FREE GROUP REPRESENTATIONS FROM VECTOR-VALUED MULTIPLICATIVE FUNCTIONS, II

M. GABRIELLA KUHN, SANDRA SALIANI, AND TIM STEGER

Abstract. Let Γ be a non-commutative free group on finitely many generators. In a previous work two of the authors have constructed the class of multiplicative representations of Γ and proved them irreducible as representation of $\Gamma \ltimes_\lambda C(\Omega)$. In this paper we analyze multiplicative representations as representations of Γ and we prove a criterium for irreducibility based on the growth of their matrix coefficients.

1. Introduction

Let Γ be a non-commutative free group on finitely many generators, Ω its boundary and $C(\Omega)$ the C^*-algebra of complex valued continuous functions on Ω. We say that a unitary representation of Γ is *tempered* if it is weakly contained in the regular representation or, alternatively, if it is a representation of $C^*_{\text{red}}(\Gamma)$, the regular C^*-algebra of Γ.

In [KS04], the first of a series of papers, two of the authors have constructed the class of *multiplicative* representations: they are acting on the completion of some space $\mathcal{H}^\infty$ of "smooth functions", which is built up from a *matrix system with inner product* denoted by (V_a, H_{ba}, B_a). This class is large enough to include all tempered representations of Γ hitherto constructed using the action of Γ on its Cayley graph. These representations are easily extendable to *boundary representations*, that is representations of the crossed product C^*-algebra $\Gamma \ltimes_\lambda C(\Omega)$. In [KS04] it has been proved that multiplicative representations are irreducible when considered as boundary representations, and criteria have been given to say exactly when two of them are equivalent.

In this paper we give conditions that ensure the irreducibility of a boundary representation *as a representation of* Γ.

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Our criteria are based on general facts concerning *boundary realizations* [KS01] as well as on the computation of the growth of matrix coefficients.

In short, a *boundary realization* of a unitary representation $(\pi, \mathcal{H})$ of Γ is a pair (ι, π') where

- π' is a representation of $\Gamma \ltimes_\lambda C(\Omega)$ on a Hilbert space $\mathcal{H}'$;
- ι is an isometric Γ-inclusion of $\mathcal{H}$ into $\mathcal{H}'$;
- $\mathcal{H}'$ is generated as a $(\Gamma, C(\Omega))$-space by $\iota(\mathcal{H})$.

If ι is unitary (i.e. $\mathcal{H}' = \mathcal{H}$), the boundary realization is called *perfect* otherwise we shall say that ι is *imperfect*.

Since Γ acts amenably (in the sense of Zimmer) on Ω, a representation $(\pi, \mathcal{H})$ of Γ admits a boundary realization if and only if it is tempered. This follows from the general considerations in [QS92]; a short proof specifically for the case at hand can be found in [IKS13].

Every multiplicative representation π provides a boundary realization of itself when considered as a representation of $\Gamma \ltimes_\lambda C(\Omega)$: are there other boundary realizations? In this paper we give a criterion, based on the growth of matrix coefficients, that ensures that there are *no other* boundary realizations.

Let us briefly explain our main tools. In 1979 Haagerup [Haa79] showed that, for a representation π of Γ having a cyclic vector v, the following conditions are equivalent:

i) π is tempered;
ii) The function $\phi_\varepsilon^v(x) =< v, \pi(x)v > e^{-\varepsilon|x|}$ is square integrable for every positive ε;
iii) $\displaystyle\sum_{|x|=n} | < v, \pi(x)v > |^2 \leq (n+1)^2 \|v\|^4$.

A consequence of iii) is

$$\|\phi_\varepsilon^v\|_2^2 = \sum_{x\in\Gamma} | < v, \pi(x)v > |^2 e^{-2\varepsilon|x|} \leq C\|v\|^4 \left(\frac{1}{\varepsilon}\right)^3 . \tag{1}$$

We shall write $\|\phi_\varepsilon^v\|_2^2 \simeq \frac{1}{\epsilon^\alpha}$ if there exist positive constants c_1 and c_2, possibly depending on v, such that

$$\frac{c_1}{\epsilon^\alpha} \leq \|\phi_\varepsilon^v\|_2^2 \leq \frac{c_2}{\epsilon^\alpha}.$$

The exponent 3 for $1/\varepsilon$ in (1) is an upper bound for the growth of the ℓ^2 norm of ϕ_ε^v which, as far as we know, is attained only in very special cases, namely for the representations corresponding to the endpoints of the isotropic/anisotropic principal series of Figà-Talamanca and Picardello [FTP82], Figà-Talamanca and Steger [FTS94] while for the endpoint representation of the series considered by Paschke [Pas01], [Pas02], one gets $1/\epsilon^2$.

In this paper we shall produce a method to compute $\|\phi_\varepsilon^v\|_2^2$ for a multiplicative representation and we continue the investigation between the existence of other boundary realizations, the irreducibility and the behavior of $\|\phi_\varepsilon^v\|_2^2$ started in [KS01].

The main results are the following

Theorem 1. *Given a multiplicative representation π, one can always find a positive integer $\alpha \leq 3$, depending only on π, such that $\|\phi_\varepsilon^v\|_2^2 \simeq \left(\frac{1}{\epsilon}\right)^\alpha$ for all smooth vectors v in $\mathcal{H}^\infty$.*

Theorem 2. *Let π be a multiplicative representation. Assume that for all $v \in \mathcal{H}^\infty$*

$$\textit{either} \quad \|\phi_\varepsilon^v\|_2^2 \simeq \frac{1}{\varepsilon^2} \quad \textit{or} \quad \|\phi_\varepsilon^v\|_2^2 \simeq \frac{1}{\varepsilon^3} \quad \textit{hold as} \quad \varepsilon \to 0,$$

then

- *There is only one boundary realization of π.*
- *π is irreducible as a Γ-representation.*

Finally we shall provide a necessary and sufficient condition (see Lemma 5.17) under which

$$\|\phi_\varepsilon^v\|_2^2 \simeq \frac{1}{\varepsilon^2} \quad \text{as} \quad \varepsilon \to 0,$$

for all vectors $v \in \mathcal{H}^\infty$.

2. Boundary Representations

Let Γ be a free group on a finite symmetric set of generators A. We shall always use the letters a, b, c, d for elements of A. The identity element is denoted by e. Every element has a unique reduced expression as $x = a_1 \dots a_n$ where $a_j a_{j+1} \neq e$. In this case the length, $|x|$, of x is n. The Cayley graph of Γ has as vertex set the elements of Γ and as undirected edges the couples $\{x, xa\}$ for $x \in \Gamma$ and $a \in A$. The distance between two vertices of the tree is defined as the number of the edges joining them, so $d(e,x) = |x|$ and $d(x,y) = |x^{-1}y|$. Two vertices x_1, x_2, of the tree are said adjacent if $d(x_1, x_2) = 1$.

The boundary Ω of Γ consists of the set of infinite reduced words $a_1 a_2 a_3 \dots$, with the topology defined by the basis

$$\Omega(x) = \{\omega \in \Omega, \text{ the reduced word for } \omega \text{ starts with } x\}.$$

The sets $\Omega(x)$ are both closed and open and Ω is a compact Hausdorff space homeomorphic to the Cantor set. Γ acts on itself by left translation. This action preserves the tree structure and extends to an action on the boundary of the tree Ω by the obvious multiplication by finite and infinite reduced words. Let $C(\Omega)$ be the C^*-algebra of continuous complex valued functions on Ω, under pointwise operations. Let $\lambda : \Gamma \to Aut(C(\Omega))$ be the action by left translation

$$(\lambda(x)F)(\omega) = F(x^{-1}\omega).$$

Definition 2.1. A boundary representation is a triple $(\pi_\Gamma, \pi_\Omega, \mathcal{H})$ where

- $\pi_\Omega : C(\Omega) \to \mathcal{B}(\mathcal{H})$ is a $*$-representation of $C(\Omega)$;
- $\pi_\Gamma : \Gamma \to \mathcal{U}(\mathcal{H})$ is a unitary representation of Γ;
- For all $x \in \Gamma$ and $F \in C(\Omega)$,

$$\pi_\Gamma(x)\pi_\Omega(F)\pi_\Gamma(x^{-1}) = \pi_\Omega(\lambda(x)F).$$

Whenever there is no confusion we shall omit the subscripts and write π for both π_Γ and π_Ω. A boundary representation is nothing else that a representation of $\Gamma \ltimes_\lambda C(\Omega)$.

Definition 2.2. A subrepresentation of a boundary representation π on $\mathcal{H}$ is a closed subspace of $\mathcal{H}$ invariant under the (restricted) action of both $\pi(\Gamma)$ and $\pi(C(\Omega))$.

A boundary representation π is irreducible if $\mathcal{H} \neq 0$ and 0 and $\mathcal{H}$ are the only subrepresentations of π.

Given another boundary representation $\pi^\sharp$ on $\mathcal{H}^\sharp$, a unitary map $\mathcal{J} : \mathcal{H} \to \mathcal{H}^\sharp$ such that $\pi^\sharp(x)\mathcal{J} = \mathcal{J}\pi(x)$, for all $x \in \Gamma$, and $\pi^\sharp(F)\mathcal{J} = \mathcal{J}\pi(F)$, for all $F \in C(\Omega)$, is called an intertwiner from π to $\pi^\sharp$. Two boundary representations are called equivalent if there exists an intertwiner between them.

2.1. **General Results on Boundary Realizations.**

Definition 2.3. Given a representation $(\pi, \mathcal{H})$, we say that a non-zero vector $w \in \mathcal{H}$ satisfies the Good Vector Bound if there exists a constant C, depending only on w, such that

$$\text{(GVB)} \qquad \sum_{|x|=n} |< v, \pi(x)w >|^2 \leq C\|v\|^2, \quad \text{for all } \; n \in \mathbb{N}, v \in \mathcal{H}.$$

Remark 2.4. We recall that, if w satisfies (GVB) and we define, for $\varepsilon > 0$, $\phi_\varepsilon^w(x) =< w, \pi(x)w > e^{-\varepsilon|x|}$, the growth condition (1) discussed in the Introduction becomes

$$\|\phi_\varepsilon^w\|_2^2 = \sum_{n=0}^{\infty} \sum_{|x|=n} |< w, \pi(x)w >|^2 e^{-2\varepsilon|x|} \leq \frac{C\|w\|^2}{1 - e^{-2\varepsilon}} \simeq \frac{1}{\varepsilon}$$

telling us that the quantity $\varepsilon\|\phi_\varepsilon^w\|_2^2$ is bounded as $\varepsilon \to 0$.

There is a deep relation between the existence of imperfect boundary realizations and the magnitude of the quantities $\|\phi_\varepsilon^v\|_2^2$, namely we have the following

Proposition 2.5. [KS01] *If a representation $(\pi, \mathcal{H})$ of Γ admits an imperfect boundary realization, then some nonzero vector $w \in \mathcal{H}$ satisfies* (GVB)*.*

Corollary 2.6. [KS01] *Let $(\pi, \mathcal{H})$ be a unitary representation of Γ and suppose that no nonzero vector satisfies* (GVB)*. Then any two boundary realizations of π are equivalent.*

Corollary 2.7. [KS01] *Let $(\pi, \mathcal{H})$ be a unitary representation of Γ and suppose that no nonzero vector satisfies* (GVB)*. Let $(\pi', \mathcal{H}')$ be a boundary realization of π and assume that π' is irreducible as a representation of $\Gamma \ltimes_\lambda C(\Omega)$. Then π is irreducible as a representation of Γ.*

3. Multiplicative Representations, Irreducibility and Inequivalence

A matrix system (system in short) (V_a, H_{ba}) consists of finite dimensional complex vector spaces V_a, for each $a \in A$, and linear maps $H_{ba} : V_a \to V_b$ for each pair $a, b \in A$, where $H_{ba} = 0$ whenever $ba = e$.

Definition 3.1. An invariant subsystem of (V_a, H_{ba}) is a collection of subspaces $W_a \subseteq V_a$ such that $H_{ba}(W_a) \subseteq W_b$, for all $a, b \in A$.

The system (V_a, H_{ba}) is called **irreducible** if it is nonzero and there are no invariant subsystems except for itself and the zero subsystem.

Definition 3.2. A map from the system (V_a, H_{ba}) to $(V_a^\sharp, H_{ba}^\sharp)$, is a tuple of linear maps (J_a), where $J_a : V_a \to V_a^\sharp$, and $H_{ba}^\sharp J_a = J_b H_{ba}$. The map (J_a) is called an equivalence if each J_a is a bijection, in that case the systems are called equivalent.

Remark 3.3. A map (J_a) between irreducible systems (V_a, H_{ba}) and $(V_a^\sharp, H_{ba}^\sharp)$, is either 0 or an equivalence. This is because the kernels (respectively the images) of the maps J_a constitute an invariant subsystem.

Definition 3.4. The triple (V_a, H_{ba}, B_a) is a system with inner products if (V_a, H_{ba}) is a matrix system, B_a is a positive definite sesquilinear form on V_a for each $a \in A$, and for any $a \in A$ and $v \in V_a$ one has

$$B_a(v, v) = \sum_{b \in A} B_b(H_{ba} v, H_{ba} v). \tag{2}$$

Every irreducible matrix system can be normalized so that it admits a unique (up to scalars) tuple (B_a) of *strictly* positive definite forms (see [KS04] Theorem 4.9). From this point on all the systems that we shall consider will be both **irreducible and normalized** so that (2) holds for a given tuple of positive definite forms.

Definition 3.5. Let (V_a, H_{ba}, B_a) be an irreducible system with inner products. A multiplicative function is a map $f : \Gamma \to \amalg_{a\in A} V_a$ satisfying the following condition: there exists $N = N(f)$ such that for any $x \in \Gamma$, $|x| \geq N$,

$$\begin{array}{rll} f(xa) \in & V_a, & \text{if } |xa| = |x|+1, \\ f(xab) = & H_{ba} f(xa), & \text{if } |xab| = |x|+2. \end{array} \tag{3}$$

Two multiplicative functions f, g are called equivalent if $f(x) = g(x)$ for all but finitely many elements of Γ. $\mathcal{H}^\infty$ denotes the quotient space of the space of multiplicative functions with respect to this equivalence relation. For any $f_1, f_2 \in \mathcal{H}^\infty$ let

$$< f_1, f_2 >= \sum_{|x|=N} \sum_{\substack{a\in A \\ |xa|=|x|+1}} B_a(f_1(xa), f_2(xa)), \tag{4}$$

where N is big enough so that both f_1 and f_2 satisfy (3).

Definition 3.6. The completion of $\mathcal{H}^\infty$ with respect to the norm induced by the inner product ((4)) will be our representation space $\mathcal{H}$.

Multiplicative functions can also be defined starting from matrix systems which are not irreducible (see [IKS]). In this case one can still find a tuple of *positive semidefinite forms* B_a such that (2) holds. Then one can proceed to define an inner product as in (4). However in this case the inner product (4) will induce a seminorm and $\mathcal{H}$ will split into the direct sum of orthogonal (with respect to to the B_a) subspaces (see [IKS] Section 5). As a consequence the corresponding multiplicative representation will be reducible and we shall not consider this possibility.

For any directed edge (x, xa) of the tree, we define

$$\Gamma(x, xa) = \{y \in \Gamma,\ d(y, xa) < d(y, x)\},$$

and we get $\Gamma = \Gamma(x, xa) \amalg \Gamma(xa, x)$. We set also $\Gamma(a) = \Gamma(e, a)$, and

$$\Gamma(x) = \{z \in \Gamma, \text{ the reduced word for } z \text{ starts with } x\}$$

$$\widetilde{\Gamma}(a) = \{y \in \Gamma, \text{ the reduced word for } y \text{ ends in } a\} .$$

The following functions can be considered, quite rightly, the bricks at the base of multiplicative functions.

Definition 3.7. For a fixed $x \in \Gamma$, $a \in A$, and $v_a \in V_a$, let $\mu[x, xa, v_a] : \Gamma \to \amalg_{b\in A} V_b$ be as follows

- $\mu[x, xa, v_a](y) = 0$, for $y \neq \Gamma(x, xa)$;
- $\mu[x, xa, v_a](xa) = v_a$;
- $\mu[x, xa, v_a](ybc) = H_{cb}\, \mu[x, xa, v_a](yb)$, if $yb, ybc \in \Gamma(x, xa)$, and $d(ybc, x) = d(y, x) + 2$.

Note that $y\Gamma(x, xa) = \Gamma(yx, yxa)$ and, modulo the equivalence relation, one has $\mu[x, xa, v_a](y^{-1}\cdot) = \mu[yx, yxa, v_a](\cdot)$.

Let $\mathbf{1}_{\Omega(y)}$, respectively $\mathbf{1}_{\Gamma(y)}$, be the characteristic function of the set $\Omega(y)$, respectively $\Gamma(y)$. The *multiplicative representation* π will act on $\mathcal{H}^\infty$ according to the rules

$$\begin{aligned}\pi_\Gamma(y)f(x) &= f(y^{-1}x)\\ \pi_\Omega(\mathbf{1}_{\Omega(y)})f &= \mathbf{1}_{\Gamma(y)}f\ .\end{aligned}$$

Observe that, modulo the equivalence relation, one has

$$\mu[yx, yxa, v_a] \;=\; \pi(y)\mu[x, xa, v_a] \quad \text{for all } y \in \Gamma\ , \tag{5}$$

irrespective of whether $|xa| = |x| + 1$. In particular, for $c \in A$ and $w \in V_{c^{-1}}$, one has
(6)

$$\mu[c, e, w](x) = \pi(c)\mu[e, c^{-1}, w](x) = \begin{cases} w, & \text{if } x = e,\\ \displaystyle\sum_{a\neq c} \mu[e, a, H_{ac^{-1}}w](x), & \text{if } x \neq e.\end{cases}$$

Fix now $y \in \Gamma$, choose $N > |y| + 1$ and write

$$f = \sum_{|x|=N} \sum_{\substack{a\in A\\ |xa|=|x|+1}} \mu[x, xa, f(xa)]$$

as an orthogonal sum of elementary multiplicative functions with disjoint supports. Since

$$\pi(y)f = \sum_{|x|=N} \sum_{\substack{a\in A\\ |xa|=|x|+1}} \mu[yx, yxa, f(xa)]$$

and the sets $\Gamma(yx, yxa)$ are also all disjoint (5) says that π is unitary.

Finally, since $C(\Omega)$ is generated by the functions $\{\mathbf{1}_{\Omega(x)},\, x \in \Gamma\}$, it is easy to verify that the pair (π_Γ, π_Ω) extends to a boundary representation of Γ on $\mathcal{H}$ that we shall simply denote by π.

4. Proof of Theorem 2

Theorem 2. *Let π be the multiplicative representation constructed from an irreducible, normalized matrix system (V_a, H_{ba}, B_a). Assume that for all $v \in \mathcal{H}^\infty$*

$$\text{either} \quad \|\phi_\varepsilon^v\|_2^2 \simeq \frac{1}{\varepsilon^2} \quad \text{or} \quad \|\phi_\varepsilon^v\|_2^2 \simeq \frac{1}{\varepsilon^3} \quad \text{hold as} \quad \varepsilon \to 0, \tag{7}$$

then

- *There is only one boundary realization of π,*
- *π is irreducible as a Γ-representation.*

Proof. Since π is irreducible as a representation of $\Gamma \ltimes C(\Omega)$ ([KS01] Theorem 5.3) by Corollaries 2.7 and 2.6 we only have to prove that no nonzero $g \in \mathcal{H}$ satisfies the Good Vector Bound. The structure of the proof is, as in Lemma 1.6 of [KS01], by contradiction. There are however crucial not straightforward differences in the choice of the main objects, due to the vector setting .

Assume that there exists a nonzero $g \in \mathcal{H}$, and a constant C depending only on g, such that for every $f \in \mathcal{H}$, and every positive integer n one has

$$\sum_{|x|=n} | < f, \pi(x)g > |^2 \leq C\|f\|^2. \tag{GVB}$$

By linearity we may assume that $\|g\| = 1$.

We shall allow the constant C to change from line to line, keeping in mind that it will always be *independent on* n.

The condition (GVB) implies that

$$\limsup_{\varepsilon\to 0^+} \quad \varepsilon \sum_{x\in\Gamma} | < f, \pi(x)g > |^2 e^{-\varepsilon|x|} = C\|f\|^2 < +\infty. \tag{8}$$

Fix $a \in A$, $v_a \in V_a$ such that $B_a(v_a, v_a) = 1$.

Let $f = \mu[e, a, v_a]$, $\quad f_y = \mathbf{1}_{\Gamma(y)} f = \pi(\mathbf{1}_{\Omega(y)})f$, and observe that

$$f = \sum_{|y|=n} f_y, \quad \text{and} \quad \|f\|^2 = \sum_{|y|=n} \|f_y\|^2 = 1, \tag{9}$$

since (9) is an orthogonal sum.

For this given f we shall compute (8). Since a finite number of terms gives a zero contribute to the lim sup, we may rewrite

$$C = \limsup_{\varepsilon\to 0^+} \varepsilon \sum_{|y|=n} \sum_{x\in\Gamma(y)} | < f, \pi(x)g > |^2 e^{-\varepsilon|x|} = \limsup_{\varepsilon\to 0^+} C_f^n(\varepsilon).$$

For each $x \in \Gamma(y)$ write $| < f, \pi(x)g > |^2 = | < f_y + f - f_y, \pi(x)g > |^2$ to get

$$\begin{aligned} C^n_f(\varepsilon) &= \varepsilon \sum_{|y|=n} \sum_{x\in\Gamma(y)} | < f_y, \pi(x)g > |^2 e^{-\varepsilon|x|} \\ &\quad + \varepsilon \sum_{|y|=n} \sum_{x\in\Gamma(y)} 2\Re e(< f_y, \pi(x)g > \overline{< f - f_y, \pi(x)g >}) e^{-\varepsilon|x|} \\ &\quad + \varepsilon \sum_{|y|=n} \sum_{x\in\Gamma(y)} | < f - f_y, \pi(x)g > |^2 e^{-\varepsilon|x|} \\ &= C^n_{1,f}(\varepsilon) + 2C^n_{2,f}(\varepsilon) + C^n_{3,f}(\varepsilon). \end{aligned}$$

By Cauchy-Schwarz inequality $|C^n_{2,f}(\varepsilon)| \le |C^n_{1,f}(\varepsilon) C^n_{3,f}(\varepsilon)|^{\frac{1}{2}}$, moreover, condition GVB and (9) imply that

$$\begin{aligned} \limsup_{\varepsilon\to0^+} C^n_{1,f}(\varepsilon) &= \limsup_{\varepsilon\to0^+} \varepsilon \sum_{|y|=n} \sum_{x\in\Gamma(y)} | < f_y, \pi(x)g > |^2 e^{-\varepsilon|x|} \\ &\le \sum_{|y|=n} \limsup_{\varepsilon\to0^+} \varepsilon \sum_{x\in\Gamma} | < f_y, \pi(x)g > |^2 e^{-\varepsilon|x|} \\ &\le C \sum_{|y|=n} \|f_y\|^2 = C\|f\|^2 = C\ . \end{aligned}$$

For sufficiently small ε one has $|C_{1,f}|^n(\varepsilon) \le 2C$, so that

$$0 \le C^n_{3,f}(\varepsilon) \le C^n_f(\varepsilon) - 2C^n_{2,f}(\varepsilon) \le C^n_f(\varepsilon) + 2\left(2C\, C^n_{3,f}(\varepsilon)\right)^{\frac{1}{2}}.$$

Hence, if $\limsup_{\varepsilon\to0^+} C^n_f(\varepsilon)$ is finite, the following

$$\limsup_{\varepsilon\to0^+} C^n_{3,f}(\varepsilon) \tag{10}$$

is also finite, say $\limsup_{\varepsilon\to0^+} C^n_{3,f}(\varepsilon) \le C$.

In the next section we shall prove (see Corollary 5.21) that, since $\mathrm{supp}(f - f_y) \subset \Gamma \setminus \Gamma(y)$ the above lim sup (10) is actually a limit, more precisely we shall prove that there exists a tuple $\widehat{B}_c$ of strictly positive definite forms on $\widehat{V}_c$, the space of antilinear functionals on $V_{c^{-1}}$, such that

$$\begin{aligned} \lim_{\varepsilon\to0^+} C^n_{3,f}(\varepsilon) &= \sum_{c\in A} \sum_{\substack{|z|=n-1 \\ |zc|=n}} \lim_{\varepsilon\to0^+} \varepsilon \sum_{x\in\Gamma(zc)} | < f - f_{zc}, \pi(x)g > |^2 e^{-\varepsilon|x|} \\ &= \sum_{c\in A} \sum_{\substack{|z|=n-1 \\ |zc|=n}} \frac{1}{k_0} \widehat{B}_c(S\pi(z^{-1})(f - f_{zc}), S\pi(z^{-1})(f - f_{zc})), \end{aligned} \tag{11}$$

where $S\pi(z^{-1})(f - f_{zc})$ is the antilinear functional on $V_{c^{-1}}$defined by the following rule

$$\langle w, S\pi(z^{-1})(f - f_{zc})\rangle =< \pi(z^{-1})(f - f_{zc}), \mu[c, e, w] >,$$

for every $w \in V_{c^{-1}}$ and $\mu[c, e, w]$ is as in (6). Hence $S\pi(z^{-1})(f - f_{zc})$ will be identified with an element of $\widehat{V}_c = \overline{V}'_{c^{-1}}$ (the reader may refer to Subsection 5.1 and Definition 5.3).

For the moment we shall assume valid (11) and we proceed with the calculations. Let $u \in \widehat{V}_c = \overline{V}'_{c^{-1}}$ and let $\|u\|_\infty = \sup_{\substack{v \in V_{c^{-1}} \\ B_{c^{-1}}(v,v)=1}} |\langle v, u\rangle|$ denote its norm as an antilinear functional on $V_{c^{-1}}$. Since $u \in \widehat{V}_c \mapsto \widehat{B}_c(u, u)^{1/2}$ also defines a norm on the same finite dimensional Banach space $\widehat{V}_c$, there exists a positive constant K_c, depending only on c and on $\widehat{B}_c$, such that for any $u \in \widehat{V}_c$ and for any unit vector $w \in V_{c^{-1}}$

$$K_c|\langle w, u\rangle| \leq K_c\|u\|_\infty \leq \widehat{B}_c(u, u)^{1/2} .$$

This yields a below estimate for each term in the sum (11) above: for any $c \in A$, $w \in V_{c^{-1}}$, $B_{c^{-1}}(w, w) = 1$ and z such that $|zc| = |z| + 1$, one has

$$\begin{aligned}(12)\quad &\widehat{B}_c(S\pi(z^{-1})(f - f_{zc}), S\pi(z^{-1})(f - f_{zc})) \\ &\qquad\geq k_c \,| < \pi(z^{-1})(f - f_{zc}), \mu[c, e, w] > |^2\end{aligned}$$

where $k_c = K_c^2 > 0$. Putting together (10), (11), and (12) we get the following

Claim.

For every $c \in A$ there exists a positive constant k_c such that, for any $n \in \mathbb{N}$ and for any choice of vectors $w_{c^{-1}} \in V_{c^{-1}}$, $B_{c^{-1}}(w_{c^{-1}}, w_{c^{-1}}) = 1$,

$$B_0^n(f) := \sum_{c\in A} \sum_{\substack{|z|=n-1 \\ |zc|=n}} k_c| < \pi(z^{-1})(f - f_{zc}), \mu[c, e, w_{c^{-1}}] > |^2$$

is uniformly bounded in n.

We fix unit vectors $w_{c^{-1}} \in V_{c^{-1}}$, with $w_a = v_a$ and we write, as we did for $C_f^n(\varepsilon)$, $B_0^n(f)$ as the sum of three terms:

$$B_0^n(f) = B_{1,f}^n - 2B_{2,f}^n + B_{3,f}^n$$

where

$$B_{1,f}^n = \sum_{c\in A} \sum_{\substack{|z|=n-1\\|zc|=n}} k_c| < f_{zc}, \pi(z)\mu[c,e,w_{c^{-1}}] > |^2$$

$$B_{2,f}^n = \sum_{c\in A} \sum_{\substack{|z|=n-1\\|zc|=n}} k_c\, 2\Re e(< f, \pi(z)\mu[c,e,w_{c^{-1}}] > \overline{< f_{zc}, \pi(z)\mu[c,e,w_{c^{-1}}] >})$$

$$B_{3,f}^n = \sum_{c\in A} \sum_{\substack{|z|=n-1\\|zc|=n}} k_c| < f, \pi(z)\mu[c,e,w_{c^{-1}}] > |^2.$$

We use again (9) to estimate $B_{1,f}^n$:

$$\begin{aligned} B_{1,f}^n &= \sum_{c\in A} \sum_{\substack{|z|=n-1\\|zc|=n}} k_c| < f_{zc}, \pi(z)\mu[c,e,w_{c^{-1}}] > |^2 \\ &\le \sum_{c\in A} k_c \sum_{\substack{|z|=n-1\\|zc|=n}} \|f_{zc}\|^2 \|\pi(z)\mu[c,e,w_{c^{-1}}]\|^2 \\ &\le k_0 \sum_{c\in A} \sum_{\substack{|z|=n-1\\|zc|=n}} \|f_{zc}\|^2 = k_0\|f\|^2, \end{aligned}$$

where $k_0 = \max_{c\in A} k_c$.

Arguing as before, we may conclude that there exists a constant C_2, possibly depending on a, but independent on n such that

$$B_{3,f}^n \le C_2.$$

The final step will consist in showing that the uniform boundedness in n of

$$B_{3,f}^n = \sum_{|z|=n-1} \sum_{\substack{c\in A\\|zc|=n}} k_c| < f, \pi(z)\mu[c,e,w_{c^{-1}}] > |^2$$

yields a contradiction.

Recall that $f = \mu[e,a,v_a]$ and call $M = M(a)$ the upper bound such that, for all n,

$$B_{3,f}^n = \sum_{|z|=n-1} \sum_{\substack{c\in A\\|zc|=n}} k_c| < f, \pi(z)\mu[c,e,w_{c^{-1}}] > |^2 \le M.$$

Setting $k_1 = \min_{c\in A} k_c > 0$, this yields

$$\sum_{|z|=n-1} \sum_{\substack{c\in A\\|zc|=n}} | < f, \pi(z)\mu[c,e,w_{c^{-1}}] > |^2 \le \frac{M}{k_1} = M_1.$$

Write

$$\sum_{|x|=n} |< f, \pi(x) f >|^2 = \sum_{\substack{|x|=n \\ x\in\Gamma(a)}} |< f, \pi(x) f >|^2 + \sum_{\substack{|x|=n \\ x\notin\Gamma(a)}} |< f, \pi(x) f >|^2$$
$$= (\mathrm{I}) + (\mathrm{II})$$

For (I) we get

$$\begin{aligned}
\sum_{\substack{|x|=n \\ x\in\Gamma(a)}} |< f, \pi(x) f >|^2 &= \sum_{\substack{|z|=n-1 \\ z\notin\Gamma(a^{-1})}} |< \pi(az) f, f >|^2 \\
&= \sum_{\substack{|z|=n-1 \\ z\notin\widetilde{\Gamma}(a)}} |< f, \pi(z\, a^{-1}) f >|^2 \\
&= \sum_{\substack{|z|=n-1 \\ z\notin\widetilde{\Gamma}(a)}} |< \mu[e,a,v_a], \pi(z)\mu[a^{-1},e,v_a] >|^2 \\
&\leq \sum_{|z|=n-1} \sum_{\substack{c\in A \\ |zc|=n}} |< f, \pi(z)\mu[c,e,w_{c^{-1}}] >|^2 \leq M_1.
\end{aligned}$$

For (II) we get

$$\begin{aligned}
\sum_{\substack{|x|=n \\ x\notin\Gamma(a)}} |< f, \pi(x) f >|^2 \quad &= \quad \sum_{\substack{|x|=n \\ x\notin\Gamma(a) \\ |xa|=n+1}} |< \mu[e,a,v_a], \pi(x)\mu[e,a,v_a] >|^2 \\
&\quad + \sum_{\substack{|x|=n \\ x\notin\Gamma(a) \\ |xa|=n-1}} |< \mu[e,a,v_a], \pi(x)\mu[e,a,v_a] >|^2.
\end{aligned}$$

The first sum in (II) is equal to zero since $x \notin \Gamma(a)$ and xa does not reduce. The second sum is

$$\begin{aligned}
&\sum_{\substack{|x|=n \\ x\notin\Gamma(a) \\ |xa|=n-1}} |< \mu[e,a,v_a], \pi(x)\mu[e,a,v_a] >|^2 \\
= &\sum_{\substack{|z|=n-1 \\ z\notin\Gamma(a) \\ |za^{-1}|=n}} |< \mu[e,a,v_a], \pi(z)\mu[a^{-1},e,v_a] >|^2 \\
\leq &\sum_{\substack{|z|=n-1 \\ z\notin\Gamma(a)}} \sum_{\substack{c\in A \\ |zc|=n}} |< \mu[e,a,v_a], \pi(z)\mu[c,e,w_{c^{-1}}] >|^2 \leq M_1.
\end{aligned}$$

Hence there exists a constant $C_3 = 2M_1 > 0$ such that

$$\sum_{|x|=n} |< \mu[e,a,v_a], \pi(x)\mu[e,a,v_a] >|^2 \leq C_3, \quad \text{for any } n,$$

and we get a contradiction since the hypothesis (7) on $\|\phi_\varepsilon^v\|_2^2$, with the choice $v = \mu[e, a, v_a]$, yields for any $\varepsilon > 0$ that either ε^{-2} or ε^{-3} is bounded by

$$\sum_{n=0}^{+\infty} \sum_{|x|=n} |<\mu[e,a,v_a], \pi(x)\mu[e,a,v_a]>|^2 \, e^{-2\varepsilon n} \leq C_3 \sum_{n=0}^{+\infty} e^{-2\varepsilon n} \simeq \frac{1}{\varepsilon} \, .$$

□

5. Computation of Matrix Coefficients

This section is devoted to the computation of the quantities

$$\|\phi_\varepsilon^{v_a,v_b}\|_2^2 = \sum_{n=0}^{+\infty} \sum_{|x|=n} |<\mu[e,a,v_a], \pi(x)\mu[e,b,v_b]>|^2 \, e^{-2\varepsilon n}.$$

In Theorem 1 we shall show that these quantities have always polinomial growth with respect to $1/\varepsilon$. Finally, in Lemma 5.20 we shall provide the exact asymptotics for $\|\phi_\varepsilon^{v_a,v_b}\|_2^2$ which are needed to prove Theorem 2.

5.1. **The Twin of the System.** Throughout the whole paper we shall use the following notation.

If V_1 and V_2 are finite dimensional complex vector spaces, $\mathscr{L}(V_1, V_2)$ is the space of linear maps $T : V_1 \to V_2$. The dual space of V is $V' = \mathscr{L}(V, \mathbb{C})$ and the duality is given, as usual, by $\langle v, v' \rangle$. If $T \in \mathscr{L}(V_1, V_2)$, then the dual map is $T' \in \mathscr{L}(V_2', V_1')$.

$\overline{V}$ is the complex conjugate vector space of V, i.e. the set V with the same addition operation, but with an altered multiplication

$$\lambda \bar{\cdot} v = \overline{\lambda} v, \quad \lambda \in \mathbb{C}, \ v \in V.$$

A map $f : V \to \mathbb{C}$ is called "antilinear" if $f : \overline{V} \to \mathbb{C}$, is linear.

The space of antilinear functionals on V is denoted by $V^* = \overline{V}'$.

We recall some identifications that will be used repeatedly in this paper.

Given finite dimensional complex vector spaces V_1 and V_2, for any $v_1 \in V_1$, $v_2 \in V_2$ and $f \in V_2'$ we consider the map

$$v_1 \otimes v_2 : V_2' \to V_1, \quad (v_1 \otimes v_2)(f) = f(v_2) \, v_1 = \langle v_2, f \rangle \, v_1 \, .$$

Then $v_1 \otimes v_2 \in \mathscr{L}(V_2', V_1)$. By linearity the above extends to an isomorphism $V_1 \otimes V_2 \cong \mathscr{L}(V_2', V_1)$.

It follows that, given $T_1 \in \mathscr{L}(V_1, V_3)$ and $T_2 \in \mathscr{L}(V_2, V_4)$, the map

$$T_1 \otimes T_2 : V_1 \otimes V_2 \to V_3 \otimes V_4$$

corresponds to the operator

$$\mathscr{L}(V_2', V_1) \to \mathscr{L}(V_4', V_3), \quad S \mapsto T_1\, S\, T_2'.$$

So we shall write

$$(T_1 \otimes T_2)S = T_1 S T_2'.$$

The duality isomorphism

$$\mathcal{L} : \mathscr{L}(V, V^*) \to \mathscr{L}(V^*, V)'$$

defines a bilinear form which can be written explicitly by means of the trace function

$$B : \mathscr{L}(V, V^*) \times \mathscr{L}(V^*, V) \to \mathbb{C},$$

$$B(T, S) := (\mathcal{L}(T))(S) = \operatorname{tr}(TS) = \operatorname{tr}(ST). \tag{13}$$

In particular, when $T = (\bar{v}_1' \otimes v_2')$ and $S = (v_3 \otimes \bar{v}_4)$ $(v_3, v_4 \in V$, $v_1', v_2' \in V')$ are elementary tensors, one has

$$\operatorname{tr}\left((\bar{v}_1' \otimes v_2')(v_3 \otimes \bar{v}_4)\right) = \overline{\langle v_4, v_1' \rangle} \langle v_3, v_2' \rangle\ . \tag{14}$$

In this case we shall omit the tr in front and we shall write, for brevity, $(\bar{v}_1' \otimes v_2')(v_3 \otimes \bar{v}_4)$.

Positive definite sesquilinear forms B_a on the space V_a are identified with maps $B_a \in \mathscr{L}(V_a, V_a^*)$, via the linear extension

$$(B_a(\lambda v))(\mu w) = B_a(\lambda v, \mu w) = \lambda \bar{\mu} B_a(v, w) = \lambda \bar{\mu} B_a(v)(w)$$

for any $v, w \in V_a$, and $\lambda, \mu \in \mathbb{C}$. Under this identification one also has $B_a^* = B_a$.

For every $a \in A$ set

$$\widehat{V}_a := V_{a^{-1}}^* = \overline{V}'_{a^{-1}}.$$

Given a matrix system (V_a, H_{ba}), H_{ba} induces an obvious linear map on the space of antilinear functionals on V_a, V_a^*, by $H_{ba}^* : V_b^* \to V_a^*$, $H_{ba}^*(f) = f \circ H_{ba}$, and also maps

$$\widehat{H}_{ba} := H_{a^{-1}b^{-1}}^* : \widehat{V}_a \to \widehat{V}_b,$$

Hence the matrix system (V_a, H_{ba}) induces another matrix system $(\widehat{V}_a, \widehat{H}_{ba})$, which is irreducible if so is (V_a, H_{ba}).

Definition 5.1. If for all $a \in A$, the bilinear form B_a is strictly positive definite, we shall say for brevity that *the tuple* $(B)_a$ *is positive definite*. If the B_a are all positive semidefinite and there exists an index $a \in A$ such that B_a *is not positive definite* we shall say that the tuple is positive semidefinite. Analogously we define negative definite tuples.

Proposition 5.2. *Assume that (V_a, H_{ba}, B_a) is an irreducible system with inner product. Then there exists a unique (up to multiple scalars) positive definite tuple $(\widehat{B}_a)$, $\widehat{B}_a : \widehat{V}_a \to \widehat{V}_a^*$ on $\widehat{V}_a$ such that the matrix system $(\widehat{V}_a, \widehat{H}_{ba}, \widehat{B}_a)$ is an irreducible system with inner products.*

Proof. Let $\mathcal{V} = \oplus_{a\in A} V_a^* \otimes V_a'$ and define $T : \mathcal{V} \to \mathcal{V}$ by the rule

$$(TC)_a = \sum_{b\in A} T_{ab} C_b, \quad T_{ab} = H_{ba}^* \otimes H_{ba}' \ .$$

Since every B_a may be regarded as an element of $V_a^* \otimes V_a$ the compatibility condition (2) can be rewritten as

$$(TB)_a = \sum_{b\in A} H_{ba}^* \otimes H_{ba}' B_b = \sum_{b\in A} H_{ba}^* B_b H_{ba} = B_a.$$

The above equation says that the tuple (B_a) is a right eigenvector for the matrix $T = (H_{ba}^* \otimes H_{ba}')_{a,b}$ corresponding to eigenvalue 1. Hence 1 is an eigenvalue for the transpose matrix $T' = \left(H_{ab} \otimes \overline{H}_{ab}\right)_{a,b}$ too. To say that $\widehat{B}_a$ is a compatible tuple for $(\widehat{V}_a, \widehat{H}_{ba})$ is equivalent to say that $\widehat{B}_a$ is a right eigenvector for the matrix $\widehat{T} = \left(\widehat{H}_{ba}^* \otimes \widehat{H}_{ba}'\right)_{a,b} = \left(H_{a^{-1}b^{-1}} \otimes \overline{H}_{a^{-1}b^{-1}}\right)_{a,b}$.

But the last matrix is obtained from

$$\left(H_{ab} \otimes \overline{H}_{ab}\right)_{a,b} = T'$$

by interchanges of rows and columns, so $\widehat{T}$ and T have the same eigenvalues. Since the matrix system (V_a, H_{ba}) is irreducible, then $(\widehat{V}_a, \widehat{H}_{ba})$ is irreducible too. Corollary 4.8 of [KS04] ensures that there exists an essentially unique eigentuple $(\widehat{B}_a)$ of strictly positive definite forms satisfying

$$\widehat{B}_a = \sum_{b\in A} \widehat{H}_{ba}^* \widehat{B}_b \widehat{H}_{ba} \ .$$

□

Definition 5.3. We shall call $(\widehat{V}_a, \widehat{H}_{ba}, \widehat{B}_a)$ the *twin system* induced by (V_a, H_{ba}, B_a).

Definition 5.4. For any $a, b \in A$, we define maps $E_{ab} : V_b \to \widehat{V}_a$ by

$$E_{ab} = \sum_{\substack{c\in A \\ c\neq a, b^{-1}}} H_{ca^{-1}}^* B_c H_{cb} = \sum_{\substack{c\in A \\ c\neq a, b^{-1}}} \widehat{H}_{ac^{-1}} B_c H_{cb},$$

where $E_{ab} = 0$ whenever $ab = e$.

We use also the following notation, for vectors $v_a \in V_a$ and $v_b \in V_b$,

$$E_{a_1 e} := H^*_{aa_1^{-1}} B_a(v_a) \in \widehat{V}_{a_1},$$

$$E_{ea_J} := \overline{H^*_{ba_J} B_b(v_b)} \in V'_{a_J}.$$

It holds $E^*_{ab} = E_{b^{-1}a^{-1}}$. Indeed by taking adjoint

$$E^*_{ab} = \sum_{\substack{c \in A \\ c \neq a, b^{-1}}} H^*_{cb} B^*_c H_{ca^{-1}} = \sum_{\substack{c \in A \\ c \neq a, b^{-1}}} H^*_{cb} B_c H_{ca^{-1}} = E_{b^{-1}a^{-1}}.$$

Let $a, b, c, d \in A$, $v_a \in V_a$ and $v_b \in V_b$, $J \geq 1$. We look for a transition matrix which rules the expression

$$\sum_{\substack{x \in \Gamma(c) \cap \widetilde{\Gamma}(d) \\ |x| = J}} | < \mu[e, a, v_a], \pi(x)\mu[e, b, v_b] > |^2.$$

It turns out that this matrix $\mathcal{D} = (D)_{i,j=1\ldots 4}$ is a 4×4 block triangular matrix obtained as tensor product of the matrix

$$\widetilde{\mathcal{D}} = \begin{pmatrix} (\widehat{H}_{ab})_{a,b} & (E_{ab})_{a,b} \\ \\ 0 & (H_{ab})_{a,b} \end{pmatrix} \tag{15}$$

by its conjugate, i.e. $\mathcal{D} = \widetilde{\mathcal{D}} \otimes \overline{\widetilde{\mathcal{D}}}$.

Note the following notation, used throughout:

$$\delta(ab) = \delta_e(ab) = \begin{cases} 1 & \text{if } \ b = a^{-1}, \\ 0 & \text{if } \ ab \neq e. \end{cases}$$

Lemma 5.5. *Let $a, b \in A$, $v_a \in V_a$, and $v_b \in V_b$. For $J \geq 1$, and a reduced word $x = a_1 a_2 \ldots a_J$ we have*

$$< \mu[e, a, v_a], \pi(a_1 \ldots a_J)\mu[e, b, v_b] >=$$

$$\begin{pmatrix} (v_b \delta(a_J b)\delta(a_J^{-1} d'))_{d' \in A} \\ \\ (E_{ea_J} \delta(a_J^{-1} d'))_{d' \in A} \end{pmatrix}^{\top} \widetilde{\mathcal{D}} \begin{pmatrix} (f^1_{J-1}(a_1 \ldots a_{J-1})\delta(a^{-1}_{J-1} c'))_{c' \in A} \\ \\ (f^2_{J-1}(a_1 \ldots a_{J-1})\delta(a^{-1}_{J-1} c'))_{c' \in A} \end{pmatrix}, \tag{16}$$

where the vectors $f^i_{j-1}(a_1 \ldots a_{J-1})$, $i = 1, 2$, $j = 1 \ldots, J-1$, are defined recursively as follows

$$f^1_1(a_1) = E_{a_1 e} \in \widehat{V}_{a_1}, \quad f^2_1(a_1) = v_a \delta(a_1 a^{-1}) \in V_{a_1},$$

$$f^1_j(a_1 \ldots a_j) = \widehat{H}_{a_j a_{j-1}} f^1_{j-1}(a_1 \ldots a_{j-1}) + E_{a_j a_{j-1}} f^2_{j-1}(a_1 \ldots a_{j-1}) \in \widehat{V}_{a_j}$$

$$f^2_j(a_1 \ldots a_j) = H_{a_j a_{j-1}} f^2_{j-1}(a_1 \ldots a_{j-1}) \in V_{a_j}.$$

Theorem 5.6. *Let $a, b, c, d \in A$, $v_a \in V_a$, and $v_b \in V_b$. Then*

$$\sum_{\substack{x\in\Gamma(c)\cap\widetilde{\Gamma}(d)\\ |x|=J}} | < \mu[e,a,v_a], \pi(x)\mu[e,b,v_b] > |^2 = R(d)\,\mathcal{D}^{J-1}\,S(c),$$

where $R(d)$ is the row vector obtained as tensor product of the vector on the left side of (16) *by its conjugate, i.e.*

$$R(d) = \begin{pmatrix} ((v_b \otimes \overline{v_b})\,\delta(db)\,\delta(d^{-1}d'))_{d'\in A} \\ ((E_{ed} \otimes \overline{v_b})\,\delta(db)\,\delta(d^{-1}d'))_{d'\in A} \\ ((v_b \otimes \overline{E_{ed}})\,\delta(db)\,\delta(d^{-1}d'))_{d'\in A} \\ ((E_{ed} \otimes \overline{E_{ed}})\,\delta(d^{-1}d'))_{d'\in A} \end{pmatrix}^{\top}, \tag{17}$$

and $S(c)$ is the column vector defined as tensor product of the vector on the right of (16)*, for $J = 2$, by its conjugate, i.e.*

$$S(c) = \begin{pmatrix} ((E_{ce} \otimes \overline{E_{ce}})\,\delta(c^{-1}c'))_{c'\in A} \\ ((v_a \otimes \overline{E_{ce}})\,\delta(c^{-1}a)\,\delta(c^{-1}c'))_{c'\in A} \\ ((E_{ce} \otimes \overline{v_a})\,\delta(c^{-1}a)\,\delta(c^{-1}c'))_{c'\in A} \\ ((v_a \otimes \overline{v_a})\,\delta(c^{-1}a)\,\delta(c^{-1}c'))_{c'\in A} \end{pmatrix}. \tag{18}$$

The interested reader can find both proofs in the version of this paper which appears in arXiv.

5.2. **The 1-Eigenspace of $\mathcal{D}$.** The following calculations do not depend on having a free group. We take an assigned indexing set A, two systems (V_a, H_{ba}) and $(V^\sharp_a, H^\sharp_{ba})$, (where $H_{ba} : V_a \to V_b$, and $H^\sharp_{ba} : V^\sharp_a \to V^\sharp_b$) and a set of linear maps $E_{ba} : V_a \to V_b{}^\sharp$. We shall denote by $\mathcal{D} = (D_{i,j})_{i,j=1,\ldots,4}$ the following matrix

$$\mathcal{D} = \begin{pmatrix} \left(H^\sharp{}_{ab} \otimes \overline{H^\sharp}_{ab}\right)_{a,b} & \left(E_{ab} \otimes \overline{H^\sharp}_{ab}\right)_{a,b} & \left(H^\sharp{}_{ab} \otimes \overline{E}_{ab}\right)_{a,b} & \left(E_{ab} \otimes \overline{E}_{ab}\right)_{a,b} \\ 0 & \left(H_{ab} \otimes \overline{H^\sharp}_{ab}\right)_{a,b} & 0 & \left(H_{ab} \otimes \overline{E}_{ab}\right)_{a,b} \\ 0 & 0 & \left(H^\sharp{}_{ab} \otimes \overline{H}_{ab}\right)_{a,b} & \left(E_{ab} \otimes \overline{H}_{ab}\right)_{a,b} \\ 0 & 0 & 0 & \left(H_{ab} \otimes \overline{H}_{ab}\right)_{a,b} \end{pmatrix} \tag{19}$$

Proposition 5.7. *Assume that we are given two normalized irreducible systems* (H_{ba}, V_a), $(V_a^\sharp, {H^\sharp}_{ba})$ *and let* $\mathcal{D}$ *be as in* (19)*. Then the spectral radius of* $\mathcal{D}$ *is* 1*. Moreover:*

A) *If the two systems are* **inequivalent**, 1 *is an eigenvalue of multiplicity* **two***;*
B) *If the two systems are* **equivalent**, 1 *is an eigenvalue of multiplicity* **four**.

Proof. Since $\mathcal{D}$ is block upper triangular, its eigenvalues are the same as the eigenvalues of the diagonal blocks $D_{j,j}$. By Lemma 4.6 of [KS04] and Theorem 3.1 of [Van68] the two blocks $D_{1,1}$ and $D_{4,4}$ have spectral radius equal to 1, and 1 is an eigenvalue irrespective of whether the two systems are equivalent. Let us turn to the other diagonal blocks. Observing that the eigenvalues of the transpose matrix ${D'}_{j,j}$ are the same as those of $D_{j,j}$, one can apply Corollary 5.4 of [KS04] to both matrices $D_{2,2} = (H^\sharp_{ab} \otimes \overline{H}_{ab})$ and $D_{3,3} = ({H^{\sharp *}}_{ab} \otimes H_{ab})$ to conclude that

(1) they both have spectral radius less or equal to 1,
(2) 1 is an eigenvalue for both if and only if the two systems are equivalent.

Let us turn now to the multiplicity of 1 for $D_{1,1}$ and $D_{4,4}$. We shall consider only $D_{1,1}$, being the other case similar. By Corollary 4.8 of [KS04], we know that there exists an essentially unique tuple $U = (U_a)$ of strictly positive definite forms such that $D_{1,1}U = U$, hence the geometric multiplicity of 1 is 1. Assume, by contradiction, that the algebraic multiplicity is more than one. Hence there exists a nonzero tuple $W = (W_a)$ satisfying

$$D_{1,1}W = W + \lambda U \quad \text{for some } \lambda \neq 0 \ .$$

We may assume that $\lambda = 1$ (if λ is negative we may replace W with $-W$). Choose t_0 big enough so that $t_0 U - W$ is positive semidefinite.

By our assumption $D^n_{1,1}(t_0 U - W)$ is also positive semidefinite for all $n \geq 0$. Since

$$D^n_{1,1}(t_0 U - W) = t_0 U - W - nU$$

when n is big enough we get a contradiction since U is *strictly* positive. To conclude the proof observe that, when the two systems are equivalent, by Remark 3.3, all the diagonal blocks $D_{j,j}$ are similar to $D_{1,1}$. □

5.3. **Proof of Theorem 1.**

Theorem 1. *Let* (V_a, H_{ba}) *be an irreducible normalized matrix system. Construct the matrix* $\mathcal{D}$ *as in* (19) *using for* $(V_a^\sharp, H_{ba}^\sharp)$ *the twin system*

$(\widehat{V}_a, \widehat{H}_{ba})$ *and let* d *be the dimension of the eigenspace of* 1 *of* $\mathcal{D}$. *For any positive* ε, $v_a \in V_a$ *and* $v_b \in V_b$ *define*

$$\|\phi_\varepsilon^{v_a,v_b}\|^2 = \sum_{x\in\Gamma} | < \mu[e,a,v_a], \pi(x)\mu[e,b,v_b] > |^2 e^{-\varepsilon|x|} .$$

Then

A) *If the two systems are* **inequivalent** *one has, as* $\varepsilon \to 0$

$$\|\phi_\varepsilon^{v_a,v_b}\|^2 \simeq \frac{1}{\varepsilon} \qquad \text{when } d = 2,$$

$$\|\phi_\varepsilon^{v_a,v_b}\|^2 \simeq \frac{1}{\varepsilon^2} \qquad \text{when } d = 1.$$

B) *If the two systems are* **equivalent**

$$\|\phi_\varepsilon^{v_a,v_b}\|^2 \simeq \frac{1}{\varepsilon} \qquad \text{when } d = 4,$$

$$\lim_{\varepsilon\to 0} \varepsilon^3 \|\phi_\varepsilon^{v_a,v_b}\|^2 \text{ exists and is finite} \qquad \text{in all other cases.}$$

Proof. Cases A) and B) correspond exactly to items A and B of Proposition 5.7. Define

$$\psi(\varepsilon, c, d) = \sum_{J=1}^{\infty} \sum_{\substack{x\in\Gamma(c)\cap\widetilde{\Gamma}(d) \\ |x|=J}} |< \mu[e,a,v_a], \pi(x)\mu[e,b,v_b] >|^2 e^{-\varepsilon J}. \tag{20}$$

It is enough to compute $\psi(\varepsilon, c, d)$ for all $c, d \in A$. By Theorem 5.6 there exist vectors $R(d)$ and $S(c)$, depending only on v_a and v_b, such that

$$\sum_{\substack{x\in\Gamma(c)\cap\widetilde{\Gamma}(d) \\ |x|=J}} | < \mu[e,a,v_a], \pi(x)\mu[e,b,v_b] > |^2 e^{-\varepsilon J} = R(d)\, \mathcal{D}^{J-1}\, S(c) e^{-\varepsilon J}$$

where $\mathcal{D} = \tilde{\mathcal{D}} \otimes \overline{\tilde{\mathcal{D}}}$ and $\tilde{\mathcal{D}}$ is as in (15). Observe that $\mathcal{D}$ is the same as the matrix of equation (19) where we set $H^\sharp_{ba} = \widehat{H}_{ba}$. Moreover, $\mathcal{D}$ depends only on the system we started with. Denote by $\mathscr{L}$ the finite dimensional vector space on which $\mathcal{D}$ acts and by K_1 the generalized eigenspace of 1. Since

$$\psi(\varepsilon, c, d) = e^{-\varepsilon} R(d)\, (I - \mathcal{D}e^{-\varepsilon})^{-1}\, S(c),$$

the growth of $\psi(\varepsilon, c, d)$ as ε goes to zero, depends only on the maximum size of the Jordan blocks J_1 relative to K_1. We recall that a Jordan block of size r will produce a leading term $\sum_{J=1}^{\infty} \binom{J-1}{r-1} e^{-\varepsilon J}$ in the computation of $\psi(\varepsilon, c, d)$. When the two systems are inequivalent, by Proposition 5.7, the dimension of K_1 is two and

$$\psi(\varepsilon, c, d) \simeq \sum_{J=1}^{\infty} e^{-\varepsilon J} \simeq \frac{1}{\varepsilon} \quad \text{when} \ \ J_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

or

$$\psi(\varepsilon, c, d) \simeq \sum_{J=1}^{\infty} (J-1) e^{-\varepsilon J} \simeq \frac{1}{\varepsilon^2} \quad \text{when} \ \ J_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

More details can be found in the proof of Lemma 5.17.

For equivalent systems there are two more possibilities for the maximum size of Jordan blocks, namely three and four. The existence of a single Jordan block of size four is ruled out by Haagerup's condition (1), since a block of size four would lead to

$$\psi(\varepsilon, c, d) \simeq \sum_{J=1}^{\infty} \binom{J-1}{3} e^{-\varepsilon J} \simeq \frac{1}{\varepsilon^4} \, .$$

The last possibility is a Jordan block of size three together with one of size one: this is exactly what happens for the representations corresponding to the endpoints of the isotropic/anisotropic principal series of Figà-Talamanca and Picardello [FTP82], Figà-Talamanca and Steger [FTS94] for which one gets

$$\psi(\varepsilon, c, d) \simeq \sum_{J=1}^{\infty} \binom{J-1}{2} e^{-\varepsilon J} \simeq \frac{1}{\varepsilon^3}.$$

□

Remark 5.8. A more accurate analysis of $\mathcal{D}$, that we shall omit here, shows that, for equivalent systems, it is not possible to have a Jordan block of length two, hence in this case there are only two possible behaviors for $\|\phi_\varepsilon^{v_a,v_b}\|_2^2$, namely $\frac{1}{\varepsilon^3}$ or $\frac{1}{\varepsilon}$.

Lemma 5.9. *Let (V_a, H_{ba}), $(V_a^\sharp, H_{ba}^\sharp)$ and $\mathcal{D}$ be as in Proposition 5.7 and let $(P_a)_a$ be the eigenvector of* 1 *of $D_{4,4}$. Assume that the two systems are inequivalent. Then for each $b \in A$, there exists a linear map $Q_b : V_b \to V_b^\sharp$ such that the vector*

$$\begin{pmatrix} (P_a Q_a^*)_a \\ (Q_a P_a)_a \\ (P_a)_a \end{pmatrix}$$

is (up to constant) the unique eigenvector of 1 *of the the principal submatrix*

$$\mathcal{D}_1 = \begin{pmatrix} \left(H_{ab}\otimes \overline{H^\sharp}_{ab}\right)_{a,b} & 0 & \left(H_{ab}\otimes \overline{E}_{ab}\right)_{a,b} \\ 0 & \left(H^\sharp_{ab}\otimes \overline{H}_{ab}\right)_{a,b} & \left(E_{ab}\otimes \overline{H}_{ab}\right)_{a,b} \\ 0 & 0 & \left(H_{ab}\otimes \overline{H}_{ab}\right)_{a,b} \end{pmatrix}$$

$$= \begin{pmatrix} D_{2,2} & 0 & D_{2,4} \\ 0 & D_{3,3} & D_{3,4} \\ 0 & 0 & D_{4,4} \end{pmatrix}.$$

obtained by deleting the rows and columns of $D_{1,1}$

Proof. We first note that, since the system (V_a, H_{ba}) is irreducible and normalized, then P_a is strictly positive definite as a form on V_a^*, and so self-adjoint when identified with the map $P_a : V_a^* \to V_a$. By Proposition 5.7, 1 is an eigenvalue of multiplicity one for $\mathcal{D}_1$. We look for a vector $R = (R_2, R_3, R_4)$ satisfying $\mathcal{D}_1 R = R$ or equivalently

$$\left\{ \begin{array}{l} D_{4,4}R_4 = R_4 \\ D_{3,3}R_3 + D_{3,4}R_4 = R_3 \\ D_{2,2}R_2 + D_{2,4}R_4 = R_2. \end{array} \right. \tag{21}$$

If $R_4 = 0$, then $R_2 = R_3 = 0$, so let us assume that $R_4 \neq 0$. The first equation in (21) yields that R_4 is proportional to $P = (P_b)_b$. Without loss of generality, we can assume that $R_4 = P$.

The second equation can be written as

$$(I - D_{3,3})R_3 = D_{3,4}P,$$

where I is the identity matrix and, given the nature of the eigenvalues of $D_{3,3}$, it has $R_3 = (I - D_{3,3})^{-1}D_{3,4}P$ as a unique solution.

Now P_b is strictly positive definite, so it is invertible as $P_b : V_b^* \to V_b$, so the map

$$Q_b = R_{3,b}\, P_b^{-1} : V_b \to V_b^\sharp, \quad R_3 = (R_{3,b})_b,$$

is linear and $R_{3,b} = Q_b\, P_b$.

Hence, the second equation in the system rewrites, for any $a \in A$,

$$\sum_{b\in A} (H^\sharp_{ab} \otimes \overline{H}_{ab})(Q_b\, P_b) + (E_{ab} \otimes \overline{H}_{ab})(P_b) = Q_a\, P_a,$$

which is equivalent to

$$\sum_{b\in A} H^\sharp_{ab} Q_b\, P_b H^*_{ab} + E_{ab} P_b H^*_{ab} = Q_a\, P_a. \tag{22}$$

Taking adjoint we get

$$\begin{aligned} &\sum_{b\in A} (H_{ab}\otimes \overline{H^\sharp}_{ab})(P_b\, Q^*_b) + (H_{ab}\otimes \overline{E}_{ab})(P_b) \\ = &\sum_{b\in A} H_{ab} P_b\, Q^*_b {H^\sharp}^*_{ab} + H_{ab} P_b E^*_{ab} = P_a Q^*_a, \end{aligned}$$

and the entry of R_2 corresponding to $a \in A$ is necessarily

$$R_{2,a} = P_a\, Q^*_a.$$

□

Theorem 5.10. *Let* (V_a, H_{ba}), $(V^\sharp_a, H^\sharp_{ba})$ *and* $P = (P_a)$ *be as in Lemma 5.9. Let* $\widetilde{P^\sharp} = (\widetilde{P^\sharp}_a)$ *be the eigenvector of* 1 *of* $D'_{1,1} = (H^\sharp_{ab} \otimes \overline{H^\sharp}_{ab})'$.

Then the eigenspace of $\mathcal{D}$ *corresponding to eigenvalue* 1 *has dimension* 2 *if and only if there exist linear maps* $Q_b : V_b \to V^\sharp_b$, $b \in A$, *satisfying* (22) *so that the quantity*

$$\begin{aligned} &{E^\sharp}_0 := \\ &\sum_{a,b\in A} \mathrm{tr}(\widetilde{P^\sharp}_a\, E_{ab}\, P_b\, Q^*_b\, {H^\sharp}^*_{ab} + \widetilde{P^\sharp}_a\, H^\sharp_{ab}\, Q_b\, P_b\, E^*_{ab} + \widetilde{P^\sharp}_a\, E_{ab}\, P_b\, E^*_{ab}), \end{aligned} \tag{23}$$

verifies ${E^\sharp}_0 = 0$.

Proof. Let $({P^\sharp}_a)$ be the right eigenvector of the block $D_{1,1}$. It is obvious that the full matrix $\mathcal{D}$ has the vector $U^\sharp = \begin{pmatrix} {P^\sharp}_a \\ 0 \\ 0 \\ 0 \end{pmatrix}$ as right eigenvector of 1. Hence the eigenspace of 1 of $\mathcal{D}$ has dimension 2 if and only if there exists an eigenvector $W = (W_j)_{j=1,\dots,4}$, not proportional to $U^\sharp$. W is a right eigenvector if and only if the three last components (W_2, W_3, W_4) satisfy (21) and the full vector satisfies

$$D_{1,1}W_1 + D_{1,2}W_2 + D_{1,3}W_3 + D_{1,4}W_4 = W_1. \tag{24}$$

Since we are looking for a vector not proportional to $U^\sharp$ we may assume $W_4 \neq 0$.

By Lemma 5.9

$$\begin{pmatrix} W_2 \\ W_3 \\ W_4 \end{pmatrix} = \begin{pmatrix} (P_a\, Q^*_a)_a \\ (Q_a\, P_a)_a \\ (P_a)_a \end{pmatrix}$$

for suitable maps $Q_a : V_a \to V_a^\sharp$.

If we denote by $W_{1,a}$ the entry of W_1 corresponding to a, equation (24) can be rewritten as

$$\begin{aligned} & W_{1,a} - \sum_{b \in A} H^\sharp_{ab} W_{1,b} {H^\sharp}^*_{ab} = \\ (25) \qquad & \sum_{b \in A} \Big[E_{ab} P_b Q_b^* {H^\sharp}^*_{ab} + {H^\sharp}_{ab} Q_b P_b E_{ab}^* + E_{ab} P_b E_{ab}^* \Big] \\ & := T_a. \end{aligned}$$

Since the submatrix $D_{1,1}$ does have the eigenvalue 1, equation (25) will have a solution if and only if the vector $T = (T_a)$ belongs to $\mathrm{Im}(I - D_{1,1})$, the image of $(I - D_{1,1})$. But

$$\mathrm{Im}(I - D_{1,1}) = (\mathrm{Ker}((I - D_{1,1})'))^\perp$$

and $\mathrm{Ker}((I - D_{1,1})')$ is the one-dimensional subspace generated by $\widetilde{P^\sharp} = (\widetilde{P^\sharp}_a)$. Hence the linear system (24) has a solution not proportional to $U^\sharp$ if and only if

$$\begin{aligned} 0 &= \mathrm{tr}(\widetilde{P^\sharp} T) \\ &= \mathrm{tr} \left(\sum_{a \in A} \widetilde{P^\sharp}_a \sum_{b \in A} \Big[E_{ab} P_b Q_b^* {H^\sharp}^*_{ab} + {H^\sharp}_{ab} Q_b P_b E_{ab}^* + E_{ab} P_b E_{ab}^* \Big] \right) \\ &= \sum_{a,b \in A} \mathrm{tr} \Big(\widetilde{P^\sharp}_a E_{ab} P_b Q_b^* {H^\sharp}^*_{ab} + \widetilde{P^\sharp}_a {H^\sharp}_{ab} Q_b P_b E_{ab}^* + \widetilde{P^\sharp}_a E_{ab} P_b E_{ab}^* \Big). \end{aligned}$$

□

We are interested in a more manageable form for ${E^\sharp}_0$. This can be achieved by an algebraic calculation.

Proposition 5.11. *The quantity* ${E^\sharp}_0$ *defined in* (23) *can be written as*

$$(26) \quad {E^\sharp}_0 = \sum_{a,b \in A} \mathrm{tr} \Big(\widetilde{P^\sharp}_a ({H^\sharp}_{ab} Q_b + E_{ab} - Q_a H_{ab}) P_b ({H^\sharp}_{ab} Q_b + E_{ab} - Q_a H_{ab})^* \Big).$$

Proof. The proof is straightforward after multiplication of all terms in the right hand side of (26).

□

We recall now a general result in linear algebra.

Lemma 5.12. *Let A, B be two strictly positive definite matrices and let C be a not necessarily square matrix. Then $tr(ACBC^*) \geq 0$, and*

$$tr(ACBC^*) = 0 \Longrightarrow C = 0.$$

Theorem 5.13. *The quantity ${E^\sharp}_0$ verifies ${E^\sharp}_0 \geq 0$, and*

$$ {E^\sharp}_0 = 0 $$

if and only if the linear maps $Q_b : V_b \to V_b^\sharp$ provided by Lemma 5.9 verify the conditions

$$ {H^\sharp}_{ab} Q_b + E_{ab} = Q_a H_{ab}, \ \forall a, b \in A. $$

Proof. Since P_a and $\widetilde{P^\sharp}_a$ are strictly positive definite for all a, the result follows from the previous Lemma 5.12. □

Corollary 5.14. *Assume that A generates a free group and that (V_a, H_{ba}, B_a) is an irreducible, normalized matrix system with inner products. Let $(\widehat{V}_a, \widehat{H}_{ba}, \widehat{B}_a)$ be the twin system as in Proposition 5.2. Assume that the two systems are* **inequivalent.** *Then the matrix $\mathcal{D}$ defined in* (19) *has two linearly independent eigenvectors of* 1 *if and only if there exist linear maps $Q_b : V_b \to \widehat{V}_b$ satisfying* (22) *and*

$$ \widehat{H}_{ab} Q_b + E_{ab} = Q_a H_{ab}, \ \forall a, b \in A. $$

Proof. Construct the matrix $\mathcal{D}$ corresponding to the two matrix systems (V_a, H_{ba}, B_a) and $(V_a^\sharp, H_{ba}^\sharp, B_a^\sharp) = (\widehat{V}_a, \widehat{H}_{ba}, \widehat{B}_a)$. Since the tuple (B_a), respectively $(\widehat{B}_a)$, is the right eigenvector for the matrix $(H_{ba}^* \otimes H_{ba}')_{a,b}$, respectively $\left(\widehat{H}_{ba}^* \otimes \widehat{H}_{ba}'\right)_{a,b}$, a direct computation shows that

- The tuple $u = (B_{b^{-1}})_b$ is the right eigenvector of 1 of the submatrix $D_{1,1}$ while the tuple $\tilde{u} = (\widehat{B}_a)_a$ is a right eigenvector of 1 for the transpose matrix $(D_{1,1})'$.
- The tuple $v = (\widehat{B}_{b^{-1}})_b$ is the right eigenvector of 1 of the submatrix $D_{4,4}$ while the tuple $\tilde{v} = (B_a)_a$ is the right eigenvector of 1 for the transpose matrix $(D_{4,4})'$.

Apply now Theorem 5.13 with $P = v = (\widehat{B}_{a^{-1}})_a$ and $\widetilde{P^\sharp} = \widetilde{u} = (\widehat{B}_a)_a$. □

Remark 5.15. Observe that the matrix $\mathcal{D}$ has

$$ U = (U_j)_{j=1,\dots,4} = \begin{pmatrix} (B_{b^{-1}})_b \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{27} $$

as right eigenvector of 1, while the dual matrix $\mathcal{D}' = ((D_{j,i})')$ has the vector

$$\tilde{V} = (\tilde{V}_j)_{j=1,\dots,4} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ (B_a)_a \end{pmatrix} \tag{28}$$

as right eigenvector of 1.

The following result is essential for the computation of (20).

Proposition 5.16. *Let (V_a, H_{ba}, B_a) be an irreducible, normalized matrix system and let $(\widehat{V}_a, \widehat{H}_{ba}, \widehat{B}_a)$ be the twin system. Assume that the two systems are inequivalent. Define $Q_b : V_b \to \widehat{V}_b$ as in Lemma 5.9 and let*

$$E_0 = \sum_{a,b\in A} \mathrm{tr}\Big(\widehat{B}_a(\widehat{H}_{ab}Q_b + E_{ab} - Q_a H_{ab})\widehat{B}_{b^{-1}}(\widehat{H}_{ab}Q_b + E_{ab} - Q_a H_{ab})^*\Big)\,. \tag{29}$$

If $E_0 \neq 0$ there exists a vector $W \neq 0$ such that

$$\mathcal{D}W = W + \lambda U, \tag{30}$$

where U is, as in (27) the right eigenvector of 1 of $\mathcal{D}$ and

$$\lambda = \frac{E_0}{\mathrm{tr}\left(\sum_{a\in A}\widehat{B}_a B_{a^{-1}}\right)}.$$

Proof. By Proposition 5.7 and Corollary 5.14 the Jordan block of $\mathcal{D}$ corresponding to eigenvalue 1 is of the form $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

We are seeking for a nonzero vector W such that $\mathcal{D}W = W + \lambda U$ for some nonzero λ.

Write

$$W = (W_i)_{i=1,\dots,4} = ((W_{i,a})_a)_{i=1,\dots,4},$$
$$U = (U_j)_{j=1,\dots,4} = (\delta_{j1}(B_{b^{-1}})_b)_{j=1,\dots,4}, \quad \delta_{j,i} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j, \end{cases}$$

and use Lemma 5.9 (with $V_a^\sharp = \widehat{V}_a$, $H_{ab}^\sharp = \widehat{H}_{ab}$, $P_a = (\widehat{B}_{a^{-1}})$) to get the three last components of W:

$$\begin{pmatrix} W_2 \\ W_3 \\ W_4 \end{pmatrix} = \begin{pmatrix} (\widehat{B}_{b^{-1}}\, Q_b^*) \\ (Q_b\, \widehat{B}_{b^{-1}}) \\ (\widehat{B}_{b^{-1}}) \end{pmatrix}$$

Let us turn to the condition about the first component W_1. Write, as in Theorem 5.10:

$$(I - D_{1,1})W_1 =$$
$$\left(-\lambda B_{a^{-1}} + \sum_{b\in A}\left[E_{ab}\widehat{B}_{b^{-1}}Q_b^*\widehat{H}_{ab}^* + \widehat{H}_{ab}Q_b\widehat{B}_{b^{-1}}E_{ab}^* + E_{ab}\widehat{B}_{b^{-1}}E_{ab}^*\right]\right)_a,$$

and require that the right hand side is perpendicular to the kernel of $(I - D_{1,1})'$, which is the one dimensional subspace generated by $\tilde{u} = (\widehat{B}_a)_a$. As in Theorem 5.10 this means to require

$$0 = -\lambda\ \mathrm{tr}\left(\sum_{a\in A}\widehat{B}_aB_{a^{-1}}\right) + E_0,$$

i.e.

$$\lambda = \frac{E_0}{\mathrm{tr}\left(\sum_{a\in A}\widehat{B}_aB_{a^{-1}}\right)} \neq 0.$$

□

Lemma 5.17. *Let (V_a, H_{ba}, B_a) be a normalized irreducible system and let $(\widehat{V}_a, \widehat{H}_{ba}, \widehat{B}_a)$ be the twin system. Assume that the two systems are inequivalent. For any $\varepsilon > 0$, $v_a \in V_a$ and $v_b \in V_b$ let*

$$\|\phi_\varepsilon^{v_a,v_b}\|_2^2 = \sum_{x\in\Gamma} | < \mu[e,a,v_a], \pi(x)\mu[e,b,v_b] > |^2 e^{-\varepsilon|x|}\ .$$

Then $\|\phi_\varepsilon^{v_a,v_b}\|_2^2 \simeq 1/\varepsilon$ if and only if the quantity E_0, defined in (29), is equal to zero. Moreover, when $E_0 \neq 0$ one has

$$\|\phi_\varepsilon^{v_a,v_b}\|_2^2 = \frac{E_0}{k_0^2}\varepsilon^{-2}B_a(v_a,v_a)B_b(v_b,v_b) + o(\varepsilon^{-2}), \quad \textit{as} \quad \varepsilon \to 0,$$

where $k_0 = \sum_{c\in A}\mathrm{tr}(\widehat{B}_cB_{c^{-1}})$.

Proof. The first assertion follows from Theorems 1 and 5.10. Assume hence that $E_0 \neq 0$ and use Theorem 5.6 to compute

$$\sum_{J=1}^{+\infty}\sum_{\substack{|x|=J\\ x\in\Gamma(c)\cap\widetilde{\Gamma}(d)}} | < \mu[e,a,v_a], \pi(x)\mu[e,b,v_b] > |^2 e^{-\varepsilon|x|} \tag{31}$$
$$= e^{-\varepsilon}R(d)\,(I - \mathcal{D}e^{-\varepsilon})^{-1}\,S(c),$$

where vectors $R(d)$ and $S(c)$ are defined in (17), (18) and depend only on v_a and v_b.

Let us estimate the quantity on the right side in the above equality. Denote by $\mathscr{L}$ the finite dimensional space on which $\mathcal{D}$ acts and by

K_1 the generalized eigenspace of 1. Use Corollary 5.14 and Proposition 5.16 to see that K_1 is spanned by the vectors U and W provided by equations (27) and (30) and take a basis of $\mathscr{L}$ which starts with U, W and ends with generalized eigenvectors of $\mathcal{D}$ corresponding to eigenvalues different from 1. With respect to this basis $\mathcal{D}$ has the following expression:

$$\mathcal{D} = \begin{pmatrix} 1 & \lambda & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & & \mathcal{F} & \\ 0 & 0 & & \dots & \end{pmatrix}$$

where the matrix $\mathcal{F}$ does not have the eigenvalue 1. Then

$$(I - \mathcal{D}e^{-\varepsilon})^{-1} = \begin{pmatrix} 0 & \lambda\varepsilon^{-2} & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix} + O(\varepsilon^{-1}), \quad \text{as} \quad \varepsilon \to 0.$$

For every vector $S \in \mathscr{L}$ write $S = s_1\,U + s_2\,W + u_{\text{other}}$, where u_{other} has a zero component in K_1. Then

$$(I - \mathcal{D}e^{-\varepsilon})^{-1}S = \lambda\varepsilon^{-2}\,s_2\,U + O(\varepsilon^{-1})(U + W + u_{\text{other}}).$$

So, for any (row) vector $R \in \mathscr{L}$,

$$\begin{aligned} e^{-\varepsilon}R\,(I - \mathcal{D}e^{-\varepsilon})^{-1}\,S &= e^{-\varepsilon}\lambda\varepsilon^{-2}\,s_2 R\,U + o(\varepsilon^{-2}) \\ &= \lambda\varepsilon^{-2}\,s_2 R\,U + o(\varepsilon^{-2}). \end{aligned}$$

Let us denote by S_2 the linear functional on $\mathscr{L}$ which associates the second coordinate s_2 in our chosen basis. According to (13), one has

$$S_2(S) = \operatorname{tr}(S_2\,S)$$

for a suitable row vector that we still denote by S_2.

We claim that S_2 is a left eigenvector of $\mathcal{D}$ corresponding to eigenvalue 1. Indeed, for $S = s_1\,U + s_2\,W + u_{\text{other}}$, we have

$$\begin{aligned} S_2(I - \mathcal{D})S &= S_2(I - \mathcal{D})(s_1\,U + s_2\,W + u_{\text{other}}) \\ &= S_2(s_2\,W - s_2\,W - s_2\lambda\,U + (I - \mathcal{D})u_{\text{other}}) \\ &= S_2(-s_2\lambda\,U + 0 \cdot W + w_{\text{other}}) \\ &= 0. \end{aligned}$$

As observed in Remark 5.15, S_2 is proportional to the transpose vector of $\tilde{V}$ as defined in (28), so that there exists $\beta \in \mathbb{C}$ such that

$$S_2 = (\ 0 \quad 0 \quad 0 \quad \beta(B_a)_{a \in A}\ ).$$

To compute β let us recall that

$$W = \begin{pmatrix} * \\ \vdots \\ * \\ (\widehat{B}_{a^{-1}})_{a \in A} \end{pmatrix},$$

hence

$$1 = S_2(0 \cdot u + W + 0 \cdot u_{\text{other}}) = \operatorname{tr}(S_2 \, W) = \beta \sum_{a \in A} \operatorname{tr}(B_a \, \widehat{B}_{a^{-1}}),$$

yields

$$\beta = \frac{1}{\sum_{a \in A} \operatorname{tr}(B_a \, \widehat{B}_{a^{-1}})}.$$

Finally, specifying $R = R(d)$ and $S = S(c)$ defined in (17) and(18),

$$R(d) \, (I - \mathcal{D}e^{-\varepsilon})^{-1} \, S(c) \quad = \quad \varepsilon^{-2} \, \frac{E_0}{k_0^2} \, \operatorname{tr}(S_2 S(c))(R(d) \, U) + o(\varepsilon^{-2}).$$

The trace is given by

$$\operatorname{tr}(S_2 S(c)) = \sum_{s \in A} \operatorname{tr}(B_s(v_a \otimes \overline{v_a}) \, \delta(c^{-1}a) \, \delta(c^{-1}s)) = B_c(v_c, v_c)\delta(c^{-1}a),$$

while

$$R(d)U = \sum_{r \in A} (v_b \otimes \overline{v_b}) \, \delta(db) \, \delta(d^{-1}r) \, B_{r^{-1}} = B_{d^{-1}}(v_{d^{-1}}, v_{d^{-1}})\delta(db).$$

By summation on $c, d \in A$ we get from (31)

$$\begin{aligned} & \sum_{x \in \Gamma} | < \mu[e, a, v_a], \pi(x)\mu[e, b, v_b] > |^2 e^{-\varepsilon |x|} \\ = \quad & \varepsilon^{-2} \, \frac{E_0}{k_0^2} \sum_{c,d \in A} B_c(v_c, v_c)\delta(c^{-1}a) \, B_{d^{-1}}(v_{d^{-1}}, v_{d^{-1}})\delta(db) + o(\varepsilon^{-2}) \\ = \quad & \varepsilon^{-2} \, \frac{E_0}{k_0^2} \, B_a(v_a, v_a) \, B_b(v_b, v_b) + o(\varepsilon^{-2}). \end{aligned}$$

□

We proceed now with the computation of the limits that are needed to prove Theorem 2.

Lemma 5.18. *Let* $c \in A$, $f_1, f_2 \in \mathcal{H}$ *such that* $\operatorname{supp} f_i \subset \Gamma \setminus \Gamma(c)$, *and* $g_1, g_2 \in \mathcal{H}$. *Then*

$$\begin{aligned} & \limsup_{\varepsilon \to 0^+} \varepsilon \sum_{x \in \Gamma(c)} | < f_1, \pi(x)g_1 >< f_2, \pi(x)g_2 > | \, e^{-\varepsilon |x|} \\ & \leq \|f_1\| \|f_2\| \|g_1\| \|g_2\| \, . \end{aligned}$$

Proof. See Lemma 3.2 of [KS01]. □

Lemma 5.19. *Let $c \in A$ be fixed and let $f \in \mathcal{H}$ such that* $\operatorname{supp} f \subset \Gamma \setminus \Gamma(c)$*. If $g \in \mathcal{H}^\infty$, $x \in \widetilde{\Gamma}(c^{-1})$, is of suitable length, then*

$$< f, \mu[c, e, g(x)] >=< f, \pi(x^{-1})g > . \tag{32}$$

Proof. By the previous Lemma we may approximate f with functions in $\mathcal{H}^\infty$ supported in $\Gamma \setminus \Gamma(c)$. For those functions one has

$$< f, \pi(x^{-1})g >= \sum_{|z|=N} \sum_{\substack{a \in A \\ |za|=N+1}} B_a(f(za), g(xza)),$$

where N is big enough so that both f and $\pi(x^{-1})g$ are multiplicative for $|z| > N$.

Since f vanishes on words starting with c and $x \in \widetilde{\Gamma}(c^{-1})$ all the words xza appearing in the above sum are reduced. Moreover, since g is multiplicative, one has

$$g(xza) = \mu[c, e, g(x)](za)$$

and (32) follows by adding up over all z. □

Lemma 5.20. *Let $c \in A$, $f \in \mathcal{H}$ such that* $\operatorname{supp} f \subset \Gamma \setminus \Gamma(c)$*; let $g \in \mathcal{H}$. Then there exists an absolute constant, $k_0 > 0$ and there exists the limit*

$$\lim_{\varepsilon \to 0^+} \varepsilon \sum_{x \in \Gamma(c)} | < f, \pi(x)g > |^2 e^{-\varepsilon|x|} = \frac{1}{k_0} \widehat{B}_c(Sf, Sf) \, \|g\|^2,$$

where Sf is the antilinear functional on $V_{c^{-1}}$ defined by the rule

$$Sf(v_{c^{-1}}) =< f, \mu[c, e, v_{c^{-1}}] >$$

and $k_0 = \sum_{a \in A} \operatorname{tr}(\widehat{B}_a B_{a^{-1}})$.

Proof. By Lemma 5.18 and density we can reduce to the case $g \in \mathcal{H}^\infty$. Identify Sf with an element of $\widehat{V}_c = \overline{V}'_{c^{-1}}$ and $Sf \otimes \overline{Sf}$ with an element of $\mathscr{L}(V_{c^{-1}}, \overline{V}'_{c^{-1}}) = \mathscr{L}(V_{c^{-1}}, V^*_{c^{-1}}) = \mathscr{L}(V^*_{c^{-1}}, V_{c^{-1}})'$. By Lemma 5.19, if $x^{-1} \in \Gamma(c)$ has length N big enough,

$$Sf(g(x)) =< f, \mu[c, e, g(x)] >=< f, \pi(x^{-1})g > \ .$$

Identify $g(x) \otimes \overline{g(x)}$ with an element of $\mathscr{L}(\bar{V}'_{c^{-1}}, V_{c^{-1}}) = \mathscr{L}(V^*_{c^{-1}}, V_{c^{-1}})$ and recall the duality expressed in (14) to get

$$(Sf \otimes \overline{Sf})(g(x) \otimes \overline{g(x)}) = | < f, \pi(x^{-1})g > |^2,$$

(here we proceed with x^{-1} instead of x by sake of calculation).

For the purpose of the limit the contribution of x such that $|x| < N$ is irrelevant, hence it in enough to compute

$$\begin{aligned} &\sum_{\substack{x^{-1}\in\Gamma(c)\\ |x|\geq N}} |< f,\pi(x^{-1})g >|^2\, e^{-\varepsilon|x|} \\ =& \sum_{\substack{x\in\widetilde{\Gamma}(c^{-1})\\ |x|\geq N}} (Sf\otimes\overline{Sf})(g(x)\otimes\overline{g(x)})\, e^{-\varepsilon|x|}\ . \end{aligned}$$

Since the trace is linear and continuous, we shall focus on

$$e^{-\varepsilon N}(Sf\otimes\overline{Sf})\sum_{n=0}^{+\infty}\left(\sum_{\substack{x\in\widetilde{\Gamma}(c^{-1}),\, x=y\cdot c^{-1}\\ |x|=|y|+1=n+N}} g(x)\otimes\overline{g(x)}\right)\, e^{-\varepsilon n}. \tag{33}$$

Now we set, for any $b\in A$, and $n\in\mathbb{N}$,

$$\beta_{n+N,b} = \sum_{\substack{x\in\widetilde{\Gamma}(b),\, x=y\cdot b\\ |x|=|y|+1=n+N}} g(x)\otimes\overline{g(x)}\in\mathscr{L}(V_b^*,V_b),$$

which defines the (column) vector $\beta_{n+N} = (\beta_{n+N,b})_b$, and

$$F_b = (Sf\otimes\overline{Sf})\,\delta(bc)\in\mathscr{L}(V_b^*,V_b)', \tag{34}$$

which defines the (row) vector $F = (F_b)_b$.

Recall from (19) the matrix $D_{4,4} = \left(H_{ab}\otimes\overline{H}_{ab}\right)_{a,b}$. We show first that $D_{4,4}\beta_N = \beta_{1+N}$. Indeed, since g is a multiplicative function, for any $a\in A$,

$$\begin{aligned} D_{4,4}\beta_N &= \sum_{b\in A}\sum_{\substack{x\in\widetilde{\Gamma}(b),\, x=y\cdot b\\ |x|=|y|+1=N}} H_{ab}g(x)\otimes\overline{H_{ab}g(x)} \\ &= \sum_{\substack{b\in A\\ b\neq a^{-1}}}\sum_{\substack{x=y\cdot b\in\widetilde{\Gamma}(b)\\ |x|=|y|+1=N}} g(yba)\otimes\overline{g(yba)} = \sum_{\substack{z=y\cdot a\in\widetilde{\Gamma}(a)\\ |z|=|y|+1=N+1}} g(z)\otimes\overline{g(z)}. \end{aligned}$$

And, by iteration, for any n we get $(D_{4,4})^n\beta_N = \beta_{n+N}$.

¿From (33) and (34) we can write

$$(Sf\otimes\overline{Sf})\sum_{\substack{x\in\widetilde{\Gamma}(c^{-1})\\ |x|\geq N}}(g(x)\otimes\overline{g(x)})\, e^{-\varepsilon|x|} \;=\; e^{-\varepsilon N}F\left(\sum_{n=0}^{+\infty}(D_{4,4}\, e^{-\varepsilon})^n\beta_N\right),$$

where the hypotheses on the matrix systems guarantee that the series converges.

The limit that we are interested in is therefore

$$\lim_{\varepsilon\to 0^+}\varepsilon\sum_{x^{-1}\in\Gamma(c)}|<f,\pi(x^{-1})g>|^2\,e^{-\varepsilon|x|}=F\left[\lim_{\varepsilon\to 0^+}\varepsilon\sum_{n=0}^{+\infty}(D_{4,4}\,e^{-\varepsilon})^n\beta_N\right],$$

the calculation of which we provide in the following claim.

Claim Let $D_\varepsilon = D_{4,4}\,e^{-\varepsilon}$. Then

$$F\left[\lim_{\varepsilon\to 0^+}\varepsilon\sum_{n=0}^{+\infty}D_\varepsilon^n\beta_N\right]=\frac{\widehat{B}_c(Sf,Sf)}{\sum_{a\in A}\operatorname{tr}(\widehat{B}_aB_{a^{-1}})}\|g\|^2. \tag{35}$$

Proof of the Claim

The quantity in bracket is a right eigenvector for the matrix $D_{4,4}$, corresponding to eigenvalue 1. Indeed, since $D_{4,4}=\lim_{\varepsilon\to 0^+}D_\varepsilon$,

$$D_{4,4}[\lim_{\varepsilon\to 0^+}\varepsilon\sum_{n=0}^{+\infty}D_\varepsilon^n\beta_N]=\lim_{\varepsilon\to 0^+}\varepsilon\sum_{n=0}^{+\infty}D_\varepsilon^{n+1}\beta_N=\lim_{\varepsilon\to 0^+}\varepsilon\sum_{n=0}^{+\infty}D_\varepsilon^n\beta_N.$$

But, up to constants, $v=(\widehat{B}_{b^{-1}})_b$ is the only right eigenvector of $D_{4,4}$ corresponding to eigenvalue 1. Hence the two vectors must be proportional, and there exists $\alpha\in\mathbb{C}$ such that the left hand side of (35) is equal to

$$F\alpha\, v=\alpha(Sf\otimes\overline{Sf})(\widehat{B}_c)=\alpha\overline{Sf}(\widehat{B}_c(Sf))=\alpha\overline{\widehat{B}_c(Sf,Sf)}=\alpha\widehat{B}_c(Sf,Sf).$$

Let us calculate α. As follows from the proof of Corollary 5.14, the transpose vector $\tilde{v}^\top$ of $\tilde{v}=(B_b)_b$ is a left eigenvector of $D_{4,4}$ corresponding to eigenvalue 1. Hence

$$\begin{aligned}\alpha\sum_{b\in A}\operatorname{tr}(B_b\widehat{B}_{b^{-1}}) &= \tilde{v}^\top\alpha v=\lim_{\varepsilon\to 0^+}\varepsilon\sum_{n=0}^{+\infty}\tilde{v}^\top D_\varepsilon^n\beta_N\\ &= \lim_{\varepsilon\to 0^+}\varepsilon\sum_{n=0}^{+\infty}e^{-n\varepsilon}\tilde{v}^\top\beta_N=\tilde{v}^\top\beta_N=\sum_{b\in A}\operatorname{tr}(B_b\,\beta_{N,b}),\end{aligned}$$

and we obtain

$$\alpha=\frac{\sum_{b\in A}\operatorname{tr}(B_b\,\beta_{N,b})}{\sum_{b\in A}\operatorname{tr}(B_b\widehat{B}_{b^{-1}})}.$$

Finally

$$\begin{aligned}\sum_{b\in A}\operatorname{tr}(B_b\,\beta_{N,b}) &= \sum_{b\in A}\operatorname{tr}(\beta_{N,b}\,B_b)=\sum_{b\in A}\sum_{\substack{x=y\cdot b\in\widetilde{\Gamma}(b)\\|x|=|y|+1=N}}B_b(g(x),g(x))\\ &= \sum_{|y|=N-1}\sum_{\substack{b\in A\\|yb|=|y|+1}}B_b(g(yb),g(yb))=<g,g>=\|g\|^2.\end{aligned}$$

This ends both the proof of the Claim and of the Lemma. □

Corollary 5.21. *Let* $y = zc \in \widetilde{\Gamma}(c)$. *Let* $f \in \mathcal{H}$ *be such that* $\mathrm{supp} f \subset \Gamma \setminus \Gamma(y)$. *Let* $g \in \mathcal{H}$. *Then*

$$\lim_{\varepsilon \to 0^+} \varepsilon \sum_{x \in \Gamma(y)} | < f, \pi(x) g > |^2 e^{-\varepsilon |x|} = \frac{1}{k_0} \widehat{B}_c(S\pi(z^{-1})f, S\pi(z^{-1})f) \, \|g\|^2,$$

where the constant k_0 *is given in the previous Lemma.*

Proof. If $x \in \Gamma(y)$ and $x = zct$ then $\pi(x) = \pi(z)\pi(ct)$. It is sufficient to apply the previous Lemma to $f_1 = \pi(z^{-1})f$ which has support in $\Gamma \setminus \Gamma(c)$. □

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Dipartimento di Matematica, Università di Milano "Bicocca", Edificio U5 Via Cozzi 53, 20125 Milano, ITALIA
E-mail address: `mariagabriella.kuhn@unimib.it`

Dipartimento di Matematica, Informatica ed Economia, Università degli Studi della Basilicata, Viale dell'ateneo lucano 10, 85100 Potenza, ITALIA
E-mail address: `sandra.saliani@unibas.it`

Matematica, Università degli Studi di Sassari, Via Piandanna 4, 07100 Sassari, ITALIA
E-mail address: `steger@uniss.it`