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Research Article

On the Regularity of Weak Contact p-Harmonic Maps

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We prove Caccioppoli type estimates and consequently establish local Hölder continuity for a class of weak contact $(2n+2)$-harmonic maps from the Heisenberg group $\mathbb{H}_n$ into the sphere S^{2m-1}.

1. Introduction

The study of *pseudoharmonic maps* was started by Barletta et al. [1] (cf. also [2, 3] for successive investigations) as a generalization of the theory of harmonic maps among Riemannian manifolds (cf., e.g., [4]) and by identifying the results of Jost and Xu [5], Zhou [6], Hajłasz and Strzelecki [7], and Wang [8] as local aspects of the theory of pseudoharmonic maps from a strictly pseudoconvex CR manifold into a Riemannian manifold (cf. also [9, pages 225-226]).

A similar class of maps, yet with values in another CR manifold, was studied in [10]. These are critical points of the functional

$$E(\phi) = \frac{1}{2}\int_M Q(\phi)\, dv, \quad \phi \in C^\infty(M,N), \tag{1}$$

where M is a compact strictly pseudoconvex CR manifold of CR dimension n, $Q(\phi) = \|(d\phi)_{H,H'}\|^2$, $dv = \theta \wedge (d\theta)^n$, and θ is a contact form on M. Also N is a contact Riemannian manifold and in particular an almost CR manifold (of CR codimension 1).

A moment's thought reveals the augmented difficulties such a theory may present. For instance, if M and N are two strictly pseudoconvex CR manifolds endowed, respectively, with contact forms θ and η, then the pseudohermitian analog of the notion of a harmonic morphism (among Riemannian manifolds) is quite obvious: one may consider continuous maps $\phi : M \to N$ such that the pullback $v \circ \phi$ of any local solution $v : U' \subseteq N \to \mathbb{R}$ to $\Delta_b^N v = 0$ in V satisfies $\Delta_b(v \circ \phi) = 0$ in $U = \phi^{-1}(U')$ in distribution sense. Here Δ_b and Δ_b^N are the sublaplacians of (M,θ) and (N,η), respectively. Unlike the situation in [2] (where the target manifold N is Riemannian and ϕ pulls back local harmonic functions on N to distribution solutions of $\Delta_b u = 0$) such ϕ is not necessarily smooth (since it is unknown whether local coordinate systems (U', x'^i) on N such that $\Delta_b^N x'^i = 0$ in U' might be produced). To give another example, should one look for a pseudohermitian analog to the Fluglede-Ishihara theorem (cf. [3] when M is CR and N is Riemannian), one would face the lack of an Ishihara type lemma (cf. [11]) as it is unknown whether $\Delta_b^N v = 0$ admits local solutions whose (horizontal) gradient and hessian have prescribed values at a point. Moreover, what would be the appropriate notion of a hessian (cf. [12] for a possible choice)?

A third example, discussed at some length in this paper, is that of the "degeneracy" of the Euler-Lagrange equations

$$\left[\left(\varphi^2\right)^i_j \circ \phi\right] \times \left\{\operatorname{div}\left[Q(\phi)^{(p-2)/2}\nabla^H \phi^j\right]\right.$$

$$+Q(\phi)^{(p-2)/2}\sum_{a=1}^{2n}\left(\Gamma'^{j}_{k\ell}\circ\phi\right)X_a\left(\phi^k\right)X_a\left(\phi^\ell\right)\Bigg\}=0,\qquad 1\le i\le 2m-1, \tag{2}$$

associated to the variational principle

$$\delta\int Q(\phi)^{p/2}dv=0, \tag{3}$$

when N is a Sasakian manifold. Indeed the $(2m-1)\times(2m-1)$ matrix $(\varphi^2)^i_j=-\delta^i_j+\xi^i\eta_j$ has but rank $2m-2$ at each point (a well-known phenomenon in contact Riemannian geometry, cf., e.g., [13]. See also [14]). Consequently, in general one may not expect regularity of weak solutions to (2). For instance, if $N=\mathbb{H}_{m-1}$ is the Heisenberg group and $\phi=(\phi',\phi^{2m-1}):U\subseteq M\to\mathbb{H}_{m-1}$ is a solution to (2), then $\phi':U\to\mathbb{R}^{2m-2}$ is subject to

$$\sum_{a=1}^{2n}X^*_a\left(\left|X\phi'\right|^{p-2}X_a\left(\phi^i\right)\right)=0,\quad 1\le i\le 2m-2, \tag{4}$$

yet ϕ^{2m-1} is an arbitrary function (cf. Section 3). For the more appealing case, where $M=\mathbb{H}_n$ is the Heisenberg group and $N=S^{2m-1}$ is the sphere, (2) may be written as

$$X^*\cdot V_A=Q(\phi)^{p/2}\phi_A,\quad 1\le A\le 2m, \tag{5}$$

(cf. Proposition 15) which is indeed the form assumed by the Euler-Lagrange equations in [7], yet unlike the situation there $X^*\cdot E_{A,B}\neq 0$ in general (cf. Proposition 16 for the notations). Although $X^*\cdot E_{A,B}$ has a quite explicit form (yielding—for a class of weak solutions $\phi:\mathbb{H}_n\to S^{2m-1}$ which are close to being horizontal maps—simple estimates on $X^*\cdot E_{A,B}$), only a weaker form of the duality inequality lemma in [7] may be proved (cf. Lemma 17) leading nevertheless (together with a hole filling argument) to Caccioppoli type estimates

$$\int_{B_X(x,r)}\left|X\phi\right|^{2n+2}dv\le Cr^\gamma, \tag{6}$$

for some $C>0$ and $0<\gamma<1$, which are known (cf., e.g., [7] for a very general argument based on work in [15]) to imply the local Hölder continuity of the given weak solution.

The paper is organized as follows. In Section 2 we recall a few conventions and basic results obtained in [10]. Sections 3 and 4 are devoted to the study of the local properties of weak contact p-harmonic maps. We show that weak contact $(2n+2)$-maps $\phi:U\subset\mathbb{H}_n\to S^{2m-1}$ are locally Hölder continuous (cf. Corollary 21) provided they are close to being horizontal maps; that is, the assumptions (96) are satisfied. The relevance of the number $p=2n+2$ stems from the facts that $\int_M\|(d\phi)_{H,H'}\|^{2n+2}dv$ is a CR invariant and $2n+2$ is the homogeneous dimension of $\mathbb{H}_n$. The authors believe that subelliptic theory should play within CR geometry, as a branch of complex analysis in several complex variables, the strong role played by elliptic theory in Riemannian geometry, and the present paper is a step in this direction.

2. Basic Conventions and Results

For all notions of CR and pseudohermitian geometry we adopt the conventions and notations in the monograph [9]. For the approach to contact structures within Riemannian geometry we rely on the presentation in Blair [13], (cf. also Tanno [16]). Given a real $(2n+1)$-dimensional C^∞ differentiable manifold M, an *almost CR structure* is a complex subbundle $T_{1,0}(M)\subset T(M)\otimes\mathbb{C}$ of the complexified tangent bundle, of complex rank n, such that $T_{1,0}(M)_x\cap T_{0,1}(M)_x=(0)$ for any $x\in M$. Here $T_{0,1}(M)=\overline{T_{1,0}(M)}$ and overbars indicate complex conjugates. The integer n is the *CR dimension* of the almost CR manifold $(M,T_{1,0}(M))$. Almost CR structures are a bundle theoretic recast of the *tangential Cauchy-Riemann operator* $\overline{\partial}_b:C^\infty(M,\mathbb{C})\to C^\infty(T_{0,1}(M)^*)$ given by $(\overline{\partial}_b f)\overline{Z}=\overline{Z}(f)$ for any $f\in C^\infty(M,\mathbb{C})$ and any $Z\in T_{1,0}(M)$. An almost CR structure is (*formally* or *Frobenius*) *integrable* if $[Z,W]\in C^\infty(U,T_{1,0}(M))$ for any $Z,W\in C^\infty(U,T_{1,0}(M))$ and any open set $U\subset M$. The tangential C-R operator may be extended to arbitrary $(0,q)$-forms on M and the resulting pseudocomplex $\overline{\partial}_b:\Omega^{0,q}(M)\to\Omega^{0,q+1}(M)$, $q\ge 0$, is a complex (i.e., $\overline{\partial}_b^2=0$) if and only if the given almost CR structure is integrable (cf. [9]). Integrable almost CR structures are commonly referred to as *CR structures* and appear mainly on real hypersurfaces of complex manifolds, as induced by the complex structure of the ambient space; that is, for any complex manifold V and any real hypersurface $M\subset V$

$$T_{1,0}(M)_x=\left[T_x(M)\otimes_{\mathbb{R}}\mathbb{C}\right]\cap T^{1,0}(V)_x,\quad x\in M, \tag{7}$$

is a CR structure on M. Here $T^{1,0}(V)\to V$ is the holomorphic tangent bundle over V (locally the span of $\{\partial/\partial z^j:1\le j\le N\}$ for any local system of complex coordinates (z^j) on V). Also N is the complex dimension of V, and then the CR dimension of M is $n=N-1$. Integrability of (7) follows from the Nijenhuis integrability of the complex structure on V. A solution f to $\overline{\partial}_b f=0$ (the *tangential C-R equations*) is a *CR function* on M and, in the context of real hypersurfaces carrying the induced CR structure (7), CR functions appear as traces on M of holomorphic functions defined on a neighborhood of M in V. Hence to say that the CR structure is given by (7) is to say that the tangential C-R equations are induced by the ordinary Cauchy-Riemann system on V. CR functions which are not traces of holomorphic functions may exist (cf., e.g., [17]). CR structures which are not given by (7), and for which there is not any embedding of M into some complex manifold V yielding (7), do exist as well (cf. again [17, page 172]). An array of geometric objects, such as pseudohermitian structures, the Levi form (cf. [9, 18]) and successively (in the nondegenerate case) contact structures, the Tanaka-Webster connection (cf. [18, 19]), the sublaplacian Δ_b and the Fefferman metric (cf. [9, 20]), springs from the given CR structure very much the way the complex structure determines the metric structure (up to a conformal invariant) on a Riemann surface and are thought of as geometric tools whose use will ultimately shed light on the properties of solutions, local and global, to the

tangential C-R equations. Integrability of $T_{1,0}(M)$ appears as a built-in ingredient of objects such as the Tanaka-Webster connection or the Fefferman metric, yet it is believed to lack the geometric meaning of involutivity of real smooth distributions on manifolds (cf., e.g., [21, page 16]). On the other hand nonintegrable examples of almost CR structures occur frequently, either on real hypersurfaces of almost complex manifolds or on contact Riemannian manifolds (cf. [13, 16]). A remedy was indicated by Tanno [16], showing that the wealth of additional structure (φ, ξ, η, g) on a given contact Riemannian manifold N compensates for the lack of integrability of $T_{1,0}(N) = \{X - i\varphi X : X \in \operatorname{Ker}(\eta)\}$ and specifically providing a generalization of the Tanaka-Webster connection to the nonintegrable context.

Given a CR manifold $(M, T_{1,0}(M))$, let $H = \operatorname{Re}\{T_{1,0}(M) \oplus T_{0,1}(M)\}$ be the *Levi*, or *maximally complex*, distribution and $J(Z + \overline{Z}) = i(Z - \overline{Z})$, $Z \in T_{1,0}(M)$, its complex structure. Let $H_x^{\perp} = \{\omega \in T_x^*(M) : \operatorname{Ker}(\omega) \supseteq H_x\}$, $x \in M$, be the conormal bundle associated to H, a real line bundle over M. Since M is assumed to be connected and orientable, the conormal bundle $H^{\perp} \to M$ is trivial. A globally defined nowhere zero section $\theta \in \Gamma^{\infty}(H^{\perp})$ is a *pseudohermitian structure* on M. For each pseudohermitian structure θ on M the *Levi form* is

$$G_{\theta}(X, Y) = (d\theta)(X, JY), \quad X, Y \in H. \tag{8}$$

Two pseudohermitian structures $\theta, \widehat{\theta} \in \Gamma^{\infty}(H^{\perp})$ are related by $\widehat{\theta} = \lambda\theta$ for some C^{∞} function $\lambda : M \to \mathbb{R} \setminus \{0\}$. If this is the case, then $G_{\widehat{\theta}} = \lambda G_{\theta}$. A CR manifold M is *nondegenerate* (resp., *strictly pseudoconvex*) if G_{θ} is nondegenerate (resp., positive definite) for some θ. If M is a nondegenerate CR manifold, of CR dimension n, then each pseudohermitian structure θ is a contact form; that is, $\theta \wedge (d\theta)^n$ is a volume form on M. If M is nondegenerate and θ is a contact form on M, there is a unique globally defined, nowhere zero, tangent vector field $T \in \mathfrak{X}^{\infty}(M)$ (the *Reeb vector* field of (M, θ)) such that $\theta(T) = 1$ and $(d\theta)(T, \cdot) = 0$. The *Webster metric* is the semi-Riemannian metric g_{θ} on M given by

$$g_{\theta}(X, Y) = G_{\theta}(X, Y), \qquad g_{\theta}(X, T) = 0, \qquad g_{\theta}(T, T) = 1, \tag{9}$$

for any $X, Y \in H$. If M is strictly pseudoconvex and θ is chosen such that G_{θ} is positive definite, then g_{θ} is a Riemannian metric on M.

Let N be a $(2m-1)$-dimensional C^{∞} manifold $(m \geq 2)$. An *almost contact structure* on N is a synthetic object (ϕ, ξ, η) consisting of a $(1,1)$-tensor field φ, a vector field $\xi \in \mathfrak{X}^{\infty}(N)$, and a 1-form $\eta \in \Omega^1(N)$ such that

$$\begin{gathered} \varphi_k^i \varphi_j^k = -\delta_j^i + \eta_j \xi^i, \qquad \eta_i \varphi_j^i = 0, \\ \varphi_j^i \xi^j = 0, \qquad \eta_i \xi^i = 1, \end{gathered} \tag{10}$$

with respect to any local coordinate system (U', x'^i) on N. A Riemannian metric g on N is *associated*, or *compatible*, to the almost contact structure (φ, ξ, η) (and (φ, ξ, η, g) is an *almost contact metric structure* on N) if

$$g_{ij} \varphi_k^i \varphi_{\ell}^j = g_{k\ell} - \eta_k \eta_{\ell}, \quad g_{ij} \xi^j = \eta_i. \tag{11}$$

Associated metrics always exist (cf. [13]). A *contact metric structure* is an almost contact metric structure (φ, ξ, η, g) such that $\Omega = d\eta$, where $\Omega \in \Omega^2(N)$ is the 2-form given by $\Omega_{ij} = g_{ik} \varphi_j^k$.

Let $\phi : M \to N$ be a C^{∞} map from a strictly pseudoconvex CR manifold M of CR dimension n into a contact Riemannian manifold $(N, \varphi, \xi, \eta, g)$. Let θ be a contact form on M such that the Levi form G_{θ} is positive definite. Let $H' = \operatorname{Ker}(\eta)$ and let us consider the vector bundle valued form $(d\phi)_{H,H'} \in \Gamma^{\infty}(H^* \otimes \phi^{-1} H')$ given by

$$\left((d\phi)_{H,H'}\right)_x = \Pi_{H',\phi(x)} \circ (d_x \phi) : H_x \longrightarrow H'_{\phi(x)}, \quad x \in M, \tag{12}$$

where $\Pi_{H'} : T(N) \to H'$ is the natural projection associated to the decomposition $T(N) = H' \oplus \mathbb{R}\xi$. Let $x \in M$ and let $\{X_a : 1 \leq a \leq 2n\}$ be a local G_{θ}-orthonormal frame of H defined on an open neighborhood $U \subseteq M$ of $x \in U$. We set

$$\begin{aligned} &Q(\phi)_x \\ &\quad = \left\|(d\phi)_{H,H'}\right\|_x^2 \\ &\quad = \sum_{a=1}^{2n} g_{\phi(x)} \left(\left((d\phi)_{H,H'}\right)_x X_{a,x}, \left((d\phi)_{H,H'}\right)_x X_{a,x}\right). \end{aligned} \tag{13}$$

Note that

$$Q(\phi) = \operatorname{trace}_{G_{\theta}} \{\Pi_H (\phi^* g)\} - \left\|\Pi_H \phi^* \eta\right\|^2. \tag{14}$$

Definition 1. Let $p \in (0, +\infty)$. A C^{∞} map $\phi : M \to N$ is said to be *contact p-harmonic* if ϕ is a critical point of the energy functional

$$E_{\Omega,p}(\phi) = \int_{\Omega} \left\|(d\phi)_{H,H'}\right\|^p \theta \wedge (d\theta)^n \tag{15}$$

for any relatively compact domain $\Omega \subseteq M$. Contact 2-harmonic maps are called *contact harmonic maps*.

Let ∇ be the Tanaka-Webster connection of (M, θ) that is the unique linear connection on M obeying to (i) H is ∇-parallel (i.e., $\nabla_X Y \in H$ for any $X \in \mathfrak{X}^{\infty}(M)$ and any $Y \in H$), (ii) $\nabla J = 0$ and $\nabla g_{\theta} = 0$, and (iii) the torsion tensor field T_{∇} of ∇ is *pure* (i.e., $T_{\nabla}(Z, W) = 0$, $T_{\nabla}(Z, \overline{W}) = 2iG_{\theta}(Z, \overline{W})T$ for any $Z, W \in T_{1,0}(M)$ and $\tau \circ J + J \circ \tau = 0$, where $\tau(X) = T_{\nabla}(T, X)$ for any $X \in \mathfrak{X}^{\infty}(M)$ (cf. Theorem 1.3 and Definition 1.25 in [9, pages 25-26]). The vector valued 1-form τ is the *pseudohermitian torsion* of ∇. Let ∇' be the generalized Tanaka-Webster connection of (N, η, g) given locally by

$$\Gamma'^i_{jk} = \Gamma^i_{jk} + \eta_j \varphi_k^i - \eta_k \nabla_j \xi^i + \xi^i \nabla_j \eta_k, \tag{16}$$

(cf., e.g., [16]), where Γ^i_{jk} are the Christoffel symbols of g_{ij}. Covariant derivatives are meant with respect to the Levi-Civita connection of (M, g). For each $X \in \mathfrak{X}^{\infty}(M)$ we consider $\phi_* X \in \Gamma^{\infty}(\phi^{-1} TN)$ given by

$$(\phi_* X)(x) = (d_x \phi) X_x \in T_{\phi(x)}(N) = \left(\phi^{-1} TN\right)_x, \quad x \in M. \tag{17}$$

Let $\nabla^{\phi} = \phi^{-1}\nabla'$ be the connection induced by ∇' in the pullback bundle $\phi^{-1}TN \to M$. We set

$$\beta_{\phi}(X,Y) = \nabla^{\phi}_{X}\phi_{*}Y - \phi_{*}\nabla_{X}Y, \quad X,Y \in \mathfrak{X}^{\infty}(M). \tag{18}$$

Let $x \in M$ and let $\{X_a : 1 \le a \le 2n\}$ be a local G_θ-orthonormal frame of H defined on an open neighborhood U of x. We define a C^∞ section $\Gamma(\phi)$ in $\phi^{-1}TN \to M$ by setting

$$\Gamma(\phi)_x = \operatorname{trace}_{G_\theta}\left\{\Pi_H\beta_\phi\right\}_x = \sum_{a=1}^{2n}\beta_\phi(X_a,X_a)_x, \tag{19}$$

where $\Pi_H\beta_\phi$ denotes the restriction of β_ϕ to $H\otimes H$. By a result in [10] the Euler-Lagrange equations associated to the variational principle $\delta E_{\Omega,p}(\phi) = 0$ are

$$\begin{aligned} &Q(\phi)^{-(p-2)/2}\left[\left(\varphi^2\right)^i_j\circ\phi\right]\operatorname{div}\left(Q(\phi)^{(p-2)/2}\nabla^H\phi^j\right) \\ &\quad = \operatorname{trace}_{G_\theta}\left\{\Pi_H\phi^*\left(\eta\otimes\tau_N\right)\right\} \\ &\qquad - \sum_{a=1}^{2n}\left[\left(\varphi^2\right)^i_j\circ\phi\right]\left(\Gamma'^{j}_{k\ell}\circ\phi\right)X_a\left(\phi^k\right)X_a\left(\phi^\ell\right), \\ &\qquad\qquad \operatorname{trace}_{G_\theta}\left\{\Pi_H\ \phi^*A_N\right\} = 0, \end{aligned} \tag{20}$$

here $\varphi^2 = -I + \eta\otimes\xi$ (cf., e.g., [13]). Also τ_N is the pseudohermitian torsion of (N,φ,ξ,η,g); that is, $\tau_N(X) = T_{\nabla'}(\xi,X)$, and $A_N(X,Y) = g(\tau_N X,Y)$ for any $X,Y\in\mathfrak{X}^\infty(N)$. Γ'^{i}_{jk} are again the local coefficients of ∇' with respect to (U',x'^i). In particular if g is a Sasakian metric, then $\phi : M \to N$ is contact p-harmonic if and only if

$$\begin{aligned} &\left[\left(\varphi^2\right)^i_j\circ\phi\right] \\ &\quad\times\left\{\operatorname{div}\left(Q(\phi)^{(p-2)/2}\nabla^H\phi^j\right)\right. \\ &\qquad\left.+Q(\phi)^{(p-2)/2}\sum_{a=1}^{2n}\left(\Gamma'^{j}_{k\ell}\circ\phi\right)X_a\left(\phi^k\right)X_a\left(\phi^\ell\right)\right\} = 0, \\ &\qquad\qquad 1\le i\le 2m-1. \end{aligned} \tag{21}$$

3. Weak Contact Harmonic Maps

Sections 3 and 4 are devoted to the study of local properties of weak critical points of the functional (15). A study of the regularity of weak solutions to subelliptic systems (such as (53)) was started by Wang [8], and Capogna and Garofalo [22], though only for maps from Carnot groups, (cf. also Zhou [23]).

Let M be a strictly pseudoconvex CR manifold and θ a contact form on M. Let $\{X_a : 1 \le a \le 2n\}$ be a local G_θ-orthonormal frame of H defined on the open set $U\subseteq M$ and X_a^* the formal adjoint of X_a; that is,

$$X_a^*u = -X_a u - f_a u, \quad u\in C_0^1(U), \tag{22}$$

where $f_a = \partial b_a^A/\partial x^A + b_a^B\Gamma^A_{AB}$ and $X_a = b_a^A\partial/\partial x^A$. Also Γ^C_{AB} are the local coefficients of the Tanaka-Webster connection of (M,θ) with respect to the local coordinate system (U,x^A) on M. Clearly $(X_a^*u,v) = (u,X_a v)$ for any $u\in C_0^1(U)$, where $(u,v) = \int_U u\bar{v}\,dv$.

Proposition 2. *Let $\phi : M \to N$ be a smooth map and g a Sasakian metric on N. Then ϕ is contact p-harmonic if and only if*

$$\begin{aligned} &\left[\left(\varphi^2\right)^i_j\circ\phi\right] \\ &\quad\times\sum_{a=1}^{2n}\left\{-X_a^*\left(Q(\phi)^{(p-2)/2}X_a\phi^j\right)\right. \\ &\qquad\left.+Q(\phi)^{(p-2)/2}\left(\Gamma'^{j}_{k\ell}\circ\phi\right)X_a\left(\phi^k\right)X_a\left(\phi^\ell\right)\right\} = 0 \end{aligned} \tag{23}$$

for any local orthonormal frame $\{X_a : 1\le a\le 2n\}$ of H.

Proof. Let us note that $\operatorname{div}(X_a) = \operatorname{trace}\{\partial_A \mapsto \nabla_{\partial_A}X_a\} = f_a$, where $\partial_A = \partial/\partial x^A$. Thus (by (22))

$$\operatorname{div}\left(Q(\phi)^{(p-2)/2}\nabla^H\phi^i\right) = -\sum_{a=1}^{2n}X_a^*\left(Q(\phi)^{(p-2)/2}X_a\phi^i\right) \tag{24}$$

on U. Then (23) follows from (21). □

Example 3 (contact p-harmonic maps into the Heisenberg group). Let $N = \mathbb{H}_{m-1}$, $m\ge 2$, be the Heisenberg group (cf., e.g., [9, pages 11–14]). Let (x^α,y^α,t) be the Cartesian coordinates on $\mathbb{R}^{2m-1}$ and let

$$X_\alpha = \frac{\partial}{\partial x^\alpha} + 2y^\alpha\frac{\partial}{\partial t}, \quad Y_\alpha = \frac{\partial}{\partial y^\alpha} - 2x^\alpha\frac{\partial}{\partial t}, \quad 1\le\alpha\le m-1. \tag{25}$$

Let φ be the $(1,1)$-tensor field on $\mathbb{H}_{m-1}$ determined by

$$\varphi(X_\alpha) = Y_\alpha, \qquad \varphi(Y_\alpha) = -X_\alpha, \qquad \varphi(\xi) = 0, \tag{26}$$

where $\xi = -\partial/\partial t$. Next the differential 1-form $\eta\in\Omega^1(\mathbb{H}_{m-1})$ given by

$$\eta = 2\sum_{\alpha=1}^{2m-2}\left(y^\alpha dx^\alpha - x^\alpha dy^\alpha\right) - dt \tag{27}$$

is a contact form on $\mathbb{H}_{m-1}$; that is, $\eta\wedge(d\eta)^{m-1}$ is a volume form. Let $H = \operatorname{Ker}(\eta)$. Finally we shall need the Riemannian metric g on $\mathbb{H}_{m-1}$ given by $g = -d\eta(\cdot,\varphi\cdot)$ on $H\otimes H$, $g(\cdot,\xi) = 0$ on H, and $g(\xi,\xi) = 1$. Then g is a Sasakian metric on $\mathbb{H}_{m-1}$ (and actually $(\mathbb{H}_{m-1},g)$ is a Sasakian space form of φ-sectional -3; cf., e.g., [13]). A calculation shows that

$$\varphi^2 : \begin{pmatrix} -\delta^\alpha_\beta & 0 & 0 \\ 0 & -\delta^\alpha_\beta & 0 \\ -2y_\beta & 2x_\beta & 0 \end{pmatrix}, \tag{28}$$

where $x_\alpha = x^\alpha$ and $y_\alpha = y^\alpha$. Let $T_\alpha = X_\alpha - iY_\alpha$ and let $T_{1,0}(\mathbb{H}_{m-1})_x$ be the span of $\{T_\alpha(x) : 1 \le \alpha \le m-1\}$ over $\mathbb{C}$. Then $T_{1,0}(\mathbb{H}_{m-1})$ is a strictly pseudoconvex CR structure on $\mathbb{H}_{m-1}$ and $\theta = -\eta$ is a contact form such that the Levi form G_θ is positive definite. Let ∇' be the Tanaka-Webster connection of $(\mathbb{H}_{m-1}, \theta)$. A calculation shows that

$$\begin{gathered}\nabla'_{\partial_\alpha}\partial_\beta = 0, \qquad \nabla'_{\partial_\alpha}\partial_{\beta+m-1} = -2\delta_{\alpha\beta}\xi, \\ \nabla'_{\partial_{\alpha+m-1}}\partial_\beta = 2\delta_{\alpha\beta}\xi, \qquad \nabla_{\partial_{\alpha+m-1}}\partial_{\beta+m-1} = 0,\end{gathered} \tag{29}$$

where $\partial_\alpha = \partial/\partial x^\alpha$ and $\partial_{\alpha+m-1} = \partial/\partial y^\alpha$ for simplicity. Hence

$$\Gamma'^{2m-1}_{\alpha,\beta+m-1} = -\Gamma^{2m-1}_{\alpha+m-1,\beta} = 2\delta_{\alpha\beta} \tag{30}$$

and the remaining connection coefficients vanish. The Webster metric g of $(\mathbb{H}_{m-1}, \theta)$ is given by

$$g : \begin{pmatrix} 2\delta_{\alpha\beta} + 4y_\alpha y_\beta & -4y_\alpha x_\beta & -2y_\alpha \\ -4x_\alpha y_\beta & 2\delta_{\alpha\beta} + 4x_\alpha x_\beta & 2x_\alpha \\ -2y_\beta & 2x_\beta & 1 \end{pmatrix}, \tag{31}$$

hence (by a straightforward calculation)

$$Q(\phi) = 2\sum_{a=1}^{2n}\sum_{i=1}^{2m-2}\left|X_a\left(\phi^i\right)\right|^2 = 2\left|X\phi'\right|^2, \tag{32}$$

where $\phi = (\phi', \phi^{2m-1}) : M \to \mathbb{H}_{m-1}$ and $\phi' = (\phi^1, \ldots, \phi^{2m-2})$. Let us substitute (28)–(32) into (23) so that to obtain

$$\sum_{a=1}^{2n} X_a^*\left(\left|X\phi'\right|^{p-2} X_a\left(\phi^i\right)\right) = 0, \quad 1 \le i \le 2m-2. \tag{33}$$

Hence if $\phi : M \to \mathbb{H}_{2m-1}$ is a contact p-harmonic map, then ϕ' is subject to (33) while ϕ^{2m-1} is an arbitrary function. Therefore, in general one may not expect regularity for a given (weak) contact p-harmonic map.

The identity (23) in Proposition 2 leads naturally to the notion of a weak solution to the contact p-harmonic map system. Indeed we may establish the following.

Lemma 4. *A smooth map $\phi : M \to N$ of a strictly pseudoconvex CR manifold M into a Sasakian manifold N is contact p-harmonic if and only if*

$$\begin{aligned}&\sum_{a=1}^{2n}\Big\{X_a^*\Big(Q(\phi)^{(p-2)/2}\left[\left(\varphi^2\right)^i_j \circ \phi\right] X_a\left(\phi^j\right)\Big) \\ &\quad -Q(\phi)^{(p-2)/2}\left[\left(\varphi^2\right)^j_k \circ \phi\right]\left(\Gamma'^i_{j\ell} \circ \phi\right) X_a\left(\phi^k\right) X_a\left(\phi^\ell\right)\Big\} \\ &= 0\end{aligned} \tag{34}$$

for any local orthonormal frame $\{X_a : 1 \le a \le 2n\}$ of H on U and any local coordinate system (U', x'^i) on N such that $\phi^{-1}(U') \supseteq U$.

Proof. Let us multiply (23) by a test function $\psi \in C_0^\infty(U)$ and integrate by parts

$$\begin{aligned}&\int Q(\phi)^{(p-2)/2}\sum_a X_a\left(\phi^j\right) X_a\left(\left(\varphi^2\right)^i_j\psi\right) dv \\ &\quad = \int Q(\phi)^{(p-2)/2}\sum_a \left(\varphi^2\right)^i_j \Gamma'^j_{k\ell} X_a\left(\phi^k\right) X_a\left(\phi^\ell\right)\psi\, dv.\end{aligned} \tag{35}$$

On the other hand (as both ξ and η are parallel with respect to ∇')

$$\frac{\partial\xi^i}{\partial x'^k} = -\Gamma'^i_{k\ell}\xi^\ell, \qquad \frac{\partial\eta_j}{\partial x'^k} = \Gamma'^\ell_{kj}\eta_\ell, \tag{36}$$

$$\frac{\partial\left(\varphi^2\right)^i_j}{\partial x'^k} = \xi^i\eta_\ell\Gamma'^\ell_{kj} - \eta_j\xi^\ell\Gamma'^i_{k\ell}, \tag{37}$$

$$\left(\varphi^2\right)^i_j\Gamma'^i_{k\ell} + \eta_\ell\xi^j\Gamma'^i_{jk} - \xi^i\eta_j\Gamma'^j_{k\ell} = \left(\varphi^2\right)^j_\ell\Gamma'^i_{jk} - T^i_{k\ell}, \tag{38}$$

where $T^i_{k\ell}$ are the coefficients of $T_{\nabla'}$ with respect to (U', x'^i). Therefore (35) may be written as

$$\begin{aligned}&\int Q(\phi)^{(p-2)/2} \\ &\quad \times \sum_a \Big\{\left(\varphi^2\right)^i_j X_a\left(\phi^j\right) X_a(\psi) \\ &\qquad -\left(\varphi^2\right)^j_k\Gamma'^i_{j\ell} X_a\left(\phi^k\right) X_a\left(\phi^\ell\right)\psi\Big\}\, dv = 0\end{aligned} \tag{39}$$

and Lemma 4 is proved. □

Let us consider the function spaces

$$W_X^{1,p}(U) = \left\{u \in L^p(U) : X_a u \in L^p(U), 1 \le a \le 2n\right\}, \tag{40}$$

where $X_a u$ are understood as weak derivatives. If $1 \le p < \infty$, then $W_X^{1,p}(U)$ are separable Banach spaces with the norms

$$\|u\|_{W_X^{1,p}(U)} = \left(\|u\|^p_{L^p(U)} + \sum_{a=1}^{2n}\|X_a u\|^p_{L^p(U)}\right)^{1/p}. \tag{41}$$

Also $W_X^{1,p}(U)$ is reflexive provided that $1 < p < \infty$. The central concept of this section may be introduced as follows. Let $\{X_a : 1 \le a \le 2n\}$ be a G_θ-orthonormal frame of H defined on the open set $U \subseteq M$. Let $U' \subseteq N$ be an open set which is relatively compact in a larger coordinate neighborhood in N.

Definition 5. A map $\phi : U \to U'$ is said to be *weak contact p-harmonic* if it is a weak solution to (34); that is, $\phi^j \in W_X^{1,p}(U)$ for any $1 \le j \le 2m-1$ and the identities (39) are satisfied for any test function $\psi \in C_0^\infty(U)$.

Let $\phi : U \to U'$ be a weak contact p-harmonic map. By (14)

$$Q(\phi) = \sum_a \left\{X_a\left(\phi^i\right) X_a\left(\phi^j\right)\left(g_{ij} \circ \phi\right) - \left[X_a(\phi^i)(\eta_i \circ \phi)\right]^2\right\} \tag{42}$$

on U, hence

$$\begin{gathered}|Q(\phi)| \leq C|X\phi|^2 \quad \text{a.e. in } U, \\ |X\phi|^2 = \sum_{a=1}^{2n}\sum_{i=1}^{2m-1}\left|X_a\left(\phi^i\right)\right|^2,\end{gathered} \tag{43}$$

where $C = \max\{\sup_{\overline{U'}}|g_{ij}|, \sup_{\overline{U'}}|\eta_i| : 1 \leq i, j \leq 2m-1\}$. Then both integrals in (39) are convergent and the adopted definition is legitimate.

Example 6 (Example 3 continued). A weak solution to (33) is a map $\phi = (\phi', \phi^{2m-1}) : U \to U' \subset\subset \mathbb{H}_{m-1}$ such that $\phi' \in W_X^{1,p}(U, \mathbb{R}^{2m-2})$ and

$$\sum_{a=1}^{2n}\int_U \left|X\phi'\right|^{p-2} X_a\left(\phi^i\right) X_a(\psi)\, dv = 0, \quad 1 \leq i \leq 2m-2, \tag{44}$$

for any $\psi \in C_0^\infty(U)$. We need to recall the following general result, due to Xu and Zuily [24]. Let $X = \{X_1, \ldots, X_m\}$ be a Hörmander system on an open set $U \subseteq \mathbb{R}^N$, $N \geq 2$, and $\Omega \subset \mathbb{R}^N$ a domain such that $U \supset \overline{\Omega}$. Let $a^{ij}(x, y)$ be a symmetric and positive definite matrix defined in $\Omega \times \mathbb{R}^\nu$. If $|f(x, y, p)| \leq a|p|^2 + b$ for any $(x, y, p) \in \Omega \times \mathbb{R}^\nu \times \mathbb{R}^{m\nu}$, then any continuous solution $\phi = (\phi^1, \ldots, \phi^\nu)$ to

$$\begin{gathered}\sum_{i,j=1}^{m} X_j^*\left(a^{ij}(x, \phi(x)) X_i\phi^\alpha(x)\right) = f^\alpha(x, \phi(x), X\phi(x)), \\ 1 \leq \alpha \leq \nu,\end{gathered} \tag{45}$$

in Ω is actually smooth. Let us assume that U is a domain such that $\overline{U}$ is contained in a coordinate neighborhood in M. By the result in [24] quoted above.

Proposition 7. *For any weak solution $\phi = (\phi', \phi^{2m}) : U \to U' \subset \mathbb{H}_{m-1}$ to the contact p-harmonic map system* (33) *if $\phi' \in C^0(U, \mathbb{R}^{2m-2})$, then $\phi' \in C^\infty(U, \mathbb{R}^{2m-2})$.*

Of course in the particular case $p = 2$ any distribution solution ϕ' is C^∞ (as the operator $\sum_{a=1}^{2n} X_a^* X_a$ is hypoelliptic).

Example 8 (contact p-harmonic maps into the sphere). Let $N = S^{2m-1} \subset \mathbb{R}^{2m}$ and let g be the canonical Sasakian metric on S^{2m-1}. Then a C^∞ contact p-harmonic map $\phi = (\phi^1, \ldots, \phi^{2m}) : M \to S^{2m-1}$ is a solution to

$$\begin{aligned}&\left[\left(\varphi^2\right)_j^i \circ \phi\right] \sum_{a=1}^{2n} X_a^*\left(Q(\phi)^{(p-2)/2}\, X_a\phi^j\right) \\ &\quad = Q(\phi)^{(p-2)/2} \\ &\qquad \times \left\{\left[\left(\varphi^2\right)_j^i \circ \phi\right] |X\phi|^2\phi^j \right. \\ &\qquad \left. + 2\sum_{a=1}^{2n} (\phi^*\eta)(X_a)\left(\varphi_j^i \circ \phi\right) X_a\left(\phi^j\right)\right\},\end{aligned} \tag{46}$$

for any $1 \leq i \leq 2m-1$. Here $|X\phi|^2 = \sum_{\beta=1}^{2m}\sum_{a=1}^{2n}|X_a\phi^\beta|^2$ and $\sum_{\beta=1}^{2m}\phi_\beta^2 = 1$ with $\phi_\beta = \phi^\beta$, $1 \leq \beta \leq 2m$. Equation (46) follows from (23) by computing the Christoffel symbols of S^{2m-1} with respect to the local coordinate system

$$\begin{gathered}\chi' : U' \to \mathbb{R}^{2m-1}, \quad \chi'(x) = x', \quad x = \left(x', x_{2m}\right) \in U', \\ U' = S^{2m-1} \cap \{x_{2m} > 0\}, \quad x' = (x_1, \ldots, x_{2m-1}),\end{gathered} \tag{47}$$

that is

$$\left|\begin{matrix} i \\ jk \end{matrix}\right| = x^i g_{jk}, \qquad g_{jk} = \delta_{jk} + \frac{x_j x_k}{1 - |x'|^2}, \tag{48}$$

so that

$$\begin{gathered}\sum_{a=1}^{2n}\left(\left|\begin{matrix} i \\ jk \end{matrix}\right| \circ \phi\right) X_a\left(\phi^j\right) X_a\left(\phi^k\right) = |X\phi|^2\, \phi^i, \\ 1 \leq i \leq 2m-1.\end{gathered} \tag{49}$$

On the other hand (cf. [9])

$$\left|\begin{matrix} i \\ jk \end{matrix}\right| = \Gamma_{jk}'^i + \omega_{jk}\xi^i + \eta_j\varphi_k^i + \eta_k\varphi_j^i \tag{50}$$

so that

$$\begin{aligned}&\left(\left|\begin{matrix} i \\ jk \end{matrix}\right| \circ \phi\right) X_a\left(\phi^j\right) X_a\left(\phi^k\right) \\ &\quad = \left(\Gamma_{jk}'^i \circ \phi\right) X_a\left(\phi^j\right) X_a\left(\phi^k\right) \\ &\qquad + 2(\phi^*\eta)(X_a)\left(\varphi_j^i \circ \phi\right) X_a\left(\phi^j\right)\end{aligned} \tag{51}$$

for any Sasakian metric g. When $N = S^{2m-1}$, the identities (49)–(51) lead to

$$\begin{aligned}&\sum_{a=1}^{2n}\left(\Gamma_{jk}'^i \circ \phi\right) X_a\left(\phi^j\right) X_a\left(\phi^k\right) \\ &\quad = |X\phi|^2\phi^i - \sum_{a=1}^{2n} 2(\phi^*\eta)(X_a)\left(\varphi_j^i \circ \phi\right) X_a\left(\phi^j\right)\end{aligned} \tag{52}$$

and then to (46) by taking into account that φ is an f-structure on S^{2m-1}; that is, $\varphi^3 + \varphi = 0$. Our next purpose in this example is to prove the following result.

Proposition 9. *Let $\phi : \mathbb{H}_n \to S^{2m-1}$ be a horizontal map. Then ϕ is contact p-harmonic if and only if ϕ is subelliptic p-harmonic with respect to the canonical Hörmander system $X = \{X_\gamma, Y_\gamma : 1 \le \gamma \le n\}$ on $\mathbb{H}_n$.*

According to [7] given a Hörmander system of vector fields $\{X_a\}$ defined on an open set $O \subseteq \mathbb{R}^N$, one may adopt the following.

Definition 10. A subelliptic p-harmonic map is a C^∞ solution $\phi : O \to \mathbb{R}^{2m}$ to the system (the formal adjoint of X_a in [7] is $-X_a^*$ under the conventions adopted in the present paper)

$$\sum_a X_a^* \left(|X\phi|^{p-2} X_a \phi^\alpha\right) = |X\phi|^p \phi^\alpha, \quad 1 \le \alpha \le 2m, \tag{53}$$

such that $\sum_{\alpha=1}^{2m} \phi_\alpha^2 = 1$.

A horizontal map is a smooth map $\phi : \mathbb{H}_n \to S^{2m-1}$ such that

$$X_a\left(\phi^i\right) \left(\eta_i \circ \phi\right) = 0, \quad 1 \le a \le 2n. \tag{54}$$

One may define weak solutions $\phi : \mathbb{H}_n \to U'$ to (54) by requiring that $\phi^i \in W_X^{1,p}(U)$ for some $1 \le p < \infty$ and that (54) holds a.e. in U. Then the statement in Proposition 9 holds for weak solutions of the relevant equations as well. In particular, by a result in [7], any weak horizontal contact p-harmonic map $\phi : \mathbb{H}_n \to U'$ is locally Hölder continuous provided that $p \ge 2n + 2$.

The proof of Proposition 9 is to write (46) in the form (53). We need the following.

Lemma 11. *Let M be a strictly pseudoconvex CR manifold. A smooth map $\phi : M \to S^{2m-1}$ is contact p-harmonic if and only if*

$$-\sum_{a=1}^{2n} X_a^* \left(Q(\phi)^{(p-2)/2} \left[\left(\varphi^2\right)_j^i \circ \phi\right] X_a\left(\phi^j\right)\right) = Q(\phi)^{p/2} \phi^i, \tag{55}$$

for any $1 \le i \le 2m-1$ and any local orthonormal frame $\{X_a : 1 \le a \le 2n\}$ of H.

By (14) if $\phi : M \to S^{2m-1}$ is a horizontal map, then $Q(\phi) = |X\phi|^2$ and one may readily check that (55) is equivalent to (53) for any $1 \le i \le 2m-1$. Of course the component ϕ_{2m} will satisfy (53) as well (as a consequence of the constraint $\sum_{\alpha=1}^{2n} \phi_\alpha^2 = 1$). To prove Lemma 11, let us multiply (46) by a test function $\psi \in C_0^\infty(U)$ and integrate over U. The left-hand side of the resulting equation is

$$\begin{aligned}
&\sum_a \int_U \left(\varphi^2\right)_j^i X_a^* \left(\rho X_a\left(\phi^j\right)\right) \psi d\nu \\
&\quad = \sum_a \int \rho X_a\left(\phi^j\right) X_a\left(\left(\varphi^2\right)_j^i \psi\right) d\nu \\
&\quad = \sum_a \int \rho \left\{X_a\left(\phi^j\right) \left(\varphi^2\right)_j^i X_a(\psi) \right. \\
&\qquad\qquad \left. + \psi X_a\left(\phi^j\right) X_a\left(\left(\varphi^2\right)_j^i \circ \phi\right)\right\} d\nu \\
&\quad = \sum_a \int \left\{X_a^*\left(\rho\left[\left(\varphi^2\right)_j^i \circ \phi\right] X_a\left(\phi^j\right)\right)\psi \right. \\
&\qquad\qquad \left. + \rho\psi X_a\left(\phi^j\right) X_a\left(\phi^k\right) \frac{\partial\left(\varphi^2\right)_j^i}{\partial x'^k}\right\} d\nu,
\end{aligned} \tag{56}$$

where $\rho = Q(\phi)^{(p-2)/2}$. Then (by (37))

$$\begin{aligned}
&\sum_a X_a\left(\phi^j\right) X_a\left(\phi^k\right) \frac{\partial\left(\varphi^2\right)_j^i}{\partial x'^k} \\
&\quad = \sum_a X_a\left(\phi^j\right) X_a\left(\phi^k\right) \left(\xi^i \eta_\ell \Gamma_{kj}^{\prime\ell} - \eta_j \xi^\ell \Gamma_{k\ell}^{\prime i}\right) \\
&\quad = (\text{by (52) and (50)}) \\
&\quad = \xi^i \eta_\ell \left(|X\phi|^2 \phi^\ell - 2\sum_a \eta_j X_a\left(\phi^j\right) \varphi_k^\ell X_a\left(\phi^k\right)\right) \\
&\qquad - \sum_a X_a\left(\phi^j\right) X_a\left(\phi^k\right) \eta_j \left(\left|{i \atop k\ell}\right| \xi^\ell - \varphi_k^i\right) \\
&\quad = \xi^i \eta_\ell \phi^\ell |X\phi|^2 \\
&\qquad + \sum_a \left\{\eta_j X_a\left(\phi^j\right) \varphi_k^i X_a\left(\phi^k\right) - X_a\left(\phi^j\right) X_a\left(\phi^k\right) \eta_j \eta_k \phi^i\right\},
\end{aligned} \tag{57}$$

hence (46) implies

$$\begin{aligned}
&\sum_a X_a^* \left(Q(\phi)^{(p-2)/2} \left[\left(\varphi^2\right)_j^i \circ \phi\right] X_a\left(\phi^j\right)\right) \\
&\quad = Q(\phi)^{(p-2)/2} \left(-|X\phi|^2 + \sum_a \left[X_a\left(\phi^j\right)\left(\eta_j \circ \phi\right)\right]^2\right) \phi^i
\end{aligned} \tag{58}$$

which yields (55) because on the sphere

$$Q(\phi) = |X\phi|^2 - \left\|\Pi_H \phi^* \eta\right\|^2. \tag{59}$$

Lemma 11 is proved.

The notion of a weak contact harmonic map as introduced above is confined to maps $\phi : M \to N$ such that the target contact Riemannian manifold N is covered by a single coordinate neighborhood. Another natural approach (customary in

the theory of harmonic maps among Riemannian manifolds, cf., e.g., [4, page 38]) is to use Nash's embedding theorem (cf. [25]) in order to embed isometrically the target manifold N into some Euclidean space $\mathbb{R}^K$ and produce an alternative first variation formula (cf. Theorem 2.22 in [26, page 139]) depending however on the embedding $N \hookrightarrow \mathbb{R}^K$.

A generalization of Nash's embedding theorem to the context of contact Riemannian geometry has been obtained by D'Ambra [27]. Let $\mathbb{H}_L \approx \mathbb{C}^L \times \mathbb{R}$ be the Heisenberg group equipped with the standard Sasakian structure $(\varphi_0, \xi_0, \eta_0, g_0)$. Let $(N, (\varphi, \xi, \eta, g))$ be a contact Riemannian manifold. By a result in [27], if N is compact and $L \geq \dim(N) + 1$, there is a C^1-embedding $\iota : N \to \mathbb{H}_L$ which is both horizontal, that is, $\iota_* H' \subset \iota^{-1} \operatorname{Ker}(\eta_0)$, and isometric in the sense that ι preserves the Levi forms

$$g_p(v, w) = g_{0,\iota(p)}\left(\left(d_p\iota\right)v, \left(d_p\iota\right)w\right), \quad v, w \in H'_p, \ p \in N. \tag{60}$$

Any contact Riemannian manifold N is in particular a sub-Riemannian manifold (in the sense of [28]); hence N carries the Carnot-Carathéodory metric $d_N : N \times N \to [0, +\infty)$ associated to the sub-Riemannian structure (H', g). In particular ι is an isometry among the metric spaces (N, d_N) and $(\mathbb{H}_L, d_X)$ (cf. Section 7 for the definition of the distance function $d_X : \mathbb{H}_L \times \mathbb{H}_L \times [0, +\infty)$). As $\mathbb{H}_L$ also possesses a linear space structure, the methods in [29] (methods of *direct infinitesimal geometry*) become available on a contact Riemannian manifold (e.g., one may merely use the balls with respect to d_N and the linear structure of the ambient space $\mathbb{H}_L$ to reformulate on N Definition 2.1 in [29, page 280]) and we conjecture that the arguments in [29] may be recovered to study the equation $\Delta_b u = 0$ on a strictly pseudoconvex CR manifold (the theory in [29] only deals with second order degenerate elliptic equations on domains in $\mathbb{R}^n$). Unfortunately the existence of C^1-embeddings of given contact structures is not sufficient for differential geometric purposes, as long as Gauss and Weingarten formulae (which require two derivatives of ι) are involved. The problem of improving D'Ambra's proof (to get a horizontal embedding of class at least C^2) is open.

4. Contact Harmonic Maps into Spheres

Let $\Omega \subset \mathbb{R}^N$ be a bounded open set and $X = \{X_1, \ldots, X_m\}$ a Hörmander system of vector fields $X_a = b_a^A(x)\partial/\partial x^A \in \mathfrak{X}(\mathbb{R}^N)$ such that $b_a^A \in C^\infty(\mathbb{R}^N) \cap \operatorname{Lip}(\mathbb{R}^N)$. We recall (cf., e.g., [9, page 261]) the following.

Definition 12. A number D is a *homogeneous dimension* relative to Ω with respect to X if there is a constant $C > 0$ such that

$$\frac{\left|B_X(x, r)\right|}{\left|B_X(x_0, r_0)\right|} \geq C\left(\frac{r}{r_0}\right)^D \tag{61}$$

for any Carnot-Carathéodory ball $B_0 = B_X(x_0, r_0)$ of center $x_0 \in \Omega$ and radius $0 < r_0 \leq \operatorname{diam}(\Omega)$ and any Carnot-Carathéodory ball $B = B_X(x, r)$ of center $x \in B_0$ and radius $0 < r \leq r_0$.

The diameter of Ω is meant with respect to the Carnot-Carathéodory metric associated to X. Hajłasz and Strzelecki [7] studied local properties of weak solutions to the system (53). Their main finding is that every weak subelliptic D-harmonic map $\phi \in W_X^{1,D}(\Omega, S^\nu)$ (i.e., every weak solution to (53) with $p = D$) is locally Hölder continuous. Maps $\phi : \Omega \to S^\nu$ with values in a unit sphere $S^\nu \subset \mathbb{R}^{\nu+1}$ have a special status due to the fact that the subelliptic harmonic map system (here (53)) may be written in a simple form using an approach commonly referred to as the *Frédéric Hélein trick* (cf. [7, page 353], see also Hélein [30]). The purpose of this section is to start a study of weak solutions to the system (55) following the ideas in [7] though confined to maps $\phi : \mathbb{H}_n \to S^{2m-1}$ which are "close to horizontal" in a sense to be made precise in the sequel.

Let $\mathbb{H}_n$ be the Heisenberg group equipped with the standard contact form $\theta = dt + i\sum_{\gamma=1}^n (z^\gamma d\overline{z}_\gamma - \overline{z}_\gamma dz^\gamma)$. Let $U \subseteq \mathbb{H}_n$ be a bounded domain. Let $\{X_a : 1 \leq a \leq 2n\} = \{X_\gamma, Y_\gamma : 1 \leq \gamma \leq n\}$ be the G_θ-orthonormal frame given by $X_\gamma = \partial/\partial x^\gamma + 2y^\gamma T$ and $Y_\gamma = \partial/\partial y^\gamma - 2x^\gamma T$, where $T = \partial/\partial t$ as in Example 3. Clearly the coefficients of the X_a's lie in $C^\infty(\mathbb{R}^{2n+1}) \cap \operatorname{Lip}(\mathbb{R}^{2n+1})$. We recall that an absolutely continuous curve $\gamma : [0, \tau] \to \mathbb{H}_n$ is *admissible* if

$$\frac{d\gamma}{dt}(t) = \sum_{a=1}^{2n} u_a(t) X_a(\gamma(t)) \tag{62}$$

for some functions $u_a(t)$ such that $\sum_{a=1}^{2n} u_a(t)^2 \leq 1$.

Definition 13. The *Carnot-Carathéodory distance* $d_X(x, y)$ among two points $x, y \in \mathbb{H}_n$ is the infimum of all $\tau > 0$ for which there exists an admissible curve $\gamma : [0, \tau] \to \mathbb{H}_n$ such that $\gamma(0) = x$ and $\gamma(\tau) = y$. Balls with respect to $d_X : \mathbb{H}_n \times \mathbb{H}_n \to [0, +\infty)$ are denoted by $B_X(x, r) = \{y \in \mathbb{H}_n : d_X(x, y) < r\}$ and referred to as *Carnot-Carathéodory balls*.

We shall characterize horizontal maps in terms of the first order differential operator

$$L_a u = u^{m+\alpha} X_a(u_\alpha) - u^\alpha X_a(u_{m+\alpha}) \tag{63}$$

defined for $u = (u_1, \ldots, u_{2m}) \in W_X^{1,p}(U, \mathbb{R}^{2m})$.

Proposition 14. *Let $\phi : U \to U' = S^{2m-1} \cap \{x_{2m} > 0\} \subset \mathbb{R}^{2m}$ be a map such that $\phi_A \in W_X^{1,p}(U)$ for any $1 \leq A \leq 2m$. Then $\phi : U \to U'$ is a (weak) horizontal map if and only if $L_a\phi = 0$ for any $1 \leq a \leq 2n$.*

Let $(z_1, \ldots, z_m)$ be the natural complex coordinates on $\mathbb{C}^m$ and set $z_\alpha = x_\alpha + iy_\alpha$ and $(x_1, \ldots, x_{2m}) = (x_1, \ldots, x_m, y_1, \ldots, y_m)$. The following conventions are adopted as to the range of indices:

$$\begin{aligned} &1 \leq A, B, \cdots \leq 2m, \qquad 1 \leq i, j, \cdots \leq 2m - 1, \\ &1 \leq \alpha, \beta, \cdots \leq m, \qquad 1 \leq r, s, \cdots \leq m - 1. \end{aligned} \tag{64}$$

Let $\nu = x^\alpha \partial/\partial x^\alpha + y^\alpha \partial/\partial y^\alpha \in \mathfrak{X}^\infty(\mathbb{R}^{2m})$ so that the pointwise restriction of ν to S^{2m-1} is a unit normal field on S^{2m-1}. Let J_0

be the complex structure on $\mathbb{C}^m$. Then $\xi \in \mathfrak{X}^\infty(S^{2m-1})$ given by $(d_x\iota)\xi_x = J_{0,x}\nu_x$ for any $x \in S^{2m-1}$ is the Reeb vector field on S^{2m-1}. Here $\iota : S^{2m-1} \to \mathbb{R}^{2m}$ is the inclusion. With respect to the local chart $\chi' = (x_1, \ldots, x_{2m-1})$ in Example 8 the Reeb vector ξ is given by

$$\xi^\alpha = -y^\alpha, \qquad \xi^{m+r} = x^r. \tag{65}$$

Then $\eta_i = g_{ij}\xi^j$ together with (48) in Example 8 leads to

$$\eta_\alpha = -y_\alpha - \frac{x_m}{y_m}x_\alpha, \qquad \eta_{m+r} = x_r - \frac{x_m}{y_m}y_r. \tag{66}$$

Finally (66) implies that $X_a(\phi^i)(\eta_i \circ \phi) = -L_a\phi$. Proposition 14 is proved. In particular $Q(\phi)$ may be written as

$$Q(\phi) = |X\phi|^2 - \sum_{a=1}^{2n} |L_a\phi|^2. \tag{67}$$

Our next task is to put (55) into a more tractable form.

Proposition 15. *Let $\phi = (\phi_1, \ldots, \phi_{2m}) : U \to U'$ such that $\phi_A \in W_X^{1,p}(U)$. Let us consider the functions*

$$\begin{aligned} V_{\alpha,a} &= Q(\phi)^{(p-2)/2}\{X_a(\phi_\alpha) - \phi_{m+\alpha}L_a\phi\}, \\ V_{n+\alpha,a} &= Q(\phi)^{(p-2)/2}\{X_a(\phi_{m+\alpha}) + \phi_\alpha L_a\phi\}, \end{aligned} \tag{68}$$

with $1 \le \alpha \le m$. Let $V_A = (V_{A,1}, \ldots, V_{A,2n})$ for any $1 \le A \le 2m$. Then $\phi : U \to U'$ is a contact p-harmonic map if and only if

$$X^* \cdot V_A = Q(\phi)^{p/2}\phi_A, \quad 1 \le A \le 2m. \tag{69}$$

Here the dot product means $X^* \cdot V_A = \sum_{a=1}^{2n} X_a^*(V_{A,a})$. Using $\varphi^2 = -I + \eta \otimes \xi$ and (65) and (66), one obtains

$$\left[\left(\varphi^2\right)_j^i\right]_{1\le i,j\le 2m-1} = \begin{bmatrix} -\delta_\beta^\alpha + y^\alpha\left(y_\beta + \frac{x_m}{y_m}x_\beta\right) & -y^\alpha\left(x_r - \frac{x_m}{y_m}y_r\right) \\ -x^s\left(y_\beta + \frac{x_m}{y_m}x_\beta\right) & -\delta_r^s + x^s\left(x_r - \frac{x_m}{y_m}y_r\right) \end{bmatrix}. \tag{70}$$

Then substitution into (55) leads to

$$\sum_{a=1}^{2n} X_a^*\left[Q(\phi)^{(p-2)/2}\left(X_a(\phi^\alpha) - \phi^{m+\alpha}L_a\phi\right)\right] = Q(\phi)^{p/2}\phi^\alpha, \tag{71}$$

$$\sum_{a=1}^{2n} X_a^*\left[Q(\phi)^{(p-2)/2}\left(X_a(\phi^{m+s}) + \phi^s L_a\phi\right)\right] = Q(\phi)^{p/2}\phi^{m+s}. \tag{72}$$

It remains to be shown that (71) and (72) imply

$$\sum_{a=1}^{2n} X_a^*\left[Q(\phi)^{(p-2)/2}\left(X_a\left(\phi^{2m}\right) + \phi^m L_a\phi\right)\right] = Q(\phi)^{p/2}\phi^{2m}. \tag{73}$$

Let us multiply (71) by $\phi^\beta\psi$, where $\psi \in C_0^\infty(U)$ is an arbitrary test function, and integrate over U so that to obtain (after integration by parts)

$$\begin{aligned} &\sum_a X_a^*\left[Q(\phi)^{(p-2)/2}\left(X_a(\phi^\alpha) - \phi^{m+\alpha}L_a\phi\right)\phi^\beta\right] \\ &\quad = Q(\phi)^{(p-2)/2} \\ &\qquad \times \left\{Q(\phi)\phi^\alpha\phi^\beta - \sum_a X_a(\phi^\alpha)X_a\left(\phi^\beta\right)\right. \\ &\qquad\qquad \left. + \sum_a \phi^{m+\alpha}X_a\left(\phi^\beta\right)L_a\phi\right\}. \end{aligned} \tag{74}$$

Similarly let us multiply (72) by $\phi^{m+r}\psi$ so that to obtain

$$\begin{aligned} &\sum_a X_a^*\left[Q(\phi)^{(p-2)/2}\left(X_a\phi^{m+s} + \phi^s L_a\phi\right)\phi^{m+r}\right] \\ &\quad = Q(\phi)^{(p-2)/2} \\ &\qquad \times \left\{Q(\phi)\phi^{m+s}\phi^{m+r} - \sum_a X_a(\phi^{m+s})X_a(\phi^{m+r})\right. \\ &\qquad\qquad \left. - \sum_a \phi^s X_a(\phi^{m+r})L_a\phi\right\}. \end{aligned} \tag{75}$$

Let us contract the indices α and β in (74) (resp., r and s in (75)), add the resulting equations, and use the identities

$$\begin{aligned} X_a(\phi^\alpha)\phi_\alpha + X_a(\phi^{m+r})\phi_{m+r} &= -X_a\left(\phi^{2m}\right)\phi_{2m}, \\ -\phi^{m+\alpha}\phi_\alpha + \phi^r\phi_{m+r} &= -\phi_m\phi_{2m}, \\ \phi^\alpha\phi_\alpha + \phi^{m+r}\phi_{m+r} &= 1 - \phi_{2m}^2, \\ \phi^{m+\alpha}X_a(\phi_\alpha) - \phi^r X_a(\phi_{m+r}) &= L_a\phi + \phi^m X_a(\phi_{2m}). \end{aligned} \tag{76}$$

We get

$$\begin{aligned} &-\sum_a X_a^*\left[Q(\phi)^{(p-2)/2}\left(X_a\left(\phi^{2m}\right) + \phi^m L_a\phi\right)\phi_{2m}\right] \\ &\quad = Q(\phi)^{(p-2)/2}\left\{Q(\phi)\left(1 - \phi_{2m}^2\right) - \sum_a X_a\left(\phi^i\right)X_a(\phi_i)\right. \\ &\qquad\qquad \left. + \sum_a \left[L_a\phi + \phi^m X_a(\phi_{2m})\right]L_a\phi\right\}. \end{aligned} \tag{77}$$

Let us use $Q(\phi) - \sum_a X_a(\phi^i)X_a(\phi_i) + \sum_a (L_a\phi)^2 = X_a(\phi_{2m})^2$ (a consequence of (67)). Finally

$$\begin{aligned} &\sum_a X_a^*\left[Q(\phi)^{(p-2)/2}\left(X_a\left(\phi^{2m}\right) + \phi^m L_a\phi\right)\phi_{2m}\right] \\ &\quad = Q(\phi)^{(p-2)/2} \\ &\qquad \times \left\{Q(\phi)\phi_{2m}^2 - \sum_a X_a(\phi_{2m})^2 - \sum_a \phi^m X_a(\phi_{2m})L_a\phi\right\}. \end{aligned} \tag{78}$$

Now the identity (73) follows from (78) and $X_a^* = -X_a - f_a$. Proposition 15 is proved.

The crucial manner of exploiting the constraint $\sum_{A=1}^{2m} \phi_A^2 = 1$ is contained in the following.

Proposition 16. *Let $U \subset \mathbb{H}_n$ be a bounded domain and $\phi : U \to S^{2m-1} \subset \mathbb{R}^{2m}$, $\phi = (\phi_1, \ldots, \phi_{2m})$, a map such that $\phi_A \in W_X^{1,p}(U)$. Then*

$$V_A = \sum_{B=1}^{2m} \phi_B E_{A,B}, \tag{79}$$

where one has set $E_{A,B} = \phi_B V_A - \phi_A V_B$. Moreover if ϕ is a contact p-harmonic map, then

$$\begin{aligned} &X^* \cdot E_{A,B} \\ &\quad = Q(\phi)^{(p-2)/2} \{\sigma_B \phi_{B+m} X(\phi_A) - \sigma_A \phi_{A+m} X(\phi_B)\} L\phi, \end{aligned} \tag{80}$$

where $\sigma_A = 1$ if $1 \le A \le m$, $\sigma_A = -1$ if $m+1 \le A \le 2m$, and the range of the indices in (80) *is meant* mod m.

The identity (79) is a consequence of the constraint alone. The identity (80) for $A = \alpha$ and $B = \beta$ follows from (74) (interchange α and β in (74) and subtract the resulting identity from (74)). In general, for any $\psi \in C_0^\infty(U)$

$$\begin{aligned} &\int_U X^* \cdot (\phi_A V_B) \psi \, dv \\ &\quad = \int_U V_B \cdot [X(\psi\phi_A) - \psi X(\phi_A)] \, dv \\ &\quad = \int_U (X^* \cdot V_B) \phi_A \psi \, dv - \int_U [V_B \cdot X(\phi_A)] \psi \, dv, \end{aligned} \tag{81}$$

hence (by (69))

$$\begin{aligned} &\int_U X^* \cdot (\phi_A V_B) \psi \, dv \\ &\quad = \int_U \left\{ Q(\phi)^{p/2} \phi_B \phi_A - Q(\phi)^{(p-2)/2} X(\phi_B) \cdot X(\phi_A) \right\} \psi \\ &\qquad - \int_U Q(\phi)^{(p-2)/2} \sigma_B \phi_{B+m} [(L\phi) \cdot X(\phi_A)] \psi \, dv. \end{aligned} \tag{82}$$

Now let us interchange A and B in (82) to produce another identity of the sort and subtract it from (82). This yields (80). Proposition 16 is proved.

Although regularity of contact p-harmonic maps cannot be expected in general (cf. Example 3), a few fundamental questions may be asked. For instance, what is the the outcome of the ordinary hole filling argument (cf., e.g., [31, pages 38–40]) and of Moser's iteration technique in regularity theory? our finding in this direction is Theorem 20. We shall need the following.

Lemma 17. *Let $U \subset \mathbb{H}_n$ be a bounded domain. Let $R_0 > 0$ and $U_1 \subset\subset U$ such that $B_X(x, 400R_0) \subset U$ for any $x \in U_1$. Let $\mathbb{B} = B_X(x_0, r)$ with $x_0 \in U_1$ be a Carnot-Carathéodory ball such that $0 < r \le R_0$ and let $\psi \in W_X^{1,2n+2}(\mathbb{B})$ be a function of compact support. Then for any contact $(2n+2)$-harmonic map $\phi : \mathbb{H}_n \to S^{2m-1}$ satisfying* (96) *for some $0 < c < 1$ and some $0 < \delta < 1$*

$$\begin{aligned} &\left| \int_{\mathbb{B}} [X^* \cdot (\phi_B E_{A,B})] \psi \, dv \right| \\ &\quad \le C \|X\psi\|_{L^D(\mathbb{B})} \left\{ \|X\phi\|_{L^D(100\mathbb{B})}^D + \|X\phi\|_{L^D(100\mathbb{B})}^{(1-\epsilon)D} \right\} \end{aligned} \tag{83}$$

for some constant $C = C(U_1, n, R_0) > 0$, where $\epsilon = (1-\delta)/D$ and $D = 2n+2$.

This is similar to Lemma 3.2 (the duality inequality) in [7, page 354] and will be proved later on in this section.

Let $U_1 \subset\subset U$ and $R_0 > 0$ as in Lemma 17. Also let $x \in U_1$ and $0 < r < R_0$ and set $\mathbb{B} = B_X(x, r)$ and $2\mathbb{B} = B_X(x, 2r)$. Let $\psi \in C_0^\infty(U)$ be a test function such that $0 \le \psi \le 1$, $\psi = 1$ on $\mathbb{B}$, $\psi = 0$ on $U \setminus 2\mathbb{B}$, and $|X\psi| \le C/r$ for some constant $C > 0$. Next let us set

$$\psi_A = [\phi_A - (\phi_A)_{2\mathbb{B}}] \psi. \tag{84}$$

Throughout if (X, μ) is a measurable space and $A \subset X$ a measurable set with $\mu(A) > 0$, we adopt the notation $u_A = (1/\mu(A)) \int_A u d\mu$. Let us take the dot product of (79) with X^*, multiply the resulting equation by ψ_A, integrate over $2\mathbb{B}$, and sum over A

$$\begin{aligned} &\sum_{A=1}^{2m} \int_{2\mathbb{B}} (X^* \cdot V_A) \psi_A dv \\ &\quad = \sum_{A,B=1}^{2m} \int_{2\mathbb{B}} [X^* \cdot (\phi_B E_{A,B})] \psi_A dv. \end{aligned} \tag{85}$$

The first line of (85) may be computed as follows:

$$\begin{aligned} &\int_{2\mathbb{B}} (X^* \cdot V_A) \psi_A dv \\ &\quad = \int_{2\mathbb{B}} V_A \cdot X(\psi_A) \, dv \\ &\quad = \int_{2\mathbb{B}} V_A \cdot \{X(\psi) [\phi_A - (\phi_A)_{2\mathbb{B}}] + \psi X(\phi_A)\} \, dv \end{aligned} \tag{86}$$

and summed over A

$$\begin{aligned} &\sum_A V_A \cdot X(\phi_A) \\ &\quad = \sum_a \{V_{\alpha,a} X_a(\psi^\alpha) + V_{m+\alpha,a} X_a(\phi^{m+\alpha})\} \\ &\quad = Q(\phi)^{(p-2)/2} \left\{ |X\phi|^2 - \sum_a (L_a\phi)^2 \right\} \\ &\quad = Q(\phi)^{p/2} \end{aligned} \tag{87}$$

by the very definition of V_A (cf. Lemma 22) and by (67). Thus (85) becomes

$$\int_{2\mathbb{B}} \psi Q(\phi)^{p/2} dv + \sum_A \int_{2\mathbb{B}} [\phi_A - (\phi_A)_{2\mathbb{B}}] V_A \cdot X(\psi)\, dv = \sum_{A,B} \int_{2\mathbb{B}} [X^* \cdot (\phi_B E_{A,B})]\, \psi_A dv. \tag{88}$$

For simplicity let $I_{A,B} = \int_{2\mathbb{B}} [X^* \cdot (\phi_B E_{A,B})] \psi_A dv$ and $C_0 = \sum_{A,B} |I_{A,B}|$. Using (88), we may perform the estimates

$$\int_{\mathbb{B}} Q(\phi)^{p/2} dv \le \int_{2\mathbb{B}} \psi Q(\phi)^{p/2} dv \le C_0 + \sum_A \int_{2\mathbb{B}} |\phi_A - (\phi_A)_{2\mathbb{B}}|\, |V_A|\, |X\psi|\, dv. \tag{89}$$

Lemma 18. *Let one set $|L\phi|^2 = \sum_{a=1}^{2n} |L_a\phi|^2$. Then $|LX| \le \sqrt{2}|X\phi|$ a.e. in U and consequently*

$$|V_A| \le \sqrt{6} Q(\phi)^{(p-2)/2} |X\phi| \tag{90}$$

a.e. in U, for any $1 \le A \le 2m$.

The inequalities in Lemma 18 follow easily from $|\phi_A| \le 1$ and $|X\phi_A| \le |X\phi|$. Using (90), we may write (89) as

$$\int_{\mathbb{B}} Q(\phi)^{p/2} dv \le C_0 + \sqrt{6} \sum_A \int_{2\mathbb{B}} Q(\phi)^{(p-2)/p} |\phi_A - (\phi_A)_{2\mathbb{B}}|\, |X\phi|\, |X\psi|\, dv. \tag{91}$$

In the following estimates C denotes some positive constant, not necessarily the same in all formulae. By Hölder's inequality

$$\begin{aligned}
&\int_{2\mathbb{B}} Q(\phi)^{(p-2)/2} |\phi_A - (\phi_A)_{2\mathbb{B}}|\, |X\phi|\, |X\psi|\, dv \\
&\le \left(\int_{2\mathbb{B}} |\phi_A - (\phi_A)_{2\mathbb{B}}|^p dv\right)^{1/p} \\
&\quad \times \left(\int_{2\mathbb{B}\setminus\mathbb{B}} \left(Q(\phi)^{(p-2)/2} |X\phi|\, |X\psi|\right)^{p/(p-1)} dv\right)^{(p-1)/p} \\
&\le C\left(\int_{2\mathbb{B}} |X\phi_A|^p dv\right)^{1/p} \\
&\quad \times \left(\int_{2\mathbb{B}\setminus\mathbb{B}} Q(\phi)^{p(p-2)/2(p-1)} |X\phi|^{p/(p-1)}\right)^{(p-1)/p},
\end{aligned} \tag{92}$$

by the Poincaré inequality

$$\left(\int_{2\mathbb{B}} |\phi_A - (\phi_A)_{2\mathbb{B}}|^p dv\right)^{1/p} \le Cr\left(\int_{2\mathbb{B}} |X\phi_A|^p dv\right)^{1/p} \tag{93}$$

and by $|X\psi| \le C/r$. Let us observe that $Q(\phi) \le |X\phi|^2$ yields

$$\left(\int_{2\mathbb{B}\setminus\mathbb{B}} Q(\phi)^{p(p-2)/2(p-1)} |X\phi|^{p/(p-1)}\right)^{(p-1)/p} \le \left(\int_{2\mathbb{B}\setminus\mathbb{B}} |X\phi|^p dv\right)^{(p-1)/p}. \tag{94}$$

Hence (by (91))

$$\int_{\mathbb{B}} Q(\phi)^{p/2} dv \le C_0 + C\left(\int_{2\mathbb{B}} |X\phi|^p dv\right)^{1/p} \times \left(\int_{2\mathbb{B}\setminus\mathbb{B}} |X\phi|^p dv\right)^{(p-1)/p}. \tag{95}$$

Let us set $I_p(r) = \int_{B_X(x,r)} |X\phi|^p dv$. Also let us restrict our considerations to maps $\phi : \mathbb{H}_n \to S^{2m-1}$ for which one may control $Q(\phi)$ from below. We adopt the following.

Definition 19. A map $\phi : \mathbb{H}_n \to S^{2m-1}$ is said to be *close to a horizontal* map if there exist constants $0 < c < 1$ and $0 < \delta < 1$ such that

$$\begin{aligned}
|L\phi| &\le c|X\phi|^\delta \quad \text{a.e. in } \{x \in \mathbb{H}_n : |X\phi|(x) \ge 1\}, \\
|L\phi| &\le c|X\phi| \quad \text{a.e. in } \{x \in \mathbb{H}_n : |X\phi|(x) < 1\}.
\end{aligned} \tag{96}$$

If $\phi : \mathbb{H}_n \to S^{2m-1}$ is close to horizontal, then (by (96))

$$Q(\phi) \ge a|X\phi|^2, \quad a = 1 - c^2 > 0. \tag{97}$$

Our main result in this section is the following.

Theorem 20. *Let $U \subset \mathbb{H}_n$ be a bounded domain in the Heisenberg group and $\overline{Z}_\alpha = \partial/\partial\overline{z}^\alpha - iz^\alpha\partial/\partial t$, $1 \le \alpha \le n$, the Lewy operators. Let $X = \{Z_\alpha + \overline{Z}_\alpha, i(Z_\alpha - \overline{Z}_\alpha) : 1 \le \alpha \le n\}$ and $U_1 \subset\subset U$. Let $\phi \in W_X^{1,2n+2}(U, S^{2m-1})$ be a map obeying to (96) for some $0 < c < 1$ and $0 < \delta < 1$. If $\phi : U \to S^{2m-1}$ is a weak contact $(2n+2)$-harmonic map, then there exist constants $r_0 > 0$, $C > 0$ and $0 < \gamma < 1$ such that*

$$\int_{B_X(x,r)} |X\phi|^{2n+2} dv \le Cr^\gamma \tag{98}$$

for any $x \in U_1$ and any $0 < r \le r_0$.

As a consequence of Theorem 20 (by applying a version of the Dirichlet growth theorem due to Macìas and Segovia [15]).

Corollary 21. *Let $U \subset \mathbb{H}_n$ be a bounded domain. Any weak contact $(2n+2)$-harmonic map $\phi : U \to S^{2m-1}$ satisfying (96) is locally Hölder continuous.*

To prove Theorem 20, we use a hole filling technique essentially due to Widman [32], (cf. also Bensoussan et al. [31,

page 38–40]). By (95) with $p = D = 2n+2$ and Lemma 17 with $\psi = \psi_A$, we have

$$\begin{aligned}\int_{\mathbb{B}} Q(\phi)^{D/2} dv &\le C\Big\{ I_D(2r)^{1/D}\big(I_D(2r) - I_D(r)\big)^{(D-1)/D} \\ &\quad + \big[I_D(200r) + I_{D/(D+1)}(200r)^{1/(D+1)}\big] \\ &\quad \times \sum_{A=1}^{2m} \|X\psi_A\|_{L^D(2\mathbb{B})}\Big\}.\end{aligned} \tag{99}$$

On the other hand, by the very definition of ψ_A, we may use the Poincaré inequality to estimate

$$\begin{aligned}&\sum_{A=1}^{2m} \|X\psi_A\|_{L^D(2\mathbb{B})} \\ &\le \sum_a \left\{\|(X\psi)[\phi_A - (\phi_A)_{2\mathbb{B}}]\|_{L^D(2\mathbb{B})} + \|\psi X\phi_A\|_{L^D(2\mathbb{B})}\right\} \\ &= \sum_A \left(\int_{2\mathbb{B}} |X\psi|^D |\phi_A - (\phi_A)_{2\mathbb{B}}|^D dv\right)^{1/D} \\ &\quad + \sum_A \left(\int_{2\mathbb{B}} |\psi|^D |X\phi_A|^D dv\right)^{1/D} \\ &\le \frac{C}{r}\sum_A \left(\int_{2\mathbb{B}} |\phi_A - (\phi_A)_{2\mathbb{B}}|^D dv\right)^{1/D} \\ &\quad + \sum_A \left(\int_{2\mathbb{B}} |X\phi_A|^D dv\right)^{1/D},\end{aligned} \tag{100}$$

that is,

$$\sum_{A=1}^{2m} \|X\psi_A\|_{L^D(2\mathbb{B})} \le C I_D(2r)^{1/D}. \tag{101}$$

Using (97) and (101), the inequality (120) yields

$$\begin{aligned}&a I_D(r) \\ &\le C\Big\{ I_D(2r)^{1/D}\big(I_D(2r) - I_D(r)\big)^{(D-1)/D} \\ &\quad + \big[I_D(200r) + I_D(200r)^{1-\epsilon}\big] I_D(2r)^{1/D}\Big\}.\end{aligned} \tag{102}$$

By the Vitali absolute continuity of the integral $I_D(200r)$, there is $r_0' > 0$ such that $I_D(200r) < 1$ for any $0 < r \le r_0'$. As a consequence of (102) we may establish the following.

Lemma 22. *There exist $0 < r_0 \le r_0'$ and $1/2 \le \lambda < 1$ such that*

$$I_D(r) \le \lambda I_D(200r)^{1-\epsilon} \tag{103}$$

for any $0 < r \le r_0$.

Proof. The proof is by contradiction. Let us assume that for any $0 < r_0 \le r_0'$ and any $1/2 \le \lambda < 1$, there is $0 < r \le r_0$ such that $I(r) > \lambda I(200r)^{1-\epsilon}$, where I is short for I_D. Note that $I(200r) \le I(200r)^{1-\epsilon}$. Then (by (102))

$$\begin{aligned}\lambda I(200r)^{1-\epsilon} &< I(r) \\ &\le C\Big\{ I(200r)(1-\lambda)^{(D-1)/D} \\ &\quad + \big[I(200r) + I(200r)^{1-\epsilon}\big] I(2r)^{1/D}\Big\} \\ &\le C I(200r)^{1-\epsilon}\left\{(1-\lambda)^{(D-1)/D} + I(2r)^{1/D}\right\}.\end{aligned} \tag{104}$$

Therefore

$$\frac{1}{2} \le \lambda < C\left\{(1-\lambda)^{(D-1)/D} + I(2r)^{1/D}\right\}. \tag{105}$$

The inequality (105) leads to

$$\left(\frac{1}{2C}\right)^D \le \int_{2\mathbb{B}} |X\phi|^D dv. \tag{106}$$

Indeed, by the contradiction assumption, we may pick a sequence $\lambda_j \in [1/2, 1)$ such that $\lambda_j \to 1$ as $j \to \infty$ and consider the corresponding radii $0 < r_j \le r_0$. By passing to a subsequence, if necessary, one may assume that $\lim_{j\to\infty} r_j = r_\infty$ for some $r_\infty \in [0, r_0]$. Let $j \to \infty$ in

$$\frac{1}{2} \le \lambda_j < C\left\{\left(1-\lambda_j\right)^{(D-1)/D} + I\left(2r_j\right)^{1/D}\right\} \tag{107}$$

and use the absolute continuity of the integral. Then either $r_\infty > 0$ (yielding (106)) or $r_\infty = 0$ and then $1/2 \le 0$, a contradiction. Finally (106) may be exploited as follows. Let $r_0 = 1/k$. By the contradiction assumption there is $0 < r \le 1/k$ such that (by (106))

$$\left(\frac{1}{2C}\right)^D \le I(2r) \le \int_{B_X(x,2/k)} |X\phi|^D dv \tag{108}$$

and the last integral tends to 0 as $k \to \infty$, a contradiction. Lemma 22 is proved.

Now we may prove the Caccioppoli type estimate (98). Let $\tau = 1/200$ so that (103) may be written as

$$I_D(\tau r) \le \lambda I_D(r)^{1-\epsilon}. \tag{109}$$

Then (by (109) and induction over m)

$$I_D(\tau^m r) \le \lambda^{[1-(1-\epsilon)^m]/\epsilon} I_D(r)^{(1-\epsilon)^m} \tag{110}$$

for any $m \in \mathbb{Z}$, $m \ge 1$. Let us consider the family of intervals $\{(\tau^m, \tau^{m-1}] : m \in \mathbb{Z}, m \ge 1\}$. It is a cover of $(0, 1]$, hence for each $0 < r \le r_0$ there is $m \in \mathbb{Z}$, $m \ge 1$, such that $\tau^m < r/r_0 \le \tau^{m-1}$. Now the inequality $r \le \tau^{m-1} r_0$ implies (by (110))

$$I_D(r) \le I_D\left(\tau^{m-1} r_0\right) \le \lambda^{[1-(1-\epsilon)^{m-1}]/\epsilon} I_D(r_0)^{(1-\epsilon)^{m-1}}. \tag{111}$$

On the other hand let us set $\gamma = (\log\lambda)/(\log\tau)$ (so that $0 < \gamma < 1$) and observe that the inequality $r/r_0 \geq \tau^m$ implies

$$\left(\frac{r}{r_0}\right)^{\gamma} > \tau^{m\gamma} = \tau^{(\log\lambda^m)/(\log\tau)} = \lambda^m, \tag{112}$$

that is, $\lambda^m < (r/r_0)^{\gamma}$. One may choose $r_0 > 0$ from the very beginning such that $I_D(r_0) < \lambda$ for any $x \in U_1$. Note that $0 < \epsilon \leq 1/2$ (by the very definition of ϵ). Then $[1-(1-\epsilon)^m]/\epsilon = \sum_{j=0}^{m-1}(1-\epsilon)^j \geq 1 + (m-1)(1/2) = (m+1)/2$, hence $\lambda^{[1-(1-\epsilon)^m]/\epsilon} \leq C\, r^{\gamma/2}$, where $C = r_0^{-\gamma/2}\sqrt{\lambda}$. Theorem 20 is proved. □

It remains that we prove Lemma 17. It suffices to prove the inequality (83) for any $\psi \in C_0^{\infty}(\mathbb{B})$. Let us consider

$$w(x) = \left(\frac{1}{4a_0}\right)|x|^{-2n}, \quad x \in \mathbb{H}_n, \tag{113}$$

where $a_0 = (2^{2-2n}\pi^{n+1}/\Gamma(n/2))^2$ and $|x| = (|z|^4 + t^2)^{1/4}$ is the Heisenberg norm of $x = (z,t)$. By a classical result of Folland, [33], $G(x,y) = w(xy^{-1})$ is a fundamental solution for the Hörmander operator $\sum_{a=1}^{2n} X_a^2$. In particular for any bounded domain $U \subset \mathbb{H}_n$ one has the representation formula

$$u(x) = \int_U X_y G(y,x)\cdot Xu(y)\, dv(y) \tag{114}$$

for any $u \in C_0^{\infty}(U)$ and any $x \in U$. By a result of Citti et al., [34], we may consider a smooth cut-off function $0 \leq \psi_0 \leq 1$ such that $\psi_0 = 1$ on $2\mathbb{B}$, $\psi_0 = 0$ on $U \setminus 4\mathbb{B}$, and $|X\psi_0| \leq C/\operatorname{diam}(\mathbb{B})$ (the diameter is meant with respect to the Carnot-Carathéodory metric on $\mathbb{H}_n$). Using (114) for $u = \psi$, one may write

$$\begin{aligned}
&\int_{\mathbb{B}} [X^*\cdot(\phi_B E_{A,B})]\,\psi\, dv \\
&\quad = \int_{\mathbb{B}} X^*\cdot(\phi_B E_{A,B})(x)\,\psi(x)\,\psi_0(x)\, dv(x) \\
&\quad = \int_{\mathbb{B}} dv(x)\,[X^*\cdot(\phi_B E_{A,B})](x)\,\psi_0(x) \\
&\qquad \times \int_{\mathbb{B}} X_y G(y,x)\cdot X\psi(y)\, dv(y) \\
&\quad = \int_{\mathbb{B}} \mathscr{A}_{A,B}\cdot(X\psi)\, dv,
\end{aligned} \tag{115}$$

where we have set

$$\mathscr{A}_{A,B}(y) = \int_{\mathbb{B}} [X^*\cdot(\phi_B E_{A,B})](x)\,\psi_0(x)\, X_y G(y,x)\, dv(x). \tag{116}$$

We wish to prove an estimate on $|\mathscr{A}(y)|$, where $\mathscr{A} = \mathscr{A}_{A,B}$ for simplicity. As it is well known, $|B_X(x,r)| = Cr^{2n+2}$ for some constant $C > 0$ and any $x \in \mathbb{H}_n$ and $r > 0$. Here $|A|$ denotes the Lebesgue measure of the set A. In particular the Lebesgue measure on $(\mathbb{H}_n, d_X)$ has the doubling property. Thus we may apply a result by Macìas and Segovia, [15], to pick a Whitney decomposition of $U_y = U \setminus \{y\}$. Precisely let $y \in \mathbb{B}$, and given $x \in U_y$, let us set $r_x = d_X(x, \mathbb{H}_n \setminus U_y)/1000$. Next let us choose among $\{B_X(x,r_x)\}_{x\in U_y}$ a maximal family of mutually disjoint balls $\{B_X(x_\alpha, r_\alpha)\}_{\alpha\in I}$. Then $U_y = \bigcup_{\alpha\in I} B_X(x_\alpha, 3r_\alpha)$ (the Whitney decomposition of U_y) and there is $N \geq 1$ such that each $x \in U$ belongs to at most N balls $B_X(x_\alpha, 6r_\alpha)$. Moreover, again by a result in [15], we may associate a partition of unity to the Whitney decomposition of U_y; that is, we may consider a family of smooth functions $\{\theta_\alpha\}_{\alpha\in I}$ such that $0 \leq \theta_\alpha \leq 1$, $\sum_{\alpha\in I}\theta_\alpha = 1$ on U_y, $\operatorname{Supp}(\theta_\alpha) \subset \mathbb{B}_\alpha = B_X(x_\alpha, 6r_\alpha)$, and $|X\theta_\alpha| \leq C/r_\alpha$. The bounds on the gradients actually follow from the work by Citti et al., [34], quoted above. Then

$$\begin{aligned}
&\mathscr{A}_a(y) \\
&\quad = \sum_{\alpha\in I}\int_{\mathbb{B}_\alpha} [X^*\cdot(\phi_B E_{A,B})](x)\,\psi_0(x) \\
&\qquad \times \theta_\alpha(x)\, X_{a,y} G(y,x)\, dv(x) \\
&\quad = \sum_{\alpha\in I}\int_{\mathbb{B}_\alpha} \left[X^*\cdot\left(\phi_B - (\phi_B)_{\mathbb{B}_\alpha}\right)E_{A,B}\right](x)\,\psi_0(x) \\
&\qquad \times \theta_\alpha(x)\, X_{a,y} G(y,x)\, dv(x) \\
&\qquad + \sum_{\alpha\in I}(\phi_B)_{\mathbb{B}_\alpha}\int_{\mathbb{B}_\alpha} (X^*\cdot E_{A,B})(x)\,\psi_0(x) \\
&\qquad\qquad \times \theta_\alpha(x)\, X_{a,y} G(y,x)\, dv(x) \\
&\quad = \mathscr{A}'_a(y) + \mathscr{A}''_a(y).
\end{aligned} \tag{117}$$

The presence of term $\mathscr{A}''_a(y)$ represents of course the main difference with respect to the proof of the so called duality inequality in [7] (there $X^*\cdot E_{A,B} = 0$). Integrating by parts,

$$\begin{aligned}
&\mathscr{A}'_a(y) \\
&\quad = \sum_{\alpha\in I}\int_{\mathbb{B}_\alpha} \left(\phi_B(x) - (\phi_B)_{\mathbb{B}_\alpha}\right) \\
&\qquad \times E_{A,B}(x)\cdot X_x\left[\psi_0(x)\,\theta_\alpha(x)\, X_{a,y} G(y,x)\right] dv(x).
\end{aligned} \tag{118}$$

Due to the explicit form of the fundamental solution $G(x,y)$, one may easily check that

$$|X_a G(x,y)| \leq C d_X(x,y)^{-2n-1}, \tag{119}$$

$$|X_a X_b G(x,y)| \leq C d_X(x,y)^{-2n-2}, \tag{120}$$

for any $x, y \in U$. Here it is irrelevant whether differentiation is performed in x or y. Estimates of the sort in the case of an arbitrary Hörmander system of vector fields have been obtained by Sánchez-Calle [35]. Estimates on $G(x,y)$ itself are available, yet only estimates on the derivatives are needed for the following calculations. Using (119)-(120) and

$$\begin{aligned}
&|X\psi_0(x)| \leq C d_X(x,y)^{-1}, \\
&|\theta_\alpha(x)| \leq C d_X(x,y)^{-1}, \quad \alpha \in I,
\end{aligned} \tag{121}$$

one has

$$\left|X_{b,x}\left[\psi_0(x)\theta_\alpha(x)X_{a,y}G(y,x)\right]\right| \le Cd_X(x,y)^{-2n-2}, \tag{122}$$

hence

$$\left|\mathscr{A}'_\alpha(y)\right| \le C\sum_{\alpha\in I}\int_{\Gamma_\alpha}\frac{\left|\phi_B(x)-(\phi_B)_{\mathbb{B}_\alpha}\right|\left|E_{A,B}(x)\right|}{d_X(x,y)^{2n+2}}dv(x), \tag{123}$$

where $\Gamma_\alpha = \mathrm{Supp}(\theta_\alpha)$. Let $x \in \mathbb{B}_\alpha = B_X(x_\alpha, 6r_\alpha)$. As $y \in \mathbb{H}_n \setminus U_y$, the very definition of r_α yields $d_X(y,x_\alpha) \ge 1000r_\alpha$; hence

$$\begin{aligned}1000r_\alpha &\le d_X(y,x_\alpha)\\ &\le d_X(y,x)+d_X(x,x_\alpha)\\ &\le d_X(x,y)+6r_\alpha\end{aligned} \tag{124}$$

and in particular $6r_\alpha \le d_X(x,y)$. Thus $|\mathbb{B}_\alpha| = Cr_\alpha^{2n+2} \le C'd_X(x,y)^{2n+2}$, where $C' = C6^{-2n-2}$; hence there is a constant $C > 0$ such that

$$d_X(x,y)^{2n+2} \ge C\left|\mathbb{B}_\alpha\right|, \quad x \in \mathbb{B}_\alpha. \tag{125}$$

Let us set $J = \{\alpha \in I : \Gamma_\alpha \cap 4\mathbb{B} \ne \emptyset\}$. Let us apply (123) and (125) and Hölder's inequality to perform the estimates

$$\begin{aligned}\left|\mathscr{A}'_a(y)\right| &\le C\sum_{\alpha\in J}\frac{1}{|\mathbb{B}_\alpha|}\int_{\mathbb{B}_\alpha}\left|\phi_B(x)-(\phi_B)_{\mathbb{B}_\alpha}\right|\ \left|E_{A,B}(x)\right|dv(x)\\ &\le C\sum_{\alpha\in J}\left(\frac{1}{|\mathbb{B}_\alpha|}\int_{\mathbb{B}_\alpha}\left|\phi_B-(\phi_B)_{\mathbb{B}_\alpha}\right|^{D^2}dv\right)^{1/D^2}\\ &\quad\times\left(\frac{1}{|\mathbb{B}_\alpha|}\int_{\mathbb{B}_\alpha}\left|E_{A,B}\right|^{D^2/(D^2-1)}dv\right)^{(D^2-1)/D^2},\end{aligned} \tag{126}$$

where we have set $D = 2n+2$ for simplicity. By (90) in Lemma 18 and $Q \le |X\phi|^2$, one has $|E_{A,B}| \le 2\sqrt{6}|X\phi|^{D-1}$; hence

$$\begin{aligned}&\left(\frac{1}{|\mathbb{B}_\alpha|}\int_{\mathbb{B}_\alpha}\left|E_{A,B}\right|^{D^2/(D^2-1)}dv\right)^{(D^2-1)/D^2}\\ &\quad\le C\left(\frac{1}{|\mathbb{B}_\alpha|}\left|X\phi\right|^{D^2/(D+1)}dv\right)^{(D^2-1)/D^2}.\end{aligned} \tag{127}$$

At this point we need to apply a version of the Sobolev inequality due to Franchi et al. [36]. Precisely, for any $1 \le p < 2n+2$ there is a constant $C > 0$ such that for any ball $B_X(x,r)$ with $x \in U$ and $0 < r \le \mathrm{diam}(U)$

$$\begin{aligned}&\left(\frac{1}{|B_X(x,r)|}\int_{B_X(x,r)}\left|u-u_{B_X(x,r)}\right|^{p^*}dv\right)^{1/p^*}\\ &\quad\le Cr\left(\frac{1}{|B_X(x,r)|}\int_{B_X(x,r)}|Xu|^p dv\right)^{1/p},\\ &\qquad p^* = \frac{2(n+1)p}{2n+2-p}.\end{aligned} \tag{128}$$

By the assumption in Theorem 20 one has $X_a\phi_B \in L^{2n+2}(U)$; hence $X_a\phi_B \in L^\nu(U)$ for any $0 < \nu \le 2n+2$. Therefore (by the Sobolev inequality above)

$$\begin{aligned}&\left(\frac{1}{|\mathbb{B}_\alpha|}\left|\phi_B-(\phi_B)_{\mathbb{B}_\alpha}\right|^{D^2}dv\right)^{1/D^2}\\ &\quad\le Cr_\alpha\left(\frac{1}{|\mathbb{B}_\alpha|}\int_{\mathbb{B}_\alpha}\left|X\phi\right|^{d^2/(D+1)}dv\right)^{(D+1)/D^2}.\end{aligned} \tag{129}$$

Collecting the information in (127) and (129),

$$\left|\mathscr{A}'_a(y)\right| \le C\sum_{\alpha\in J}r_\alpha\left(\frac{1}{|\mathbb{B}_\alpha|}\int_{\mathbb{B}_\alpha}\left|X\phi\right|^{D^2/(D+1)}dv\right)^{(D+1)/D}. \tag{130}$$

In the sequel we write briefly $a \approx b$ whenever $a/C \le b \le Ca$ for some constant $C \ge 1$. Let $\alpha \in J$. If there is $k \in \mathbb{Z}$ such that $x_\alpha \in B_X(y,2^{k-1}) \setminus B_X(y,2^{k-2})$, then $r_\alpha \approx 2^k$ and $\mathbb{B}_\alpha \subset B_X(y,2^k)$ (our arguments follow closely those in [7, page 356]). Moreover

$$\frac{|\mathbb{B}_\alpha|}{|B_X(y,2^k)|} = \left(\frac{6r_\alpha}{2^k}\right)^{2n+2}, \tag{131}$$

hence $|\mathbb{B}_\alpha| \approx |B_X(y,2^k)|$. Consequently

$$\begin{aligned}&r_\alpha\left(\frac{1}{|\mathbb{B}_\alpha|}\int_{\mathbb{B}_\alpha}\left|X\phi\right|^{D^2/(D+1)}dv\right)^{(D+1)/D}\\ &\quad\le C2^k\left(\frac{1}{|B_X(y,2^k)|}\int_{B_X(y,2^k)}\left|X\phi\right|^{D^2/(D+1)}dv\right)^{(D+1)/D}.\end{aligned} \tag{132}$$

Also $\{\alpha \in J : x_\alpha \in B_X(y,2^{k-1}) \setminus B_X(x_\alpha,2^{k-2})\} = \emptyset$ whenever $2^{k-2} \ge \mathrm{diam}(8\mathbb{B})$ and the estimate (130) may be written as

$$\begin{aligned}&\left|\mathscr{A}'_a(y)\right|\\ &\le C\sum_{2^k\le 4\,\mathrm{diam}(8\mathbb{B})}2^k\left(\frac{1}{|B_X(y,2^k)|}\int_{B_X(y,2^k)}\left|X\phi\right|^{D^2/(D+1)}dv\right)^{(D+1)/D}.\end{aligned} \tag{133}$$

Next we shall express the estimate on $|\mathscr{A}'_a(y)|$ in terms of Riesz potentials and then use the general estimates on L^p

norms of Riesz potentials as obtained by Hàjlasz and Koskela [37]. To recall the needed result, let (X,ρ) be a metric space endowed with a Borel measure μ such that $\mu(B)>0$ for any ball $B\subset X$. Let $A\subset X$ be a bounded open set and let us consider the numbers $q>0$, $\sigma\geq 1$, and $h>0$.

Definition 23. An (*abstract*) Riesz potential operator $J_{h,q}^{\sigma,A}$ is given by

$$\left(J_{h,q}^{\sigma,A}g\right)(x) = \sum_{2^k\leq 2\sigma\,\mathrm{diam}(A)} 2^{kh}\left(\frac{1}{\left|B\left(x,2^k\right)\right|}\int_{B(x,2^k)}\left|g\left(z\right)\right|^q d\mu(z)\right)^{1/q}. \tag{134}$$

The estimate (133) implies

$$\left|\mathscr{A}_a'\left(y\right)\right|\leq C\left(J_{1,q}^{2,8\mathbb{B}}\left|X\phi\right|\right)(y),\quad q=\frac{D}{D+1}. \tag{135}$$

The needed result in [37] holds for an arbitrary metric space (X,ρ) endowed with a Borel measure μ such that $\mu(B)>0$ for any ball $B\subset X$. Let $A\subset X$ be a bounded open set such that μ is doubling on

$$V=\{x\in X:\mathrm{dist}\left(x,A\right)<2\sigma\,\mathrm{diam}\left(A\right)\}. \tag{136}$$

Let us assume that there are constants $b>0$ and $D>0$ such that

$$\mu\left(B\left(x,R\right)\right)\geq b\left(\frac{R}{\mathrm{diam}(A)}\right)^D\mu\left(A\right) \tag{137}$$

for any $x\in A$ and any $0<R\leq 2\sigma\,\mathrm{diam}(A)$. Moreover let $h>0$ and $0<q\leq s<D/h$. Then (cf. [37])

$$\left\|J_{h,q}^{\sigma,A}g\right\|_{L^{s^*}(A,\mu)}\leq C\left(\frac{\mathrm{diam}\left(A\right)}{\mu(A)^{1/D}}\right)^h\|g\|_{L^s(V,\mu)}, \tag{138}$$

where $s^*=sD/(D-hs)$ and the constant $C>0$ depends only on h,σ,q,s,b,D, and the doubling constant. Then (by Hölder's inequality with $1/(2n+2)+1/D'=1$, resp., with $1/\mu+1/\mu'=1$)

$$\begin{aligned}\left|\int_{\mathbb{B}}\left[X^*\cdot\left(\phi_B E_{A,B}\right)\right]\psi dv\right| &\leq \sum_{a=1}^{2n}\|X\psi\|_{L^{2n+2}(\mathbb{B})}\\ &\quad\times\left(\int_{\mathbb{B}}\left|\mathscr{A}_a'\left(y\right)\right|^{2(n+1)/(2n+1)}dv(y)\right)^{(2n+1)/[2(n+1)]}\\ &\quad+\sum_a\|X\psi\|_{L^\mu(\mathbb{B})}\left(\int_{\mathbb{B}}\left|\mathscr{A}_a''\left(y\right)\right|^{\mu'}dv(y)\right)^{1/\mu'}\end{aligned} \tag{139}$$

with $1<\mu<D$ to be determined later on. At this point we need an estimate on $|\mathscr{A}_a''(y)|$. By (80) in Proposition 16 if $\phi:\mathbb{H}_n\to S^{2m-1}$ is a contact $(2n+2)$-harmonic map obeying to our assumptions (96), then

$$\left|X^*\cdot E_{A,B}\right|\leq 2Q(\phi)^{(D-2)/2}\left|X\phi\right|\left|L\phi\right|\leq\left|2cX\phi\right|^{D-1+\delta}, \tag{140}$$

hence (by (119))

$$\begin{aligned}\left|\mathscr{A}_a''\left(y\right)\right| &\leq \sum_{\alpha\in I}\int_{\mathbb{B}_\alpha}\left|\left(X^*\cdot E_{A,B}\right)(x)\right|\left|\psi_0\left(x\right)\right|\\ &\quad\times\left|\theta_\alpha\left(x\right)\right|\left|X_{a,y}G\left(y,x\right)\right|dv\left(x\right)\\ &\leq C\sum_{\alpha\in I}\int_{\Gamma_\alpha}\frac{\left|X\phi\right|^\nu}{d_X(x,y)^{2n+1}}dv\left(x\right),\end{aligned} \tag{141}$$

where $\nu=D-1+\delta$ and $0<\nu<D$. By $d_X(x,y)^{2n+1}\geq C|\mathbb{B}_\alpha|/r_\alpha$ for any $x\in\mathbb{B}_\alpha$ one obtains

$$\begin{aligned}\left|\mathscr{A}_a''\left(y\right)\right| &\leq C\sum_{\alpha\in J}r_\alpha\frac{1}{|\mathbb{B}_\alpha|}\int_{\mathbb{B}_\alpha}\left|X\phi\right|^\nu dv\\ &\leq C\sum_{\alpha\in J}\frac{r_\alpha}{|\mathbb{B}_\alpha|}\left(\int_{\mathbb{B}_\alpha}\left|X\phi\right|^D\right)^{\nu/D}|\mathbb{B}_\alpha|^{(D-\nu)/D},\end{aligned} \tag{142}$$

that is,

$$\left|\mathscr{A}_a''\left(y\right)\right|\leq C\sum_{2^k\leq 4\,\mathrm{diam}(8\mathbb{B})}2^k\left(\frac{1}{\left|B_X\left(y,2^k\right)\right|}\int_{B_X(y,2^k)}\left|X\phi\right|^D dv\right)^{\nu/D}, \tag{143}$$

hence

$$\left|\mathscr{A}_a''\left(y\right)\right|\leq C\left(J_{1,D/\nu}^{2,8\mathbb{B}}\left|X\phi\right|^\nu\right)(y). \tag{144}$$

Therefore (by (135) and (144))

$$\begin{aligned}\left(\int_{\mathbb{B}}\left|\mathscr{A}_a'\right|^{D/(D-1)}dv\right)^{(D-1)/D} &\leq C\left\|J_{1,D/(D+1)}^{2,8\mathbb{B}}\left|X\phi\right|^D\right\|_{L^{D/(D-1)}(8\mathbb{B})}\\ &\leq C\frac{\mathrm{diam}\left(8\mathbb{B}\right)}{|8\mathbb{B}|^{1/D}}\left\|\left|X\phi\right|^D\right\|_{L^1(V)},\end{aligned} \tag{145}$$

that is,

$$\left(\int_{\mathbb{B}}\left|\mathscr{A}_a'\right|^{D/(D-1)}dv\right)^{(D-1)/D}\leq C\|X\phi\|_{L^D(100\mathbb{B})}^D, \tag{146}$$

where $V=\{x\in\mathbb{H}_n:\mathrm{dist}(x,8\mathbb{B})\leq 4\ \mathrm{diam}(8\mathbb{B})\}$, respectively,

$$\begin{aligned}\left(\int_{\mathbb{B}}\left|\mathscr{A}_a''\right|^{\mu/(\mu-1)}dv\right)^{(\mu-1)/\mu} &\leq C\left\|J_{1,D/\nu}^{2,8\mathbb{B}}\left|X\phi\right|^\nu\right\|_{L^{\mu/(\mu-1)}(8\mathbb{B})}\\ &\leq C\frac{\mathrm{diam}\left(8\mathbb{B}\right)}{|8\mathbb{B}|^{1/D}}\left\|\left|X\phi\right|^\nu\right\|_{L^s(V)}\\ &\leq C\left\|\left|X\phi\right|^\nu\right\|_{L^s(100\mathbb{B})},\end{aligned} \tag{147}$$

where

$$0 < \frac{D}{\gamma} \le s < D, \qquad \frac{\mu}{\mu - 1} = s^* = \frac{sD}{D - s}. \tag{148}$$

Therefore it must be that

$$1 < \mu \le \frac{D}{2 - \delta}, \qquad s = \frac{\mu D}{(D + 1)\mu - D}. \tag{149}$$

On the other hand

$$\begin{aligned} &\left\| |X\phi|^{\gamma} \right\|_{L^s(100\mathbb{B})} \\ &\quad = \left(\int_{100\mathbb{B}} \left(|X\phi|^D \right)^{\gamma\mu/[(D+1)\mu - D]} dv \right)^{[(D+1)\mu - D]/(\mu D)} \end{aligned} \tag{150}$$

and we may choose μ such that $\gamma\mu/[(D + 1)\mu - D] = 1$; that is, $\mu = D/(2 - \delta)$. Consequently

$$\begin{aligned} &\frac{(D + 1)\mu - D}{\mu D} = 1 - \epsilon, \quad \epsilon = \frac{1 - \delta}{D}, \\ &\left\| |X\phi|^{\gamma} \right\|_{L^s(100\mathbb{B})} = \left(\int_{100\mathbb{B}} |X\phi|^D dv \right)^{1-\epsilon}, \\ &\left(\int_{\mathbb{B}} \left| \mathscr{A}_a'' \right|^{\mu/(\mu-1)} dv \right)^{(\mu-1)/\mu} \le C \left(\int_{100\mathbb{B}} |X\phi|^D dv \right)^{1-\epsilon}. \end{aligned} \tag{151}$$

Also

$$\begin{aligned} \|X\psi\|_{L^\mu(\mathbb{B})}^{\mu} &= \int_{\mathbb{B}} |X\psi|^{\mu} dv \\ &\le \left(\int_{\mathbb{B}} |X\psi|^{\mu(2-\delta)} dv \right)^{1/(2-\delta)} |\mathbb{B}|^{(1-\delta)/(2-\delta)} \\ &\le C \left(\int_{\mathbb{B}} |X\psi|^D dv \right)^{1/(2-\delta)}, \end{aligned} \tag{152}$$

that is, $\|X\psi\|_{L^\mu(\mathbb{B})} \le C\|X\psi\|_{L^D(\mathbb{B})}$. Summing up (by (139) and (146) and (151)),

$$\begin{aligned} &\left| \int_{\mathbb{B}} \left[X^* \cdot (\phi_B E_{A,B}) \right] \psi \, dv \right| \\ &\quad \le \|X\psi\|_{L^D(\mathbb{B})} \left\{ \|X\phi\|_{L^D(100\mathbb{B})}^{D} + \|X\phi\|_{L^D(100\mathbb{B})}^{(1-\epsilon)D} \right\} \end{aligned} \tag{153}$$

which is (83). Lemma 17 is proved.

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