Reports of Enlarged Session of the
Seminar of I. Vekua Institute
of Applied Mathematics
Volume 28, 2014

ON THE POTENTIAL THEORY IN THE LINEAR THEORY OF VISCOELASTIC MATERIALS WITH VOIDS

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Abstract. By using an indirect boundary integral method, the solution of the first (second) BVP of steady vibrations related to the linear theory of viscoelasticity for Kelvin-Voigt materials with voids is represented by means of a simple (double) layer elastopotential.

Keywords and phrases: Boundary integral equations, layer potentials, viscoelasticity, Kelvin-Voigt material with voids.

AMS subject classification: 31B10, 35C15, 74D05.

As is well known, the classical indirect method of Fredholm gives the solution of the Dirichlet problem for the n-dimensional Laplacian in terms of a double layer potential $u(x) = \int_\Sigma \varphi(y) \frac{\partial}{\partial \nu_y} s(x-y) d\sigma_y$, where $\nu(x) = (\nu_1(x), \ldots, \nu_n(x))$ denotes the outward unit normal vector at the point $x = (x_1, \ldots, x_n) \in \Sigma$ and s is the fundamental solution of the Laplace equation. There is another approach, which consists in looking for the solution in the form of a simple layer potential $u(x) = \int_\Sigma \varphi(y) s(x-y) d\sigma_y$; in this case an integral equation of the first kind arises

$$\int_\Sigma \varphi(y) s(x-y) d\sigma_y = g(x), \qquad x \in \Sigma. \tag{1}$$

Muskhelishvili ([1], p.184) gave a method for solving (1) when $n = 2$, which leads to the study of a singular integral equation. Even if such approach is based on the theory of holomorphic functions of one complex variable and uses the derivative with respect to the arc length, in [2] it was generalized to the case of n real variables by one of the authors. The main idea consists in replacing holomorphic functions by conjugate differential forms and the derivative with respect to the arc length by the exterior differential operator d. This method hinges on the theory of reducible operators and on the theory of differential forms; it does not require the use of pseudo-differential operators nor the use of hypersingular integrals. The approach has been applied to different BVPs for other PDEs in simply and multiply connected domains (see [3-14]).

Here, we consider the application of the method to the study of the two basic BVPs for the homogeneous equations of steady vibrations in the linear theory of viscoelasticity for Kelvin-Voigt materials with voids, i.e. for the system

$$\begin{cases} \mu_1 \Delta u + (\lambda_1 + \mu_1) \text{grad div } u + b_1 \text{grad } \varphi + \rho\omega^2 u = 0, \\ (\alpha_1 \Delta + \xi_2)\varphi - \nu_1 \text{div } u = 0, \end{cases} \tag{2}$$

where $u = (u_1, u_2, u_3)$ is a complex time-independent vector function, φ is a complex time-independent function, ρ is the reference mass density ($\rho > 0$), ω is the oscillation frequency ($\omega > 0$), $\lambda_1 = \lambda - i\omega\lambda^*$, $\mu_1 = \mu - i\omega\mu^*$, $b_1 = b - i\omega b^*$, $\alpha_1 = \alpha - i\omega\alpha^*$,

$\nu_1 = b - i\omega\nu^*$, $\xi_1 = \xi - i\omega\xi^*$, $\xi_2 = \rho_0\omega^2 - \xi_1$, $\rho_0 = \rho k$, k being the equilibrated inertia ($k > 0$), and $\lambda, \mu, b, \alpha, \xi, \lambda^*, \mu^*, b^*, \alpha^*, \nu^*, \xi^*$ are (real) constitutive coefficients (see [15]).

We denote the matrix of fundamental solution of the homogeneous system (2) by $\Gamma = (\Gamma_{pq})_{4\times4}$ (see [15]). Moreover, the fundamental solution of the system

$$\begin{cases} \mu_1\Delta u + (\lambda_1 + \mu_1)\text{grad div } u = 0, \\ \alpha_1\Delta\varphi = 0 \end{cases}$$

is the matrix $\Psi = (\Psi_{pq})_{4\times4}$, whose entries are $\Psi_{lj}(x) = -\frac{1}{8\pi}\left(\frac{1}{\mu_1}\Delta\delta_{lj} - \frac{\lambda_1+\mu_1}{\mu_1\mu_2}\frac{\partial^2}{\partial x_l\partial x_j}\right)|x|$ ($\mu_2 = \lambda_1 + 2\mu_1$), $\Psi_{44}(x) = \frac{1}{\alpha_1}s(x)$, $\Psi_{l4}(x) = \Psi_{4j}(x) = 0$, $l, j = 1, 2, 3$.

Lemma 1. ([15], Theorem 4.2) *If $\alpha_1\mu_1\mu_2 \neq 0$, then the relations*

$$\Psi_{pq}(x) = \mathcal{O}\left(\frac{1}{|x|}\right), \qquad \Gamma_{pq}(x) - \Psi_{pq}(x) = \mathcal{O}(1 + |x|),$$
$$\frac{\partial^m}{\partial x_1^{m_1}\partial x_2^{m_2}\partial x_3^{m_3}}[\Gamma_{pq}(x) - \Psi_{pq}(x)] = \mathcal{O}\left(\frac{1}{|x|^{m-1}}\right)$$

hold in a neighborhood of the origin, where $m = m_1 + m_2 + m_3$, $m \geq 1$, $m_l \geq 0$, $l = 1, 2, 3$ and $p, q = 1, 2, 3, 4$.

Therefore, $\Psi(x)$ is the singular part of the matrix $\Gamma(x)$.

In what follows, $\Omega \subset \mathbb{R}^3$ is a bounded simply connected domain (i.e. $\mathbb{R}^3 \setminus \overline{\Omega}$ is connected) such that its boundary $\Sigma = \partial\Omega$ is a Lyapunov hypersurface (i.e. Σ has a uniformly Hölder continuous normal field of some exponent $l \in (0, 1]$); p indicates a real number such that $p \in]1, +\infty[$. Let us consider the first BVP in the class $\mathcal{S}^p$ of the simple layer elastopotentials $U[\phi](x) = \int_\Sigma \Gamma(x - y)\phi(y)d\sigma_y$ with density in $[L^p(\Sigma)]^4$:

$$\begin{cases} U \in \mathcal{S}^p, & \\ A(D_x)U = 0 & \text{in } \Omega, \\ U = F & \text{on } \Sigma, \quad F \in [W^{1,p}(\Sigma)]^4 \end{cases} \tag{3}$$

(by $W^{1,p}(\Sigma)$ we denote the usual Sobolev space), where $U = (u, \varphi)$ and $A(D_x) = (A_{pq}(D_x))_{4\times4}$ is the matrix whose entries are $A_{lj}(D_x) = (\mu_1\Delta + \rho\omega^2)\delta_{lj} + (\lambda_1 + \mu_1)\frac{\partial^2}{\partial x_l\partial x_j}$, $A_{l4}(D_x) = b_1\frac{\partial}{\partial x_l}$, $A_{4l}(D_x) = -\nu_1\frac{\partial}{\partial x_l}$, $A_{44}(D_x) = \alpha_1\Delta + \xi_2$, $l, j = 1, 2, 3$ (δ_{lj} being the Kronecker delta).

Imposing the boundary condition we get the integral system of the first kind

$$\int_\Sigma \Gamma(x - y)\phi(y)\, d\sigma_y = F(x) \tag{4}$$

on Σ. Following the approach introduced in [2], we take the differential of both sides of (4), obtaining the following singular integral system

$$\int_\Sigma d_x[\Gamma(x - y)]\phi(y)\, d\sigma_y = dF(x). \tag{5}$$

In (5) the unknown is the vector $(\phi_1, ..., \phi_4)$ whose components are scalar functions, while the data is the vector $(dF_1, \ldots, dF_4)$ whose components are differential forms of degree 1. We are going to show that the singular integral system (5) can be reduced to an equivalent Fredholm one.

It is possible to prove (see ([4], Lemma 5.3)) that the singular integral operator[1]

$$S_0 : [L^p(\Sigma)]^4 \longrightarrow [L^p_1(\Sigma)]^4, \qquad S_0(\phi)(x) = \int_\Sigma d_x[\Psi(x-y)]\phi(y)d\sigma_y$$

can be reduced on the left. This means that there exists a linear and continuous operator $S' : [L^p_1(\Sigma)]^4 \longrightarrow [L^p(\Sigma)]^4$ such that $S'S_0$ is a Fredholm operator from $[L^p(\Sigma)]^4$ into itself. Let us define the singular integral operator

$$S : [L^p(\Sigma)]^4 \longrightarrow [L^p_1(\Sigma)]^4, \qquad S\phi(x) = \int_\Sigma d_x[\Gamma(x-y)]\phi(y)d\sigma_y.$$

Since $S - S_0$ is compact by Lemma 1 and we can write $S = (S - S_0) + S_0$, we obtain that $S'S$ is a Fredholm operator. We thus obtain the next claim.

Proposition 1. ([4], Proposition 5.1) *The singular integral operator S can be reduced on the left.*

We deduce that the integral system (5) admits a solution if, and only if, $\int_\Sigma \gamma_i \wedge \overline{dF}_i = 0$, $i = 1,2,3,4$, for every $\gamma \in [L^q_1(\Sigma)]^4$ solution of the homogeneous adjoint system $S^*_j\gamma(x) = \displaystyle\int_\Sigma \gamma_i(y) \wedge d_y[\Gamma_{ij}(x-y)] = 0$, a.e. $x \in \Sigma$, $j = 1,2,3,4$. If

$$\mu^* > 0, \quad 3\lambda^* + 2\mu^* > 0, \quad \alpha^* > 0, \quad (3\lambda^* + 2\mu^*)\xi^* > \frac{3}{4}(b^* + \nu^*)^2, \tag{6}$$

one can prove that $S^*\gamma = 0$ if, and only if, γ_i is a weakly closed 1-form (see ([4], Theorem 5.1)). Consequently, the singular integral system $S\phi = dF$ is always solvable.

Since any solution of a Dirichlet problem with constant datum can be represented by means of a simple layer elastopotential (see ([4], Lemma 5.5)), we have the following representation for the solution of the first problem.

Theorem 1. ([4], Theorem 5.2) *If conditions* (6) *hold, the first BVP* (3) *admits a unique solution U. In particular, the density ϕ of U can be written as $\phi = \phi_0 + \psi_0$, where ϕ_0 solves the singular integral system* (5) *and ψ_0 is the density of a simple layer elastopotential which is constant on Σ.*

We remark that, the obtained reduction is not an equivalent reduction[2]. However, we still have an equivalence between (5) and the Fredholm equation $S'S\phi = S'(dF)$. In fact, as in ([16], pp.253-254), one can show that $N(S'S) = N(S)$. This implies that, if G is such that there exists a solution g of the system $Sg = G$, then this system is satisfied if and only if $S'Sg = S'G$. Since we know that the system $Sg = dF$ is solvable, we have that $Sg = dF$ if, and only if, $S'Sg = S'(dF)$ (see ([4], Theorem 5.3)).

These results permit also to represent the solution of the second BVP of steady vibrations related to the linear theory of viscoelasticity for Kelvin-Voight materials

[1] The symbol $L^p_k(\Sigma)$ stands for the space of the differential forms of degree k defined on Σ whose components belong to $L^p(\Sigma)$ in a coordinate system of class C^1 and then in every coordinate system of class C^1.

[2] A left reduction is said to be equivalent if $\mathrm{N(S')} = \{0\}$, where $\mathrm{N(S')}$ denotes the kernel of S'. This implies that $S\alpha = \beta$ if, and only if, $S'S\alpha = S'\beta$.

with voids by means of the double layer elastopotential. For the details we refer to ([4], Section 6).

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Received 23.05.2014; revised 22.11.2014; accepted 29.12.2014.

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