A Nyström method for integral equations with fixed singularities of Mellin type in weighted L^p spaces☆

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Abstract

We consider integral equations of the second kind with fixed singularities of Mellin type. According to the behavior of the Mellin kernel, we first determine suitable weighted L^p spaces where we look for the solution. Then, for its approximation, we propose a numerical method of Nyström type based on a Gauss-Jacobi quadratura formula. Actually, a slight modification of the classical procedure is introduced in order to prove convergence results in weighted L^p spaces. Moreover, a preconditioning technique allows us to solve well conditioned linear systems. We show the efficiency of the proposed method through some numerical tests.

Keywords: Mellin kernel, integral equation of Mellin type, Nyström method, Lagrange interpolation, Gaussian rule
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1. Introduction

This paper is concerned with the numerical solution of second kind integral equations with fixed singularities of Mellin convolution type given by

$$f(y) + \int_0^1 k(x,y)f(x)dx + \int_0^1 h(x,y)f(x)dx = g(y), \quad y \in (0,1], \tag{1}$$

where f is the unknown, $h(x,y)$ and $g(y)$ are sufficiently smooth functions on $[0,1] \times [0,1]$ and $[0,1]$, respectively, and

$$k(x,y) = \frac{1}{x}\bar{k}\left(\frac{y}{x}\right) \tag{2}$$

is the Mellin kernel defined by means of a function $\bar{k} : [0,+\infty) \to [0,+\infty)$ satisfying suitable assumptions. Letting

$$(Kf)(y) = \int_0^1 k(x,y)f(x)dx, \tag{3}$$

$$(Hf)(y) = \int_0^1 h(x,y)f(x)dx \tag{4}$$

and denoting by I the identity operator, we can rewrite the equation (1) as follows

$$(I + K + H)f = g. \tag{5}$$

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The increasing interest in numerical methods for such type of integral equations, usually indicated as of *Mellin type*, comes from their large range of applications, particularly in engineering. They usually arise from a boundary integral equation reformulation of linear elliptic PDEs on nonsmooth domains. For example, equations of type (1) with

$$\bar{k}(t) = \frac{1}{\pi} \frac{|\sin \alpha|}{1 - 2t \cos \alpha + t^2}$$

occur when the single layer potential is used to solve the Neumann problem for the Laplace's equation on domains forming corners with interior angle α (see [1, 2]).

Because of the fixed singularity of the kernel $k(x, y)$ at the point $x = y = 0$, the Mellin convolution operator K could be not bounded with respect to the uniform norm. As a further consequence, the solution of the integral equation (1) or one of its derivatives could be singular at the origin. In order to deal with these singularities, some Authors (see, for example, [3, 4, 5, 6, 7, 8, 9]) proposed to introduce suitable smoothing changes of variables which improve the behavior of the unknown function f and let to carry out the analysis of the problem in L^p spaces. Then, in order to achieve the numerical solution, suitable approximation schemes involving pieciewise polynomials, global algebraic polynomials or trigonometric ones (see, for instance, [10, 11, 6, 7, 12, 3, 9, 13, 14, 4, 5, 8]) are applied, often combined with these regularization techniques.

An alternative strategy can be represented by the study of (1) directly in weighted L^p spaces, without it being necessary to make use of any smoothing transformation. In such a case, the purpose of a numerical method will be to approximate the searched solution in this type of space.

The latter is the approach which we are going to follow in the present paper. Once we have determined a suitable weighted space $L^p_{v^{-\beta}}$, with $v^{-\beta}(x) = x^{-\beta}$, $-1 + \frac{1}{p} < \beta < \frac{1}{p}$, where we look for the solution of the integral equation (1), we apply a Nyström method which uses the Gauss-Jacobi quadrature rule with respect to the weight function $v^{\beta}(x) = x^{\beta}$. As it is usual in the analysis of numerical procedure for the solution of Mellin convolution equations, stability can be proved only by slightly modifying the classical method in a neighborhood of the singular point.

Generalizing some recent results given in [15] (where the analysis of the method was performed only in the L^2 space), we propose a procedure based on a suitable modification of the adopted integration formula. It is employed in the numerical computation of the transform $(Kf)(y)$ when the evaluation point y lies very close to the origin. By introducing such a modification, if the stability of the method is assumed, we are able to give an error estimate which shows how the convergence order depends on the solution behaviour close to the singularity 0. Anyway, the stability and convergence are amply demonstrated through several numerical tests.
Moreover, for the computation of the Nyström approximation of the solution f, inspired by a preconditioning technique in [16], we can reduce to solve well conditioned linear systems, such that the condition numbers of the system matrix do not increase with the matrix dimension.

The contents of the paper are so organized. Section 2 provides some preliminaries useful in the sequel. In Section 3 the discrete operators approximating the integral ones K and H are introduced. In Section 4 the numerical procedure is described. Section 5 is dedicated to the discussion of some computational aspects (the preconditioning of the linear systems and the choice of the free parameters involved in the numerical method). Section 6 contains the proof of the theoretical results and in Section 7 we present some numerical tests which show the efficiency of the proposed procedure.

2. Preliminaries

2.1. Notation and basic facts

In the sequel $\mathcal{C}$ denotes a positive constant which may have different values in different formulas. We will write $\mathcal{C}(a, b, \ldots)$ to say that $\mathcal{C}$ depends only on the parameters $a, b, \ldots$ and $\mathcal{C} \neq \mathcal{C}(a, b, \ldots)$ to say that $\mathcal{C}$ is independent of the parameters $a, b, \ldots$.
Moreover, we will use the following notation for the kernels

$$k(x, y) = k_x(y) = k_y(x), \qquad h(x, y) = h_x(y) = h_y(x).$$

Let us introduce the function spaces where we are going to study the equation (5).

With $v^\rho(x) = x^\rho$, $\rho > -\frac{1}{p}$, $1 \le p < +\infty$, we denote by $L^p_{v^\rho}$, the weighted space of all real-valued measurable functions F on $[0,1]$ such that

$$\|F\|_{L^p_{v^\rho}} = \|v^\rho F\|_p = \left(\int_0^1 |v^\rho(x)F(x)|^p dx\right)^{\frac{1}{p}} < +\infty.$$

Moreover, we consider the following weighted Sobolev type spaces

$$W^p_r(v^\rho) = \left\{F \in L^p_{v^\rho} :\ F^{(r-1)} \in AC(0,1),\ \|v^\rho F^{(r)}\varphi^r\|_p < +\infty\right\},$$

where r is a positive integer, $\varphi(t) = \sqrt{t(1-t)}$ and $AC(0,1)$ denotes the collection of all functions which are absolutely continuous on every closed subset of $(0,1)$, endowed with the norm

$$\|F\|_{W^p_r(v^\rho)} = \|v^\rho F\|_p + \|v^\rho F^{(r)}\varphi^r\|_p.$$

The notation L^p and W^p_r refers to the case $\rho = 0$.

Let us denote by

$$E_m(F)_{v^\rho,p} = \inf_{P\in\mathbb{P}_m} \|v^\rho(F-P)\|_p$$

the error of best approximation of the function $F \in L^p_{v^\rho}$ by means of polynomials of degree at most m ($\mathbb{P}_m$ is the set of all polynomials of degree at most m). For functions F belonging to $W^p_r(v^\rho)$ the following estimate

$$E_m(F)_{v^\rho,p} \le \frac{\mathcal{C}}{m^r}\|v^\rho F^{(r)}\varphi^r\|_p, \quad \mathcal{C} \ne \mathcal{C}(m,F), \tag{6}$$

holds true (see, for example, [17, (2.5.22), p. 172]).

Now, since the proposed method will be of Nyström type we need a quadrature formula to approximate the integrals. We will use a Gauss-Jacobi quadrature rule on the interval $[0,1]$ w.r.t. a weight v^ρ (see, for instance, [18])

$$\int_0^1 F(x)v^\rho(x)dx = \sum_{j=1}^m \lambda_{m,j}F(x_{m,j}) + e_m(F), \tag{7}$$

where $x_{m,j}, j = 1,\dots,m$, are the zeros of the m-th Jacobi polynomial orthogonal w.r.t. the weight v^ρ, $\lambda_{m,j}, j = 1,\dots,m$, are the corresponding Christoffel numbers and $e_m(F)$ is the remainder term. We recall that for functions F belonging to $W^1_r(v^\rho)$ the following estimate of the quadrature error

$$|e_m(F)| \le \frac{\mathcal{C}}{m^r}\int_0^1 v^\rho(x)|F^{(r)}(x)|\varphi^r(x)dx, \quad \mathcal{C} \ne \mathcal{C}(m,F), \tag{8}$$

holds true (see, for example, [17, Theorem 5.1.8, p. 338] and [17, (2.5.22), p. 172]).

2.2. Mapping properties of the operators K and H

Next theorem establishes the continuity of the operator K defined in (3) in suitable weighted L^p spaces.

Theorem 2.1. *If the function $\bar{k}(t)$ in (2) satisfies*

$$\int_0^{+\infty} t^{\frac{1}{p}-1-\beta}\bar{k}(t)dt < +\infty, \tag{9}$$

with $1 \le p < +\infty$ and $\beta < \frac{1}{p}$, then the operator $K : L^p_{v^{-\beta}} \longrightarrow L^p_{v^{-\beta}}$ is continuous and

$$\|K\|_{L^p_{v^{-\beta}}\to L^p_{v^{-\beta}}} \le \int_0^{+\infty} t^{\frac{1}{p}-1-\beta}\bar{k}(t)dt.$$

Concerning the operator H given in (4), we state the following theorem.

Theorem 2.2. *If the kernel $h(x,y)$ satisfies*

$$\sup_{0\le x\le 1} \|h_x\|_{W_r^p(v^{-\beta})} < +\infty, \tag{10}$$

with $1 \le p < +\infty$ and $-1+\frac{1}{p} < \beta < \frac{1}{p}$, then the operator $H : L^p_{v^{-\beta}} \longrightarrow L^p_{v^{-\beta}}$ is compact.

Using the above theorems, the following result is a consequence of the Neumann series theorem and [2, Corollary 3.8].

Theorem 2.3. *Let $1 \le p < +\infty$ and $-1+\frac{1}{p} < \beta < \frac{1}{p}$, and let us assume that $\mathrm{Ker}(I+K+H) = \{0\}$ in $L^p_{v^{-\beta}}$. If the function $\bar{k}(t)$ in (2) satisfies*

$$\int_0^{+\infty} t^{\frac{1}{p}-1-\beta}\bar{k}(t)dt < 1 \tag{11}$$

and the kernel $h(x,y)$ satisfies (10), then the equation (5) admits a unique solution f in $L^p_{v^{-\beta}}$ for each right-hand side $g \in L^p_{v^{-\beta}}$.

Since, as one can deduce from the above theorem, the unique solution f of the equation (5) belongs to the space $L^p_{v^{-\beta}}$, we can find a function $\bar{f} \in L^p$ such that $f = v^\beta \bar{f}$.

It is easy to prove that the equation $(I+K+H)f = g$ is equivalent to the following equation

$$(\bar{I}+\bar{K}+\bar{H})\bar{f} = g, \tag{12}$$

where $\bar{I} = v^\beta I$,

$$(\bar{K}\bar{f})(y) = \int_0^1 k(x,y)\bar{f}(x)v^\beta(x)dx,$$

and

$$(\bar{H}\bar{f})(y) = \int_0^1 h(x,y)\bar{f}(x)v^\beta(x)dx,$$

in the sense that if f is the unique solution of $(I+K+H)f = g$ then $\bar{f} = v^{-\beta}f$ is the unique solution of $(\bar{I}+\bar{K}+\bar{H})\bar{f} = g$ and vice versa.

Moreover, the following result is a consequence of Theorems 2.1-2.3

Corollary 2.1. *Let $1 \le p < +\infty$, $-1+\frac{1}{p} < \beta < \frac{1}{p}$ and let us assume that $\mathrm{Ker}(\bar{I}+\bar{K}+\bar{H}) = \{0\}$ in L^p. If the function $\bar{k}(t)$ defined in (2) satisfies (11) and the kernel $h(x,y)$ satisfies (10), then the equation (12) admits a unique solution $\bar{f}$ in L^p for each right-hand side $g \in L^p_{v^{-\beta}}$.*

3. The discrete operators

In order to introduce a Nyström type method for the numerical solution of the integral equation (12), we consider two suitable discrete operators for approximating the operators $\bar{K}$ and $\bar{H}$, respectively.
From now on we will assume $1 \le p < +\infty$, $-1+\frac{1}{p} < \beta < \frac{1}{p}$, $q = 1-\frac{1}{p}$ and $r \ge 1$.

We approximate the operator $\bar{H}$ by means of the following operator

$$(\bar{H}_m\bar{f})(y) = \sum_{j=1}^{m} \lambda_{m,j} h(x_{m,j}, y)\bar{f}(x_{m,j}),$$

obtained applying to $(\bar{H}\bar{f})(y)$ the Gauss-Jacobi quadrature rule (7) w.r.t. the weight v^β.

The following theorems hold true.

Theorem 3.1. *If the kernel $h(x,y)$ satisfies*

$$\sup_{0\le x\le 1}\|h_x\|_{L^p_{v^{-\beta}}}<+\infty \tag{13}$$

then

$$\sup_m\|\bar{H}_m\|_{W^p_r\to L^p_{v^{-\beta}}}<+\infty. \tag{14}$$

Moreover, if the kernel $h(x,y)$ satisfies (10) *then*

$$\sup_m\|\bar{H}_m\|_{W^p_r\to W^p_r(v^{-\beta})}<+\infty. \tag{15}$$

Theorem 3.2. *Let us assume that the kernel $h(x,y)$ satisfies* (10) *and*

$$\sup_{0\le y\le 1}\|h_y\|_{W^q_r(v^{\beta})}<+\infty. \tag{16}$$

Then

$$\lim_m\|(\bar{H}-\bar{H}_m)F\|_{L^p_{v^{-\beta}}}=0,\quad \forall F\in W^p_r, \tag{17}$$

and the sequence of operators $\{\bar{H}_m\}_m$, as maps from W^p_r into $L^p_{v^{-\beta}}$, satisfies

$$\lim_M\sup_m\sup_{\|F\|_{W^p_r}=1}E_M(\bar{H}_mF)_{v^{-\beta},p}=0, \tag{18}$$

i.e., it is collectively compact.

As done for the operator $\bar{H}$, we could approximate the integral $(\bar{K}\bar{f})(y)$ by means of

$$(\bar{K}_m\bar{f})(y)=\sum_{j=1}^{m}\lambda_{m,j}k(x_{m,j},y)\bar{f}(x_{m,j}), \tag{19}$$

i.e., using the quadrature rule (7) w.r.t. the Jacobi weight v^{β}. However, the following lemma, giving the estimate of the corresponding remainder term $e_m(k_y\bar{f})$, shows that the behaviour of the Gaussian formula becomes worse and worse when y is closer and closer to 0.

Lemma 3.1. *Assuming that the kernel $k(x,y)$ given in* (2) *satisfies*

$$\left\|v^{\beta}\frac{\partial^j k_y}{\partial x^j}\varphi^j\right\|_q\le \mathcal{C}y^{\beta-\frac{j}{2}-\frac{1}{p}},\quad j=0,1,\dots,r,\quad \mathcal{C}\ne\mathcal{C}(y), \tag{20}$$

then, for all $F\in W^p_r$, we have

$$|e_m(k_yF)|\le\frac{\mathcal{C}}{m^r}\|F\|_{W^p_r}y^{\beta-\frac{r}{2}-\frac{1}{p}},\quad \mathcal{C}\ne\mathcal{C}(m,F,y). \tag{21}$$

We observe that by assumption (20), we require that the function $k(\cdot,y)$ belongs to the weighted Sobolev space $W^q_r(v^{\beta})$, for any fixed $y\in(0,1]$, but not uniformly with respect to $y\in[0,1]$. On the other hand, the uniform boundedness of the norm in (20) cannot be assumed for kernels of Mellin type, because of their fixed strong singularity at $x=y=0$.
The estimate (21) suggests us to approximate $\bar{K}$ by means of a new "modified" discrete operator $\widetilde{K}_m$ defined as follows.
We choose the points $y_m=\frac{c}{m^{2-2\varepsilon}}$ and $\bar{y}_m=\frac{c}{m^{2-2\mu\varepsilon}}$, with c a fixed positive constant, ε an arbitrarily small positive quantity and μ a parameter chosen in the interval $(1,\frac{1}{\varepsilon})$ such that $\bar{y}_m$ is sufficiently close to y_m. Then, we introduce the operator

$$(\widetilde{K}_m\bar{f})(y)=\begin{cases}(\bar{K}_m\bar{f})(y), & y_m\le y\le 1\\ L_N(\bar{K}_m\bar{f},y), & 0<y<y_m\end{cases}, \tag{22}$$

where $(\bar{K}_m\bar{f})(y)$ is given by (19) and

$$L_N(\bar{K}_m\bar{f},y) = \sum_{i=1}^{N} l_i(y)(\bar{K}_m\bar{f})(z_i), \quad l_i(y) = \prod_{\substack{j=1\\ j\neq i}}^{N} \frac{y-z_j}{z_i-z_j},$$

is the Lagrange polynomial interpolating the function $\bar{K}_m\bar{f}$ at the equispaced points $z_i = \bar{y}_m + \frac{(i-1)\bar{y}_m}{N}$, $i = 1,\dots,N$, $N \ll m$.

In this way, we approximate $(\bar{K}\bar{f})(y)$ by the Gauss-Jacobi quadrature rule only when y is sufficiently far from 0. Conversely, when y is very close to 0 we use the Lagrange polynomial that interpolates $(\bar{K}_m\bar{f})(y)$ at the knots z_i, $i = 1,\dots,N$.

Concerning the operator $\widetilde{K}_m$ we state the following theorem.

Theorem 3.3. *Assuming that the kernel* $k(x,y)$ *given in* (2) *satisfies* (9) *and* (20)*, we have*

$$\sup_m \|\widetilde{K}_m\|_{W_r^p \to L^p_{v^{-\beta}}} \le \mathcal{C}, \quad \mathcal{C} \neq \mathcal{C}(m), \tag{23}$$

$$\limsup_m \|\widetilde{K}_m\|_{W_r^p \to L^p_{v^{-\beta}}} \le \int_0^{+\infty} t^{\frac{1}{p}-1-\beta}\bar{k}(t)dt, \tag{24}$$

and

$$\lim_m \|(\bar{K} - \widetilde{K}_m)F\|_{L^p_{v^{-\beta}}} = 0, \quad \forall F \in W_r^p. \tag{25}$$

We point out that the choice of the interpolation knots in the interval $(\bar{y}_m, 2\bar{y}_m)$ is crucial for proving the relations (24) and (25).

The stated properties of the operators $\widetilde{K}_m$ and $\bar{H}_m$, encourage us to employ them in the application of a Nyström-type method for the numerical solution of (12).

4. The numerical method

The numerical method we propose is a Nyström type method that consists in computing the solution $\bar{f}_m$ of the following finite dimensional equation

$$(\bar{I} + \widetilde{K}_m + \bar{H}_m)\bar{f}_m = g \tag{26}$$

as an approximation of the solution $\bar{f}$ of (12). It can be regarded as a "modification" of the classical method just based on the Gauss-Jacobi quadrature rule.

In order to derive a linear system in some sense equivalent to (26), we collocate it at the zeros $x_{m,i}$, $i = 1,\dots,m$, of the Jacobi polynomial of degree m that is orthogonal with respect to the weight function v^β. Then, letting

$$x_{m,s} = \min_{i=1,\dots,m}\{x_{m,i} \ : \ x_{m,i} > y_m\},$$

and taking into account that

$$L_N(\bar{K}_m\bar{f},y) = \sum_{i=1}^{N} l_i(y)(\bar{K}_m\bar{f})(z_i) = \sum_{j=1}^{m} \lambda_{m,j} L_N(k(x_{m,j},\cdot),y)\bar{f}(x_{m,j}),$$

we get, for $i = 1,\dots,m$,

$$\begin{cases} \displaystyle\sum_{j=1}^{m} \left[v^\beta(x_{m,i})\delta_{i,j} + \lambda_{m,j}\left(k(x_{m,j},x_{m,i}) + h(x_{m,j},x_{m,i})\right)\right] a_j = g(x_{m,i}), & i \ge s, \\ \displaystyle\sum_{j=1}^{m} \left[v^\beta(x_{m,i})\delta_{i,j} + \lambda_{m,j}\left(L_N(k(x_{m,j},\cdot),x_{m,i}) + h(x_{m,j},x_{m,i})\right)\right] a_j = g(x_{m,i}), & i < s, \end{cases} \tag{27}$$

that is a linear system of order m in the unknowns $a_j = \bar{f}_m(x_{m,j})$, $j = 1, \dots, m$.

Solving the above system we can construct the Nyström interpolant

$$\bar{f}_m(y) = \begin{cases} v^{-\beta}(y) \left[g(y) - \sum_{j=1}^{m} \lambda_{m,j} \left(k(x_{m,j}, y) + h(x_{m,j}, y) \right) a_j \right], & y_m \le y \le 1, \\ \\ v^{-\beta}(y) \left[g(y) - \sum_{j=1}^{m} \lambda_{m,j} \left(L_N(k(x_{m,j}, \cdot), y) + h(x_{m,j}, y) \right) a_j \right], & 0 \le y < y_m. \end{cases}$$

Note that each solution $\bar{f}_m$ of (26) furnishes a solution of (27): merely evaluating it at the node points $(x_{m,j})_{j=1,\dots,m}$. Conversely, if $\mathbf{a} = (a_j, \dots,\ a_m)^T \in \mathbb{R}^m$ is solution of (27) then $\bar{f}_m$ is the unique solution of (26) that agrees with $\mathbf{a}$ at the node points.

The following theorem gives an error estimate for the proposed method.

Theorem 4.1. *Let us assume that* $\mathrm{Ker}(\bar{I} + \bar{K} + \bar{H}) = \{0\}$ *in* L^p*, the kernel* $k(x, y)$ *given in* (2) *satisfies* (11) *and* (20) *and the kernel* $h(x, y)$ *satisfies* (10) *and* (16)*. If, for sufficiently large* m*, the operators* $\bar{I} + \widetilde{K}_m + \bar{H}_m : W_r^p \to L^p_{v^{-\beta}}$ *are invertible and their inverses are uniformly bounded, under the assumptions* $\bar{f} \in W_r^p$ *and* $g \in L^p_{v^{-\beta}}$*, we get the following error estimate*

$$\|\bar{f} - \bar{f}_m\|_{L^p} \le \mathcal{C} \left[\left(\int_0^{y_m} |v^{-\beta}(y)(\bar{K}\bar{f})(y)|^p dy \right)^{\frac{1}{p}} + \frac{\|\bar{f}\|_{W_r^p}}{m^{\nu\varepsilon}} \right], \tag{28}$$

with $\nu = \min\left\{ r, 2(\mu - 1)(-\beta + \frac{1}{p}) \right\}$ *and* $\mathcal{C} \neq \mathcal{C}(m, \bar{f})$.

Setting $f_m = v^{\beta} \bar{f}_m$, it is easily seen that

$$\|f - f_m\|_{L^p_{v^{-\beta}}} = \|\bar{f} - \bar{f}_m\|_{L^p}, \tag{29}$$

then (28) actually provides an error estimate for the approximation of the solution f of the initial problem (1) by means of the function f_m.

5. Computational aspects

5.1. Conditioning of the linear system

In the paper [16] the authors propose some new techniques for obtaining well conditioned linear systems in the numerical solution of Fredholm integral equations of the second kind by means of projection type methods. Inspired by such techniques we propose to solve in place of (27) a new preconditioned linear system. First, we multiply both sides in (27) by the following nonsingular diagonal matrix

$$D = \begin{pmatrix} \sqrt{\Delta x_{m,1}} v^{-\beta}(x_{m,1}) & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \sqrt{\Delta x_{m,m}} v^{-\beta}(x_{m,m}) \end{pmatrix},$$

where $\Delta x_{m,j} = x_{m,j+1} - x_{m,j}$, $j = 1, \dots, m-1$, and $\Delta x_{m,m} = 1 - x_{m,m}$. Then, choosing as new unknowns $a_j^* = \sqrt{\Delta x_{m,j}} \bar{f}_m(x_{m,j})$, $j = 1, \dots, m$, we get the following new system

$$\begin{cases} \sum_{j=1}^{m} \left[\delta_{i,j} + \sqrt{\frac{\Delta x_{m,i}}{\Delta x_{m,j}}} v^{-\beta}(x_i) \lambda_{m,j} \left(k(x_{m,j}, x_{m,i}) + h(x_{m,j}, x_{m,i}) \right) \right] a_j^* = \sqrt{\Delta x_{m,i}} v^{-\beta}(x_{m,i}) g(x_{m,i}), \\ \qquad i \geq s, \\ \sum_{j=1}^{m} \left[\delta_{i,j} + \sqrt{\frac{\Delta x_{m,i}}{\Delta x_{m,j}}} v^{-\beta}(x_i) \lambda_{m,j} \left(L_N(k(x_{m,j}, \cdot), x_{m,i}) + h(x_{m,j}, x_{m,i}) \right) \right] a_j^* = \sqrt{\Delta x_{m,i}} v^{-\beta}(x_{m,i}) g(x_{m,i}), \\ \qquad i < s. \end{cases} \tag{30}$$

After solving (30) we can construct the Nyström interpolant $\bar{f}_m(y)$ as follows

$$\bar{f}_m(y) = \begin{cases} v^{-\beta}(y) \left[g(y) - \sum_{j=1}^{m} \lambda_{m,j} \left(k(x_{m,j}, y) + h(x_{m,j}, y) \right) \frac{a_j^*}{\sqrt{\Delta x_{m,j}}} \right], & y_m \leq y \leq 1, \\ v^{-\beta}(y) \left[g(y) - \sum_{j=1}^{m} \lambda_{m,j} \left(L_N(k(x_{m,j}, \cdot), y) + h(x_{m,j}, y) \right) \frac{a_j^*}{\sqrt{\Delta x_{m,j}}} \right], & 0 \leq y < y_m. \end{cases}$$

As one can see in the examples showed in Section 7, the linear systems (30) are better conditioned than the linear systems (27).

5.2. The choice of the parameters

The choice of the parameters c, ε and μ involved in the definition of the points $y_m = \frac{c}{m^{2-2\varepsilon}}$ and $\bar{y}_m = \frac{c}{m^{2-2\mu\varepsilon}}$ can be made according to the solution behavior near the origin.
In [19] (see Theorem 1.10) it has been proved that, if the Mellin kernel

$$k(x, y) = \frac{1}{x} \bar{k} \left(\frac{y}{x} \right)$$

satisfies the conditions

$$\int_0^\infty t^{\frac{1}{p} - 1 - \sigma} \left| t^j \bar{k}^{(j)}(t) \right| dt < \infty, \quad j = 0, 1, \dots, l, \tag{31}$$

for some $1 \leq p < \infty$, $\sigma > 0$ and $l \in \mathbb{N}_0$, then, under suitable assumptions on k, h, and g, the solution f of (1) belongs to the space $X_\sigma^{p,l} + \mathbb{P}_d$, with $d < \min\{l, \sigma - 1/2\}$, being

$$X_\sigma^{p,l} = \left\{ f \ : \ x^{j-\sigma} f^{(j)}(x) \in L^p, \ j = 0, 1 \dots, l \right\},$$

and $\mathbb{P}_d$ the space of the polynomials of degree $\leq d$ if $d \geq 0$, $\mathbb{P}_d = \{0\}$ if $d < 0$.
Assuming that (31) holds true, we can suppose that the solution of the equation (1) takes the form

$$f(x) = x^\sigma f_0(x) + f_1(x), \tag{32}$$

with $x^j f_0^{(j)}(x) \in L^p$, $j = 0, 1, \dots, l$, and $f_1(x)$ a smoother function in $[0, 1]$.
Furthermore, we suppose that the condition (11) is fulfilled for some $-1 + \frac{1}{p} < \beta < \frac{1}{p}$ and $\beta < \sigma$.

Under the previous assumptions, (32) implies that $\bar{f} \in W_r^p$ with $r \geq \lfloor 2(\sigma - \beta) \rfloor$ ($\lfloor \tau \rfloor$ denotes the integer part of τ). Moreover, since the Gaussian quadrature rule $(\bar{K}_m \bar{f})(y)$ in (19) could suffer a loss of accuracy for y very close to 0, especially for larger values of r (see the estimate (21)), we propose to take $c = \gamma \cdot 10^{2(\sigma - \beta)}$, with $0 < \gamma < 1$, and the small positive number ε such that $10^{-1} \leq \varepsilon \leq 10^{-3}$. This choice of the couple (c, ε) guarantees that the breaking point y_m is sufficiently far from 0.

We also suggest to choose $\mu = 2(\sigma - \beta) + 1$. With this value of μ the error estimate (28) combined with (29) becomes

$$\|f - f_m\|_{L^p_{v^{-\beta}}} \leq \mathcal{C} \left[\left(\int_0^{y_m} |v^{-\beta}(y)(Kf)(y)|^p dy \right)^{\frac{1}{p}} + \frac{1}{m^{\nu\varepsilon}} \right],$$

with $\nu = \min \left\{ r, 4(\sigma - \beta) \left(-\beta + \frac{1}{p} \right) \right\}$ and $\mathcal{C} \neq \mathcal{C}(m)$.
Finally, when $\mu = 2(\sigma - \beta) + 1$, from Theorem 4.1 we can deduce the following result.

Corollary 5.1. *Under the same assumptions of Theorem 4.1, if the condition* (31) *is fulfilled for some* $\sigma > 0$ *and the solution* f *of* (5) *takes the form* (32)*, then the approximating solution* f_m *satisfies*

$$\|f - f_m\|_{L^p_{v^{-\beta}}} \leq \mathcal{C} \left(\frac{1}{m^{2(\sigma-\beta)(1-\varepsilon)}} + \frac{1}{m^{\nu\varepsilon}} \right), \tag{33}$$

with $\nu = \min \left\{ r, 4(\sigma - \beta) \left(-\beta + \frac{1}{p} \right) \right\}$ *and* $\mathcal{C} \neq \mathcal{C}(m)$.

We remark that the number N of the equidistant knots $z_i = \bar{y}_m + \frac{(i-1)\bar{y}_m}{N}$, $i = 2, \dots, N$, has to be small in order to guarantee a sufficiently accurate approximation of $(\bar{K}\bar{f})(y)$ by means of the interpolating polynomial $L_N(\bar{K}_m\bar{f}, y)$, when $y < y_m$. The numerical evidence shows that an efficient choice is $N = 4, 5$.

The numerical tests in Section 7 highlights also that the theoretical rate of convergence provided by the error estimate (33) sometimes seems to be a bit pessimistic if compared with the true convergence order.

6. Proofs

Proof of Theorem 2.1. The result stated in this theorem is well known (see, for instance, [20, p. 173]). For the reader's convenience, here we give its proof. We have

$$\begin{aligned}
\|v^{-\beta}(KF)\|_p &= \left(\int_0^1 \left| v^{-\beta}(y) \int_0^1 \frac{1}{x} \bar{k}\left(\frac{y}{x}\right) F(x) dx \right|^p dy \right)^{\frac{1}{p}} \\
&= \left(\int_0^1 \left| v^{-\beta}(y) \int_0^\infty \frac{\bar{k}(t)}{t} F\left(\frac{y}{t}\right) \chi_{[y,\infty]}(t) dt \right|^p dy \right)^{\frac{1}{p}},
\end{aligned}$$

where $\chi_{[y,\infty]}$ denotes the characteristic function of the interval $[y, \infty]$. Applying the Minkowski inequality, we get

$$\begin{aligned}
\|v^{-\beta}(KF)\|_p &\leq \int_0^\infty \frac{\bar{k}(t)}{t} \left(\int_0^1 v^{-\beta p}(y) \left| F\left(\frac{y}{t}\right) \chi_{[y,\infty]}(t) \right|^p dy \right)^{\frac{1}{p}} dt \\
&= \int_0^\infty t^{\frac{1}{p}-1-\beta} \bar{k}(t) \left(\int_0^{\frac{1}{t}} v^{-\beta p}(z) \left| F(z) \chi_{[0,1]}(z) \right|^p dz \right)^{\frac{1}{p}} dt \\
&= \|v^{-\beta} F\|_p \int_0^\infty t^{\frac{1}{p}-1-\beta} \bar{k}(t) dt.
\end{aligned}$$

□

Proof of Theorem 2.2. Applying the Minkowski and the Hölder inequalities, we get

$$\|v^{-\beta}(HF)\|_p \leq \int_0^1 |F(x)| \left(\int_0^1 |v^{-\beta}(y) h(x,y)|^p dy \right)^{\frac{1}{p}} dx \leq \mathcal{C} \sup_{0 \leq x \leq 1} \|h_x\|_{L^p_{v^{-\beta}}} \|F\|_{L^p_{v^{-\beta}}}$$

and, analogously,

$$\|v^{-\beta}(HF)^{(r)}\varphi^r\|_p \le \mathcal{C} \sup_{0\le x\le 1} \left\| \frac{\partial^r}{\partial y^r} h_x \varphi^r \right\|_{L^p_{v^{-\beta}}} \|F\|_{L^p_{v^{-\beta}}}.$$

Then, under the assumption (10), the operator $H : L^p_{v^{-\beta}} \to W^p_r(v^{-\beta})$ is continuous. Moreover, using the inequality (6) we have

$$E_m(HF)_{v^{-\beta},p} \le \frac{\mathcal{C}}{m^r}\|v^{-\beta}(HF)^{(r)}\varphi^r\|_p \le \frac{\mathcal{C}}{m^r}\|F\|_{L^p_{v^{-\beta}}}, \quad \mathcal{C} \ne \mathcal{C}(m,F),$$

and, then,

$$\lim_m \sup_{\|F\|_{L^p_{v^{-\beta}}}=1} E_m(HF)_{v^{-\beta},p} = 0. \tag{34}$$

Now, from (34) we deduce the compactness of the operator H as a map from $L^p_{v^{-\beta}}$ into itself (see [21, p. 44]). □

Proof of Theorem 3.1. Applying the Minkowski inequality, we have

$$\begin{aligned} \|v^{-\beta}\bar{H}_m F\|_p &\le \sum_{j=1}^m \lambda_{m,j}|F(x_{m,j})| \left(\int_0^1 |v^{-\beta}(y)h(x_{m,j},y)|^p dy\right)^{\frac{1}{p}} \\ &\le \sup_{0\le x\le 1}\|h_x\|_{L^p_{v^{-\beta}}} \sum_{j=1}^m \lambda_{m,j}|F(x_{m,j})|. \end{aligned}$$

Taking into account that (see, [17, pp. 338-339])

$$\sum_{j=1}^m \lambda_{m,j}|F(x_{m,j})| \le \mathcal{C}\left(\int_0^1 |F(x)|v^\beta(x)dx + \frac{1}{m}\int_0^1 |F'(x)|\varphi(x)v^\beta(x)dx\right), \quad \mathcal{C} \ne \mathcal{C}(m,F),$$

applying the Hölder inequality and Lemma 2.1 in [22], we get

$$\|v^{-\beta}\bar{H}_m F\|_p \le \mathcal{C}\sup_{0\le x\le 1}\|h_x\|_{L^p_{v^{-\beta}}}\|F\|_{W^p_r}, \quad \mathcal{C} \ne \mathcal{C}(m,h_x,F). \tag{35}$$

Thus, from the hypothesis (13) we can deduce (14). Now, proceeding as before, we obtain

$$\|v^{-\beta}(\bar{H}_m F)^{(r)}\varphi^r\|_p \le \mathcal{C}\sup_{0\le x\le 1}\left\|\frac{\partial^r}{\partial y^r}h_x\varphi^r\right\|_{L^p_{v^{-\beta}}}\|F\|_{W^p_r}, \quad \mathcal{C} \ne \mathcal{C}(m,h_x,F), \tag{36}$$

and, then, combining (35) with (36), under the assumption (10), we get (15), too. □

Proof of Theorem 3.2. We note that $|(\bar{H}F)(y) - (H_m F)(y)| = |e_m(h_y F)|$, where $e_m(h_y F)$ is the remainder term of the Gauss-Jacobi rule (7) w.r.t. the weight v^β. Then, one has

$$\|v^{-\beta}(\bar{H} - \bar{H}_m)F\|_p = \left(\int_0^1 |v^{-\beta}(y)e_m(h_y F)|^p dy\right)^{\frac{1}{p}}.$$

On the other hand, using (8), the Hölder inequality and Lemma 2.1 in [22], we get

$$\begin{aligned} |e_m(h_y F)| &\le \frac{\mathcal{C}}{m^r}\int_0^1 v^\beta(x)\left|\frac{\partial^r}{\partial x^r}[h(x,y)F(x)]\right|\varphi^r(x)dx \\ &\le \frac{\mathcal{C}}{m^r}\sum_{j=0}^r \binom{r}{j}\int_0^1 v^\beta(x)\left|\frac{\partial^j h_y(x)}{\partial x^j}\varphi^j(x)\right|\left|F^{(r-j)}(x)\varphi^{r-j}(x)\right|dx \\ &\le \frac{\mathcal{C}}{m^r}\|F\|_{W^p_r}\|h_y\|_{W^q_r(v^\beta)}, \end{aligned} \tag{37}$$

where $\mathcal{C} \neq \mathcal{C}(m, h_y, F)$. Therefore, we can conclude that

$$\|v^{-\beta}(\bar{H} - \bar{H}_m)F\|_p \leq \frac{\mathcal{C}}{m^r}\|F\|_{W_r^p} \sup_{0\leq y\leq 1} \|h_y\|_{W_r^q(v^\beta)}, \quad \mathcal{C} \neq \mathcal{C}(m, h_y, F), \tag{38}$$

and then, under the assumption (16), (17) follows.

Now we prove (18). Since, by Theorem 3.1, $\bar{H}_m F \in W_r^p(v^{-\beta})$, $\forall F \in W_r^p$, using (6), we get

$$E_M(\bar{H}_m F)_{v^{-\beta},p} \leq \frac{\mathcal{C}}{M^r}\|v^{-\beta}(\bar{H}_m F)^{(r)}\varphi^r\|_p, \quad \mathcal{C} \neq \mathcal{C}(M, \bar{H}_m F).$$

Moreover, using (36), under the assumption (10), we deduce

$$E_M(\bar{H}_m F)_{v^{-\beta},p} \leq \frac{\mathcal{C}}{M^r}\|F\|_{W_r^p}, \quad \mathcal{C} \neq \mathcal{C}(M, m, F),$$

and, then, (18) easily follows. Finally, using a result in [21, p. 44], it is easy to prove that the condition (18) is equivalent to the collectively compactness of the sequence of operators $\{\bar{H}_m\}_m$ as maps from W_r^p into $L^p_{v^{-\beta}}$. □

Proof of Lemma 3.1. Proceeding as done for the estimate (37), we get

$$|e_m(k_y F)| \quad \leq \quad \frac{\mathcal{C}}{m^r}\|F\|_{W_r^p}\, \|k_y\|_{W_r^q(v^\beta)}, \quad \mathcal{C} \neq \mathcal{C}(m, k_y, F),$$

and, under the assumption (20), (21) follows. □

Proof of Theorem 3.3. We first prove (23) and (24). We have

$$\begin{aligned} \|v^{-\beta}\widetilde{K}_m F\|_p \quad &\leq \quad \left(\int_0^{y_m} |v^{-\beta}(y)(\widetilde{K}_m F)(y)|^p dy\right)^{\frac{1}{p}} + \left(\int_{y_m}^1 |v^{-\beta}(y)(\widetilde{K}_m F)(y)|^p dy\right)^{\frac{1}{p}} \\ &=: \quad A_1 + A_2. \end{aligned} \tag{39}$$

Recalling (22) and (7), we get

$$\begin{aligned} A_2 \quad &= \quad \left(\int_{y_m}^1 \left|v^{-\beta}(y)\left[(\bar{K}F)(y) - e_m(k_y F)\right]\right|^p dy\right)^{\frac{1}{p}} \\ &\leq \quad \|v^{-\beta}\bar{K}F\|_p + \left(\int_{y_m}^1 \left|v^{-\beta}(y) e_m(k_y F)\right|^p dy\right)^{\frac{1}{p}}. \end{aligned} \tag{40}$$

From Theorem 2.1 it is easy to deduce that, for $F \in W_r^p$,

$$\|\bar{K}F\|_{L^p_{v^{-\beta}}} \leq \|F\|_p \int_0^{+\infty} t^{\frac{1}{p}-1-\beta}\bar{k}(t)dt. \tag{41}$$

Moreover, using Lemma 3.1 and recalling that $y_m = \frac{c}{m^{2-2\varepsilon}}$, we obtain

$$\begin{aligned} \left(\int_{y_m}^1 \left|v^{-\beta}(y) e_m(k_y F)\right|^p dy\right)^{\frac{1}{p}} \quad &\leq \quad \frac{\mathcal{C}}{m^r}\|F\|_{W_r^p}\left(\int_{y_m}^1 y^{\left(\beta-\frac{r}{2}-\frac{1}{p}\right)p-\beta p} dy\right)^{\frac{1}{p}} \\ &\leq \quad \frac{\mathcal{C}}{m^r} y_m^{-\frac{r}{2}}\|F\|_{W_r^p} \\ &\leq \quad \frac{\mathcal{C}}{m^{r\varepsilon}}\|F\|_{W_r^p}, \end{aligned} \tag{42}$$

where $\mathcal{C} \neq \mathcal{C}(m, F, k_y)$. Therefore, substituting (41) and (42) into (40), we deduce

$$A_2 \leq \|F\|_{W_r^p} \left(\int_0^{+\infty} t^{\frac{1}{p}-1-\beta} \bar{k}(t) dt + \frac{\mathcal{C}}{m^{r\varepsilon}} \right), \quad \mathcal{C} \neq \mathcal{C}(m, F, k_y). \tag{43}$$

Concerning A_1, using (22) and the Minkowski inequality, we have

$$\begin{aligned} A_1 &= \left(\int_0^{y_m} \left| v^{-\beta}(y) \sum_{i=1}^{N} l_i(y) (\bar{K}_m F)(z_i) \right|^p dy \right)^{\frac{1}{p}} \\ &\leq \left(\sum_{i=1}^{N} |(\bar{K}_m F)(z_i)| \right) \max_{1 \leq i \leq N} \max_{0 \leq y \leq y_m} |l_i(y)| \left(\int_0^{y_m} y^{-\beta p} \right)^{\frac{1}{p}}. \end{aligned} \tag{44}$$

On the other hand, recalling that $z_i = \bar{y}_m + \frac{(i-1)\bar{y}_m}{N}$, $i = 1, \ldots, N$, we have $z_j - y \leq \bar{y}_m \left(\frac{N+j-1}{N} \right)$ and $|z_i - z_j| = \frac{\bar{y}_m}{N} |i - j| \geq \frac{\bar{y}_m}{N}$, then

$$|l_i(y)| = \prod_{\substack{j=1 \\ j \neq i}}^{N} \frac{z_j - y}{|z_i - z_j|} \leq \prod_{\substack{j=1 \\ j \neq i}}^{N} (N + j - 1) = \frac{(N)_N}{N + i - 1} \leq \frac{(N)_N}{N}, \tag{45}$$

where $(a)_k = \frac{(a+k-1)!}{(a-1)!}$ denotes the Pochhammer symbol. Moreover, we have

$$|(\bar{K}_m F)(z_i)| \leq |(\bar{K} F)(z_i)| + |e_m(k_{z_i} F)|.$$

By Lemma 3.1 and taking into account that, using the Hölder inequality and the assumptions on k, we get

$$|(\bar{K} F)(z_i)| \leq \int_0^1 |k(x, z_i)| |F(x)| v^{\beta}(x) dx \leq \|F\|_{L^p} \|v^{\beta} k_{z_i}\|_q \leq \|F\|_{W_r^p} z_i^{\beta - \frac{1}{p}},$$

we obtain

$$|(\bar{K}_m F)(z_i)| \leq \mathcal{C} \|F\|_{W_r^p} \left[\frac{z_i^{\beta - \frac{r}{2} - \frac{1}{p}}}{m^r} + z_i^{\beta - \frac{1}{p}} \right], \tag{46}$$

where $\mathcal{C} \neq \mathcal{C}(m, F, k_y)$. Substituting (45) and (46) into (44) and recalling the definitions of z_i, $i = 1, \ldots, N$, y_m and $\bar{y}_m$, we get

$$\begin{aligned} A_1 &\leq \mathcal{C} \|F\|_{W_r^p} \, y_m^{-\beta + \frac{1}{p}} \sum_{i=1}^{N} z_i^{\beta - \frac{1}{p}} \left[1 + \frac{z_i^{-\frac{r}{2}}}{m^r} \right] \\ &\leq \mathcal{C} \|F\|_{W_r^p} \frac{1}{m^{2\varepsilon(\mu-1)\left(-\beta + \frac{1}{p}\right)}} \left[1 + \frac{1}{m^{r\mu\varepsilon}} \right] \\ &\leq \frac{\mathcal{C}}{m^{2\varepsilon(\mu-1)\left(-\beta + \frac{1}{p}\right)}} \|F\|_{W_r^p}, \end{aligned} \tag{47}$$

where $\mathcal{C} \neq \mathcal{C}(m, F, k)$. Finally, combining (47) and (43) with (39), we have

$$\|\widetilde{K}_m F\|_{L^p_{v^{-\beta}}} \leq \|F\|_{W_r^p} \left(\int_0^{+\infty} t^{\frac{1}{p}-1-\beta} \bar{k}(t) dt + \frac{\mathcal{C}}{m^{r\varepsilon}} + \frac{\mathcal{C}}{m^{2\varepsilon(\mu-1)\left(-\beta + \frac{1}{p}\right)}} \right),$$

where $\mathcal{C} \neq \mathcal{C}(m, F, k)$, and (23) and (24) easily follow.

It remains to prove (25). For $F \in W^p_r$, we have

$$\begin{aligned} \|v^{-\beta}(\bar{K}-\widetilde{K}_m)F\|_p &\leq \left(\int_0^{y_m} |v^{-\beta}(y)[(\bar{K}F)(y)-(\widetilde{K}_mF)(y)]|^p dy\right)^{\frac{1}{p}} \\ &+ \left(\int_{y_m}^1 |v^{-\beta}(y)[(\bar{K}F)(y)-(\widetilde{K}_mF)(y)]|^p dy\right)^{\frac{1}{p}} \\ &=: B_1+B_2. \end{aligned} \tag{48}$$

By (22) and (19) and recalling (7) with $\rho=\beta$, we get

$$B_2 = \left(\int_{y_m}^1 \left|v^{-\beta}(y)e_m(k_yF)\right|^p dy\right)^{\frac{1}{p}}$$

and, using (42), we deduce

$$B_2 \leq \frac{\mathcal{C}}{m^{r\varepsilon}}\|F\|_{W^p_r}, \quad \mathcal{C} \neq \mathcal{C}(m,F,k_y). \tag{49}$$

Concerning B_1, we have

$$B_1 \leq \left(\int_0^{y_m} |v^{-\beta}(y)(\bar{K}F)(y)|^p dy\right)^{\frac{1}{p}} + \left(\int_0^{y_m} |v^{-\beta}(y)(\widetilde{K}_mF)(y)|^p dy\right)^{\frac{1}{p}}$$

and, by (47), we get

$$B_1 \leq \left(\int_0^{y_m} |v^{-\beta}(y)(\bar{K}F)(y)|^p dy\right)^{\frac{1}{p}} + \frac{\mathcal{C}}{m^{2\varepsilon(\mu-1)\left(-\beta+\frac{1}{p}\right)}}\|F\|_{W^p_r}, \tag{50}$$

with $\mathcal{C} \neq \mathcal{C}(m,F,k)$. Therefore, combining (50) and (49) with (48), we obtain

$$\begin{aligned} \|(\bar{K}-\widetilde{K}_m)F\|_{L^p_{v^{-\beta}}} &\leq \left(\int_0^{y_m} |v^{-\beta}(y)(\bar{K}F)(y)|^p dy\right)^{\frac{1}{p}} \\ &+ \left(\frac{\mathcal{C}}{m^{r\varepsilon}} + \frac{\mathcal{C}}{m^{2\varepsilon(\mu-1)\left(-\beta+\frac{1}{p}\right)}}\right)\|F\|_{W^p_r}, \end{aligned} \tag{51}$$

where $\mathcal{C} \neq \mathcal{C}(m,F,k)$. Since $\{y_m\}_m$ is a monotonic decreasing sequence with $\lim_m y_m = 0$, we deduce that $\left\{\left(\int_0^{y_m} |v^{-\beta}(y)(\bar{K}F)(y)|^p dy\right)^{\frac{1}{p}}\right\}_m$ is a monotonic decreasing sequence with

$$\inf_m \left(\int_0^{y_m} |v^{-\beta}(y)(\bar{K}F)(y)|^p dy\right)^{\frac{1}{p}} = 0$$

and, then,

$$\lim_m \left(\int_0^{y_m} |v^{-\beta}(y)(\bar{K}F)(y)|^p dy\right)^{\frac{1}{p}} = 0.$$

Consequently, from (51) the thesis follows. □

Proof of Theorem 4.1. Under the hypotheses, by standard arguments, we can deduce

$$\|\bar{f}-\bar{f}_m\|_p \leq \mathcal{C}\left(\|(\bar{K}-\widetilde{K}_m)\bar{f}\|_{L^p_{v^{-\beta}}} + \|(\bar{H}-\bar{H}_m)\bar{f}\|_{L^p_{v^{-\beta}}}\right)$$

and, using (38) and (51), the estimate (28) easily follows. □

Proof of Corollary 5.1. Under the assumptions, we have

$$\begin{aligned}
\left(\int_0^{y_m} |v^{-\beta}(y)(\bar{K}\bar{f})(y)|^p dy\right)^{\frac{1}{p}} &= \left(\int_0^{y_m} y^{-\beta p}\left|\int_0^1 \frac{1}{x}\bar{k}\left(\frac{y}{x}\right)\bar{f}(x)x^{\beta}dx\right|^p dy\right)^{\frac{1}{p}} \\
&= \left(\int_0^{y_m}\left|\int_y^{\infty}\frac{\bar{k}(t)}{t^{\beta+1}}\bar{f}\left(\frac{y}{t}\right)dt\right|^p dy\right)^{\frac{1}{p}} \\
&= \left(\int_0^{y_m}\left|\int_0^{\infty}\frac{\bar{k}(t)}{t^{\beta+1}}\bar{f}\left(\frac{y}{t}\right)\chi_{[y,\infty)}(t)dt\right|^p dy\right)^{\frac{1}{p}},
\end{aligned}$$

where $\chi_{[y,\infty)}$ denotes the characteristic function on the interval $[y,\infty)$.
Now, by using Minkowski's inequality and recalling that, under our assumptions, $\bar{f} = v^{-\beta}f$ with $f \in X_{\sigma}^{p,l}$, we get

$$\begin{aligned}
\left(\int_0^{y_m} |v^{-\beta}(y)(\bar{K}\bar{f})(y)|^p dy\right)^{\frac{1}{p}} &\le \int_0^{\infty}\frac{\bar{k}(t)}{t^{\beta+1}}\left(\int_0^{y_m}\left|\bar{f}\left(\frac{y}{t}\right)\right|^p\left|\chi_{[y,\infty)}(t)\right|^p dy\right)^{\frac{1}{p}} dt \\
&\le \int_0^{\infty}\frac{\bar{k}(t)}{t^{\beta+1}}\left(\int_0^{y_m}\left(\frac{y}{t}\right)^{(\sigma-\beta)p}\left|\left(\frac{y}{t}\right)^{-\sigma} f\left(\frac{y}{t}\right)\right|^p\left|\chi_{[y,\infty)}(t)\right|^p dy\right)^{\frac{1}{p}} dt \\
&\le y_m^{\sigma-\beta}\int_0^{\infty}\frac{\bar{k}(t)}{t^{\sigma+1}}\left(\int_0^{1}\left|\left(\frac{y}{t}\right)^{-\sigma} f\left(\frac{y}{t}\right)\right|^p\left|\chi_{[y,\infty)}(t)\right|^p dy\right)^{\frac{1}{p}} dt \\
&= y_m^{\sigma-\beta}\int_0^{\infty}\frac{\bar{k}(t)}{t^{\sigma+1-\frac{1}{p}}}\left(\int_0^{\frac{1}{t}}\left|z^{-\sigma}f(z)\right|^p\left|\chi_{[0,1]}(z)\right|^p dz\right)^{\frac{1}{p}} dt \\
&= y_m^{\sigma-\beta}\|v^{-\sigma}f\|_p\int_0^{\infty}\frac{\bar{k}(t)}{t^{\sigma+1-\frac{1}{p}}}dt,
\end{aligned}$$

from which the thesis immediately follows. □

7. Numerical results

In this section we show by some numerical tests the effectiveness of the proposed method. For each example, in the tables we will report the absolute errors

$$e_m(y) = |f(y) - f_m(y)|, \quad y \in (0,1],$$

or the weighted absolute errors

$$\bar{e}_m(y) = v^{-\beta}(y)|f(y) - f_m(y)|, \quad y \in (0,1],$$

in case the solution f belongs to some space $L^2_{v^{-\beta}}$ with $\beta < 0$.

Moreover, we will get approximate values err_m for the weighted norm error $\|f - f_m\|_{L^2_{v^{-\beta}}}$ without a further computational effort but only using the solutions of the solved linear systems (30). To this end, we write

$$\|f - f_m\|_{L^2_{v^{-\beta}}} = \left(\int_0^1 |\bar{f}(x) - \bar{f}_m(x)|^2 v^{-\beta}(x)v^{\beta}(x)dx\right)^{\frac{1}{2}}$$

and, applying the Gaussian rule w.r.t the weight v^{β}, we compute

$$err_m = \left(\sum_{j=1}^{m}\lambda_{m,j}\left(\bar{f}(x_{m,j}) - \bar{f}_m(x_{m,j})\right)^2 v^{-\beta}(x_{m,j})\right)^{\frac{1}{2}}.$$

When the exact solution f is not known we will retain as exact the approximating one f_{2048}.

In the following tables we will also show the estimated order of convergence

$$EOC_m = \frac{\log\left(err_m/err_{2m}\right)}{\log 2}.$$

As one can see the convergence order appears to be much higher than the theoretical expectations.

Finally, we will report the condition numbers $\mathrm{cond}(A_m)$ and $\mathrm{cond}(A_m^*)$, in the spectral norm, of the matrices A_m and A_m^* associated with the linear systems (27) and (30), respectively. As one will be able to see, the values $\mathrm{cond}(A_m^*)$ related to the preconditioned systems (30) are smaller and do not increase with m.

The kernels $h(x,y)$ and $k(x,y)$ considered in the numerical tests satisfies the assumptions (10), (16) and (20) for any $r \in \mathbb{N}_0$. We will specify for which values of the parameters β and σ the assumptions (11) and (31) are fulfilled. Furthermore, for each example, we will report the values chosen for the parameters c, ε and μ involved in the method. Such choices have been performed according to the criteria explained in Subsection 5.2.

Example 7.1. *We consider the second kind equation of Mellin convolution type* (1) *with*

$$k(x,y) = \sqrt{\frac{xy^3}{(x^2+y^2)^3}}, \quad h(x,y) = xy+1, \quad g(y) = \frac{e^y+1}{1+y^2},$$

whose exact solution is unknown. The function $\bar{k}(t) = \sqrt{\frac{t^3}{(1+t^2)^3}}$ *defining the kernel* k*, satisfies condition* (31) *with* $-1 < \sigma < 2$*. Moreover,* (11) *is fulfilled taking* $\beta = -0.49$*. Then, when* $\sigma = 1.99$ *and* $\beta = -0.49$ *we get* $f(x) = x^{-0.49}\bar{f}(x)$ *with* $\bar{f}$ *belonging at least at* W_4^2*. In Tables 1-3 we report the results obtained by applying the proposed method. The numerical evidence shows that the condition numbers of the preconditioned linear systems* (30) *do not increase when the dimension* m *grows up, differently from what happens for the initial linear systems* (27)*.*

Table 1: **Example 7.1** $c = 0.0001 \cdot 10^{2 \cdot 2.48}$, $\mu = 5.96$, $\epsilon = 10^{-1}$, $N = 4$

m	$\bar{e}_m(0.1)$	$\bar{e}_m(0.3)$	$\bar{e}_m(0.5)$	$\bar{e}_m(0.7)$	$\bar{e}_m(0.9)$
8	1.51e-02	1.33e-02	1.03e-02	8.26e-03	7.08e-03
16	1.19e-02	3.58e-03	2.23e-03	1.90e-03	1.78e-03
32	9.09e-04	2.89e-04	3.07e-04	3.48e-04	3.82e-04
64	1.67e-05	2.05e-05	2.69e-05	3.19e-05	3.56e-05
128	1.82e-07	4.67e-07	6.18e-07	7.31e-07	8.14e-07
256	1.71e-07	2.89e-07	3.72e-07	4.38e-07	4.88e-07
512	5.33e-08	9.03e-08	1.16e-07	1.37e-07	1.52e-07
1024	1.15e-08	1.95e-08	2.52e-08	2.96e-08	3.31e-08

Example 7.2. *Let us assume that the known functions in the Mellin integral equation* (1) *are the following*

$$k(x,y) = \frac{1}{4\pi}\left(\frac{y}{x}\right)^{\frac{4}{5}} \frac{1}{x+y}, \quad h(x,y) = (xy+y)^2, \quad g(y) = ye^{y^2}.$$

The exact solution is unknown. The function $\bar{k}$ *satisfies* (31) *with* $0.3 < \sigma < 1.3$*, then we choose* $\sigma = 1.25$*. Taking* $\beta = 0.45$ *also condition* (11) *holds true and we get* $f(x) = x^{0.45}\bar{f}(x)$ *with* $\bar{f}$ *belonging at least at* W_1^2*. In Tables 4-6 we show the results obtained applying our numerical method.*

Table 2: **Example 7.1** $c = 0.0001 \cdot 10^{2 \cdot 2.48}$, $\mu = 5.96$, $\epsilon = 10^{-1}$, $N = 4$

m	$\bar{e}_m(10^{-2})$	$\bar{e}_m(10^{-5})$	$\bar{e}_m(10^{-7})$	$\bar{e}_m(10^{-9})$	$\bar{e}_m(10^{-11})$
8	6.88e-05	3.10e-05	3.28e-06	3.43e-07	3.59e-08
16	1.55e-02	5.65e-04	5.92e-05	6.20e-06	6.49e-07
32	1.17e-02	4.30e-04	4.51e-05	4.72e-06	4.94e-07
64	7.51e-04	1.72e-04	1.80e-05	1.89e-06	1.98e-07
128	9.95e-06	5.13e-05	5.40e-06	5.65e-07	5.92e-08
256	6.05e-08	1.12e-05	1.20e-06	1.26e-07	1.32e-08
512	1.86e-08	1.40e-06	1.74e-07	1.82e-08	1.91e-09
1024	3.78e-09	2.18e-07	4.04e-09	4.23e-10	4.43e-11

Table 3: **Example 7.1** $c = 0.0001 \cdot 10^{2 \cdot 2.48}$, $\mu = 5.96$, $\epsilon = 10^{-1}$, $N = 4$

m	err_m	EOC_m	cond(A_m^*)	cond(A_m)
8	1.49e-002	0.96	3.8086	1.0555e+01
16	7.67e-003	2.02	4.2625	1.9767e+01
32	1.88e-003	3.20	4.4943	3.8391e+01
64	2.04e-004	3.69	4.7014	7.5352e+01
128	1.57e-005	4.03	4.8722	1.4836e+02
256	9.64e-007	2.92	5.0145	2.9242e+02
512	1.26e-007	2.34	5.1544	5.7657e+02
1024	2.50e-008		5.2644	1.1370e+03

Example 7.3. *We consider the second kind integral equation of type* (1) *with kernels*

$$k(x, y) = \frac{1}{16\pi^2} \frac{1}{\sqrt{x(x+y)}}, \qquad h(x, y) = (x + y)^2,$$

having as exact solution the function $f(y) = y^{\frac{9}{2}}$. *In Tables 7-9 we show the numerical results obtained applying the described method with* $\sigma = 0.49$ *and* $\beta = 0.1$. *Let us note that, with this choice of the parameters* σ *and* β, *the kernel* k *satisfies both* (31) *and* (11). *Moreover, one has that* $\bar{f} \in W_9^2$.

Example 7.4. *We consider the Mellin convolution equation* (1) *with*

$$k(x, y) = \frac{1}{\pi} \frac{x \sin \alpha}{x^2 - 2xy \cos \alpha + y^2}, \qquad \alpha = \frac{5}{4}\pi, \qquad h(x, y) = x^{\frac{7}{2}}(x^2 + y^2), \qquad f(y) = \frac{1}{y^{\frac{1}{4}}}.$$

Let us observe that the solution belongs to the space L^2 *but does not belong to* W_r^2 *for some* $r \geq 1$. *In this case the method proposed in [15] cannot be applied and it becomes nececessary to study the problem in a suitable weighted* L^2 *space. Since the function* $\bar{k}$ *satisfies* (31) *with* $-0.5 < \sigma < 0.5$, *taking* $\beta = -0.49$, *we get* (11) *and* $\bar{f} \in W_1^2$. *In Tables 10-12 we show the results provided by our numerical method with* $\sigma = 0.49$ *and* $\beta = -0.49$.

Conclusions

Several boundary integral equations on domains with corners turn into equations of Mellin type through a parametric representation of the boundary. Therefore numerical methods for approximating their solutions are of increasing interest in applications.

Table 4: **Example 7.2** $c = 0.1 \cdot 10^{2 \cdot 0.8}$, $\mu = 2.6$, $\epsilon = 10^{-3}$, $N = 5$

m	$e_m(0.1)$	$e_m(0.3)$	$e_m(0.5)$	$e_m(0.7)$	$e_m(0.9)$
8	1.71e-07	1.96e-06	2.04e-06	2.26e-06	2.63e-06
16	6.38e-06	4.35e-06	3.00e-06	1.51e-06	2.50e-07
32	2.30e-06	1.57e-06	1.07e-06	5.22e-07	1.37e-07
64	6.37e-07	4.35e-07	2.97e-07	1.44e-07	3.87e-08
128	1.56e-07	1.07e-07	7.32e-08	3.55e-08	9.43e-09
256	3.59e-08	2.46e-08	1.68e-08	8.16e-09	2.15e-09
512	7.72e-09	5.28e-09	3.61e-09	1.75e-09	4.60e-10
1024	1.38e-09	9.46e-10	6.47e-10	3.14e-10	8.23e-11

Table 5: **Example 7.2** $c = 0.1 \cdot 10^{2 \cdot 0.8}$, $\mu = 2.6$, $\epsilon = 10^{-3}$, $N = 5$

m	$e_m(10^{-2})$	$e_m(10^{-5})$	$e_m(10^{-7})$	$e_m(10^{-9})$	$e_m(10^{-11})$
8	6.58e-04	2.50e-03	2.52e-03	2.52e-03	2.52e-03
16	1.61e-05	7.79e-04	7.94e-04	7.94e-04	7.94e-04
32	4.51e-06	2.31e-04	2.43e-04	2.44e-04	2.44e-04
64	1.22e-06	6.41e-05	7.39e-05	7.41e-05	7.41e-05
128	2.99e-07	1.49e-05	2.21e-05	2.23e-05	2.23e-05
256	6.85e-08	2.22e-06	6.45e-06	6.59e-06	6.59e-06
512	1.47e-08	1.39e-07	1.75e-06	1.85e-06	1.85e-06
1024	2.63e-09	1.95e-08	3.76e-07	4.26e-07	4.27e-07

In this paper we consider integral equations whose Mellin kernels satisfy condition (9) for some $-1+\frac{1}{p} < \beta < \frac{1}{p}$. We highlight that when β is negative the solution is singular at the origin and necessarily belongs to weighted spaces. To our knowledge, this case has not been extensively investigated in literature.

We propose a new Nyström type method based on a Gauss-Jacobi formula. Since this quadrature rule is inefficient for approximating the Mellin integral operator (see Lemma 3.1), we suitably modify the discrete operator close to the singularity.

The well conditioning of the linear system arising from the discretization of the integral equation is a crucial issue in the computation of the approximate solution. A preconditioning technique, which takes into account the function spaces where we solve the equation, allow us to solve well conditioned systems.

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Table 6: **Example 7.2** $c = 0.1 \cdot 10^{2 \cdot 0.8}$, $\mu = 2.6$, $\epsilon = 10^{-3}$, $N = 5$

m	err_m	EOC_m	cond(A_m^*)	cond(A_m)
8	6.95e-05	1.17	2.1322	5.9317e+00
16	3.08e-05	1.28	2.1064	1.0759e+01
32	1.26e-05	1.45	2.0933	1.9815e+01
64	4.60e-06	1.57	2.0891	3.6733e+01
128	1.54e-06	1.66	2.0883	6.8307e+01
256	4.86e-07	1.74	2.0884	1.2721e+02
512	1.45e-07	1.95	2.0888	2.3710e+02
1024	3.74e-08		2.0891	4.4207e+02

Table 7: **Example 7.3** $c = 0.1 \cdot 10^{2 \cdot 0.39}$, $\mu = 1.78$, $\epsilon = 10^{-1}$, $N = 4$

m	$e_m(0.1)$	$e_m(0.3)$	$e_m(0.5)$	$e_m(0.7)$	$e_m(0.9)$
8	3.57e-011	2.26e-011	4.99e-012	2.86e-011	7.92e-011
16	2.02e-014	1.36e-014	2.77e-017	2.32e-014	5.61e-014
32	3.07e-017	3.20e-017	2.77e-017	8.32e-017	2.22e-016

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Table 8: **Example 7.3** $c = 0.1 \cdot 10^{2 \cdot 0.39}$, $\mu = 1.78$, $\epsilon = 10^{-1}$, $N = 4$

m	$e_m(10^{-2})$	$e_m(10^{-3})$	$e_m(10^{-4})$	$e_m(10^{-5})$	$e_m(10^{-6})$
8	1.57e-10	8.48e-10	9.76e-10	9.89e-10	9.91e-10
16	7.06e-14	7.87e-12	1.25e-11	1.31e-11	1.32e-11
32	5.26e-19	2.31e-14	1.36e-13	1.56e-13	1.58e-13
64	8.27e-17	4.63e-17	1.16e-15	1.83e-15	1.85e-15
128	2.82e-17	9.17e-18	2.87e-17	2.77e-17	5.55e-17

Table 9: **Example 7.3** $c = 0.1 \cdot 10^{2 \cdot 0.39}$, $\mu = 1.78$, $\epsilon = 10^{-1}$, $N = 4$

m	$err_{2,m}$	EOC	cond(A_m^*)	cond(A_m)
8	4.58e-11	10.31	2.89	3.57
16	3.59e-14	7.53	2.79	4.08
32	1.94e-16		2.75	4.68

Table 10: **Example 7.4** $c = 0.01 \cdot 10^{2 \cdot 0.98}$, $\mu = 2.96$, $\epsilon = 10^{-3}$, $N = 5$

m	$\bar{e}_m(0.1)$	$\bar{e}_m(0.3)$	$\bar{e}_m(0.5)$	$\bar{e}_m(0.7)$	$\bar{e}_m(0.9)$
8	1.37e-04	6.14e-05	3.05e-05	1.76e-05	1.00e-05
16	3.55e-05	8.41e-06	4.06e-06	2.31e-06	1.30e-06
32	4.47e-06	1.03e-06	4.95e-07	2.82e-07	1.59e-07
64	5.33e-07	1.22e-07	5.88e-08	3.35e-08	1.89e-08
128	6.26e-08	1.43e-08	6.90e-09	3.94e-09	2.22e-09
256	7.30e-09	1.67e-09	8.06e-10	4.60e-10	2.59e-10
512	8.51e-10	1.95e-10	9.39e-11	5.35e-11	3.01e-11
1024	9.89e-11	2.27e-11	1.09e-11	6.23e-12	3.51e-12

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Table 11: **Example 7.4** $c = 0.01 \cdot 10^{2 \cdot 0.98}$, $\mu = 2.96$, $\epsilon = 10^{-3}$, $N = 5$

m	$\bar{e}_m(10^{-2})$	$\bar{e}_m(10^{-3})$	$\bar{e}_m(10^{-4})$	$\bar{e}_m(10^{-5})$	$\bar{e}_m(10^{-6})$
8	2.44e-02	1.19e-02	3.92e-02	3.43e-02	2.35e-02
16	9.20e-04	1.75e-02	2.07e-02	2.79e-02	2.15e-02
32	7.42e-05	7.77e-03	4.71e-04	1.92e-02	1.86e-02
64	1.07e-05	8.19e-05	1.11e-02	7.81e-03	1.45e-02
128	1.28e-06	2.43e-05	1.55e-03	3.70e-03	9.02e-03
256	1.50e-07	3.04e-06	3.08e-05	4.94e-03	2.11e-03
512	1.74e-08	3.56e-07	7.03e-06	1.80e-04	3.39e-03
1024	2.03e-09	4.15e-08	8.44e-07	1.40e-05	1.45e-03

Table 12: **Example 7.4** $c = 0.01 \cdot 10^{2 \cdot 0.98}$, $\mu = 2.96$, $\epsilon = 10^{-3}$, $N = 5$

m	err_m	EOC_m	cond(A_m^*)	cond(A_m)
8	4.92e-03	1.42	1.8316	8.2229e+00
16	1.83e-03	1.45	1.8181	1.6207e+01
32	6.71e-04	1.46	1.8080	3.1762e+01
64	2.42e-04	1.47	1.8031	6.2424e+01
128	8.72e-05	1.47	1.8010	1.2294e+02
256	3.12e-05	1.48	1.8000	2.4241e+02
512	1.12e-05	1.48	1.7994	4.7824e+02
1024	4.01e-06		1.7991	9.4374e+02