f(0) + x f(1) \tag{1.19}$$

($f \in C([0,1])$, $0 \leqslant x \leqslant 1$).

2. Differential operators associated with Markov operators

In this section we shall recall some results from [6] which will be useful in the sequel. Actually, given a Markov operator T on $C(K)$, we shall consider a second-order differential operator W_T on $C^2(K)$ which is strictly connected with the Bernstein–Schnabl operators B_n defined by (1.2). Under suitable assumptions on T, the operator $(W_T, C^2(K))$ is the (pre)-generator of a Markov semigroup on $C(K)$ which can be approximated by means of suitable iterates of the B_n's. Some examples will be also discussed.

From now on fix a Markov operator $T : C(K) \to C(K)$ satisfying (1.3), that is,

$$T(h) = h \quad \text{for every } h \in \{\mathbf{1}, pr_1, \ldots, pr_d\},$$

K being a convex compact subset $\mathbf{R}^d$, $d \geqslant 1$, whose interior is assumed to be non-empty.

For every $m \geqslant 1$, we denote by $P_m(K)$ the linear subspace of the (restrictions to K of) polynomials of degree no greater than m.

Clearly, $P_m(K) \subset P_{m+1}(K)$ and $P_\infty(K) := \bigcup_{m\geqslant 1} P_m(K)$ is a subalgebra of $C(K)$ which separates the points of K and which is dense in $C(K)$.

Now consider the differential operator $W_T : C^2(K) \to C(K)$ defined by

$$W_T(u) := \frac{1}{2}\sum_{i,j=1}^{d} \alpha_{ij} \frac{\partial^2 u}{\partial x_i\, \partial x_j} \tag{2.1}$$

$(u \in C^2(K))$, where, for each $i, j = 1, \dots, d$ and $x = (x_i)_{1\leqslant i\leqslant d} \in K$,

$$\alpha_{ij}(x) := T(pr_i pr_j)(x) - (pr_i pr_j)(x) = T\big((pr_i - x_i)(pr_j - x_j)\big)(x). \tag{2.2}$$

Accordingly, if $\xi_1, \dots, \xi_d \in \mathbf{R}$, then

$$\sum_{i,j=1}^{d} \alpha_{ij}(x)\xi_i\xi_j = T\Bigg(\Bigg(\sum_{i=1}^{d} \xi_i(pr_i - x_i)\Bigg)^2\Bigg)(x) \geqslant 0,$$

which implies that W_T is elliptic. Moreover, it degenerates on $\partial_T K$ (see (1.8)) and, in particular, on $\partial_e K$ (see Remark 1.2, 2) because $\alpha_{ij} = 0$ on $\partial_T K$ for every $i, j = 1, \dots, d$.

The operator W_T will be referred to as the *elliptic second-order differential operator associated with the Markov operator* T.

Note also that, for each $i, j = 1, \dots, d$,

$$W_T(pr_i pr_j) = \alpha_{ij} = T(pr_i pr_j) - pr_i pr_j$$

and hence, if $P \in P_2(K)$, then $W_T(P) = T(P) - P$. Therefore, if T is a Markov projection and $T(P_2(K)) \subset P_2(K)$, then

$$W_T\big(T(P)\big) = 0 \quad \text{for every } P \in P_2(K). \tag{2.3}$$

Differential operators of the form (2.1) are of concern in the study of diffusion problems arising from different areas such as biology, mathematical finance, physics. In the special case where T is a positive projection, a rather complete overview on them can be found in Chapter 6 of [5].

It turns out that the differential operator W_T is generated by an asymptotic formula for Bernstein–Schnabl operators (see [6, Theorem 4.2]).

Theorem 2.1. *For every $u \in C^2(K)$,*

$$\lim_{n\to\infty} n\big(B_n(u) - u\big) = W_T(u) \quad \textit{uniformly on } K. \tag{2.4}$$

In [6, Theorem 4.3] we also show that, under some additional assumptions on the Markov operator T on $C(K)$, the differential operator W_T is closable in $C(K)$ and its closure generates a Markov semigroup on $C(K)$. Moreover, this semigroup is obtained as a limit of suitable iterates of Bernstein–Schnabl operators associated with T.

Before stating this result, we recall that a *core* for a linear operator $A : D(A) \to C(K)$ is a linear subspace D_0 of $D(A)$ which is dense in $D(A)$ with respect to the graph norm $\|u\|_A := \|A(u)\|_\infty + \|u\|_\infty$ $(u \in D(A))$.

Moreover, if $(A, D(A))$ is the generator of a semigroup $(T(t))_{t\geqslant 0}$ on $C(K)$, then it determines the semigroup uniquely and, if $(B, D(B))$ is a closed operator and if there exists a linear subspace $D_0 \subset D(A) \cap D(B)$ which is a core for $(A, D(A))$ and $A = B$ on D_0, then $(B, D(B)) = (A, D(A))$.

After these preliminaries, we state the following result (see [6, Theorem 4.3 and Corollary 4.4]).

Theorem 2.2. *Let K be a convex compact subset of $\mathbf{R}^d$, $d \geqslant 1$, having non-empty interior, and consider a Markov operator T on $C(K)$ satisfying (1.3). Furthermore, assume that*

$$T\big(P_m(K)\big) \subset P_m(K) \quad \text{for every } m \geqslant 2. \tag{2.5}$$

Then, the differential operator $(W_T, C^2(K))$ is closable and its closure $(A_T, D(A_T))$ generates a Markov semigroup $(T(t))_{t\geqslant 0}$ on $C(K)$ such that, for every $t \geqslant 0$ and for every sequence $(k(n))_{n\geqslant 1}$ of positive integers satisfying $\lim_{n\to\infty} k(n)/n = t$, one gets

$$T(t)(f) = \lim_{n\to\infty} B_n^{k(n)}(f) \quad \text{uniformly on } K \tag{2.6}$$

for every $f \in C(K)$.

Moreover, $P_\infty(K)$ is a core for $(A_T, D(A_T))$ and, if $u, v \in C(K)$ and $\lim_{n\to\infty} n(B_n(u) - u) = v$ uniformly on K, then $u \in D(A_T)$ and $A_T(u) = v$.

In particular, if $\lim_{n\to\infty} n(B_n(u) - u) = 0$ uniformly on K, then $u \in D(A_T)$ and $A_T(u) = 0$.

The representation formula (2.6) can be useful to investigate several qualitative and quantitative properties of both the semigroups $(T(t))_{t\geqslant 0}$ (i.e., of the solutions to the initial-boundary value problems associated with the generator A_T) and the transition functions of the corresponding Markov processes. These aspects will be carefully treated in [7].

Below we show some examples of Markov operators satisfying (2.5) together with their relevant differential operators.

Examples 2.3. 1. (See [10].) Consider a Markov operator T on $C([0,1])$ satisfying (1.3), i.e.,

$$T(e_1) = e_1, \tag{2.7}$$

where $e_1(x) := x$ $(0 \leqslant x \leqslant 1)$.

Then, for every $u \in C^2([0,1])$ and $x \in [0,1]$,

$$W_T(u)(x) = \frac{\alpha(x)}{2} u''(x), \tag{2.8}$$

with

$$\alpha(x) := T(e_2)(x) - x^2 \tag{2.9}$$

and $e_2(x) := x^2$ $(0 \leqslant x \leqslant 1)$.

Examples of Markov operators on $C([0,1])$ which, in addition, satisfy (2.5) can be easily achieved. Consider, for instance, for a given $n \geqslant 1$, the n-th Bernstein operator

$$B_n(f)(x) := \sum_{k=0}^{n} \binom{n}{k} f\left(\frac{k}{n}\right) x^k (1-x)^{n-k}$$

($f \in C([0,1])$, $0 \leqslant x \leqslant 1$). In this case, $\alpha(x) = \frac{x(1-x)}{n}$ ($0 \leqslant x \leqslant 1$).

2. (See [2], [5, Chapter 6].) The differential operator associated with the canonical projection T_d on the d-dimensional simplex K_d (see (1.15) and (1.18)) is given by

$$\begin{aligned} W_{T_d}(u)(x) &:= \frac{1}{2} \sum_{i,j=1}^{d} x_i(\delta_{ij} - x_j) \frac{\partial^2 u}{\partial x_i \partial x_j}(x) \\ &= \frac{1}{2} \sum_{i=1}^{d} x_i(1-x_i) \frac{\partial^2 u}{\partial x_i^2}(x) - \sum_{1 \leqslant i < j \leqslant d} x_i x_j \frac{\partial^2 u}{\partial x_i \partial x_j}(x) \end{aligned} \tag{2.10}$$

($u \in C^2(K_d)$, $x = (x_i)_{1 \leqslant i \leqslant d} \in K_d$), where δ_{ij} stands for the Kronecker symbol.

The operator (2.10) falls into the class of the so-called Fleming–Viot operators, which were studied by Feller in [17] in the description of a stochastic process associated with a diffusion approximation of a gene frequency model in population genetics; subsequently, they have been object of investigation by several authors (see, e.g., [1,13,16] and the references quoted therein).

The coefficients of W_{T_d} vanish on the vertices of the simplex. Furthermore, in this case $T_d(P_m(K_d)) \subset P_1(K_d)$ for every $m \geqslant 2$ and hence (2.5) holds true.

3. (See [6].) As above, let K_d be the canonical simplex of $\mathbf{R}^d$ (see (1.15)) and consider the Markov operator $S : C(K_d) \to C(K_d)$ defined by

$$S(f)(x) := \begin{cases} (1 - \frac{x_1}{1-\sum_{i=2}^d x_i}) f(0, x_2, \ldots, x_d) + \frac{x_1}{1-\sum_{i=2}^d x_i} f(1 - \sum_{i=2}^d x_i, x_2, \ldots, x_d) & \text{if } \sum_{i=2}^d x_i \neq 1; \\ f(0, x_2, \ldots, x_d) & \text{if } \sum_{i=2}^d x_i = 1 \end{cases}$$

($f \in C(K_d)$, $x = (x_i)_{1 \leqslant i \leqslant d} \in K_d$).

Then,

$$S(pr_1 pr_j) = \begin{cases} (1 - \sum_{i=2}^d pr_i) pr_1 & \text{if } j = 1; \\ pr_1 pr_j & \text{if } 1 < j \leqslant d \end{cases} \tag{2.11}$$

and $S(pr_i pr_j) = pr_i pr_j$ for every $1 < i \leqslant j \leqslant d$.

Therefore, the differential operator associated with S is given by

$$W_S(u)(x) = \frac{1}{2} x_1 \left(1 - \sum_{i=1}^{d} x_i\right) \frac{\partial^2 u}{\partial x_1^2}(x) \tag{2.12}$$

($u \in C^2(K_d)$, $x = (x_i)_{1 \leqslant i \leqslant d} \in K_d$).

Thus, W_S degenerates on the faces $\{x = (x_i)_{1 \leqslant i \leqslant d} \in K_d \mid x_1 = 0\}$ and $\{x = (x_i)_{1 \leqslant i \leqslant d} \in K_d \mid \sum_{i=1}^d x_i = 1\}$.

Moreover, $S(P_m(K_d)) \subset P_m(K_d)$ for every $m \geqslant 2$, because, if $m_1, \ldots, m_d$ are positive integers, then

$$S\left(pr_1^{m_1} \cdots pr_d^{m_d}\right) = \begin{cases} pr_2^{m_2} \cdots pr_d^{m_d} & \text{if } m_1 = 0; \\ (1 - \sum_{i=2}^d pr_i)^{m_1 - 1} pr_1 pr_2^{m_2} \cdots pr_d^{m_d} & \text{if } m_1 \geqslant 1. \end{cases}$$

Finally, if we consider the Markov operator $Z := (T_d + S)/2$, where T_d is given by (1.18), the differential operator associated with it is given by

$$W_Z(u)(x) = \frac{1}{4}\left(\left(2x_1(1-x_1) - x_1\sum_{i=2}^{d} x_i\right)\frac{\partial^2 u}{\partial x_1^2}(x) + \sum_{i=2}^{d} x_i(1-x_i)\frac{\partial^2 u}{\partial x_i^2}(x) - \sum_{1\leqslant i<j\leqslant d} x_i x_j \frac{\partial^2 u}{\partial x_i \partial x_j}(x)\right) \tag{2.13}$$

($u \in C^2(K_d)$, $x = (x_i)_{1\leqslant i\leqslant d} \in K_d$).

Therefore, W_Z degenerates on the vertices of K_d and Z satisfies (2.5) as well.

We end this section with a further important example. Consider a symmetric matrix $(a_{ij})_{1\leqslant i,j\leqslant d}$ of Hölder continuous functions on $\mathrm{int}(K)$ with exponent $\beta \in \,]0,1[$. Let L be the differential operator

$$L(u)(x) := \sum_{i,j=1}^{d} a_{ij}(x)\frac{\partial^2 u}{\partial x_i\,\partial x_j}(x) \tag{2.14}$$

($u \in C^2(\mathrm{int}(K))$, $x \in \mathrm{int}(K)$) and assume that it is strictly elliptic, i.e., for every $x \in \mathrm{int}(K)$ the matrix $(a_{ij}(x))_{1\leqslant i,j\leqslant d}$ is positive-definite and, denoted by $\sigma(x)$ its smallest eigenvalue, we have $\sigma(x) \geqslant \sigma_0 > 0$, for some $\sigma_0 \in \mathbf{R}$.

Denote by $T_L : C(K) \to C(K)$ the *Poisson operator associated with* L. Thus, for every $f \in C(K)$, $T_L(f)$ denotes the unique solution to the Dirichlet problem

$$\begin{cases} Lu = 0 & \text{on } \mathrm{int}(K),\ u \in C(K) \cap C^2\big(\mathrm{int}(K)\big); \\ u = f & \text{on } \partial K. \end{cases} \tag{2.15}$$

T_L is a Markov projection satisfying (1.3) and $\partial_T K = \partial K$.

For instance, if K is the closed unit ball (with respect to the Euclidean norm $\|\cdot\|_2$) of center the origin of $\mathbf{R}^d$ and L is the Laplace operator Δ, i.e.,

$$\Delta u = \sum_{i=1}^{d} \frac{\partial^2 u}{\partial x_i^2} \tag{2.16}$$

($u \in C^2(\mathrm{int}(K))$), then

$$T_\Delta(f)(x) = \begin{cases} \frac{1-\|x\|_2^2}{\sigma_d}\int_{\partial K} \frac{f(z)}{\|z-x\|_2^d}\,d\sigma(z) & \text{if } \|x\|_2 < 1; \\ f(x) & \text{if } \|x\|_2 = 1 \end{cases} \tag{2.17}$$

($f \in C(K)$, $x \in K$), where σ_d and σ denote the surface area of the unit sphere in $\mathbf{R}^d$ and the surface measure on ∂K, respectively.

In order to determine (in some particular cases) the differential operator associated with the Poisson operator we need to point out some preliminary results.

Consider a convex compact subset K of $\mathbf{R}^d$, $d \geqslant 2$, such that its boundary ∂K is an ellipsoid, i.e., there exist a real symmetric and positive-definite matrix $R = (r_{ij})_{1\leqslant i,j\leqslant d}$ and $\bar{x} = (\bar{x}_i)_{1\leqslant i\leqslant d} \in \mathbf{R}^d$ such that

$$\partial K = \left\{ x \in \mathbf{R}^d \;\middle|\; Q(x-\bar{x}) := \sum_{i,j=1}^{d} r_{ij}(x_i - \bar{x}_i)(x_j - \bar{x}_j) = 1 \right\}. \tag{2.18}$$

Furthermore, consider a strictly elliptic differential operator

$$L(u)(x) := \sum_{i,j=1}^{d} c_{ij}\frac{\partial^2 u}{\partial x_i\,\partial x_j}(x) \tag{2.19}$$

$(u \in C^2(\mathrm{int}(K)), x \in \mathrm{int}(K))$ associated with a real symmetric and positive-definite matrix $C = (c_{ij})_{1\leqslant i,j\leqslant d}$ and denote by T_L the relevant Poisson operator on $C(K)$ (see (2.15)).

We can now describe the differential operator W_{T_L}, that we shall briefly denote by W_L, defined according to (2.1) and (2.2) (see [19,2]).

Theorem 2.4. *Let K and L be as in (2.18) and (2.19). Then the differential operator W_L associated with T_L is given by*

$$W_L(u)(x) = \begin{cases} (2\sum_{i,j=1}^d r_{ij}c_{ij})^{-1}(1-Q(x))L(u)(x) & \text{if } x \in \mathrm{int}(K); \\ 0 & \text{if } x \in \partial K \end{cases}$$

$(u \in C^2(K),\ x \in K)$. Moreover, for every $m \geqslant 1$, T_L maps $P_m(K)$ into $P_m(K)$.

In particular, if K is the closed ball (with respect to the Euclidean norm $\|\cdot\|_2$) with center $\bar{x} \in \mathbf{R}^d$ and radius $r > 0$ and if $L = \Delta$, then

$$W_\Delta(u)(x) = \begin{cases} \frac{r^2-\|x-\bar{x}\|_2^2}{2d}\Delta(u)(x) & \text{if } \|x-\bar{x}\|_2 < r; \\ 0 & \text{if } \|x-\bar{x}\|_2 = r \end{cases} \tag{2.20}$$

$(u \in C^2(K),\ x \in K)$ and T_Δ maps $P_m(K)$ into $P_m(K)$ for every $m \geqslant 1$.

Remark 2.5. Replacing, if necessary, each coefficient c_{ij} of the matrix C in (2.19) by $c_{ij}(\sum_{i,j=1}^d r_{ij}c_{ij})^{-1}$, $i, j = 1, \dots, d$, we may always assume that $\sum_{i,j=1}^d r_{ij}c_{ij} = 1$. In this case, the differential operator W_L turns into

$$W_L(u)(x) = \begin{cases} \frac{1-Q(x)}{2}L(u)(x) & \text{if } x \in \mathrm{int}(K); \\ 0 & \text{if } x \in \partial K \end{cases}$$

$(u \in C^2(K),\ x \in K)$.

3. Markov operators preserving polynomials

The main assumption in Theorem 2.2 involves the invariance under T of the spaces of polynomials of degree m, $m \geqslant 1$. Such a property, that seems to have its own independent interest, will be discussed below in more details.

As a first simple remark, note that, if T satisfies (1.3) and (2.5), then for every $\lambda \in [0,1]$ the operator $U_\lambda := \lambda T + (1-\lambda)I$ satisfies the same property.

We begin by presenting a counterexample to (2.5).

Example 3.1. Let $K = K_2$ be the canonical simplex of $\mathbf{R}^2$ (see (1.15)) and consider the Poisson operator $T_\Delta : C(K_2) \to C(K_2)$ associated with the Laplace operator $\Delta u(x,y) := \frac{\partial^2 u}{\partial x^2}(x,y) + \frac{\partial^2 u}{\partial y^2}(x,y)$ $(u \in C^2(\mathrm{int}(K_2)),\ (x,y) \in \mathrm{int}(K_2))$ (see (2.16) and (2.17)). Then $T_\Delta(P_2(K_2)) \not\subset P_2(K_2)$.

Indeed, consider the function $f(x,y) = x^2$ $((x,y) \in K_2)$ and assume that $T_\Delta(f) \in P_2(K_2)$, i.e., there exist $a, b, c, m, n, p \in \mathbf{R}$ such that

$$T_\Delta(f)(x,y) = ax^2 + bxy + cy^2 + mx + ny + p$$

for every $(x,y) \in K_2$.

Since $T_\Delta(f) = f$ on ∂K_2, we have

$$T_\Delta(f)(0,y) = f(0,y) \quad \big(y \in [0,1]\big),$$
$$T_\Delta(f)(x,0) = f(x,0) \quad \big(x \in [0,1]\big)$$

and

$$T_\Delta(f)(x,1-x) = f(x,1-x) \quad \big(x \in [0,1]\big).$$

Accordingly, we get $b = c = m = n = p = 0$ and $a = 1$. Thus, $T_\Delta(f) = f$ and this is not possible because f is not harmonic on $\operatorname{int}(K_2)$.

Below we shall consider another property similar to (2.5), namely,

$$T\big(P_2(K)\big) \subset P_1(K), \quad \text{i.e.,} \quad T(h_1 h_2) \in P_1(K) \quad \text{for every } h_1, h_2 \in P_1(K). \tag{3.1}$$

Note that assumption (3.1) is satisfied when K is a simplex and T is the canonical projection on $C(K)$ (see Theorem 1.4).

Next we show that this is the only case where (3.1) can occur.

Theorem 3.2. *Assume that there exists a Markov operator T on $C(K)$ satisfying (1.3) and (3.1). Then K is a simplex and T is the canonical projection associated with it.*

In particular, $T(P_m(K)) \subset P_1(K)$ for every $m \geqslant 2$.

Proof. Setting M as in (1.6), from (1.3) it follows that $P_1(K) \subset M$. Thus, M separates the points of K and, by (3.1),

$$T\big(P_1(K)^2\big) \subset T\big(P_2(K)\big) \subset P_1(K) \subset M.$$

Therefore, by Proposition 1.3, T is a projection and $\partial_T K = \partial_H K$, where $H := T(C(K))$ (see (1.8)).

We now proceed by induction to show that $T(P_m(K)) \subset P_1(K)$ for every $m \geqslant 2$. Indeed, assume that the inclusion holds true for some $m \geqslant 2$ and fix $u \in P_{m+1}(K)$ having the form $u = \prod_{i=1}^{m+1} h_i$, with $h_1, \ldots, h_{m+1} \in P_1(K)$.

Setting $v := \prod_{i=1}^{m} h_i \in P_m(K)$, we have that $u = v h_{m+1}$ and $T(v) \in P_1(K)$, so that $h_{m+1} T(v) \in P_2(K)$ and $T(h_{m+1} T(v)) \in P_1(K)$.

But $h_{m+1} T(v) = h_{m+1} v = u$ on $\partial_T K = \partial_H K$, and hence, by (1.14), $T(u) = T(h_{m+1} T(v)) \in P_1(K)$.

From the above, it follows that $T(P_\infty(K)) \subset P_1(K)$ and hence, by continuity, $T(C(K)) = P_1(K)$. From Theorem 1.4 it turns out that K is a simplex and T is its canonical projection. $\square$

From Theorem 2.4 it follows that, if K is an ellipsoid, then several classes of Poisson operators associated with strictly elliptic operators verify (2.5).

The next result, which generalizes Theorem 3.5 of [11], shows that the inclusion $T(P_2(K)) \subset P_2(K)$ characterizes the ellipsoids between those convex compact subsets of $\mathbf{R}^d$ that are *strictly convex*, i.e., $\partial_e K = \partial K$. In such a case, necessarily $\operatorname{int}(K) \neq \emptyset$ unless K is *trivial*, i.e., K reduces to a singleton.

Theorem 3.3. *Given a non-trivial strictly convex compact subset K of $\mathbf{R}^d$, $d \geqslant 2$, the following statements are equivalent:*

(i) *There exists a non-trivial Markov operator T on $C(K)$, i.e., $T \neq I$, satisfying (1.3) and (2.5).*
(ii) *There exists a non-trivial Markov operator T on $C(K)$ satisfying (1.3) such that*

$$T\big(P_2(K)\big) \subset P_2(K). \tag{3.2}$$

(iii) *There exists a non-trivial Markov operator T on $C(K)$ satisfying (1.3) such that*

$$T(\Phi) \in P_2(K), \tag{3.3}$$

where $\Phi := \sum_{i=1}^{d} pr_i^2 = \|\cdot\|_2^2$.
(iv) *∂K is an ellipsoid defined by a quadratic form $Q(x-\bar{x}) := \sum_{i,j=1}^{d} r_{ij}(x_i-\bar{x}_i)(x_j-\bar{x}_j)$ ($x = (x_i)_{1\leqslant i\leqslant d} \in \mathbf{R}^d$) with center $\bar{x} = (\bar{x}_i)_{1\leqslant i\leqslant d} \in \mathbf{R}^d$ (see (2.18)).*

Moreover, if T is a non-trivial Markov projection on $C(K)$ satisfying (1.3) and (3.2) (or (3.3)), then one and only one of the following statements holds true:

(a) *T is the Poisson operator associated with a suitable strictly elliptic differential operator of the form (2.19), whose coefficients $(c_{ij})_{1\leqslant i,j\leqslant d}$ are constant and satisfy $\sum_{i,j=1}^{d} r_{ij}c_{ij} = 1$.*
(b) *For every $x \in \mathrm{int}(K)$ the support Supp$(\tilde{\mu}_x^T)$ (see (1.1)) is contained in an affine hyperplane R_x through x and hence, for every $f \in C(K)$,*

$$T(f)(x) = \int\limits_{\partial K \cap R_x} f \, d\tilde{\mu}_x^T. \tag{3.4}$$

Proof. The implications (i) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (iii) are obvious.

(iii) $\Rightarrow$ (iv). From (3.3) we infer that $T(\Phi) - \Phi$ is the restriction to K of a polynomial P of degree at most two. Consider the hypersurface

$$S := \big\{x \in \mathbf{R}^d \;\big|\; P(x) = 0\big\}.$$

From Theorem 1.1 we get that

$$\partial_T K = \big\{x \in K \;\big|\; T(\Phi)(x) = \Phi(x)\big\} = K \cap S. \tag{1}$$

Now we proceed to show that

$$\partial K = \partial_T K = S. \tag{2}$$

Indeed, (1.12) implies that $\partial K = \partial_e K \subset \partial_T K \subset S$. Before showing the converse inclusion, we first observe that $S \neq K$, otherwise, by (1), $\partial_T K = K$ and, as a consequence of the classical Korovkin theorem (see, e.g., [5, Theorem 4.2.7]), we should have $T = I$. On account of this preliminary remark, we deduce that $\mathrm{int}(K) \not\subset S$ and hence we can choose $x_0 \in \mathrm{int}(K) \setminus S \subset \mathrm{int}(K) \setminus \partial_T K$.

Now, in order to complete the proof of (2), assume, on the contrary, that there exists $\bar{y} \in S \setminus \partial K$.

Then the straight line R through $\bar{y}$ and x_0 cannot be contained in S (because $x_0 \notin S$) and hence, since P is a polynomial of degree at most two, $R \cap S$ contains at most two points.

On the other hand, $R \cap S$ contains exactly two points because K is strictly convex, and $R \cap \partial K \subset R \cap \partial_T K \subset R \cap S$, so that $R \cap S = R \cap \partial K = R \cap \partial_T K$ and hence $\bar{y} \in \partial K$, a contradiction.

Having now (2) at our disposal, assume that

$$P(x) = \sum_{i,j=1}^{d} a_{ij}x_i x_j + \sum_{i=1}^{d} b_i x_i + c \tag{3}$$

($x = (x_i)_{1\leqslant i\leqslant d} \in \mathbf{R}^d$), where the matrix $A := (a_{ij})_{1\leqslant i,j\leqslant d}$ is symmetric and $b_1, \ldots, b_d, c \in \mathbf{R}$. Since ∂K is a bounded quadric, then it is an ellipsoid and, in particular, $\det A \neq 0$. Therefore, considering the point $\bar{x} = (\bar{x}_i)_{1\leqslant i\leqslant d}$ whose coordinates are the unique solutions to the system

$$2\sum_{j=1}^{d} a_{ij}\xi_j + b_i = 0, \quad i = 1, \ldots, d, \tag{4}$$

then

$$P(x) := \sum_{i,j=1}^{d} a_{ij}(x_i - \bar{x}_i)(x_j - \bar{x}_j) - \gamma$$

($x = (x_i)_{1\leqslant i\leqslant d} \in \mathbf{R}^d$), where $\gamma := \sum_{i,j=1}^{d} a_{ij}\bar{x}_i\bar{x}_j - c$.

Therefore,

$$\partial K = \left\{ x \in \mathbf{R}^d \;\middle|\; \sum_{i,j=1}^{d} a_{ij}(x_i - \bar{x}_i)(x_j - \bar{x}_j) = \gamma \right\}.$$

Since ∂K is an ellipsoid, then $\gamma \neq 0$ and the matrix $(r_{ij})_{1\leqslant i,j\leqslant d}$ is positive-definite, where $r_{ij} := \frac{a_{ij}}{\gamma}$ ($i, j = 1, \ldots, d$). Thus,

$$\partial K = \{ x \in \mathbf{R}^d \mid Q(x - \bar{x}) = 1 \}, \tag{5}$$

where $Q(x - \bar{x}) := \sum_{i,j=1}^{d} r_{ij}(x_i - \bar{x}_i)(x_j - \bar{x}_j)$ ($x = (x_i)_{1\leqslant i\leqslant d} \in \mathbf{R}^d$).

(iv) $\Rightarrow$ (i). It is a consequence of Theorem 2.4.

In order to show the last part of the statement, consider a Markov projection T on $C(K)$ satisfying (1.3) and (3.2). In case (b), for every $x \in \operatorname{int}(K)$, since $\mathit{Supp}(\tilde{\mu}_x^T) \subset \partial K$ (see (1.13) and the preceding formula (2)), we get

$$T(f)(x) = \int_K f\, d\tilde{\mu}_x^T = \int_{\partial K \cap R_x} f\, d\tilde{\mu}_x^T$$

for every $f \in C(K)$.

Suppose that case (b) does not occur and fix $\bar{z} \in \operatorname{int}(K)$ such that $\mathit{Supp}(\tilde{\mu}_{\bar{z}}^T)$ is not contained in any affine hyperplane through $\bar{z}$.

Without loss of generality we can assume that ∂K is an ellipsoid with center the origin of $\mathbf{R}^d$, so that we can consider a positive definite quadratic form $Q(x) = \sum_{i,j=1}^{d} r_{ij}x_i x_j$ ($x = (x_i)_{1\leqslant i\leqslant d} \in \mathbf{R}^d$) such that

$$K = \{ x \in \mathbf{R}^d \mid Q(x) \leqslant 1 \}.$$

Given $i, j = 1, \ldots, d$, since the function $\alpha_{ij} := T(pr_i pr_j) - pr_i pr_j$ is a polynomial of degree at most two which vanishes on $\partial_T K = \partial K = \{x \in \mathbf{R}^d \mid Q(x) = 1\}$, by Hilbert's Nullstellen Satz there exists $c_{ij} \in \mathbf{R}$ such that

$$\alpha_{ij} = c_{ij}(\mathbf{1} - Q). \tag{6}$$

Note that the matrix $(\alpha_{ij}(\bar{z}))_{1\leqslant i,j\leqslant d}$ is positive-definite. Indeed, if $\xi = (\xi_i)_{1\leqslant i\leqslant d} \in \mathbf{R}^d \setminus \{0\}$, then the quantity

$$\sum_{i,j=1}^{d} \alpha_{ij}(\bar{z})\xi_i\xi_j = T\left(\left(\sum_{i=1}^{d} \xi_i\bigl(pr_i - pr_i(\bar{z})\bigr)\right)^2\right)(\bar{z})$$

is strictly positive, otherwise we should have

$$\sum_{i=1}^{d} \xi_i\bigl(pr_i - pr_i(\bar{z})\bigr) = 0 \quad \text{on } Supp\bigl(\tilde{\mu}_{\bar{z}}^T\bigr)$$

i.e.,

$$Supp\bigl(\tilde{\mu}_{\bar{z}}^T\bigr) \subset \bigl\{y \in \mathbf{R}^d \bigm| \langle \xi, y - \bar{z}\rangle = 0\bigr\},$$

a contradiction (here $\langle\cdot,\cdot\rangle$ denotes the canonical scalar product on $\mathbf{R}^d$).

As a consequence, from (6) it follows that the matrix $(c_{ij})_{1\leqslant i,j\leqslant d}$ is symmetric and positive-definite.

Let us consider the differential operator W_T defined by (2.1). From (2.3) and (6) it follows that, for every $i, j = 1, \dots, d$,

$$W_T\bigl(pr_i pr_j + c_{ij}(\mathbf{1} - Q)\bigr) = W_T\bigl(T(pr_i pr_j)\bigr) = 0,$$

so that

$$W_T(Q) = \sum_{i,j=1}^{d} r_{ij} W_T(pr_i pr_j) = \sum_{i,j=1}^{d} r_{ij} c_{ij} W_T(Q).$$

On the other hand, since $Q = \mathbf{1}$ on $\partial K = \partial_T K$, from (1.14) it follows that $T(Q) = \mathbf{1}$ and, by (2.3), we have that

$$W_T(Q) = T(Q) - Q = \mathbf{1} - Q.$$

Thus, $W_T(Q)$ does not vanish on $\operatorname{int}(K)$, so that $\sum_{i,j=1}^{d} r_{ij}c_{ij} = 1$ and the proof is now complete. □

A special case of the previous result is worth being stated separately.

Corollary 3.4. *Given a non-trivial strictly convex compact subset K of $\mathbf{R}^d$, $d \geqslant 2$, the following statements are equivalent*:

(i) *There exists a non-trivial Markov operator T on $C(K)$ satisfying* (1.3), *such that*

$$T(\Phi) - (1 + \lambda)\Phi \in P_1(K) \tag{3.5}$$

for some $\lambda \in \mathbf{R}, \lambda \neq 0$, where $\Phi := \sum_{i=1}^{d} pr_i^2$.

(ii) *K is a ball with respect to the Euclidean norm $\|\cdot\|_2$ on $\mathbf{R}^d$.*

Moreover, if

$$T(\Phi) = (1 + \lambda)\Phi + \sum_{i=1}^{d} b_i x_i + c$$

with $\lambda \in \mathbf{R}, \lambda \neq 0$, *and* $b_1, \dots, b_d, c \in \mathbf{R}$, *then* K *is the ball of center* $\bar{x} = (\bar{x}_i)_{1 \leqslant i \leqslant d} \in \mathbf{R}^d$ *and radius* r, *where*

$$\bar{x}_i := -\frac{b_i}{2\lambda} \quad \textit{for every } 1 \leqslant i \leqslant d \quad \textit{and} \quad r := \sqrt{\|\bar{x}\|_2^2 - \frac{c}{\lambda}}.$$

Proof. (i) $\Rightarrow$ (ii). It is enough to apply the same reasoning as in the proof of the implication (iii) $\Rightarrow$ (iv) of Theorem 3.3 with $P(x) = \sum_{i=1}^d \lambda {x_i}^2 + \sum_{i=1}^d b_i x_i + c$ $(x = (x_i)_{1 \leqslant i \leqslant d} \in \mathbf{R}^d)$ (see also (3)–(5)).

(ii) $\Rightarrow$ (i). Assume that K is the ball of center $\bar{x} = (\bar{x}_i)_{1 \leqslant i \leqslant d} \in \mathbf{R}^d$ and radius r and consider the Poisson operator T_Δ associated with the Laplace operator Δ (see (2.15) and (2.16)). Then T_Δ is a Markov operator satisfying (1.3) and

$$T_\Delta(\Phi) = 2\sum_{i=1}^d \bar{x}_i pr_i + r^2 - \|\bar{x}\|_2^2. \qquad \square$$

Remark 3.5. In [11] the reader can find a complete description of those convex compact subsets K of $\mathbf{R}^2$ such that there exists a Markov projection T on $C(K)$ satisfying (1.3) and (2.5).

We proceed further to study condition (2.5) in the setting of product spaces.

Consider a finite family $(K_i)_{1 \leqslant i \leqslant d}$ of convex compact subsets having non-empty interior, each contained in some $\mathbf{R}^{s_i}$, $s_i \geqslant 1$, $i = 1, \dots, d$. For every $i = 1, \dots, d$, let $T_i : C(K_i) \to C(K_i)$ be a Markov operator satisfying (1.3) and (2.5). Setting $K := \prod_{i=1}^d K_i$ and denoting by $T := \bigotimes_{i=1}^d T_i$ the *tensor product* of $(T_i)_{1 \leqslant i \leqslant d}$ (see [5, pp. 32–36]), then T is a Markov operator on $C(K)$ which satisfies (1.3).

For every $i = 1, \dots, d$, set

$$A_i := A_{T_i} \tag{3.6}$$

and, for $j = 1, \dots, d$,

$$A_{i,j} := \begin{cases} A_i & \text{if } j = i; \\ I_{D(A_j)} & \text{if } j \neq i. \end{cases} \tag{3.7}$$

Moreover, for every $i = 1, \dots, d$, denote by $(T_i(t))_{t \geqslant 0}$ the Markov semigroup on $C(K_i)$ generated by $(A_i, D(A_i))$. Then we can define a linear operator $A : \bigotimes_{i=1}^d D(A_i) \to C(K)$ such that, for every $(u_i)_{1 \leqslant i \leqslant d} \in \prod_{i=1}^d D(A_i)$,

$$A\left(\bigotimes_{i=1}^d u_i\right) = \bigotimes_{i=1}^d A_i(u_i), \tag{3.8}$$

where $\bigotimes_{i=1}^d D(A_i)$ denotes the linear subspace generated by

$$\left\{ \bigotimes_{i=1}^d u_i \;\middle|\; u_i \in D(A_i),\ 1 \leqslant i \leqslant d \right\}.$$

The operator A will be denoted by $\bigotimes_{i=1}^d A_i$ and it will be again referred to as the tensor product of the family $(A_i)_{1 \leqslant i \leqslant d}$.

Finally, consider the Bernstein–Schnabl operators B_n $(n \geqslant 1)$ associated with $T = \bigotimes_{i=1}^d T_i$ and, for every $i = 1, \dots, d$, let $B_{n,i}$ be the Bernstein–Schnabl operators associated with T_i (see (1.2)). From the commutativity as well as the associativity properties of tensor products of measures (see [14, Vol. I, Section 13]), it follows that

$$B_n\left(\bigotimes_{i=1}^d f_i\right) = \bigotimes_{i=1}^d B_{n,i}(f_i), \tag{3.9}$$

for every $\bigotimes_{i=1}^d f_i \in \prod_{i=1}^d C(K_i)$.

Theorem 3.6. *The Markov operator $T = \bigotimes_{i=1}^d T_i$ satisfies (2.5). Moreover, if $(A_T, D(A_T))$ is the generator of the semigroup $(T(t))_{t\geqslant 0}$ as in Theorem 2.2, then*

(i) *$T(t) = \bigotimes_{i=1}^d T_i(t)$ for every $t \geqslant 0$.*
(ii) *The subspace $\bigotimes_{i=1}^d D(A_i)$ is contained in $D(A_T)$, it is a core for $(A_T, D(A_T))$ and*

$$A_T = \sum_{i=1}^d \bigotimes_{j=1}^d A_{i,j} \quad \text{on} \ \bigotimes_{i=1}^d D(A_i)$$

(see (3.8)).
(iii) *$\bigotimes_{i=1}^d C^2(K_i)$ is a core for $(A_T, D(A_T))$ and, if $(u_i)_{1\leqslant i\leqslant d} \in \prod_{i=1}^d C^2(K_i)$, then*

$$A_T\left(\bigotimes_{i=1}^d u_i\right) = \sum_{i=1}^d u_1 \otimes \cdots \otimes u_{i-1} \otimes W_{T_i}(u_i) \otimes u_{i+1} \otimes \cdots \otimes u_d.$$

Proof. Note that, given $(u_i)_{1\leqslant i\leqslant d}, (v_i)_{1\leqslant i\leqslant d} \in \prod_{i=1}^d P_1(K_i)$, then, for every $i = 1, \ldots, d$,

$$(u_i v_i) \circ pr_i = (u_i \circ pr_i)(v_i \circ pr_i) \in P_2(K)$$

and

$$T\big((u_i \circ pr_i)(v_i \circ pr_i)\big) = T_i(u_i v_i) \circ pr_i \in P_2(K).$$

Moreover, for $i, j = 1, \ldots, d$, $i \neq j$,

$$T\big((u_i \circ pr_i)(v_j \circ pr_j)\big) = \big(T_i(u_i) \circ pr_i\big)\big(T_j(v_j) \circ pr_j\big) = (u_i \circ pr_i)(u_j \circ pr_j) \in P_2(K).$$

On the other hand, for every $u \in P_1(K)$, there exist $\alpha_0, \alpha_1, \ldots, \alpha_d \in \mathbf{R}$ and $(u_i)_{1\leqslant i\leqslant d} \in \prod_{i=1}^d P_1(K_i)$ such that $u = \alpha_0 + \sum_{i=1}^d \alpha_i(u_i \circ pr_i)$. Therefore, on account of the preceding identities, it follows that $T(P_2(K)) \subset P_2(K)$.

By induction, it is now easy to show that $T(P_m(K)) \subset P_m(K)$ for every $m \geqslant 2$. According to Theorem 2.2 we can consider the Markov semigroup $(T(t))_{t\geqslant 0}$ on $C(K)$, along with its generator $(A_T, D(A_T))$.

Looking at the family of generators $((A_i, D(A_i)))_{1\leqslant i\leqslant d}$ defined by (3.6), from [18, Section A-I-3.7, p. 23] it follows that the operator $\sum_{i=1}^d \bigotimes_{j=1}^d A_{i,j}$ defined on $\bigotimes_{i=1}^d D(A_i)$ is closable on $C(K)$ and its closure $(B, D(B))$ generates a C_0-semigroup $(S(t))_{t\geqslant 0}$ on $C(K)$ given by

$$S(t) = \bigotimes_{i=1}^d T_i(t) \quad (t \geqslant 0).$$

Moreover, $\bigotimes_{i=1}^d D(A_i)$ is a core for $(B, D(B))$.

We now proceed to show that

$$\bigotimes_{i=1}^d C^2(K_i) \subset D(A_T) \quad \text{and} \quad A_T = B \quad \text{on} \ \bigotimes_{i=1}^d C^2(K_i). \tag{1}$$

Indeed, given $(u_i)_{1\leqslant i\leqslant d} \in \bigotimes_{i=1}^d C^2(K_i)$ and considered the sequence $(B_n)_{n\geqslant 1}$ of Bernstein–Schnabl operators associated with T (see (1.2)), then, on account of (3.9) and Theorem 2.1 for $u = \bigotimes_{i=1}^d u_i$, we get

$$\begin{aligned}
\lim_{n\to\infty} n\big(B_n(u) - u\big) &= \lim_{n\to\infty} n\left(\bigotimes_{i=1}^d B_{n,i}(u_i) - \bigotimes_{i=1}^d u_i\right) \\
&= \lim_{n\to\infty} n\left(\sum_{i=1}^d B_{n,1}(u_1)\otimes\cdots\otimes B_{n,i-1}(u_{i-1})\otimes\big(B_{n,i}(u_i)-u_i\big)\otimes u_{i+1}\otimes\cdots\otimes u_d\right) \\
&= \sum_{i=1}^d \lim_{n\to\infty} B_{n,1}(u_1)\otimes\cdots\otimes B_{n,i-1}(u_{i-1})\otimes\big[n\big(B_{n,i}(u_i)-u_i\big)\big]\otimes u_{i+1}\otimes\cdots\otimes u_d \\
&= \sum_{i=1}^d u_1\otimes\cdots\otimes u_{i-1}\otimes W_{T_i}(u_i)\otimes u_{i+1}\otimes\cdots\otimes u_d = B(u).
\end{aligned}$$

Therefore, by Theorem 2.2, $u \in D(A_T)$ and $A_T(u) = B(u)$.

On the other hand, for every $m \geqslant 1$,

$$P_m(K) \subset \bigcup_{m_1+\cdots+m_d\leqslant m} \bigotimes_{i=1}^d P_{m_i}(K_i) \subset \bigotimes_{i=1}^d C^2(K_i),$$

which implies that $P_\infty(K) \subset \bigotimes_{i=1}^d C^2(K_i) \subset \bigotimes_{i=1}^d D(A_i)$ and $A_T = B$ on $P_\infty(K)$ by virtue of (1). Furthermore, due to Theorem 2.2, $P_\infty(K)$ is a core for $(A_T, D(A_T))$ and hence $(B, D(B)) = (A_T, D(A_T))$ (see the remarks before Theorem 2.2). In particular, $T(t) = S(t)$ for every $t \geqslant 0$ and the proof is now complete. □

The special case where $K_i = [0,1]$ for every $i = 1,\dots,d$, is worth being studied separately.

Let $Q_d := [0,1]^d$, $d \geqslant 1$, and for every $i = 1,\dots,d$ consider a Markov operator T_i on $C([0,1])$ satisfying (2.7) and (2.5).

If $T := \bigotimes_{i=1}^d T_i : C(Q_d) \to C(Q_d)$, then, for every $u \in C^2(Q_d)$ and $x = (x_i)_{1\leqslant i\leqslant d} \in Q_d$,

$$W_T(u)(x) = \frac{1}{2}\sum_{i=1}^d \alpha_i(x)\frac{\partial^2 u}{\partial x_i^2}(x), \tag{3.10}$$

where $\alpha_i(x) := T_i(e_2)(x_i) - x_i^2$ $(1 \leqslant i \leqslant d)$.

Finally note that, if $T_i = T_1$ for any $i = 1,\dots,d$ (see (1.19)), then

$$W_T(u)(x) = \frac{1}{2}\sum_{i=1}^d x_i(1-x_i)\frac{\partial^2 u}{\partial x_i^2}(x) \tag{3.11}$$

$(u \in C^2(Q_d),\ x = (x_i)_{1\leqslant i\leqslant d} \in Q_d)$.

Corollary 3.7. *Under the preceding assumptions, the operator T maps $P_m(Q_d)$ into $P_m(Q_d)$ for every $m \geqslant 1$.*

Therefore, the differential operator $(W_T, C^2(Q_d))$ is closable and its closure generates a Markov semigroup $(T(t))_{t\geqslant 0}$ on $C(Q_d)$ satisfying all the properties stated in Theorems 2.2 and 3.6.

We end the paper by discussing property (2.5) for Markov operators which are the *convex convolution product of* two given ones.

Consider two Markov operators S and T on $C(K)$ satisfying (1.3) and the relevant selections of probability Borel measures $(\tilde{\mu}_x^S)_{x\in K}$ and $(\tilde{\mu}_x^T)_{x\in K}$ associated with them according to (1.1).

Considering the mapping $\pi_2 : K \times K \to K$ defined by

$$\pi_2(x_1, x_2) := \frac{x_1 + x_2}{2} \quad \bigl((x_1, x_2) \in K \times K\bigr), \tag{3.12}$$

for each $f \in C(K)$, we define the following function on K by setting

$$U(f)(x) := \int\limits_{K\times K} f \circ \pi_2 \, d\tilde{\mu}_x^S \otimes d\tilde{\mu}_x^T = \int\limits_K \int\limits_K f\left(\frac{x_1 + x_2}{2}\right) d\tilde{\mu}_x^S(x_1)\, d\tilde{\mu}_x^T(x_2) \tag{3.13}$$

$(x \in K)$. Then $U(f) \in C(K)$ and the operator U is a Markov operator satisfying (1.3). Moreover, $\partial_S K \cap \partial_T K \subset \partial_U K$.

The operator U will be called the *convex convolution product of S and T*. The Bernstein–Schnabl operators associated with U are given by

$$B_{n,U}(f)(x) = B_{n,S}(B_{n,T,f,x})(x) \quad \bigl(f \in C(K),\ x \in K\bigr), \tag{3.14}$$

where

$$B_{n,T,f,x}(x_1) := B_{n,T}\bigl(f\bigl(\pi_2(x_1, \cdot)\bigr)\bigr)(x) \quad (x_1 \in K). \tag{3.15}$$

We finally mention that the differential operator W_U associated with the Markov operator (3.13) is given by

$$W_U = \frac{W_T + W_S}{4}. \tag{3.16}$$

Theorem 3.8. *Let K be a convex compact subset of $\mathbf{R}^d$, $d \geqslant 1$, having non-empty interior, and consider two Markov operators S and T on $C(K)$ satisfying (1.3) and (2.5). Then the convex convolution product U of S and T defined by (3.13) maps $P_m(K)$ into $P_m(K)$ for each $m \geqslant 1$.*

Therefore, the differential operator $W_U = (W_S + W_T)/4$ (see (3.16)) defined on $C^2(K)$ is closable and its closure generates a Markov semigroup on $C(K)$ satisfying all the properties stated in Theorem 2.2.

Proof. The case $m = 1$ being obvious, we can assume $m \geqslant 2$. Consider $h_1, \dots, h_m \in P_1(K)$ and $x_1, x_2 \in K$. Denote by $F(m, 2)$ the set of all mappings $\sigma : \{1, \dots, m\} \to \{1, 2\}$ and, for $\sigma \in F(m, 2)$, set $R_1^\sigma := \{i = 1, \dots, m \mid \sigma(i) = 1\}$ and $R_2^\sigma := \{i = 1, \dots, m \mid \sigma(i) = 2\}$. Therefore,

$$\left(\prod_{i=1}^m h_i\right)\left(\frac{x_1 + x_2}{2}\right) = \frac{1}{2^m} \sum_{\sigma\in F(m,2)} \prod_{i\in R_1^\sigma} h_i(x_1) \prod_{i\in R_2^\sigma} h_i(x_2),$$

where the product $\prod_{i\in R_k^\sigma} h_i$ is, by convention, equal to 1 if $R_k^\sigma = \emptyset$ for some $k = 1, 2$. Then, from (3.13), it follows that

$$U\left(\prod_{i=1}^m h_i\right) = \frac{1}{2^m} \sum_{\sigma\in F(m,2)} S\left(\prod_{i\in R_1^\sigma} h_i\right) T\left(\prod_{i\in R_2^\sigma} h_i\right) \in P_m(K),$$

because of the assumptions on S and T and since $\operatorname{card} R_1^\sigma + \operatorname{card} R_2^\sigma = m$. $\square$

Remark 3.9. From Theorem 3.8 it turns out that the sum $W_S + W_T = 4W_U$, defined on $C^2(K)$, is closable and its closure generates a Markov semigroup $(T(t))_{t\geqslant 0}$, which is the rescaled semigroup with parameter 4 (see, e.g., [15, Chapter III, Section 1]) of the semigroup generated by the closure of $(W_U, C^2(K))$.

This result is not trivial because, in general, the investigation of the generation property of the sum of two generators is a delicate problem (see, e.g., [15, Chapter III, Section 1]).

However, the sum W_S+W_T is also equal to $2W_{\frac{S+T}{2}}$ and $\frac{S+T}{2}$ is a Markov operator on $C(K)$ satisfying (1.3) and (2.5). Therefore, the semigroup $(T(t))_{t\geqslant 0}$ also coincides with the rescaled semigroup with parameter 2 generated by the closure of $(W_{\frac{S+T}{2}}, C^2(K))$. Thus it can be represented as in (2.6) in terms of iterates of Bernstein–Schnabl operators associated with U or with $\frac{S+T}{2}$. We refer to [7] for more details in this respect.

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