Approximation of Hilbert and Hadamard transforms on $(0, +\infty)$ ☆

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Dedicated to Professor Francesco A. Costabile on his 70th birthday

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Abstract

The authors propose a numerical method for computing Hilbert and Hadamard transforms on $(0, +\infty)$ by a simultaneus approximation process involving a suitable Lagrange polynomial of degree s and " truncated" Gaussian rule of order m, with $s \ll m$. The proposed procedure is convergent and pointwise error estimates are given. Finally, some numerical tests confirming the theoretical error estimates are presented.

Keywords: Hypersingular integrals, Hadamard finite part, Lagrange Interpolation, Orthogonal Polynomials, Approximation by Polynomials.

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1. Introduction

Denoting by $w(x) := w_{\alpha,\beta}(x) = e^{-x^\beta} x^\alpha$ a Generalized Laguerre weight of parameters $\alpha \geq 0$, $\beta > \frac{1}{2}$, let

$$\mathcal{H}_0(fw, t) := \int_0^{+\infty} \frac{f(x)}{x-t} w(x) dx, \tag{1.1}$$

where the integral is in the Cauchy principal value sense and let

$$\mathcal{H}_1(fw, t) \quad := \quad \int_0^{+\infty} \frac{f(x)}{(x-t)^2} w(x) dx = \frac{d}{dt}\mathcal{H}_0(fw, t), \quad \alpha \geq 0 \tag{1.2}$$

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the Finite Part (FP) of the integral in the Hadamard sense.

In the present paper we propose a method for approximating $\mathcal{H}_0(fw,t)$ and $\mathcal{H}_1(fw,t)$, by a process of simultaneous approximation. Integrals of both the types are of interest in many contexts, such as singular and hypersingular boundary integral equations, which are tools for modeling many phenomena arising in different areas of physics (see [1], [2], [3], [4], [5] and the references there in).

While the literature on the topic is rich in the case of bounded intervals [6], [3],[7], [8], [9], [10], [11], [1], [12] less attention has been paid to the case of unbounded ones. On the other hand, FP integrals over unbounded ranges, reducible to the type (1.1), are employed in the solution of hypersingular integral equations coming from Neumann 2D elliptic problems on semiplanes by using a Petrov-Galerkin infinite BEM approach [4].

Start from

$$\begin{aligned}\mathcal{H}_0(fw,t) &= \int_0^{+\infty} \frac{f(x)-f(t)}{x-t} w(x)dx + f(t) \fint_0^{+\infty} \frac{w(x)}{x-t} dx \\ &=: \mathcal{F}(fw,t) + f(t)\mathcal{H}_0(w,t).\end{aligned}$$

Moreover, in view of (1.2)

$$\mathcal{H}_1(fw,t) = \frac{d}{dt}\mathcal{F}(fw,t) + f(t)\mathcal{H}_1(w,t) + f'(t)\mathcal{H}_0(w,t).$$

Since for many choices of α, β an analytical expression for $\mathcal{H}_0(w,t)$ is known, we approximate $\mathcal{F}(fw,t)$ by the Lagrange polynomial $\mathcal{L}_{s+1}(\mathcal{F}(fw))$ of degree s (s "small") and $\mathcal{F}(fw,t)'$ by its derivative. To compute the samples $\mathcal{F}(fw)$ at the interpolation knots we use a "truncated" Gaussian rule based on the zeros $\{x_{m,k}\}_{k=1}^{m}$ of the generalized Laguerre polynomial $p_m(w)$. Then we get

$$\mathcal{H}_0(fw,t) \simeq \mathcal{L}_{s+1}(\mathcal{F}(fw),t) + f(t)\mathcal{H}_0(w,t),$$

$$\mathcal{H}_1(fw,t) \simeq \frac{d}{dt}\left(\mathcal{L}_{s+1}(\mathcal{F}(fw),t)\right) + f(t)\mathcal{H}_1(w,t) + f'(t)\mathcal{H}_0(w,t).$$

The previous scheme can be applied for m large enough, s.t. $t \in (x_{m,k}, x_{m,k+1})$, for some k. When t is "large", a careful use of the Gaussian rule gives satisfactory results. "Intermediate" cases can be treated combining both of the aforesaid procedures. An additional advantage we propose here is to avoid the computation of $f'(t)$, if it is required, by reusing the same samples employed in the approximation of $\mathcal{F}(fw)$, with the same rate of convergence. We prove that our procedure is convergent, giving error estimates for f in some Sobolev spaces. Moreover, we propose some numerical tests, confirming the theoretical error estimates.

The plan of the paper is the following: next section contains some preliminary results and notations; Section 3 includes the new method and the corresponding estimate of the error. Section 4 contains some numerical tests, while Section 5 includes the proofs.

2. Notations and basic tools

Along all the paper the constant $\mathcal{C}$ will be used several times, having different meaning in different formulas. From now on we will write $\mathcal{C} \neq \mathcal{C}(a, b, \ldots)$ in order to say that $\mathcal{C}$ is a positive constant independent of the parameters $a, b, \ldots$, and $\mathcal{C} = \mathcal{C}(a, b, \ldots)$ to say that $\mathcal{C}$ depends on $a, b, \ldots$. Moreover, if $A, B \geq 0$ are quantities depending on some parameters, we will write $A \sim B$, if there exists a constant $0 < \mathcal{C} \neq \mathcal{C}(A, B)$ such that $\frac{B}{\mathcal{C}} \leq A \leq \mathcal{C}B$.

For any bivariate function $h(x, y)$, h_x and h_y will denote the function in the single variable y or x, respectively.

$\mathbb{P}_m$ will denote the space of the algebraic polynomials of degree at most m.

Let $\{p_m(w)\}_m$ be the sequence of the orthonormal polynomials w.r.t. w with positive leading coefficients. By $a_m(w)$ we will denote the $m-$th Mhaskar-Rachmanoff-Saff number w.r.t w (in the sequel M-R-S number) and denoted by $x_{m,k}, k = 1, \ldots, m$, the zeros of $p_m(w)$, we recall [13]

$$\frac{\mathcal{C}a_m(w)}{m^2} < x_{m,1} < x_{m,2} < \cdots < x_{m,m} < a_m(w) \left(1 - \frac{\mathcal{C}}{m^{\frac{2}{3}}}\right),$$

$$a_m(w) = C(\alpha, \beta) m^{\frac{1}{\beta}} \sim m^{\frac{1}{\beta}}. \tag{2.3}$$

From now on, for any fixed $0<\theta<1$, $j:=j(m)$ will be defined as

$$j=\min_{k=1,2,..,m}\{k : x_{m,k}\geq a_m\theta\}\,. \tag{2.4}$$

Setting $\Delta x_{m,k}=x_{m,k+1}-x_{m,k}$, we recall that uniformly in $m\in\mathbb{N}$, [14]

$$\Delta x_{m,k}\sim\frac{\sqrt{a_m(w)}}{m}\sqrt{x_{m,k}}\sim\Delta x_{m,k-1},\quad k=1,2,\dots,j, \tag{2.5}$$

2.1. Functional Spaces

With $u(x):=u_{\gamma,\beta}(x)=x^\gamma e^{-x^\beta/2}, \gamma\geq 0,\ \beta>\frac{1}{2}$, we will consider

$$C_u=\begin{cases}\left\{f\in C^0((0,\infty)) : \lim\limits_{\substack{x\to+\infty\\ x\to 0^+}}(fu)(x)=0\right\}, & \gamma>0,\\ \left\{f\in C^0([0,\infty)) : \lim\limits_{x\to+\infty}(fu)(x)=0\right\}, & \gamma=0,\end{cases}$$

equipped with the norm

$$\|f\|_{C_u}=:\|fu\|=\sup_{x\geq 0}|(fu)(x)|\,,$$

where $C^0(E)$ is the space of continuous functions on the set E. Sometimes, for the sake of brevity, we will use $\|f\|_E=\sup_{x\in E}|f(x)|$.

For smoother functions, let us consider the Sobolev-type spaces of order $r\in\mathbb{N}$

$$W_r(u)=\left\{f\in C_u\ :\ f^{(r-1)}\in AC(0,+\infty)\text{ and }\ \|f^{(r)}\varphi^r u\|<+\infty\right\},\quad \varphi(x)=\sqrt{x},$$

where $AC((0,+\infty))$ is the set of all the functions which are absolutely continuous on every closed subset of $(0,+\infty)$, equipped with the norm

$$\|f\|_{W_r(u)}:=\|fu\|+\|f^{(r)}\varphi^r u\|.$$

Denoting by

$$E_m(f)_u=\inf_{P\in\mathbb{P}_m}\|(f-P)u\|$$

the error of best polynomial approximation in C_u, for any function $f\in W_r(u)$ the following estimate holds [15]

$$E_m(f)_u\leq\mathcal{C}\left(\frac{\sqrt{a_m}}{m}\right)^r\|f^{(r)}\varphi^r u\|,\quad \mathcal{C}\neq\mathcal{C}(m,f), \tag{2.6}$$

where $a_m:=a_m(u)$ denotes the M-R-S number w.r.t u. Since by (2.3) $a_m(u)\sim a_m(w)$, throughout we employ the same symbol a_m to denote both of them.

2.2. Truncated Gaussian rule

The so called "truncated" Gauss-Laguerre rule ([16], [17]) is based on the first j zeros of $p_m(w)$, with j defined in (2.4), i.e.

$$\int_0^{+\infty} f(x)w(x)dx = \sum_{k=1}^{j} f(x_{m,k})\lambda_{m,k} + R_m(f), \tag{2.7}$$

where $\{\lambda_{m,k}\}_{k=1}^m$ are the Christoffel numbers w.r.t. w and $R_m(f)$ is the remainder term. For all $f \in W_r(u)$, under the assumption $\alpha - \gamma > -1$ [16, Proposition 2.3]

$$|R_m(f)| \le \mathcal{C}\left(\frac{\sqrt{a_m}}{m}\right)^r \|f^{(r)}\varphi^r u\|, \; 0 < \mathcal{C} \neq \mathcal{C}(m, f). \tag{2.8}$$

Other estimates of (2.7) can be found in [18].

2.3. Definition of Hadamard finite part integrals over $(0, +\infty)$

We recall that many properties fulfilled by finite-part integrals over bounded intervals can be found in [7] (see also [6], [2]). We recall that assuming $w(x) = e^{-x^\beta}x^\alpha$, $\alpha > -1$

$$\mathcal{H}_1(w,t) := \mathop{=\!\!\!\!\!\!\int}_0^{+\infty} \frac{w(x)}{(x-t)^2}dx,$$

is defined as the FP of the integral in the Hadamard sense, since w is a generalized Hölder-continuous function on $(0, \infty)$, i.e. w is Hölder continuous in any closed subinterval of $(0, \infty)$, with an integrable singularity [19] (see also [20]). By using standard arguments (see for instance [21]), the following equivalent definition holds, under the assumption $\alpha \ge 0$

$$\mathcal{H}_1(w,t) = \frac{d}{dt}\mathop{-\!\!\!\!\!\!\int}_0^{+\infty} \frac{w(x)}{x-t}dx, \quad \alpha \ge 0. \tag{2.9}$$

Finally, assuming $f \in W_2(u)$ and $\alpha \ge 0$,

$$\begin{aligned} \mathcal{H}_1(fw,t) \quad &:= \quad \mathop{=\!\!\!\!\!\!\int}_0^{+\infty} \frac{f(x)}{(x-t)^2}w(x)dx = \mathop{=\!\!\!\!\!\!\int}_0^{+\infty} \frac{f(x)-f(t)-f'(t)(x-t)}{(x-t)^2}w(x)dx \\ &+ \quad f(t)\mathop{=\!\!\!\!\!\!\int}_0^{+\infty} \frac{w(x)}{(x-t)^2}dx + f'(t)\mathop{-\!\!\!\!\!\!\int}_0^{+\infty} \frac{w(x)}{(x-t)}dx, \end{aligned} \tag{2.10}$$

where the first right-hand integral in (2.10) exists for any fixed $t \in [a,b] \subset (0,\infty)$.

3. The method

In what follows we set

$$\mathcal{F}(fw,t) := \int_0^{+\infty} \frac{f(x)-f(t)}{x-t}w(x)dx,$$

$$\mathcal{F}_1(fw,t) := \mathcal{F}(fw,t)' = \int_0^{+\infty} \frac{f(x)-f(t)-f'(t)(x-t)}{(x-t)^2} w(x)dx.$$

Let us start from

$$\mathcal{H}_0(fw,t) = \mathcal{F}(fw,t) + f(t)\mathcal{H}_0(w,t)$$

focusing our attention on $\mathcal{F}(fw,t)$, since $\mathcal{H}_0(w,t)$ can be efficiently computed with at least the same accuracy of $\mathcal{F}(fw,t)$ (see Section 4).

Let $0 < \theta < 1$ be fixed and let j be the index defined in (2.4). Then for any fixed $t \in [a,b] \subset (0, a_m\theta)$, there exists $k \in \{1,2,\dots,j-1\}$ s.t. $t \in [x_{m,k}, x_{m,k+1}]$. Setting

$$t_{i+[\frac{s}{2}]} = \frac{x_{m,k+i} + x_{m,k+i+1}}{2}, \quad i = -\left[\frac{s}{2}\right], \dots, \left[\frac{s+1}{2}\right], \tag{3.11}$$

let $\mathcal{L}_{s+1}(\mathcal{F}(fw))$ be the Lagrange polynomial interpolating $\mathcal{F}(fw)$ at $t_0, t_1, \dots, t_s$, i.e.

$$\mathcal{L}_{s+1}(\mathcal{F}(fw),t) = \sum_{i=0}^{s} \ell_i(t)\mathcal{F}(fw,t_i), \quad \ell_i(t) = \prod_{k=0,k\neq i}^{s} \frac{t-t_k}{t_i-t_k}.$$

In the general case the quantities $\mathcal{F}(fw,t_i)$, $i = 0,1,\dots,s$, cannot be computed exactly and so we will use the truncated Gaussian rule in (2.7) to approximate them, i.e. setting

$$\mathcal{G}_m(fw,t_i) \quad = \quad \sum_{k=1}^{j} \frac{f(x_{m,k})-f(t_i)}{x_{m,k}-t_i}\lambda_{m,k}, \quad i = 0,1,\dots,s, \tag{3.12}$$

we use

$$\mathcal{F}(fw,t_i) \sim \mathcal{G}_m(fw,t_i), \quad i = 0,1,\dots,s.$$

Thus we have

$$\mathcal{H}_0(fw,t) = \mathcal{L}_{s+1}(\mathcal{G}_m(fw),t) + f(t)\mathcal{H}_0(w,t) + \rho_{s,m}^{(0)}(f,t). \tag{3.13}$$

Since

$$\mathcal{H}_1(fw,t) = \mathcal{F}_1(fw,t) + f(t)\mathcal{H}_1(w,t) + f'(t)\mathcal{H}_0(w,t),$$

it seems natural to approximate $\mathcal{F}_1(fw,t)$ by $\mathcal{L}_{s+1}(\mathcal{G}_m(fw),t)'$, i.e.

$$\begin{aligned} \mathcal{H}_1(fw,t) \quad &= \quad \mathcal{L}_{s+1}(\mathcal{G}_m(fw),t)' + f(t)\mathcal{H}_1(w,t) \\ &+ \quad f'(t)\mathcal{H}_0(w,t) + \rho_{s,m}^{(1)}(f,t). \end{aligned} \tag{3.14}$$

The previous work-scheme is essentially a simultaneous approximation process of the function $\mathcal{F}(fw)$ and its first derivative, by means of a “local” Lagrange polynomial interpolating $\mathcal{G}_m(fw)$. Of course, the major advantage is taken whenever both the integrals $\mathcal{H}_0(fw,t)$ and $\mathcal{H}_1(fw,t)$ have to be computed for the same value t.

About the error estimate we are able to prove the following

Theorem 3.1. *For any $f \in W_{r+1}(u)$, $r \geq 1$, $0 < \gamma \leq \alpha$, and for $s \geq r+1$, we have*

$$|\rho_{s,m}^{(i)}(f,t)| \leq \mathcal{C}\left(\frac{\sqrt{a_m}}{m}\right)^{r-i} \|f\|_{W_{r+1}(u)}, \quad i \in \{0,1\}, \tag{3.15}$$

where $\mathcal{C} \neq \mathcal{C}(m,f)$.

Remark 3.1. *The degree s of the Lagrange polynomial depends on the smoothness of the function f, since by (3.15) we have to choose $s \geq r+1$, when $f \in W_{r+1}(u)$ in order to obtain a convergent process. In all the cases, we have to choose the degree $s \ll m$, since the unweighted Lebesgue constants $\{\|\mathcal{L}_s\|_{C_u([a,b])}\}_{s \in \mathbb{N}}$ could grow "enough", so that their contribute could not be ignored.*

Remark 3.2. *Other choices of interpolation knots are possible, provided that they are sufficiently far from the quadrature nodes $\{x_{m,k}(w)\}_{k=1}^{j}$ to avoid numerical cancellation (see [22].*

3.1. Approximation of $\mathcal{H}_1(fw)$ without computing $f'(t)$

Till now we have assumed that the derivative $f'(t)$ should be easily computable. On the other hand, whenever the computation of $f'(t)$ has to be avoided, our procedure can be again performed reusing all the samples employed in the Gaussian rule (3.12).

For any fixed $0 < \theta < 1$, let $x_{m,j}$ be the zero defined in (2.4) and let χ_j be the characteristic function of the segment $(0, x_{m,j})$. For a given function f the Lagrange polynomial

$$L_{m+1}^*(w,f,x) := L_{m+1}(w,\chi_j f,x) = \sum_{k=1}^{j} l_{m+1,k}(x) f(x_{m,k}),$$

$$l_{m+1,k}(x) = \frac{p_m(w,x)(a_m - x)}{p_m'(x_{m,k})(a_m - x_{m,k})(x - x_{m,k})}, \quad k \leq j,$$

interpolates f at the zeros of $p_m(w,x)(a_m - x)$ [16] (see also [23]). The polynomial $L_{m+1}(w,f)$ belongs to the subspace $\mathcal{P}_m^*$ of $\mathbb{P}_m$

$$\mathcal{P}_m^* = \{q \in \mathbb{P}_m : q(x_{m,k}) = q(a_m) = 0, \quad k > j\} \subset \mathbb{P}_m$$

and $L_{m+1}(w)$ projects C_u onto $\mathcal{P}_m^*$. Choosing m as in the truncated Gaussian rule (3.12), all the samples of f involved in the construction of the Lagrange polynomial have been already computed.

About the error committed in approximating f' by $L^*_{m+1}(w,f)'$, the following result holds (see [24]) :

Theorem 3.2. *With $w(x) = e^{-x^\beta} x^\alpha$ and $u(x) = e^{-x^\beta/2} x^\gamma$, assuming that α and γ satisfy $\frac{\alpha}{2} + \frac{1}{4} \le \gamma \le \frac{\alpha}{2} + \frac{5}{4}$, then for any $f \in W_{r+1}(u)$,*

$$\|(f - L^*_{m+1}(w,f))'\varphi u\| \quad \le \quad \mathcal{C} \log m \left(\frac{\sqrt{a_m}}{m}\right)^r \|f\|_{W_{r+1}(u)}, \quad \varphi(x) = \sqrt{x}, \tag{3.16}$$

where $0 < \mathcal{C} \neq \mathcal{C}(m,f)$.

By estimates (3.16) and (3.15), for the error in approximating f' is negligible w.r.t. the error in approximating $\mathcal{F}(fw)'$.

3.2. The case t "large"

The above introduced method is applicable for m sufficiently large so that $t < \theta a_m$. On the other hand, for "large" t, for instance, for $t = 1000, \beta = 1$ and $\theta = \frac{1}{8}$, we must take $m > 2000$. This expensive procedure can be avoided by an easy different approach. To be more precise, for any fixed t, we propose to apply the Gaussian rule for values of m such that

$$t > x_{m,j} + 1,$$

where j is defined in (2.4). Thus, setting $G_{t,i}(x) = \frac{f(x)}{(x-t)^{1+i}}$, $i \in \{0,1\}$, we have

$$\mathcal{H}_i(fw,t) = \sum_{k=1}^{j} G_{t,i}(x_{m,k})\lambda_{m,k} + R_m(G_{t,i}). \tag{3.17}$$

Since f and $G_{t,i}(f)$ have the same smoothness for $t > x_{m,j} + 1$, assuming $f \in W_{r+1}(u), r \ge 0$, by (2.8) it follows

$$|R_m(G_{t,i})| \le \mathcal{C} \left(\frac{\sqrt{a_m}}{m}\right)^{r+1} \|G^{(r+1)}_{t,i} \varphi^{r+1} u\|, \quad i \in \{0,1\}, \tag{3.18}$$

where $0 < \mathcal{C} \neq \mathcal{C}(m, G_{t,i}, t)$.

We remark that, obviously, the error bound in (3.18) holds for a fixed m and therefore the limit of $R_m(G_{t,k})$ as $m \to \infty$ has no meaning.

Remark 3.3. *In conclusion, by using (3.14) or (3.17), we can compute $\mathcal{H}_0(fw,t)$, $\mathcal{H}_1(fw,t)$ for a "wide" range of t. Roughly speaking, for "small" value of t and m large enough, t lies between two consecutive zeros of $p_m(w)$ and we will use (3.14). For t "large " and especially for f sufficiently smooth, we look for m such that t lies* *outside the range $[x_{m,1}, x_{m,j}]$ and then we use (3.17). Finally, in the "middle" case (t is not "too small" or "too large"), we can use a suitable combination of both of the rules (3.17) and (3.14).*

4. Numerical experiments

First we give some details about the computation of $\mathcal{H}_0(w)$ in some relevant cases. In the case $\boxed{\beta = 1}$ it is [25, p.325, n. 16]

$$\int_0^{+\infty} \frac{e^{-x}}{x-t} dx = -e^{-t} Ei(t), \qquad \alpha = 0, \tag{4.19}$$

$$\int_0^{+\infty} \frac{e^{-x} x^\alpha}{x-t} dx = -\pi t^\alpha e^{-t} \cot((1+\alpha)\pi) + \Gamma(\alpha) {}_1F_1(1, 1-\alpha, -t), \quad \alpha \neq 0, \tag{4.20}$$

where Ei is the Exponential Integral function and ${}_1F_1$ is the Confluent Hypergeometric function.

Let $\boxed{\beta \in \mathbb{N}, \beta > 1.}$ We will use

$$\int_0^{+\infty} \frac{e^{-x^\beta} x^\alpha}{x-t} dx = \frac{1}{\beta} \int_0^{+\infty} \frac{y^{\frac{\alpha+1}{\beta}-1}}{y^{\frac{1}{\beta}} - t} e^{-y} dy = \frac{1}{\beta} \sum_{k=0}^{\beta-1} t^{\beta-1-k} \int_0^{+\infty} \frac{y^{\frac{\alpha+1+k}{\beta}-1}}{y - t^\beta} e^{-y} dy, \tag{4.21}$$

combined with (4.19) or (4.20) according to the case $\alpha = 0$ or $\alpha \neq 0$, respectively. Some details in the case $\beta = \frac{p}{q}$, $p,\ q \in \mathbb{N},\ q > 1$ can be found in [26].
About the computation of $\mathcal{H}_1(w,t)$, in view of (2.9), it can be performed combining (4.19)-(4.21) with the following relations [27, p. 1086, 9.213]

$$\frac{d}{dt} Ei(t) = -\frac{d}{dt} \int_{-t}^{+\infty} \frac{e^{-x}}{x} dx = \frac{e^t}{t}, \quad \frac{d}{dt} {}_1F_1(a,b;t) = \frac{a}{b} {}_1F_1(a+1, b+1, t).$$

We note that $w(x) = e^{-x^\beta} x^\alpha$, $\alpha \geq 0$ and $\beta > \frac{1}{2}$, is not a classical weight when $\beta \neq 1$ and, then, the coefficients of the three-term recurrence relation of the corresponding orthonormal polynomials are unknown. In such a case, we have built a suitable algorithm for the computation of the zeros of the polynomials $p_m(w)$ and of the Christoffel numbers. Essentially, such an algorithm consists in computing the moments

$$\mu_k = \int_0^{+\infty} x^k w(x) dx, \quad k = 0, 1, \ldots,$$

in extended arithmetic with high accuracy and, subsequently, in using the functions `aChebyshevAlgorithm` and `aGaussianNodesWeights` of the software package `OrthogonalPolynomials` (see [28]).

Now we show the numerical results obtained by implementing the above introduced procedure. In the next we will denote by $\Phi_{i,m,s}(f,t)$, $i \in 0,1$ the results obtained by means of the rules (3.13) and/or (3.17) for $i=0$ and those by means of (3.14) and/or (3.17) for $i=1$, as well as described in Remark 3.3, i.e. about the approximations

$$\mathcal{H}_0(fw,t) \sim \Phi_{0,m,s}(f,t) := \begin{cases} \mathcal{L}_{s+1}(\mathcal{G}_m(fw),t) + f(t)\mathcal{H}_0(w,t), & t \in [a,b] \subset (x_{m,1}, a_m\theta) \\ \\ \displaystyle\sum_{i=1}^{j} G_{t,0}(x_{m,i})\lambda_{m,i}, & t > x_{m,j}+1. \end{cases}$$

$$\mathcal{H}_1(fw,t) \sim \Phi_{1,m,s}(f,t) := \begin{cases} \mathcal{L}_{s+1}(\mathcal{G}_m(fw),t)' + f(t)\mathcal{H}_1(w,t) & +f'(t)\mathcal{H}_0(w,t), \\ & t \in [a,b] \subset (x_{m,1}, a_m\theta) \\ \\ \displaystyle\sum_{i=1}^{j} G_{t,1}(x_{m,i})\lambda_{m,i}, & t > x_{m,j}+1. \end{cases}$$

In each example for any t we report the values computed for different choices of s, m with the only settled digits and for any m the index $j(m)$, depending on θ, empirically detected under the criterion

$$j := \max_{0\le k\le s} \max_{i=1,\dots,m} \left\{ i \ : \ \left| \lambda_{m,i} \frac{f(x_{m,i}) - f(t_k)}{x_{m,i} - t_k} \right| \ge eps \right\},$$

where eps is the machine precision. We remark that j is the effective number of terms of the Gaussian sum (3.12) and therefore to compute simultaneously $\Phi_{i,m,s}(f,t)$, $i \in 0,1$, $j(s+1)$ function's evaluations are required. This data is especially relevant when $j \ll m$, for instance for bounded or decreasing functions f.

The algorithm for the computation of $\Phi_{i,m,s}(f,t)$, $i \in 0,1$, was implemented in Wolfram Mathematica Language. All the computations were performed in Wolfram Mathematica 9.0 on a PC with a Intel Core i7-6700HQ 3.5 GHz processor and 4 GB of memory. In each table, in the column labelled by T we reported the time in second required for the simultaneous computation of $\Phi_{i,m,s}(f,t)$, $i \in 0,1$ (computed by the function `Timing`). As one can see, for $\beta \neq 1$, the timings are a little bit longer due to the more expensive procedure used for the computation of the zeros of $p_m(w)$ and of the corresponding Christoffel numbers.

Example 4.1.

$$\mathcal{H}_i(fw_{\frac{1}{2}},t) = \fint_0^{+\infty} \frac{|x-2|^{\frac{7}{2}}}{(x-t)^{i+1}}\sqrt{x}e^{-x}dx, \quad f(x) = |x-2|^{\frac{7}{2}}, \quad i = 0, 1.$$

We have $f \in W_3(u)$ with $\gamma < \frac{1}{2}$ and, according to the theoretical estimate in (3.15), the error behaves like $\frac{1}{m^{\left(1-\frac{i}{2}\right)}}$, $\quad i \in \{0,1\}$. Then, with $m = 300$ we can expect 2 exact digits for $i = 0$ and 1 exact digit for $i = 1$. By inspecting Tables 2, we observe that when t is close to the critical point 2 the numerical errors are comparable with the theoretical ones, while better behaviors are attained when $|t-2|$ is "large" enough.

	m	j	$\Phi_{0,m,s}(f,0.1)$	$\Phi_{1,m,s}(f,0.1)$	T
s=5	300	74	6.51129	−47.10118	2.046875
	500	92	6.511296	−47.1011860	5.546875

	m	j	$\Phi_{0,m,s}(f,2.00000001)$	$\Phi_{1,m,s}(f,2.00000001)$	T
s=7	100	49	−0.1	1.1	0.25
	300	85	−0.108	1.1	2.078125
	700	129	−0.1088	1.18	10.906550

	m	j	$\Phi_{0,m,s}(f,110.1)$	$\Phi_{1,m,s}(f,110.1)$	T
s=5	60	36	−0.05436	0.0005177	0.09375
	100	61	−0.054360	0.00051772	0.234375
	110	67	−0.054360100096815	0.000517721419080	0.296875

Table 1: Example 4.1: $t = 0.1$, $t = 2.00000001$ and $t = 110.1$

Example 4.2.

$$\begin{aligned} \mathcal{H}_i(fw_{\frac{5}{2}},t) &= \fint_0^{+\infty} \frac{\sinh\left(\frac{x}{8}\right)\left|x-\frac{1}{2}\right|^{9/2}}{(x-t)^{i+1}} x^{\frac{5}{2}} e^{-x} dx, \\ f(x) &= \sinh\left(\frac{x}{8}\right)\left|x-\frac{1}{2}\right|^{9/2}, \quad i = 0, 1. \end{aligned}$$

Since $f \in W_4(u), \gamma < \frac{5}{2}$, according to Theorem 3.1, the error behaves like $\frac{1}{(\sqrt{m})^{3-i}}$, $\quad i \in \{0,1\}$. Note that for "large" m almost a 77% reduction in function's evaluations is taken.

	m	j	$\Phi_{0,m,s}(f, 0.4999901)$	$\Phi_{1,m,s}(f, 0.4999901)$	T
s=5	100	57	594.1586	84.66	0.265625
	300	100	594.15864	84.6632	2.078125
s=7	400	115	594.158641	84.66324	3.734375
	m	j	$\Phi_{0,m,s}(f, 3)$	$\Phi_{1,m,s}(f, 3)$	T
s=4	200	81	984.51	256.4	0.953125
	300	100	984.518	256.42	2.093750
	600	141	984.518	256.426	7.906225
s=5	400	115	984.5180	256.4269	3.609375
s=7	200	81	984.518022	256.42690	0.968750
	400	115	984.5180225	256.426907	3.703125
	600	141	984.51802252	256.42690786	8.015625

Table 2: Example 4.2: $t = 0.4999901$ and $t = 3$

Example 4.3.

$$H_i(fw_{\frac{5}{2}}, t) = \fint_0^{+\infty} \frac{|\sin(x-2)|^{\frac{13}{2}}}{(x-t)^{i+1}} x^{\frac{5}{2}} e^{-x^3} dx, \quad f(x) = |\sin(x-2)|^{\frac{13}{2}}, \quad i = 0, 1.$$

Taking into account that $f \in W_6(u), \gamma < \frac{5}{2}, \ \beta = 3$, in view of (3.15) the errors behave like $\frac{1}{m^{25/6}}$ and $\frac{1}{m^{20/6}}$ for $\mathcal{H}_0(fw)$ and $\mathcal{H}_1(fw)$, respectively. We emphasize a satisfactory accuracy achieved by interpolating polynomials of low degrees as 5 and 7.

Example 4.4.

$$\mathcal{H}_i(fw_{\frac{3}{2}}, t) = \fint_0^{+\infty} \frac{|x-5|^{\frac{9}{2}}}{(x-t)^{i+1}} x^{\frac{3}{2}} e^{-x^2} dx, \quad f(x) = |x-5|^{\frac{9}{2}}, \quad i = 0, 1.$$

In this case $f \in W_4(u_\gamma), \gamma < \frac{3}{2}, \beta = 2$, and according to the theoretical estimate, the errors behave like $\frac{1}{m^{9/4}}$ and $\frac{1}{m^{3/2}}$ for $\mathcal{H}_0(fw)$ and $\mathcal{H}_1(fw)$, respectively. Also in this case, the numerical results agree with the predicted estimates, especially when $|t-5|$ is "small".

5. The proofs

Letting $F_t(x) = \frac{f(x)-f(t)}{x-t}$, we prove the following

	m	j	$\Phi_{0,m,s}(f,0.5)$	$\Phi_{1,m,s}(f,0.5)$	T
s=5	100	74	0.4120349	-0.97005	0.406250
	200	123	0.4120349580	-0.97005	1.515625
	400	210	0.41203495807	-0.97005804	5.484375
	700	327	0.412034958073	-0.970058045	17.906250
s=7	300	168	0.4120349580732	-0.9700580458	3.140625
	400	210	0.41203495807324	-0.97005804584	5.578125
	500	251	0.41203495807324	-0.970058045849	8.796875
s=9	800	363	0.412034958073247	-0.9700580458499	23.843750
	m	j	$\Phi_{0,m,s}(f,2.5)$	$\Phi_{1,m,s}(f,2.5)$	T
s=5	100	75	-0.0789852	0.046	0.406250
	300	173	-0.078985249	0.0463015	3.171875
	500	257	-0.0789852498	0.04630157	8.828125
s=7	400	216	-0.07898524983	0.046301573	5.625
	500	257	-0.078985249831	0.0463015733	8.843750
s=9	500	257	-0.0789852498311	0.046301573325	8.859375

Table 3: Example 4.3: $t = 0.5$ and $t = 2.5$

Lemma 5.1. *Let $\gamma \geq 0$ and $1 \leq q \in \mathbb{N}$. If $f \in W_q(u)$ then, for any fixed $t \in [a,b] \subset (0, a_m\theta)$, $F_t \in W_{q-1}(u)$.*

proof Under the assumption on f it follows $F_t \in C_u$, since both the limit conditions hold. Moreover, by a result in ([19]) $F_t \in C^{(q-1)}([a,b]) \subset W_{q-1}(u)$. Thus we have to prove

$$\|F_t\|_{W_{q-1}(u)} \leq \mathcal{C}\|f\|_{W_q(u)}, \tag{5.22}$$

where $\mathcal{C} \neq \mathcal{C}(m,f)$. By the change of variable $\tau = t + z(x-t)$ we get

$$F_t(x) = \int_0^1 f'(t+z(x-t))dz, \quad F_t^{(q-1)}(x) = \int_0^1 f^{(q)}(t+z(x-t))z^{q-1}dz,$$

	m	j	$\Phi_{0,m,s}(f, 0.25)$	$\Phi_{1,m,s}(f, 0.25)$	T
s=5	100	68	615.79093312872	-466.691155632	0.390625
	200	109	615.790933128721	-466.6911556321	1.453125
	m	j	$\Phi_{0,m,s}(f, 4.999)$	$\Phi_{1,m,s}(f, 4.999)$	T
s=5	100	67	-60.716482	14.7554	0.390625
	800	283	-60.7164828	14.75541	23.656280
s=7	800	283	-60.71648281	14.755418	23.703125

Table 4: Example 4.4: $t = 0.25$ and $t = 4.999$

$$\begin{aligned}\varphi^{q-1}(x)u(x)|F_t^{(q-1)}(x)| &= \varphi^{q-1}(x)u(x)\left|\int_0^1 f^{(q)}(t+z(x-t))z^{q-1}dz\right| \\ &\leq \|f^{(q)}\varphi^q u\|\int_0^1 \frac{\varphi^{q-1}(x)u(x)z^{q-1}}{\varphi^q(t+z(x-t))u(t+z(x-t))}dz \\ &= \|f^{(q)}\varphi^q u\|\int_t^x \frac{\varphi^{q-1}(x)u(x)}{\varphi^q(\tau)u(\tau)}\frac{(\tau-t)^{q-1}}{(x-t)^q}d\tau.\end{aligned}$$

Now, taking into account the assumption $t \in [a, b] \subset (0, +\infty)$, if $x > t$

$$\int_t^x \frac{\varphi^{q-1}(x)u(x)}{\varphi^q(\tau)u(\tau)}\frac{(\tau-t)^{q-1}}{\sqrt{\tau}(x-t)^q}d\tau \quad \leq \quad \frac{\mathcal{C}}{\sqrt{t}}\frac{1}{(x-t)^q}\int_t^x (\tau-t)^{q-1}d\tau \leq \mathcal{C},$$

since $\varphi^q(\tau)u(\tau) > \varphi^q(x)u(x)$ and, for $x < t$

$$\int_x^t \frac{\varphi^{q-1}(x)u(x)}{\varphi^q(\tau)u(\tau)}\frac{(t-\tau)^{q-1}}{(t-x)^q}d\tau \leq \frac{\varphi^{q-1}(x)u(x)}{\varphi^q(t)u(t)}\int_x^t \frac{(t-\tau)^{q-1}}{(t-x)^q}d\tau \leq \frac{e^{\frac{t^\beta - x^\beta}{2}}}{\sqrt{t}} \leq \mathcal{C},$$

since $\varphi^q(\tau)u(\tau) > \varphi^q(t)u(t)$. Therefore, for any $x \geq 0$, (5.22) follows. □

Lemma 5.2. *[24, Lemma 6.3] Let $0 < \gamma \leq \alpha$ and $1 \leq q \in \mathbb{N}$. If $f \in W_q(u)$ then $\mathcal{F}(fw) \in W_{q-1}(u)$ and, for any $t > 0$,*

$$|\mathcal{F}(fw)^{(q-1)}(t)\varphi^{q-1}(t)u(t)| \leq \mathcal{C}\|f\|_{W_q(u)}, \ \mathcal{C} \neq \mathcal{C}(m, f). \tag{5.23}$$

Proof of Theorem 3.1 Denoting by $\delta_{k,s} \equiv [t_0, t_s]$, where t_i are defined in (3.11), by (2.5), we have

$$|\delta_{k,s}| \leq \left(x_{m,k+\left[\frac{s+1}{2}\right]+1} - x_{m,k-\left[\frac{s}{2}\right]}\right) \sim (s+1)\frac{\sqrt{a_m}}{m}\sqrt{t}, \quad t \in \delta_{k,s}. \tag{5.24}$$

We prove (3.15) only for $i = 1$, since the case $i = 0$ is simpler. We have

$$\begin{aligned} |\rho^{(1)}_{s,m}(f,t)| &\leq |\mathcal{F}(fw,t)' - \mathcal{L}_{s+1}(\mathcal{F}(fw),t)'| + |\mathcal{L}_{s+1}(\mathcal{F}(fw) - \mathcal{F}_m(fw),t)'| \\ &=: I_1(t) + I_2(t). \end{aligned} \tag{5.25}$$

Using [29, Theorem 3.19]

$$\|I_1\|_{\delta_{k,s}} \leq \mathcal{C}|\delta_{k,s}|^{r-1}\omega_{s-r}(\mathcal{F}(fw)^{(r)}, |\delta_{k,s}|),$$

where the function $\omega_k(g,t)$ denotes the $k-$th modulus of continuity of a given function $g \in C^0([a,b])$. Since by Lemma 5.1 $\mathcal{F}(fw) \in C^r([a,b])$ for any closed $[a,b] \subset (0, a_m\theta)$, we have

$$\|I_1\|_{\delta_{k,s}} \leq \mathcal{C}|\delta_{k,s}|^{r-1}\|\mathcal{F}(fw)^{(r)}\|_{\delta_{k,s}}.$$

Thus, by (5.23) and (5.24), we obtain

$$\|I_1\|_{\delta_{k,s}} \leq \mathcal{C}\left(\frac{\sqrt{a_m}}{m}\right)^{r-1}\|\mathcal{F}(fw)^{(r)}\varphi^r u\| \leq \mathcal{C}\left(\frac{\sqrt{a_m}}{m}\right)^{r-1}\|f\|_{W_{r+1}(u)}. \tag{5.26}$$

Using the Markov-Bernstein inequality [30, p.236]

$$\|p_n'\|_{[a,b]} \leq \frac{2n^2}{b-a}\|p_n\|_{[a,b]}, \quad \forall p_n \in \mathbb{P}_n,$$

and taking into account (5.24) again, we get

$$\begin{aligned} \|I_2\|_{\delta_{k,s}} &\leq \frac{2s^2}{|\delta_{k,s}|}\|\mathcal{L}_{s+1}(\mathcal{F}(fw) - \mathcal{F}_m(fw))\|_{\delta_{k,s}} \\ &\leq \frac{\mathcal{C}}{\sqrt{t}}\frac{m}{\sqrt{a_m}}\|\mathcal{L}_{s+1}\|_{\delta_{k,s}}\|\mathcal{F}(fw) - \mathcal{F}_m(fw)\|_{\delta_{k,s}}, \ \mathcal{C} = \mathcal{C}(s), \end{aligned}$$

where $\|\mathcal{L}_{s+1}\|_{\delta_{k,s}} = \sup_{\|f\|_{\delta_{k,s}}=1}\|\mathcal{L}_{s+1}(f)\|_{\delta_{k,s}}$ denotes the Lebesgue constant in C_u. Since $\mathcal{L}_{s+1}(g)$ is a local Lagrange polynomial on $s+1$ knots, s fixed, $s \ll m$, it is $\|\mathcal{L}_{s+1}\|_{\delta_{k,s}} \leq \mathcal{C}$, $\mathcal{C} = \mathcal{C}(s)$. Moreover, by Lemma 5.1 $F_t \in W_r(u)$ for any fixed t and then, by (2.8) and (2.6), we deduce

$$\|\mathcal{F}(fw) - \mathcal{F}_m(fw)\|_{\delta_{k,s}} \leq \mathcal{C}\left(\frac{\sqrt{a_m}}{m}\right)^{r}\|F_t\varphi^r u\| \leq \mathcal{C}\left(\frac{\sqrt{a_m}}{m}\right)^{r}\|f\|_{W_{r+1}(u)},$$

where $\mathcal{C} \neq \mathcal{C}(m,f)$. Consequently

$$\|I_2\|_{\delta_{k,s}} \leq \mathcal{C}\left(\frac{\sqrt{a_m}}{m}\right)^{r-1}\|f\|_{W_{r+1}(u)}. \tag{5.27}$$

The thesis follows combining (5.26) and (5.27) with (5.25).□

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