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# A note on the elliptic Kirchhoff equation in $\mathbb{R}^N$ perturbed by a local nonlinearity

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In this note, we complete the study made in [The elliptic Kirchhoff equation in $\mathbb{R}^N$ perturbed by a local nonlinearity, *Differential Integral Equations* **25** (2012) 543–554] on a Kirchhoff type equation with a Berestycki–Lions nonlinearity. We also correct Theorem 0.6 inside.

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## 0. Introduction

In this note, we consider the nonlinear Kirchhoff equation

$$-\left(a + b\int_{\mathbb{R}^N} |\nabla u|^2\right)\Delta u = g(u) \quad \text{in } \mathbb{R}^N, \;\; N \geq 3, \tag{1}$$

where we assume general hypotheses on $g$. We will investigate the existence of a solution depending on the value of the positive parameters $a$ and $b$. We will fix an incorrect sentence contained in [1] and complete that paper with additional results.

We refer to [1] and the references within for a justification of our study and a bibliography on the problem.

## 1. Existence and Characterization of the Solutions

Assume that

**(g1)** $g \in C(\mathbb{R}, \mathbb{R})$, $g(0) = 0$;
**(g2)** $-\infty < \liminf_{s\to 0^+} g(s)/s \leq \limsup_{s\to 0^+} g(s)/s = -m < 0$;
**(g3)** $-\infty \leq \limsup_{s\to +\infty} g(s)/s^{2^*-1} \leq 0$;
**(g4)** there exists $\zeta > 0$ such that $G(\zeta) := \int_0^\zeta g(s)\, ds > 0$.

It is well-known that the previous assumptions coincide with those in [2], where the problem

$$-\Delta v = g(v) \quad \text{in } \mathbb{R}^N, \;\; N \geq 3, \tag{2}$$

was studied and solved.

First of all, we present the following general result which provides a characterization of the solutions of (1).

**Theorem 1.1.** *$u \in C^2(\mathbb{R}^N) \cap \mathcal{D}^{1,2}(\mathbb{R}^N)$ is a solution to (1) if and only if there exists $v \in C^2(\mathbb{R}^N) \cap \mathcal{D}^{1,2}(\mathbb{R}^N)$ solution to (2) and $t > 0$ such that $t^2 a + t^{4-N} b \times \int_{\mathbb{R}^N} |\nabla v|^2 = 1$ and $u(\cdot) = v(t\cdot)$.*



**Proof.** We first prove the "if" part. Suppose $v \in C^2(\mathbb{R}^N) \cap \mathcal{D}^{1,2}(\mathbb{R}^N)$ and $t > 0$ are as in the statement of the theorem and set $u(\cdot) = v(t\cdot) = v_t(\cdot) \in C^2(\mathbb{R}^N) \cap \mathcal{D}^{1,2}(\mathbb{R}^N)$. We compute

$$\begin{aligned}
-\Delta u(x) &= -\Delta v_t(x) = -t^2 \Delta v(tx) \\
&= t^2 g(v(tx)) = t^2 g(u(x)) = \frac{g(u(x))}{a + bt^{2-N} \displaystyle\int_{\mathbb{R}^N} |\nabla v|^2} \\
&= \frac{g(u(x))}{a + b \displaystyle\int_{\mathbb{R}^N} |\nabla u|^2}.
\end{aligned}$$

Now we prove the "only if" part. Suppose $u \in C^2(\mathbb{R}^N) \cap \mathcal{D}^{1,2}(\mathbb{R}^N)$ is a solution of (1) and set $h = \sqrt{a + b \int_{\mathbb{R}^N} |\nabla u|^2}$, $v(\cdot) = u(h\cdot) = u_h(\cdot)$. Of course $v \in C^2(\mathbb{R}^N) \cap \mathcal{D}^{1,2}(\mathbb{R}^N)$ and $u(\cdot) = v(\frac{1}{h}\cdot)$. Moreover,

$$-\Delta v(x) = -h^2 \Delta u(hx) = h^2 \frac{g(u(hx))}{a + b \displaystyle\int_{R^N} |\nabla u|^2} = g(u_h(x)) = g(v(x))$$

and, if we set $t = \frac{1}{h}$,

$$t^2 = \frac{1}{a + b \displaystyle\int_{\mathbb{R}^N} |\nabla u|^2} = \frac{1}{a + bt^{2-N} \displaystyle\int_{\mathbb{R}^N} |\nabla v|^2}. \qquad \square$$

**Remark 1.2.** Assume **(g1)–(g4)**. By the existence result contained in [2], it is obvious that for $N = 3$ there exists a solution of (1) for any $a$ and $b$ positive numbers.

For $N = 4$, we should have a solution if and only if there exists $v$ solution of (2) such that $b \int_{\mathbb{R}^N} |\nabla v|^2 < 1$. Taking into account the computations in [2, Sec. 4.3], we know that the ground state solution of Eq. (2) has the minimal value of the $\mathcal{D}^{1,2}(\mathbb{R}^N)$ norm among all the solutions of the equation. Then, for $N = 4$ we conclude that Eq. (1) has a solution if and only if the ground state solution $\bar{v}$ of (2) is such that $b \int_{\mathbb{R}^N} |\nabla \bar{v}|^2 < 1$.

Consider the functional of the action related with Eq. (1)

$$I(u) = \frac{1}{2}\left(a + \frac{b}{2}\int_{\mathbb{R}^N} |\nabla u|^2\right)\int_{\mathbb{R}^N} |\nabla u|^2 - \int_{\mathbb{R}^N} G(u),$$

where $G(s) = \int_0^s g(z)\,dz$, being $g$ possibly modified as in [2] in order to make $I$ a $C^1$ functional on $H^1(\mathbb{R}^N)$. We observe that, for small dimensions, the value of the action computed in the solutions increases as the $\mathcal{D}^{1,2}(\mathbb{R}^N)$ norm increases according to the following.

**Proposition 1.3.** *Assume $N = 3$ or $N = 4$. If $v_1$ and $v_2$ are solutions of* (2) *and $\int_{\mathbb{R}^N} |\nabla v_1|^2 < \int_{\mathbb{R}^N} |\nabla v_2|^2$ and, for $N = 4$, we also have $b\int_{\mathbb{R}^N} |\nabla v_2|^2 < 1$, then, calling $t_1$ and $t_2$ the positive numbers such that respectively $v_1(t_1\cdot)$ and $v_2(t_2\cdot)$ are solutions of* (1), *we have $t_2 < t_1$ and $I(v_1(t_1\cdot)) < I(v_2(t_2\cdot))$.*

**Proof.** By Theorem 1.1, it is immediate to see that $t_2 < t_1$. Now observe that any solution of (1) satisfies the Pohozaev identity

$$a\frac{N-2}{2N}\int_{\mathbb{R}^N} |\nabla u|^2 + b\frac{N-2}{2N}\left(\int_{\mathbb{R}^N} |\nabla u|^2\right)^2 - \int_{\mathbb{R}^N} G(u) = 0. \tag{3}$$

As a consequence the action computed in any solution of (1) is

$$I(u) = a\frac{1}{N}\int_{\mathbb{R}^N} |\nabla u|^2 + b\frac{4-N}{4N}\left(\int_{\mathbb{R}^N} |\nabla u|^2\right)^2$$

and then, if $v$ and $t > 0$ are related with $u$ as in Theorem 1.1, we have that

$$I(u) = a\frac{t^{2-N}}{N}\int_{\mathbb{R}^N} |\nabla v|^2 + b\frac{4-N}{4N}t^{4-2N}\left(\int_{\mathbb{R}^N} |\nabla v|^2\right)^2.$$

Since $t^2 a + t^{4-N} b \int_{\mathbb{R}^N} |\nabla v|^2 = 1$, we can cancel the dependence of $I$ from $b$

$$\begin{aligned} I(u) &= \frac{\int_{\mathbb{R}^N} |\nabla v|^2}{Nt^N}\left(at^2 + \frac{4-N}{4}(1 - at^2)\right) \\ &= \frac{a}{4}\frac{\int_{\mathbb{R}^N} |\nabla v|^2}{t^{N-2}} + \frac{4-N}{4N}\frac{\int_{\mathbb{R}^N} |\nabla v|^2}{t^N}. \end{aligned} \tag{4}$$

The conclusion easily follows from (4). □

In the following corollary, we establish the conditions which guarantee the existence of a ground state solution for $N \geq 3$. In particular, we correct [1, Theorem 0.6] for what concerns the dimension $N = 4$.

**Corollary 1.4.** *Assume* (**g1**)–(**g4**).

*If $N = 3$ then Eq.* (1) *has a ground state solution.*

*If $N = 4$ then Eq.* (1) *has a ground state solution if and only if $b\int_{\mathbb{R}^N} |\nabla \bar{v}|^2 < 1$, being $\bar{v}$ a ground state solution of* (2).

*If* $N \geq 5$ *then Eq.* (1) *has a solution if and only if*

$$a \leq \left(\frac{N-4}{N-2}\right)^{\frac{N-2}{N-4}} \left(\frac{2}{(N-4)b\int_{\mathbb{R}^N} |\nabla \bar{v}|^2}\right)^{\frac{2}{N-4}}, \tag{5}$$

*being* $\bar{v}$ *a ground state solution of* (2). *Moreover, the functional attains the infimum.*

**Proof.** Since the functional of the action related with Eq. (2), when computed in the solutions of the equation, is directly proportional to the $\mathcal{D}^{1,2}(\mathbb{R}^N)$ norm of the solutions (see Remark 1.2), the conclusion for cases $N = 3$ and $N = 4$ follows immediately from Proposition 1.3 and [2].

If $N \geq 5$, by Theorem 1.1 we have to show that there exists a solution $v$ of Eq. (2) and $t > 0$ such that $t^2 a + t^{4-N} b \int_{\mathbb{R}^N} |\nabla v|^2 = 1$. Of course such a couple $(v, t)$ exists if only if there exists $t_0 > 0$ such that

$$t_0^2 a + t_0^{4-N} b \int_{\mathbb{R}^N} |\nabla \bar{v}|^2 = 1 \tag{6}$$

for $\bar{v}$ a ground state solution of (2). By studying the function $f(t) = at^2 + b_0 t^{4-N}$ for $t > 0$, being $b_0 = b \int_{\mathbb{R}^N} |\nabla \bar{v}|^2$, we observe that (6) holds for some $t_0$ if and only if

$$\min_{t>0} f(t) \leq 1. \tag{7}$$

An easy computation leads to (5). As a remark we point out that if (7) holds with the strict inequality, then we can find two values $t_1 < t_2$ which solve (6) and two corresponding distinct solutions $\bar{v}_{t_1}$ and $\bar{v}_{t_2}$ to Eq. (1).

Now we prove that the functional $I$ attains the minimum. For $i = 1, 2$, define $g_i$ and $G_i$ as in [2]. Observe that, by [2, (3.4) and (3.5)],

$$\begin{aligned} I(u) &= \frac{1}{2}\left(a + \frac{b}{2}\int_{\mathbb{R}^N} |\nabla u|^2\right)\int_{\mathbb{R}^N} |\nabla u|^2 + \int_{\mathbb{R}^N} G_2(u) - \int_{\mathbb{R}^N} G_1(u) \\ &\geq \frac{1}{2}\left(a + \frac{b}{2}\int_{\mathbb{R}^N} |\nabla u|^2\right)\int_{\mathbb{R}^N} |\nabla u|^2 + (1-\varepsilon)\int_{\mathbb{R}^N} G_2(u) - C_\varepsilon \int_{\mathbb{R}^N} |u|^{\frac{2N}{N-2}} \\ &\geq \frac{1}{2}\left(a + \frac{b}{2}\int_{\mathbb{R}^N} |\nabla u|^2\right)\int_{\mathbb{R}^N} |\nabla u|^2 + \frac{1-\varepsilon}{2} m \int_{\mathbb{R}^N} |u|^2 - C_\varepsilon \int_{\mathbb{R}^N} |u|^{\frac{2N}{N-2}}, \end{aligned}$$

where $\varepsilon < 1$ and $C_\varepsilon > 0$ are suitable constants.

Since $\mathcal{D}^{1,2}(\mathbb{R}^N) \hookrightarrow L^{\frac{2N}{N-2}}(\mathbb{R}^N)$, for a suitable positive constant $C$ we have

$$\begin{aligned} I(u) &\geq \frac{1}{2}\left(a + \frac{b}{2}\int_{\mathbb{R}^N} |\nabla u|^2\right)\int_{\mathbb{R}^N} |\nabla u|^2 \\ &\quad + \frac{1-\varepsilon}{2} m \int_{\mathbb{R}^N} |u|^2 - C\left(\int_{\mathbb{R}^N} |\nabla u|^2\right)^{\frac{N}{N-2}}, \end{aligned}$$

and, since for $N \geq 5$ we have $1 < \frac{N}{N-2} < 2$, we deduce that the functional is bounded below and coercive with respect to $H^1(\mathbb{R}^N)$ norm.

Now, since for every $u \in H^1(\mathbb{R}^N)$ and its corresponding Schwarz symmetrization $u^*$ we have

$$\int_{\mathbb{R}^N} |\nabla u^*|^2 \leq \int_{\mathbb{R}^N} |\nabla u|^2, \quad \int_{\mathbb{R}^N} G(u^*) = \int_{\mathbb{R}^N} G(u),$$

we can look for a minimizer of $I$ in $H^1_r(\mathbb{R}^N)$, the set of radial functions in $H^1(\mathbb{R}^N)$. As in [2], we can prove that the functional

$$u \in H^1_r(\mathbb{R}^N) \mapsto \int_{\mathbb{R}^N} G_1(u) \in \mathbb{R}$$

is compact, so, by standard arguments based on Weierstrass Theorem, $I$ attains the infimum. □

**Remark 1.5.** Suppose $N \geq 5$. By previous corollary we have that if (5) does not hold, then $I$ is nonnegative in $H^1(\mathbb{R}^N)$.

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