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Mixed gravitational field equations on globally hyperbolic spacetimes

Elisabetta Barletta[1], **Sorin Dragomir**[1], **Vladimir Rovenski**[2] **and Marc Soret**[3]

[1] Dipartimento di Matematica, Informatica ed Economia, Università degli Studi della Basilicata, Viale dell'Ateneo Lucano 10, Campus Macchia Romana, I-85100 Potenza, Italy
[2] Department of Mathematics, Faculty of Science and Science Education, University of Haifa, Haifa, Israel
[3] Laboratoire de Mathématiques et Physique Théorique, Université François Rabelais, Tours, France

E-mail: elisabetta.barletta@unibas.it, sorin.dragomir@unibas.it, rovenski@math.haifa.ac.il and Marc.Soret@lmpt.univ-tours.fr

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Abstract

For every globally hyperbolic spacetime M, we derive new *mixed* gravitational field equations embodying the smooth Geroch infinitesimal splitting $T(M) = \mathcal{D} \oplus \mathbb{R}\nabla\mathcal{T}$ of M, as exhibited by Bernal and Sánchez (2005 *Commun. Math. Phys.* **257** 43–50). We give sufficient geometric conditions (e.g. $\mathcal{T}$ is isoparametric and $\mathcal{D}$ is totally umbilical) for the existence of exact solutions $-\beta\, d\mathcal{T} \otimes d\mathcal{T} + \overline{g}$ to mixed field equations in free space. We linearize and solve the mixed field equations $\mathrm{Ric}_{\mathcal{D}}(g)_{\mu\nu} - \rho_{\mathcal{D}}(g)\, g_{\mu\nu} = 0$ for empty space, where $\rho_{\mathcal{D}}(g)$ is the mixed scalar curvature of foliated spacetime $(M, \mathcal{D})$ (due to Rovenski (2010 arXiv:1010.2986 v1[math.DG])). If $g_\epsilon = g_0 + \epsilon\gamma$ is a solution to the linearized field equations, then each leaf of $\mathcal{D}$ is totally geodesic in $(\mathbb{R}^4 \setminus \mathbb{R}, g_\epsilon)$ to order $O(\epsilon)$. We derive the equations of motion of a material particle in the gravitational field $g_{\mu\nu}$ governed by the mixed field equations $\mathrm{Ric}_{\mathcal{D}}(g)_{\mu\nu} - \rho_{\mathcal{D}}(g)\, \omega_\mu\omega_\nu - \Lambda\, g_{\mu\nu} = 2\pi\kappa c^{-2}\left\{T_{\mu\nu} - \frac{1}{2} T\, g_{\mu\nu}\right\}$. In the weak field ($\epsilon \ll 1$) and low velocity ($\|\mathbf{v}\|/c \ll 1$) limit, the motion equations are $d^2\mathbf{r}/dt^2 = \nabla\phi + \mathbf{F}$, where $\phi = (\epsilon/2)c^2\gamma_{00}$.

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1. Smooth Geroch splitting

A spacetime M is *globally hyperbolic* if it is strongly causal[4] and $J^+(p) \cap J^-(q)$ is compact for any $p, q \in M$ (cf definition 2.11 in [2], p 30). A *topological hypersurface* is a three-dimensional topological manifold without boundary, embedded in M. A *Cauchy hypersurface*

[4] By a recent result in [6] 'strong causality' may be replaced by 'causality'.

is a subset $\mathcal{S} \subset M$ such that $\mathcal{S}$ is met exactly once by every inextendible (cf convention 2.8 in [2], p 27) timelike curve in M. A classical theorem by Geroch (cf [14] and theorem 2.13 in [2], p 31) is that global hyperbolicity is equivalent to the existence of a Cauchy hypersurface $\mathcal{S}$. Precisely, the result in [14] is that there is a continuous time function (cf [2], p 29) $\mathcal{T} : M \to \mathbb{R}$, whose level sets $\{\mathcal{T}^{-1}(c) : c \in \mathbb{R}\}$ are topological Cauchy hypersurfaces and M is homeomorphic to $\mathbb{R} \times \mathcal{S}$. Although apparently confined to the category of topological spaces and continuous maps, Geroch's result does admit a smooth version, as recently found by Bernal and Sánchez, [3–5]. Indeed, for every globally hyperbolic spacetime (M, g), there is (cf [4]) a C^∞ diffeomorphism $F : \mathbb{R} \times \mathcal{S} \to M$ such that $F^* g = -\beta \, d\mathcal{T} \otimes d\mathcal{T} + \overline{g}$, where $\mathcal{S}$ is a smooth spacelike Cauchy hypersurface, $\mathcal{T} : \mathbb{R} \times \mathcal{S} \to \mathbb{R}$ is the natural projection, $\beta : \mathbb{R} \times \mathcal{S} \to (0, +\infty)$ is a C^∞ function and $\overline{g}$ is a symmetric $(0, 2)$-tensor field on $\mathbb{R} \times \mathcal{S}$, such that the synthetic object $(\mathcal{S}, \mathcal{T}, \beta, \overline{g})$ obeys that (i) $\nabla\left(\mathcal{T} \circ F^{-1}\right)$ is timelike and past-pointing everywhere in M, (ii) for each level set $\mathcal{S}_t = \mathcal{T}^{-1}(t)$, the manifold $F(\mathcal{S}_t)$ is a Cauchy hypersurface and the pullback $\overline{g}_t = i_t^* \overline{g}$ is a Riemannian metric on $\mathcal{S}_t$, (iii) the radical of $\left(F^{-1}\right)^* \overline{g}$ at every $x \in M$ is Span $\left\{\nabla\left(\mathcal{T} \circ F^{-1}\right)\right\}_x$. Here, $i_t : \mathcal{S}_t \to \mathbb{R} \times \mathcal{S}$ is the inclusion. In particular, $\mathcal{T} \circ F^{-1}$ is a time function and each hypersurface $F(\mathcal{S}_t)$ is spacelike. Therefore, globally hyperbolic spacetimes come equipped with a built-in additional structure, i.e. a pair $(\mathcal{D}, \mathcal{D}^\perp)$ of orthogonal Pfaffian systems, both completely integrable in the case at hand, i.e. $\mathcal{D} = T(\mathcal{F})$ where $\mathcal{F}$ is the foliation of M given by the Pfaffian equation $d(\mathcal{T} \circ F^{-1}) = 0$. Although smoothing of time function (C^1 smoothing) was previously proved in particular cases (e.g. for deterministic globally hyperbolic spacetimes, cf [7]), the full solution to the smoothing problem belongs to [3–5] (especially the proof proposed by Seifert [27] is believed to be incomplete). To emphasize on the relevance of the results in [3–5] (enabling one to apply differential geometric methods, and in particular foliation theory, to the study of globally hyperbolic spacetimes), we adopt the following terminology. Let (M, g) be a spacetime equipped with a C^∞ time function $\mathcal{T} : M \to \mathbb{R}$ such that the metric tensor is given by $g = -\beta \, (d\mathcal{T}) \otimes (d\mathcal{T}) + \overline{g}$ for some C^∞ function $\beta : M \to (0, +\infty)$ and some $(0, 2)$-tensor field $\overline{g}$ such that $\mathrm{Rad}(T(M), \overline{g}) = \mathbb{R}\nabla\mathcal{T}$. The foliation $\mathcal{F}$ of M by level hypersurfaces of $\mathcal{T}$ is referred to as the *Bernal–Sánchez foliation*.

On the other hand, geometric objects such as mixed sectional and scalar curvature have been recently exploited (though confined to the case where g is a Riemannian metric cf [23, 24]) in order to embody $(\mathcal{D}, \mathcal{D}^\perp)$ into a theory aiming to find critical metrics for various actions (cf [22], p 19–53). In this spirit, our task in this work is to derive new gravitational field equations on M as the Euler–Lagrange equations of the variational principle $\delta \int_M \rho_{\mathcal{D}}(g) \, d\,\mathrm{vol}(g) = 0$ (where $\rho_{\mathcal{D}}(g)$ is the mixed scalar curvature, cf [24]) which embody the infinitesimal decomposition $T(M) = \mathcal{D} \oplus \mathbb{R}\nabla\mathcal{T}$.

The paper is organized as follows. In section 2, we derive the first variation formula for the action (cf (2) below)

$$S[g] = \int_\Omega \left\{ \frac{1}{2\mathfrak{a}} \left(\rho_{\mathcal{D}}(g) - 2\Lambda\right) + \mathcal{L} \right\} d\,\mathrm{vol}(g), \quad g \in \mathcal{M}(\mathcal{D}, Z),$$

where $\mathfrak{a}$ is a coupling constant to be determined in the classical limit of the mixed field equations

$$\mathrm{Ric}_{\mathcal{D}}(g)_{\mu\nu} - \rho_{\mathcal{D}}(g)\omega_\mu\omega_\nu - \Lambda g_{\mu\nu} = \mathfrak{a}\{T_{\mu\nu} - (1/2)T g_{\mu\nu}\}$$

(cf (22) and (63) below).

In section 3, we study the geometry of a spacetime M endowed with a Bernal–Sánchez foliation $\mathcal{F}$ (by following the work of Tondeur [31] in the positive definite case) and pinpoint an exact solution (of the form $-\beta \, (d\mathcal{T}) \otimes d\mathcal{T} + \overline{g}$) to mixed field equations $\mathrm{Ric}_{\mathcal{D}}(g)_{\mu\nu} - \rho_{\mathcal{D}}(g) \, \omega_\mu\omega_\nu = 0$ in free space under additional geometric assumptions, i.e.

when the time function $\mathcal{T} : M \to \mathbb{R}$ is isoparametric and the Bernal–Sánchez foliation is totally umbilical. When $\lambda \equiv [-g(\nabla\mathcal{T}, \nabla\mathcal{T})]^{1/2} \in \mathbb{R}$, a necessary and sufficient condition is that time function $\mathcal{T}$ be a solution to the nonlinear hyperbolic equation $\Box\mathcal{T} = \lambda^2/(\mathcal{T} + b\lambda)$ for some $b \in \mathbb{R}$.

Further consequences of the field equations (63) are examined in section 4. There we derive the motion equations

$$2\varphi^\alpha_\mu\left(\frac{\mathrm{d}u^\mu}{\mathrm{d}s} + \Gamma^\mu_{\sigma\rho}u^\sigma u^\rho\right) = \frac{\mathrm{d}}{\mathrm{d}s}[\varphi^{\alpha 0}(u^0)^{-1}u^\mu u_\mu] - u^\mu u_\mu\left[\varphi^{\alpha\sigma}{}_{|\sigma} + \varphi^{\alpha\sigma}\Gamma^\rho_{\rho\sigma}\right],$$

$$\varphi^\alpha_\mu \equiv \delta^\alpha_\mu + Z^\alpha\omega_\mu, \quad u^\mu \equiv \frac{\mathrm{d}x^\mu}{\mathrm{d}s},$$

(cf (70) below) of a point particle in a gravitation field governed by (63), in the spirit of classical work by Einstein, Infeld and Hoffmann [11].

Section 5 is devoted to the examination of the mixed field equations (63) in the classical limit, i.e. under the weak field assumption. Together with the linearization about the Minkowski metric of motion equations, this leads to the determination of the coupling constant in (2) as $\mathfrak{a} = 2\pi\kappa/c^2$.

2. Mixed Einstein–Hilbert action

Einstein's identification of the permanent character of gravitation with the geometry of a region of spacetime, where gravitational effects are present, need not be confined to the metric tensor alone. Additional first-order G-structures, intrinsic to spacetime, may be treated on an equal footing and included in the conceptual equation

$$\begin{pmatrix}\text{tensor representing}\\ \text{geometry of space}\end{pmatrix} = \begin{pmatrix}\text{tensor representing}\\ \text{energy content of space}\end{pmatrix}. \tag{1}$$

A globally hyperbolic spacetime M possesses, besides the metric tensor g, an integrable Pfaffian system $\mathcal{D} : \mathrm{d}\mathcal{T} = 0$ to whom g is intimately related, i.e. $g = -\beta\, \mathrm{d}\mathcal{T} \otimes \mathrm{d}\mathcal{T} + \overline{g}$ for some C^∞ function $\beta : M \to (0, +\infty)$ and some $(0, 2)$-tensor field $\overline{g}$ whose null space is precisely $\mathbb{R}\nabla\mathcal{T}$ and for each leaf $i_\mathcal{S} : \mathcal{S} \hookrightarrow M$ of $\mathcal{D}$ the pullback $i^*_\mathcal{S}\overline{g}$ is a Riemannian metric on $\mathcal{S}$. It is then natural, in the context of a spacetime endowed with a Bernal–Sánchez foliation, to look for field equations (1) whose left-hand member embodies both g and $\mathcal{D}$.

Let (M, g) be a four-dimensional Lorentzian manifiold and $\mathcal{D}$ be a real rank p Pfaffian system on M $(1 \leqslant p \leqslant 3)$ such that $\mathcal{D}_x$ is nondegenerate in $(T_x(M), g_x)$ for any $x \in M$. Let $\mathcal{D}^\perp$ be the orthogonal complement of $\mathcal{D}$. Let R^∇ be the curvature tensor field of (M, g). Let $\{E_\alpha : 0 \leqslant \alpha \leqslant 3\}$ be a local g-orthonormal frame of $T(M)$, defined on the open set $U \subset M$, adapted to $(\mathcal{D}, \mathcal{D}^\perp)$, i.e. $E_{3-p+a} \in \mathcal{D}$ and $E_k \in \mathcal{D}^\perp$ for any $1 \leqslant a \leqslant p$ and $0 \leqslant k \leqslant 3 - p$.

Definition 1. *The mixed scalar curvature is (cf [23, 24])*

$$\rho_\mathcal{D}(g) = \sum_{a=1}^{p}\sum_{k=0}^{3-p}\epsilon_{3-p+a}\epsilon_k g(R^\nabla(E_{3-p+a}, E_k)E_k, E_{3-p+a})$$

on U, where $\epsilon_\alpha = g(E_\alpha, E_\alpha) \in \{\pm 1\}$.

Clearly, $\rho_\mathcal{D}(g)$ is globally defined and $\rho_\mathcal{D}(g) \in C^\infty(M, \mathbb{R})$.

Definition 2. *The mixed Einstein–Hilbert action is*

$$S[g] \equiv S_\Omega[g] = \int_\Omega \left\{\frac{1}{2\mathfrak{a}}\,(\rho_\mathcal{D}(g) - 2\Lambda) + \mathcal{L}\right\}\, \mathrm{d}\,\mathrm{vol}(g), \tag{2}$$

where $\Omega \subset M$ is a relatively compact domain and g is thought of as varied in the space of all Lorentzian metrics such that $\mathcal{D}$ is nondegenerate and its orthogonal complement is the same for every g.

Locally,

$$\mathrm{d}\ \mathrm{vol}(g) = \sqrt{-\mathfrak{g}}\ \mathrm{d}x^0 \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3, \quad \mathfrak{g} = \det[g_{\alpha\beta}],$$

where $g_{\alpha\beta} = g(\partial/\partial x^\alpha\ ,\ \partial/\partial x^\beta)$ with respect to a local coordinate system (U, x^α) on M. Also, Λ is a cosmological constant and $\mathcal{L} = \mathcal{L}(g_{\mu\nu})$ is a Lagrangian describing the matter contents of spacetime. One may vary the connection ∇ as well (leading to Palatini-type variational principles $\delta S[g, \nabla] = 0$, cf e.g. [12] and [18]). This issue will be addressed in further work. We confine ourselves to the case where $\mathcal{D}$ is a spacelike Pfaffian system of codimension 1 ($p = 3$) and $\mathcal{D}^\perp$ is spanned by the globally defined timelike vector field $Z \in \mathfrak{X}(M)$. Then we may choose $E_0 = [-g(Z, Z)]^{-\frac{1}{2}} Z$ (so that $g(E_0, E_0) = -1$) and

$$\rho_{\mathcal{D}}(g) = -g^{\mu\nu} g(R^\nabla(\partial_\mu, E_0)E_0\ ,\ \partial_\nu) = g(Z, Z)^{-1}\mathrm{Ric}_\nabla(Z, Z), \tag{3}$$

where Ric_∇ is the Ricci curvature of (M, g).

Let $\mathbf{t} = |g(Z, Z)|^{-1/2}Z = t^\mu\, \partial_\mu$. The physical meaning of $\rho_{\mathcal{D}}(g)$ is tied to the fact that $\mathrm{Ric}_\nabla(\mathbf{t}, \mathbf{t})$ measures the average gravitational attraction. As suggested by the reviewer, a parallel to the work by Szabados (on quasi-local mass–energy–momentum constructions in general relativity, cf [30]) yields a further physical interpretation, i.e. $R_{\mu\nu}t^\mu t^\nu$ may be thought of as the average gravitational attraction as perceived by the local observer t^α, and if $\Sigma \subset \mathcal{T}^{-1}(t_0)$ is a relatively compact domain for some $t_0 \in \mathbb{R}$, then (with the notations and conventions in [30], p 14)

$$\int_\Sigma t_\mu R^{\mu\nu} \frac{1}{3!}\epsilon_{\nu\alpha\beta\gamma}\mathrm{d}x^\alpha \wedge \mathrm{d}x^\beta \wedge \mathrm{d}x^\gamma$$

is the quasi-local average gravitational attraction as perceived by the fleet of observers being at rest with respect to Σ.

Action (2) is thought of as a functional $S : \mathcal{M}(\mathcal{D}, Z) \to \mathbb{R}$ defined on the space $\mathcal{M}(\mathcal{D}, Z)$ consisting of all Lorentzian metrics $g \in \mathrm{Lor}(M)$ such that $\mathcal{D}$ is spacelike (in particular nondegenerate) with respect to g and $g(X, Z) = 0$ for every $X \in \mathcal{D}$. Let $\{g_t\}_{|t|<\delta} \subset \mathcal{M}(\mathcal{D}, Z)$ be a smooth one-parameter variation of $g_0 = g$ such that the induced infinitesimal variation $h = (\partial g_t/\partial t)_{t=0}$ is supported in Ω (i.e. $\mathrm{Supp}(h) \subset \Omega$). We adopt the notations

$$\dot{\Gamma}^\alpha_{\beta\gamma} = (\partial\Gamma^\alpha_{\beta\gamma}/\partial t)_{t=0}, \quad \dot{R}^\alpha_{\beta\mu\nu} = (\partial R^\alpha_{\beta\mu\nu}/\partial t)_{t=0},$$

where $\Gamma^\alpha_{\beta\gamma}(t)$ and $R^\alpha_{\beta\mu\nu}(t)$ are respectively the Christofell symbols of $g_{\mu\nu}(t)$ and the local components of R^{∇^t}. Also ∇^t is the Levi-Civita connection of (M, g_t). If $f \in C^\infty(M)$, we set $f_{|\alpha} = \partial f/\partial x^\alpha$ for simplicity. The derivative of $R^\alpha_{\beta\mu\nu} = \Gamma^\alpha_{\mu\nu|\beta} - \Gamma^\alpha_{\beta\nu|\mu} + \Gamma^\sigma_{\mu\nu}\Gamma^\alpha_{\beta\sigma} - \Gamma^\sigma_{\beta\nu}\Gamma^\alpha_{\mu\sigma}$ at $t = 0$ is

$$\dot{R}^\alpha_{\beta\mu\nu} = \dot{\Gamma}^\alpha_{\mu\nu|\beta} - \dot{\Gamma}^\alpha_{\beta\nu|\mu} + \dot{\Gamma}^\sigma_{\mu\nu}\Gamma^\alpha_{\beta\sigma} + \Gamma^\sigma_{\mu\nu}\dot{\Gamma}^\alpha_{\beta\sigma} - \dot{\Gamma}^\sigma_{\beta\nu}\Gamma^\alpha_{\mu\sigma} - \Gamma^\sigma_{\beta\nu}\dot{\Gamma}^\alpha_{\mu\sigma}\,. \tag{4}$$

Let $[g^{\mu\nu}(t)]$ be the inverse of $[g_{\mu\nu}(t)]$. Taylor expansion $g_{\mu\nu}(t) = g_{\mu\nu} + t\, h_{\mu\nu} + O(t^2)$ together with $g_{\mu\nu}(t)g^{\nu\sigma}(t) = \delta^\sigma_\mu$ for every $|t| < \delta$ yields

$$g^{\mu\nu}(t) = g^{\mu\nu} - th^{\mu\nu} + O(t^2), \quad h^{\mu\nu} = h_{\alpha\beta}g^{\alpha\mu}g^{\beta\nu}.$$

Since

$$\begin{aligned}\Gamma_{\beta\gamma\sigma}(t) &= \frac{1}{2}(g_{\beta\sigma|\gamma} + g_{\gamma\sigma|\beta} - g_{\beta\gamma|\sigma}) \\ &= \Gamma_{\beta\gamma\sigma} + \frac{t}{2}(h_{\beta\sigma|\gamma} + h_{\gamma\sigma|\beta} - h_{\beta\gamma|\sigma}) + O(t^2),\end{aligned}$$

$$\nabla_\gamma h_{\beta\sigma} = h_{\beta\sigma|\gamma} - \Gamma^\alpha_{\gamma\beta} h_{\alpha\sigma} - \Gamma^\alpha_{\gamma\sigma} h_{\beta\alpha},$$

it follows that

$$\Gamma^\alpha_{\beta\gamma}(t) = g^{\alpha\sigma}(t)\Gamma_{\beta\gamma\sigma}(t)$$
$$= \Gamma^\alpha_{\beta\gamma} + \frac{t}{2} g^{\alpha\sigma}(h_{\beta\sigma|\gamma} + h_{\gamma\sigma|\beta} - h_{\beta\gamma|\sigma}) - t h^\alpha_\nu \Gamma^\nu_{\beta\gamma} + O(t^2),$$

so that

$$\dot\Gamma^\alpha_{\beta\gamma} = \tfrac{1}{2} g^{\alpha\sigma}(\nabla_\gamma h_{\beta\sigma} + \nabla_\beta h_{\gamma\sigma} - \nabla_\sigma h_{\beta\gamma}). \tag{5}$$

Therefore (by (5)) $\dot\Gamma^\alpha_{\beta\gamma}$ is a $(1, 2)$-tensor field on M. Its covariant derivative is given by

$$\nabla_\beta \dot\Gamma^\alpha_{\mu\nu} = \dot\Gamma^\alpha_{\mu\nu|\beta} + \dot\Gamma^\sigma_{\mu\nu}\Gamma^\alpha_{\beta\sigma} - \Gamma^\sigma_{\beta\mu}\dot\Gamma^\alpha_{\sigma\nu} - \Gamma^\sigma_{\beta\nu}\dot\Gamma^\alpha_{\mu\sigma}\,. \tag{6}$$

Let us interchange β and μ and subtract the resulting equation from (6). Then (by (4))

$$\nabla_\beta \dot\Gamma^\alpha_{\mu\nu} - \nabla_\mu \dot\Gamma^\alpha_{\beta\nu} = \dot R^\alpha_{\beta\mu\nu}\,. \tag{7}$$

Moreover (by (2)),

$$\frac{\mathrm{d}}{\mathrm{d}t}\{S[g_t]\}_{t=0} = \int_\Omega \left\{ \frac{1}{2\mathfrak{a}} \left(\dot\rho_{\mathcal{D}}(g) + \frac{\rho_{\mathcal{D}}(g) - 2\Lambda}{\sqrt{-\mathfrak{g}}} \frac{\partial}{\partial t}\{\sqrt{-\mathfrak{g}_t}\}_{t=0} \right) + \frac{1}{\sqrt{-\mathfrak{g}}} \frac{\partial}{\partial t}\{\mathcal{L}\sqrt{-\mathfrak{g}_t}\}_{t=0} \right\} \mathrm{d}\,\mathrm{vol}(g),$$

where $\dot\rho_{\mathcal{D}}(g) = \{\partial\rho_{\mathcal{D}}(g_t)/\partial t\}_{t=0}$ and $\mathfrak{g}_t = \det[g_{\mu\nu}(t)]$. We recall that

$$\frac{\partial \mathfrak{g}}{\partial g_{\mu\nu}} = \mathfrak{g}\, g^{\mu\nu}, \quad \Gamma^\alpha_{\alpha\sigma} = \frac{1}{2} g^{\mu\nu} g_{\mu\nu|\sigma} = (\log\sqrt{-\mathfrak{g}})_{|\sigma}\,. \tag{8}$$

By (8) and $\{\partial\mathcal{L}(g_t)/\partial t\}_{t=0} = (\partial\mathcal{L}/\partial g_{\mu\nu})\, h_{\mu\nu}$, one obtains

$$\frac{1}{2\mathfrak{a}} \frac{\rho_{\mathcal{D}}(g) - 2\Lambda}{\sqrt{-\mathfrak{g}}} \frac{\partial}{\partial t}\{\sqrt{-\mathfrak{g}}\}_{t=0} + \frac{1}{\sqrt{-\mathfrak{g}}} \frac{\partial}{\partial t}\{\mathcal{L}\sqrt{-\mathfrak{g}}\}_{t=0}$$
$$= \frac{1}{2\mathfrak{a}} \left(\frac{\rho_{\mathcal{D}}(g)}{2} - \Lambda \right) g^{\mu\nu} h_{\mu\nu} + \left(\frac{\partial\mathcal{L}}{\partial g_{\mu\nu}} + \frac{1}{2} g^{\mu\nu}\mathcal{L} \right) h_{\mu\nu}. \tag{9}$$

It remains that we compute the derivative of the mixed scalar curvature (of $(M, \mathcal{D}, g_t)$). Contraction of α and β in (7) leads to

$$\dot R_{\mu\nu} = \nabla_\alpha \dot\Gamma^\alpha_{\mu\nu} - \nabla_\mu \dot\Gamma^\alpha_{\alpha\nu}\,. \tag{10}$$

Let $Z_t = [-g_t(Z, Z)]^{-\frac{1}{2}} Z$ so that Z_t is a unit $(g_t(Z_t, Z_t) = -1)$ normal on $\mathcal{D}$ in (M, g_t) for every $|t| < \delta$. Thus (by (3) and (10)), one has $\rho_{\mathcal{D}}(g_t) = -\mathrm{Ric}_{\nabla_t}(Z_t, Z_t) = -Z^\mu(t)Z^\nu(t)R_{\mu\nu}(t)$ and

$$\dot\rho_{\mathcal{D}}(g) = -2\dot Z^\mu Z^\nu R_{\mu\nu} - Z^\mu Z^\nu \left(\nabla_\alpha \dot\Gamma^\alpha_{\mu\nu} - \nabla_\mu \dot\Gamma^\alpha_{\alpha\nu} \right), \tag{11}$$

where $\dot Z^\mu = \{\partial Z^\mu(t)/\partial t\}_{t=0}$ and $Z_0 = Z^\alpha \partial_\alpha$. Moreover,

$$Z^\mu Z^\nu \left(\nabla_\alpha \dot\Gamma^\alpha_{\mu\nu} - \nabla_\mu \dot\Gamma^\alpha_{\alpha\nu} \right) = \nabla_\alpha X^\alpha - 2Z^\nu \dot\Gamma^\alpha_{\mu\nu} \nabla_\alpha Z^\mu + (Z^\nu \nabla_\mu Z^\mu + Z^\mu \nabla_\mu Z^\nu)\dot\Gamma^\alpha_{\alpha\nu}, \tag{12}$$

where $X^\alpha = \left(Z^\mu \dot\Gamma^\alpha_{\mu\nu} - Z^\alpha \dot\Gamma^\mu_{\mu\nu} \right) Z^\nu$. Also,

$$\nabla_\alpha X^\alpha = \frac{1}{\sqrt{-\mathfrak{g}}} (\sqrt{-\mathfrak{g}} X^\alpha)_{|\alpha} = \mathrm{div}(X),$$

where $X \in \mathfrak{X}(M)$ is the tangent vector field locally given by $X = X^\alpha \partial_\alpha$. Differentiating in $Z^\mu(t) = \left[-g_{\alpha\beta}(t) Z^\alpha Z^\beta \right]^{-\frac{1}{2}} Z^\mu$ at $t = 0$ yields

$$\dot Z^\mu = \tfrac{1}{2}(h_{\alpha\beta} Z^\alpha Z^\beta) Z^\mu\,. \tag{13}$$

Then (by (3) and (12)–(13)), equation (11) becomes

$$\dot\rho_{\mathcal D}(g) = h(Z_0\,,\, Z_0)\rho_{\mathcal D}(g) - \mathrm{div}(X) - (Z^\nu \nabla_\mu Z^\mu + Z^\mu \nabla_\mu Z^\nu)\dot\Gamma^\alpha_{\alpha\nu} + 2Z^\nu \dot\Gamma^\alpha_{\mu\nu}\nabla_\alpha Z^\mu. \tag{14}$$

As a consequence of (5)

$$\dot\Gamma^\alpha_{\alpha\nu} = \tfrac12 g^{\alpha\sigma}\nabla_\nu h_{\alpha\sigma}$$

and (14) may be written as

$$\begin{aligned}\dot\rho_{\mathcal D}(g) = {} & h(Z_0\,, Z_0)\rho_{\mathcal D}(g) - \mathrm{div}(X) + Z^\nu(\nabla_\nu h_{\mu\sigma} + \nabla_\mu h_{\nu\sigma} - \nabla_\sigma h_{\mu\nu})\nabla^\sigma Z^\mu \\ & -\tfrac12 (Z^\nu \nabla_\mu Z^\mu + Z^\mu \nabla_\mu Z^\nu)\nabla_\nu h^\sigma_\sigma,\end{aligned} \tag{15}$$

where $\nabla^\sigma = g^{\alpha\sigma}\nabla_\alpha$ and $h^\alpha_\sigma = g^{\alpha\beta}h_{\beta\sigma}$. To represent (15) as an inner product with h, one needs to integrate several times by parts. Let us consider the $(1, 1)$-tensor field $A^\alpha_\beta = \nabla_\beta Z^\alpha$. Its covariant derivative is

$$\nabla_\alpha A^\gamma_\beta = A^\gamma_{\beta|\alpha} + A^\rho_\beta \Gamma^\gamma_{\alpha\rho} - \Gamma^\rho_{\alpha\beta}A^\gamma_\rho.$$

Then

$$Z^\nu(\nabla_\mu Z^\mu)(\nabla_\nu h_{\alpha\sigma}) = \nabla_\nu B^\nu_{\alpha\sigma} - \left[\left(A^\mu_\mu\right)^2 + Z^\nu\, \nabla_\nu \nabla_\mu Z^\mu\right] h_{\alpha\sigma}\,, \tag{16}$$

$$Z^\mu(\nabla_\mu Z^\nu)(\nabla_\nu h_{\alpha\sigma}) = \nabla_\nu C^\nu_{\alpha\sigma} - \left[A^\nu_\mu A^\mu_\nu + Z^\mu\, \nabla_\nu \nabla_\mu Z^\nu\right] h_{\alpha\sigma}\,, \tag{17}$$

$$Z^\nu(\nabla_\alpha Z^\mu)(\nabla_\nu h_{\mu\sigma}) = \nabla_\nu D^\nu_{\alpha\sigma} - \left[A^\nu_\nu A^\mu_\alpha + Z^\nu\, \nabla_\nu \nabla_\alpha Z^\mu\right] h_{\mu\sigma}\,, \tag{18}$$

$$Z^\nu(\nabla_\alpha Z^\mu)(\nabla_\mu h_{\nu\sigma}) = \nabla_\mu E^\mu_{\alpha\sigma} - \left[A^\nu_\mu A^\mu_\alpha + Z^\nu\, \nabla_\mu \nabla_\alpha Z^\mu\right] h_{\nu\sigma}\,, \tag{19}$$

$$Z^\nu(\nabla_\alpha Z^\mu)(\nabla_\sigma h_{\mu\nu}) = \nabla_\sigma F_\alpha - \left[A^\nu_\sigma A^\mu_\alpha + Z^\nu\, \nabla_\sigma \nabla^\mu_\alpha\right] h_{\mu\nu}\,, \tag{20}$$

where

$$B^\nu_{\alpha\sigma} = Z^\nu A^\mu_\mu h_{\alpha\sigma},\quad C^\nu_{\alpha\sigma} = Z^\mu A^\nu_\mu h_{\alpha\sigma},\quad D^\nu_{\alpha\sigma} = Z^\nu A^\mu_\alpha h_{\mu\sigma},$$

$$E^\mu_{\alpha\sigma} = Z^\nu A^\mu_\alpha h_{\nu\sigma},\quad F_\alpha = Z^\nu A^\mu_\alpha h_{\mu\nu}.$$

Let us set $Y^\nu = g^{\alpha\sigma}\left[\tfrac12\left(B^\nu_{\alpha\sigma} + C^\nu_{\alpha\sigma}\right) - D^\nu_{\alpha\sigma} - E^\nu_{\alpha\sigma}\right] + g^{\alpha\nu}F_\alpha$ and substitute from (16)–(20) into (15). One obtains

$$\begin{aligned}\dot\rho_{\mathcal D}(g) = {} & h(Z_0\,,\, Z_0)\,\rho_{\mathcal D}(g) - \mathrm{div}(X+Y) \\ & + \tfrac12 h^\alpha_\alpha\left[\left(\mathrm{trace}\, A\right)^2 + Z_0(\mathrm{trace}\, A) + \mathrm{trace}\,(A^2) + Z^\mu \nabla_\nu A^\nu_\mu\right] \\ & - h^\alpha_\mu\left[(\mathrm{trace}\, A)A^\mu_\alpha + Z^\nu\nabla_\nu A^\mu_\alpha + A^\mu_\nu A^\nu_\alpha + Z^\mu\nabla_\nu A^\nu_\alpha\right] + g^{\alpha\sigma}\left[A^\nu_\sigma A^\mu_\alpha + Z^\nu\nabla_\sigma A^\mu_\alpha\right]h_{\mu\nu}.\end{aligned} \tag{21}$$

Let $T_{\mu\nu} = -2\,\partial\mathcal L/\partial g^{\mu\nu} + g_{\mu\nu}\,\mathcal L$ be the stress–energy tensor so that[5]

$$g_{\alpha\mu}g_{\beta\nu}\frac{\partial\mathcal L}{\partial g_{\alpha\beta}} + \frac12 g_{\mu\nu}\mathcal L = \frac12 T_{\mu\nu}.$$

The pointwise inner product of the $(0, 2)$-tensor fields B and C is the function $\langle B, C\rangle : M \to \mathbb R$ locally given by $\langle B, C\rangle = g^{\alpha\lambda}g^{\beta\mu}B_{\alpha\beta}C_{\lambda\mu}$. Then

$$h(Z_0, Z_0) = \langle \omega\otimes\omega, h\rangle,\quad h^\alpha_\alpha = \langle g, h\rangle,$$

[5] The sign 'discrepancy' is due to $\partial\mathcal L/\partial g_{\alpha\beta} = -g_{\alpha\mu}g_{\beta\nu}\,\partial\mathcal L/\partial g^{\mu\nu}$ (a consequence of $g_{\mu\nu}g^{\nu\alpha} = \delta^\alpha_\mu$).

$$\left(\frac{\partial\mathcal{L}}{\partial g_{\mu\nu}} + \frac{1}{2}g^{\mu\nu}\mathcal{L}\right)h_{\mu\nu} = \frac{1}{2}\langle T, h\rangle,$$

$$h^{\alpha}_{\mu}\left[(\text{trace } A)A^{\mu}_{\alpha} + Z^{\nu}\nabla_{\nu}A^{\mu}_{\alpha} + A^{\mu}_{\nu}A^{\nu}_{\alpha} + Z^{\mu}\nabla_{\nu}A^{\nu}_{\alpha}\right] = \big\langle\big[(\text{trace } A)\nabla_{\mu}\omega_{\nu} + Z^{\sigma}\nabla_{\sigma}\nabla_{\mu}\omega_{\nu} + A^{\sigma}_{\mu}\nabla_{\sigma}\omega_{\nu} + \omega_{\nu}\nabla_{\sigma}A^{\sigma}_{\mu}\big]\mathrm{d}x^{\mu}\otimes\mathrm{d}x^{\nu}, h\big\rangle,$$

$$g^{\alpha\sigma}\left[A^{\nu}_{\sigma}A^{\mu}_{\alpha} + Z^{\nu}\nabla_{\sigma}A^{\mu}_{\alpha}\right]h_{\mu\nu} = \langle(\nabla_{\sigma}\omega_{\mu}\cdot\nabla^{\sigma}\omega_{\nu} + \omega_{\mu}\nabla_{\sigma}\nabla^{\sigma}\omega_{\nu})\mathrm{d}x^{\mu}\otimes\mathrm{d}x^{\nu}, h\rangle,$$

where $\omega_{\alpha} = g_{\alpha\beta}Z^{\beta}$. We established

Theorem 1. *Let $g \in \mathcal{M}(\mathcal{D}, Z)$ be a critical point of S, i.e. $\{\mathrm{d}S\,[g_t]\,/\mathrm{d}t\}_{t=0} = 0$ for any smooth one-parameter variation of g supported in Ω. Then g is a solution to the Euler–Lagrange equations*

$$\mathrm{Ric}_{\mathcal{D}}(g)_{\mu\nu} - \tfrac{1}{2}\,\mathrm{Scal}_{\mathcal{D}}(g)\,g_{\mu\nu} - \rho_{\mathcal{D}}(g)\left(\omega_{\mu}\omega_{\nu} + \tfrac{1}{2}g_{\mu\nu}\right) + \Lambda g_{\mu\nu} = \mathfrak{a}T_{\mu\nu}, \qquad (22)$$

where $\mathrm{Ric}_{\mathcal{D}}(g)$ and $\mathrm{Scal}_{\mathcal{D}}(g)$ are defined by

$$\mathrm{Ric}_{\mathcal{D}}(g)_{\mu\nu} = (\text{trace } A)\nabla_{\mu}\omega_{\nu} + Z^{\sigma}\,\nabla_{\sigma}\nabla_{\mu}\omega_{\nu} + A^{\sigma}_{\mu}\nabla_{\sigma}\omega_{\nu} + \omega_{\nu}\nabla_{\sigma}A^{\sigma}_{\mu} - (\nabla_{\sigma}\omega_{\mu})(\nabla^{\sigma}\omega_{\nu}) - \omega_{\mu}\nabla_{\sigma}\nabla^{\sigma}\omega_{\nu}, \qquad (23)$$

and

$$\mathrm{Scal}_{\mathcal{D}}(g) = Z^{\mu}\nabla_{\nu}A^{\nu}_{\mu} + (\text{trace } A)^2 + \text{trace } (A^2) + Z_0(\text{trace } A).$$

Definition 3. *The* $(0, 2)$*-tensor field* $\mathrm{Ric}_{\mathcal{D}}(g)$ *is referred to as the mixed Ricci curvature of* $g \in \mathcal{M}(\mathcal{D}, Z)$. *Equations (22) are the mixed gravitational field equations.*

Let $\Box$ be the wave operator of (M, g), i.e.

$$\Box f = -\nabla^{\sigma}\nabla_{\sigma}f = -\frac{1}{\sqrt{-\mathfrak{g}}}(\sqrt{-\mathfrak{g}}g^{\mu\nu}f_{|\nu})_{|\mu}$$

for any $f \in C^2(M)$. To see how $\mathrm{Ric}_{\mathcal{D}}(g)$ and $\mathrm{Scal}_{\mathcal{D}}(g)$ relate to each other, one computes the contraction

$$g^{\mu\nu}\mathrm{Ric}_{\mathcal{D}}(g)_{\mu\nu} = -(\nabla_{\sigma}Z^{\rho})(\nabla^{\sigma}\omega_{\rho}) - Z^{\rho}\nabla_{\sigma}\nabla^{\sigma}\omega_{\rho} + (\text{trace } A)\nabla^{\rho}\omega_{\rho} + Z^{\nu}\nabla_{\nu}\nabla^{\rho}\omega_{\rho} + \omega_{\rho}\nabla_{\nu}\nabla^{\rho}Z^{\nu} + (\nabla^{\rho}Z^{\nu})(\nabla_{\nu}\omega_{\rho}). \qquad (24)$$

Also,

$$\begin{aligned}(\nabla_{\sigma}Z^{\rho})(\nabla^{\sigma}\omega_{\rho}) + Z^{\rho}\nabla_{\sigma}\nabla^{\sigma}\omega_{\rho} &= \nabla_{\sigma}\left(Z^{\rho}\nabla^{\sigma}\omega_{\rho}\right)\\ &= \nabla_{\sigma}\{\nabla^{\sigma}(Z^{\rho}\omega_{\rho}) - \omega_{\rho}\nabla^{\sigma}Z^{\rho}\}\\ &= -\Box(Z^{\rho}\omega_{\rho}) - \nabla_{\sigma}(\omega_{\rho}\nabla^{\sigma}Z^{\rho});\end{aligned}$$

hence $2\nabla_{\sigma}\left(Z^{\rho}\nabla^{\sigma}\omega_{\rho}\right) = 0$ (because of $Z^{\rho}\omega_{\rho} = -1$) and (24) simplifies accordingly

$$g^{\mu\nu}\mathrm{Ric}_{\mathcal{D}}(g)_{\mu\nu} = \omega_{\rho}\nabla_{\nu}\nabla^{\rho}Z^{\nu} + (\text{trace } A)^2 + \text{trace } (A^2) + Z_0(\text{trace } A),$$

so that

$$g^{\mu\nu}\mathrm{Ric}_{\mathcal{D}}(g)_{\mu\nu} = \mathrm{Scal}_{\mathcal{D}}(g). \qquad (25)$$

Let now g be a solution to the field equations (22) in the absence of matter

$$\mathrm{Ric}_{\mathcal{D}}(g)_{\mu\nu} = (1/2)\,\mathrm{Scal}_{\mathcal{D}}(g)\,g_{\mu\nu} + \rho_{\mathcal{D}}(g)\,(\omega_{\mu}\omega_{\nu} + (1/2)\,g_{\mu\nu}). \qquad (26)$$

Contraction of the right-hand side of (26) with $g^{\mu\nu}$ is $2\,\mathrm{Scal}_{\mathcal{D}}(g) + \rho_{\mathcal{D}}(g)$; hence (by (26) and (25)) $\mathrm{Scal}_{\mathcal{D}}(g) = -\rho_{\mathcal{D}}(g)$.

Corollary 1. *Gravitational field equations in free space are*

$$\mathrm{Ric}_{\mathcal{D}}(g)_{\mu\nu} - \rho_{\mathcal{D}}(g)\,\omega_\mu\omega_\nu = 0, \tag{27}$$

where $\omega_\mu = [-g(Z,Z)]^{\frac{1}{2}}\; g_{\mu\nu} Z^\nu$.

The purpose of the following section is to exhibit an exact solution to mixed field equations (27) in free space.

3. Geometry of Bernal–Sánchez foliation

Let M be a spacetime, with a Bernal–Sánchez foliation as in section 1. As $\mathcal{T}$ is a time function (in particular, $\mathcal{T}$ has no critical points, i.e. $\mathrm{Crit}(\mathcal{T}) = \emptyset$), one may set $\lambda = [-g(\nabla\mathcal{T}\,,\ \nabla\mathcal{T})]^{1/2}$ so that $Z = \lambda^{-1}\nabla\mathcal{T}$ is a unit (i.e. $g(Z,Z) = -1$) normal field. We recall (cf [3]) that $\overline{g}$ is a lightlike metric on M whose radical

$$\mathrm{Rad}\,(T(M),\overline{g}) = \{X \in T(M) : \overline{g}(X,Y) = 0, \quad Y \in T(M)\}$$

is the span of $\nabla\mathcal{T}$. Hence

$$\begin{aligned} g(\nabla\mathcal{T},\nabla\mathcal{T}) &= -\,\beta(\mathrm{d}\mathcal{T}\otimes\mathrm{d}\mathcal{T})(\nabla\mathcal{T},\nabla\mathcal{T}) \\ &= -\,\beta(\nabla\mathcal{T})(\mathcal{T})^2 = -\beta g(\nabla\mathcal{T},\nabla\mathcal{T})^2 \end{aligned}$$

so that $\lambda = 1/\sqrt{\beta}$. The second fundamental form of $\mathcal{F}$ in (M,g) is

$$\alpha(X,Y) = \pi\nabla_X Y, \quad X,Y \in \mathfrak{X}_{\mathcal{F}},$$

where $\pi : T(M) \to Q = T(M)/T(\mathcal{F})$ is the projection. Also $\mathfrak{X}_{\mathcal{F}} = C^\infty(T(\mathcal{F}))$ (the Lie algebra of vector fields tangent to $\mathcal{F}$). Let $\Omega^p_B(\mathcal{F})$ be the space of all basic p-forms ($p \in \{0,1\}$) on $(M,\mathcal{F})$. In particular, $\Omega^0_B(\mathcal{F})$ is the space of all basic functions, i.e. each $u \in \Omega^0_B(\mathcal{F})$ is a real-valued C^∞ function on M which is constant along the leaves of $\mathcal{F}$. As $T(\mathcal{F})$ is nondegenerate,

$$\sigma : Q \to T(\mathcal{F})^\perp, \quad \sigma(s) = Y^\perp, \quad Y \in T(M), \quad \pi(Y) = s,$$

is a vector bundle isomorphism. Let g_Q be the bundle metric induced by g on Q, i.e.

$$g_Q(r,s) = g(\sigma(r),\sigma(s)), \quad r,s \in Q.$$

The Weingarten map $W(Z) : T(\mathcal{F}) \to T(\mathcal{F})$ is given by

$$g(W(Z)X,Y) = g_Q(\alpha(X,Y),\pi Z), \quad X,Y \in \mathfrak{X}_{\mathcal{F}}.$$

Let $\mathrm{Hess}_{\mathcal{T}} = \nabla\mathrm{d}\mathcal{T}$ be the Hessian of the time function $\mathcal{T}$. One has

$$\begin{aligned} \mathrm{Hess}_{\mathcal{T}}(X,Y) &= g(\nabla_X\nabla\mathcal{T},Y) = g(X(\lambda)Z + \lambda\nabla_X Z, Y) \\ &= \lambda g(\nabla_X Z,Y) = -\lambda g(Z,\nabla_X Y) = -\lambda g(Z,(\nabla_X Y)^\perp) \\ &= -\,\lambda g_Q(\pi Z,\pi\nabla_X Y) = -\lambda g_Q(\pi Z,\alpha(X,Y)); \end{aligned}$$

hence

$$\mathrm{Hess}_{\mathcal{T}}(X,Y) = -\lambda g(W(Z)X,Y). \tag{28}$$

Then

Lemma 1. *The second fundamental form of the Bernal–Sánchez foliation is given by*

$$\alpha(X,Y) = \frac{1}{\lambda}\,\mathrm{Hess}_{\mathcal{T}}(X,Y)\,\pi Z \tag{29}$$

for any $X,Y \in \mathfrak{X}_{\mathcal{F}}$.

Taking traces in (28), the mean curvature form $\mathbf{k} \in \Omega^1(M)$ is (locally)

$$\mathbf{k}(Z) = \operatorname{trace} W(Z) = \sum_{i=1}^{3} g(W(Z)E_i, E_i)$$

or

$$\mathbf{k}(Z) = -\sqrt{\beta} \sum_{i=1}^{3} \mathrm{Hess}_{\mathcal{T}}(E_i\,,\, E_i) \tag{30}$$

for any local g-orthonormal frame $\{E_i : 1 \leqslant i \leqslant 3\}$ of $T(\mathcal{F})$, and of course $\mathbf{k}(X) = 0$ for any $X \in \mathfrak{X}_{\mathcal{F}}$. Then (by taking $E_0 = Z$)

$$\Box \mathcal{T} = -\sum_{\alpha=0}^{3} \epsilon_\alpha \left\{E_\alpha^2(\mathcal{T}) - (\nabla_{E_\alpha} E_\alpha)\mathcal{T}\right\} = -\sum_{\alpha} \epsilon_\alpha \mathrm{Hess}_{\mathcal{T}}(E_\alpha, E_\alpha);$$

hence (by (30))

Lemma 2. *The mean curvature form of* $(M, \mathcal{F})$ *is given by*

$$\mathbf{k}(Z) = \sqrt{\beta}\,\{\Box\,\mathcal{T} - \mathrm{Hess}_{\mathcal{T}}(Z, Z)\}\,. \tag{31}$$

Let $\overset{\circ}{\nabla}$ be the Bott connection of $\mathcal{F}$ (cf e.g. (5.1) in [31], p 46). The inner product g_x on $T(\mathcal{F})_x^{\perp}$ is a negative definite for any $x \in M$; hence the bundle metric $-g_Q$ is Riemannian. The Bernal–Sánchez foliation is (transversally) Riemannian if g_Q is holonomy invariant, i.e. $\overset{\circ}{\nabla}_X g_Q = 0$ for any $X \in \mathfrak{X}_{\mathcal{F}}$. If this is the case, then the Lorentzian metric g is *bundle-like*. Let us set $\tau^{\perp} = \nabla_Z Z \in \mathfrak{X}_{\mathcal{F}}$. By slightly generalizing theorem 7.3 in [31], p 75 (to the case of a Lorentzian ambient space), one has the following.

Lemma 3. $(\mathcal{F}, -g_Q)$ *is Riemannian if and only if* $\tau^{\perp} = 0$.

For any $X \in \mathfrak{X}_{\mathcal{F}}$,

$$\mathrm{Hess}_{\mathcal{T}}(X, \nabla\mathcal{T}) = (\nabla_X \nabla\mathcal{T})(\mathcal{T}) = g(\nabla\mathcal{T}, \nabla_X \nabla\mathcal{T})$$

or

$$\mathrm{Hess}_{\mathcal{T}}(X, \nabla\mathcal{T}) = \tfrac{1}{2} X\,(g(\nabla\mathcal{T}, \nabla\mathcal{T}))\,. \tag{32}$$

Yet

$$\tau^{\perp} = \nabla_Z(\lambda^{-1}\nabla\mathcal{T}) = Z(\lambda^{-1})\nabla\mathcal{T} + \lambda^{-1}\nabla_Z\nabla\mathcal{T};$$

hence (by (32))

$$\begin{aligned} g(\tau^{\perp}, X) &= \lambda^{-1} g(\nabla_Z \nabla\mathcal{T}, X) \\ &= \lambda^{-2}\mathrm{Hess}_{\mathcal{T}}(\nabla\mathcal{T}, X) = -\frac{X(\lambda^2)}{2\lambda^2} = \frac{X(\beta)}{2\beta}. \end{aligned}$$

We may conclude that (by (31))

Theorem 2. $(\mathcal{F}, -g_Q)$ *is Riemannian if and only if* $\beta \in \Omega_B^0(\mathcal{F})$, *i.e.* β *is a basic function. Also,* $\mathcal{F}$ *is harmonic (i.e. each leaf of* $\mathcal{F}$ *is minimal in* (M, g)*) if and only if*

$$(1/\beta)\,\Box\,\mathcal{T} = \mathrm{Hess}_{\mathcal{T}}\,(\nabla\mathcal{T}\,,\,\nabla\mathcal{T}). \tag{33}$$

As $Z = \lambda^{-1}\nabla\mathcal{T}$, one has $\omega = \lambda^{-1}d\mathcal{T}$. Let $\pi^{\perp} : T(M) \to T(\mathcal{F})$ be the natural projection. If (x^{α}) is a local coordinate system on M, then

$$\partial_{\alpha} = \pi^{\perp}\partial_{\alpha} - \lambda^{-1}\mathcal{T}_{|\alpha} Z. \tag{34}$$

Also,

$$\nabla_{\alpha}\omega_{\mu} = \omega_{\mu|\alpha} - \Gamma^{\sigma}_{\alpha\mu}\omega_{\sigma} = (\lambda^{-1})_{|\alpha}\mathcal{T}_{|\mu} + \lambda^{-1}\mathcal{T}_{|\mu|\alpha} - \lambda^{-1}\Gamma^{\sigma}_{\alpha\mu}\mathcal{T}_{|\sigma},$$

i.e.

$$\nabla_{\alpha}\omega_{\mu} = (\sqrt{\beta})_{|\alpha}\mathcal{T}_{|\mu} + \sqrt{\beta}\,\mathrm{Hess}_{\mathcal{T}}(\partial_{\alpha}\,,\,\partial_{\mu}). \tag{35}$$

The Hessian term in (35) may be evaluated by using the decomposition (34). Note first that for any vector field $Y \in \mathfrak{X}(M)$ (by $\nabla g = 0$),

$$\mathrm{Hess}_{\mathcal{T}}(Y, \nabla\mathcal{T}) = \tfrac{1}{2}Y(g(\nabla\mathcal{T}, \nabla\mathcal{T}));$$

hence (by $g(\nabla\mathcal{T}, \nabla\mathcal{T}) = -1/\beta$),

$$\mathrm{Hess}_{\mathcal{T}}(Y, \nabla\mathcal{T}) = -\tfrac{1}{2}Y\left(\lambda^{2}\right). \tag{36}$$

Next (by (34) and (36) with $Y \in \{\partial_{\alpha}\,,\,\partial_{\mu}\}$),

$$\mathrm{Hess}_{\mathcal{T}}(\partial_{\alpha}\,,\,\partial_{\mu}) = \mathrm{Hess}_{\mathcal{T}}(\pi^{\perp}\partial_{\alpha}\,,\,\pi^{\perp}\partial_{\mu}) + \frac{1}{2\lambda^{2}}\{(\pi^{\perp}\partial_{\alpha})(\lambda^{2})\mathcal{T}_{|\mu} + (\pi^{\perp}\partial_{\mu})\mathcal{T}_{|\alpha}\}$$
$$+\frac{1}{\lambda^{4}}\mathrm{Hess}_{\mathcal{T}}(\nabla\mathcal{T}, \nabla\mathcal{T})\mathcal{T}_{|\alpha}\mathcal{T}_{|\mu}. \tag{37}$$

Using the following identity (together with (36) for $Y = \nabla\mathcal{T}$ and (28))

$$\frac{1}{2\lambda^{2}}(\pi^{\perp}\partial_{\alpha})(\lambda^{2}) = -\frac{1}{2\beta}\{\beta_{|\alpha} + \beta\mathcal{T}_{|\alpha}(\nabla\mathcal{T})(\beta)\},$$

one may rewrite (37) as

$$\mathrm{Hess}_{\mathcal{T}}(\partial_{\alpha}\,,\,\partial_{\mu}) = -\lambda\, g(W(Z)\pi^{\perp}\partial_{\alpha}\,,\,\pi^{\perp}\partial_{\mu})$$
$$-\tfrac{1}{2}\{(\log\beta)_{|\alpha}\mathcal{T}_{|\mu} + (\log\beta)_{|\mu}\mathcal{T}_{|\alpha} + (\nabla\mathcal{T})(\beta)\mathcal{T}_{|\alpha}\mathcal{T}_{|\mu}\}. \tag{38}$$

Finally, let us substitute (38) into (35) (and observe simplification of terms based on $(\lambda^{-1})_{|\alpha} = \frac{1}{2}\lambda^{-1}(\log\beta)_{|\alpha}$). This gives

$$\nabla_{\alpha}\omega_{\mu} = -g(W(Z)\pi^{\perp}\partial_{\alpha}\,,\,\pi^{\perp}\partial_{\mu}) - \frac{1}{2\lambda}\{(\log\beta)_{|\mu} + (\nabla\mathcal{T})(\beta)\mathcal{T}_{|\mu}\}\mathcal{T}_{|\alpha}. \tag{39}$$

Let $\{E_i : 1 \leqslant i \leqslant 3\}$ be a local g-orthonormal frame of $\mathcal{D} = T(\mathcal{F})$. Then (for some C^{∞} functions E_i^{α})

$$E_i = E_i^{\alpha}\partial_{\alpha} = E_i^{\alpha}\{\pi^{\perp}\partial_{\alpha} - \lambda^{-1}\mathcal{T}_{|\alpha}Z\} = E_i^{\alpha}\pi^{\perp}\partial_{\alpha}$$

by (34) and $E_i(\mathcal{T}) = 0$. Let us complete $\{E_i : 1 \leqslant i \leqslant 3\}$ to a local g-orthonormal frame adapted to $\mathcal{D}$ by setting $E_0 = Z$. Consequently (as $\epsilon_0 = -1$),

$$\mathbf{k}(Z) = \mathrm{trace}\, W(Z) = \sum_{i=1}^{3} g(W(Z)E_i, E_i)$$
$$= \sum_i E_i^{\alpha}E_i^{\mu} g(W(Z)\pi^{\perp}\partial_{\alpha}, \pi^{\perp}\partial_{\mu})$$
$$= \left(g^{\alpha\mu} + E_0^{\alpha}E_0^{\mu}\right)g(W(Z)\pi^{\perp}\partial_{\alpha}, \pi^{\perp}\partial_{\mu})$$

or (as $\pi^{\perp}E_0 = 0$)

$$\mathbf{k}(Z) = g^{\alpha\mu}g(W(Z)\pi^{\perp}\partial_{\alpha}\,,\,\pi^{\perp}\partial_{\mu}). \tag{40}$$

Moreover (by (39)–(40))

$$\begin{aligned}\text{trace }(A) &= A^\alpha_\alpha = \nabla_\alpha Z^\alpha = g^{\alpha\mu}\nabla_\alpha \omega_\mu \\ &= -\mathbf{k}(Z) - \frac{1}{2\lambda} g^{\alpha\mu} \mathcal{T}_{|\alpha}\{(\log\beta)_{|\mu} + (\nabla\mathcal{T})(\beta)\mathcal{T}_{|\mu}\} \\ &= -\mathbf{k}(Z) - \frac{1}{2\lambda}\{g(\nabla\mathcal{T}, \nabla\log\beta) + g(\nabla\mathcal{T}, \nabla\mathcal{T})(\nabla\mathcal{T})(\beta)\}.\end{aligned}$$

The geometric meaning of A^α_μ is clarified by the following lemma.

Lemma 4. *Let A be the $(1,1)$-tensor field on M locally given by $A\partial_\alpha = A^\nu_\alpha \partial_\nu$. Then (i) $AX = -W(Z)X$ for any $X \in \mathfrak{X}_{\mathcal{F}}$. Also, (ii)*

$$AZ = \tfrac{1}{2}\{\nabla\log\beta + (\nabla\mathcal{T})(\beta)\,\nabla\mathcal{T}\} = -\lambda^{-1}\{\nabla\lambda + \lambda^{-1}Z(\lambda)\,\nabla\mathcal{T}\}. \tag{41}$$

Consequently, (iii)

$$\text{trace }(A) = -\mathbf{k}(Z), \tag{42}$$

and $AZ \in \mathfrak{X}_{\mathcal{F}}$. (iv) If $\lambda \in \Omega^0_B(\mathcal{F})$, then $AZ = 0$.

Note that (39) may be also written as

$$\nabla_\alpha\omega_\mu = -g(W(Z)\pi^\perp\partial_\alpha\,,\ \pi^\perp\partial_\mu) + \beta\,(\pi^\perp\partial_\mu)(\lambda)\,\mathcal{T}_{|\alpha}\,. \tag{43}$$

Consequently,

$$A\pi^\perp\partial_\alpha = -g^{\gamma\mu}g(W(Z)\pi^\perp\partial_\alpha, \pi^\perp\partial_\mu)\partial_\gamma + \beta g^{\gamma\mu}(\pi^\perp\partial_\mu)(\lambda)\mathcal{T}_{|\alpha}\partial_\gamma + \lambda^{-1}\mathcal{T}_{|\alpha}AZ$$

which together with

$$g(\pi^\perp\partial_\alpha, \pi^\perp\partial_\mu) = g_{\alpha\mu} + \lambda^{-2}\mathcal{T}_{|\alpha}\mathcal{T}_{|\mu}$$

yields (i). We have (by (39) and $\pi^\perp Z = 0$ and $Z(\mathcal{T}) = -\lambda$)

$$\begin{aligned}AZ = Z^\alpha g^{\gamma\mu}(\nabla_\alpha\omega_\mu)\partial_\gamma &= -g^{\gamma\mu}\Big\{g(W(Z)\pi^\perp Z, \pi^\perp\partial_\mu) \\ &\quad + \frac{1}{2\lambda}[(\log\beta)_{|\mu} + (\nabla\mathcal{T})(\beta)\mathcal{T}_{|\mu}]Z(\mathcal{T})\Big\}\partial_\gamma \\ &= \frac{1}{2}\{\nabla\log\beta + (\nabla\mathcal{T})(\beta)\nabla\mathcal{T}\}\end{aligned}$$

and (41) is proved. Next (by (41)) $g(AZ, \nabla\mathcal{T}) = 0$ (hence $AZ \in \mathfrak{X}_{\mathcal{F}}$). Statement (iv) in lemma 4 follows from (41) and the identity

$$g^{\alpha\mu}(\pi^\perp\partial_\mu)(\lambda)\partial_\alpha = \nabla\lambda + \lambda^{-1}Z(\lambda)\nabla\mathcal{T}.$$

Definition 4. *The time function $\mathcal{T} : M \to \mathbb{R}$ is isoperimetric if*

$$X(g(\nabla\mathcal{T}, \nabla\mathcal{T})) = 0, \quad X(\Box\mathcal{T}) = 0,$$

for any $X \in \mathfrak{X}_{\mathcal{F}}$.

The adopted terminology mimics the Riemannian case (cf e.g. [31], p 116, when the concept is due to Cartan) and no immediate geometric interpretation is claimed at this point (except that λ and $\Box\mathcal{T}$ are basic functions). It is also unknown whether Münzner's structure theorems (cf [19, 20]) carry over to the case of a spacetime endowed with the Bernal–Sánchez foliation. By (31)

$$\mathbf{k}(Z) = \sqrt{\beta}\,\left\{\Box\mathcal{T} + \frac{1}{\lambda}\,g(\nabla\mathcal{T}\,,\ \nabla\lambda)\right\}. \tag{44}$$

Indeed,

$$\begin{aligned}\mathrm{Hess}_{\mathcal{T}}(Z,Z) &= g(\nabla_Z \nabla \mathcal{T}, Z) = \lambda^{-2} g(\nabla_{\nabla \mathcal{T}} \nabla \mathcal{T}, \nabla \mathcal{T}) \\ &= \frac{1}{2\lambda^2}(\nabla \mathcal{T})(g(\nabla \mathcal{T}, \nabla \mathcal{T})) = -\frac{1}{2\lambda^2}(\nabla \mathcal{T})(\lambda^2) \\ &= -\frac{1}{\lambda}(\nabla \mathcal{T})(\lambda) = -\frac{1}{\lambda} g(\nabla \mathcal{T}, \nabla \lambda)\end{aligned}$$

(and one may substitute into (6)).

Theorem 3. *If the Bernal–Sánchez foliation* $(\mathcal{F}, -g_Q)$ *is Riemannian and the time function* $\mathcal{T}$ *is isoperimetric, then each leaf of* $\mathcal{F}$ *is a constant mean curvature hypersurface in* (M, g).

The proof is an adaptation (to spacetimes carrying Bernal–Sánchez foliations) of the arguments in [31, pp 115, 116]. To start with, we establish

Lemma 5. *For every* $X \in \mathfrak{X}_{\mathcal{F}}$

$$g([X, Z], Z) = g(X, \tau^{\perp}). \tag{45}$$

In particular, if $(\mathcal{F}, -g_Q)$ *is Riemannian, then* $[X, Z] \in \mathfrak{X}_{\mathcal{F}}$ *for any* $X \in \mathfrak{X}_{\mathcal{F}}$.

Indeed (as ∇ is torsion-free and $\nabla g = 0$)

$$\begin{aligned}g([X, Z], Z) &= g(\nabla_X Z, Z) - g(\nabla_Z X, Z) \\ &= \tfrac{1}{2} X(g(Z, Z)) - Z(g(X, Z)) + g(X, \nabla_Z Z)\end{aligned}$$

(implying (45)). The second statement follows from lemma 3. The assumption that $\mathcal{T}$ is isoperimetric yields (by $X(\lambda) = 0$)

$$[X, \nabla \mathcal{T}] \in \mathfrak{X}_{\mathcal{F}}, \quad X \in \mathfrak{X}_{\mathcal{F}}. \tag{46}$$

Then (under the assumptions of theorem 3),

$$\begin{aligned}X(g(\nabla \mathcal{T}, \nabla \lambda)) &= \frac{1}{2\lambda} X(g(\nabla \mathcal{T}, \nabla \lambda^2)) = \frac{1}{2\lambda} X((\nabla \mathcal{T})(\lambda^2)) \\ &= \frac{1}{2\lambda}\{[X, \nabla \mathcal{T}](\lambda^2) + (\nabla \mathcal{T})(X\lambda^2)\},\end{aligned}$$

i.e. (by (46))

$$X(g(\nabla \mathcal{T}, \nabla \lambda)) = 0, \quad X \in \mathfrak{X}_{\mathcal{F}}. \tag{47}$$

Finally, we may apply X to (44) to conclude that (by (47))

$$X(\mathbf{k}(Z)) = \sqrt{\beta} X(\Box \mathcal{T}) = 0,$$

i.e. $\mathbf{k}(Z) \in \Omega^0_B(\mathcal{F})$. On the other hand, as is well known, the pointwise restriction of $\mathbf{k}(Z)$ to a leaf $L \in M/\mathcal{F}$ is the mean curvature of L.

Through the remainder of section 3, we assume that $\mathcal{T}$ is isoperimetric and $\mathcal{F}$ is totally umbilical in (M, g), i.e. $\mathcal{F}$ is transversally Riemannian ($\tau^{\perp} = 0$) and

$$W(Z)X = rX, \quad X \in \mathfrak{X}_{\mathcal{F}}, \tag{48}$$

where we have set $r \equiv k(Z)/3$. By a result of Ponge and Reckziegel (cf [21]), $\mathcal{F}$ is already totally umbilical if for instance $M = \mathbb{R} \times \mathcal{S}$ for some leaf $\mathcal{S} = \pi_1^{-1}(t_0) \in M/\mathcal{F}$ and $\overline{g} = \pi_2^* g_{\mathcal{S}}$, where $g_{\mathcal{S}}$ is the first fundamental form of $\mathcal{S}$ and $\pi_1 : M \to \mathbb{R}$ and $\pi_2 : M \to \mathcal{S}$ are projections. One has (by (43))

$$\nabla_\alpha \omega_\mu = -r(g_{\alpha\mu} + \omega_\alpha \omega_\mu). \tag{49}$$

Consequently,

$$A^\sigma_\mu \nabla_\sigma \omega_\nu - (\nabla_\sigma \omega_\mu)(\nabla^\sigma \omega_\nu) = 0, \tag{50}$$

$$\nabla_\sigma \nabla_\mu \omega_\nu = -r_{|\sigma}(g_{\mu\nu} + \omega_\mu \omega_\nu) + r^2(\omega_\nu g_{\sigma\mu} + \omega_\mu g_{\sigma\nu} + 2\,\omega_\sigma \omega_\mu \omega_\nu), \tag{51}$$

$$Z^\sigma \nabla_\sigma \nabla_\mu \omega_\nu = -Z(r)(g_{\mu\nu} + \omega_\mu \omega_\nu), \tag{52}$$

$$\nabla_\sigma A^\sigma_\mu = -r_{|\mu} - \{Z(r) - 3r^2\}\,\omega_\mu\,, \tag{53}$$

$$\nabla_\sigma \nabla^\sigma \omega_\nu = -r_{|\nu} - \{Z(r) - 3r^2\}\,\omega_\nu \tag{54}$$

and

$$\mathrm{trace}\,(A)\,\nabla_\mu \omega_\nu = 3r^2(g_{\mu\nu} + \omega_\mu \omega_\nu). \tag{55}$$

Substitution from (50)–(55) into (23) furnishes

$$\mathrm{Ric}_{\mathcal{D}}(g)_{\mu\nu} = -\{Z(r) - 3r^2\}\,(g_{\mu\nu} + \omega_\mu \omega_\nu). \tag{56}$$

Thus in free space (by (25))

$$\rho_{\mathcal{D}}(g) = -g^{\mu\nu}\mathrm{Ric}_{\mathcal{D}}(g)_{\mu\nu} = 3\{Z(r) - 3r^2\}. \tag{57}$$

Finally (by (56)–(57)), $g = -\beta\,(d\mathcal{T})^2 + \overline{g}$ is a solution to field equations (27) if and only if $Z(r) - 3r^2 = 0$ or (by $r \in \Omega^0_B(\mathcal{F})$ and $r_{|\alpha} = -\lambda^{-1}\mathcal{T}_{|\alpha} Z(r)$)

$$r_{|\alpha} + 3\omega_\alpha r^2 = 0. \tag{58}$$

We have established

Theorem 4. *Let (M, g) be a spacetime with a Bernal–Sánchez foliation $\mathcal{F}$. Assume that $\mathcal{F}$ is totally umbilical and $\mathcal{T} : M \to \mathbb{R}$ is isoparametric. The mean curvature $k(Z) \in \Omega^0_B(\mathcal{F})$ obeys to*

$$\mathrm{d}\,k(Z) + \lambda^{-1}\,k(Z)^2\,\mathrm{d}\mathcal{T} = 0, \tag{59}$$

if and only if $-\beta\,(d\mathcal{T})^2 + \overline{g}$ is an exact solution to the field equations (27) in free space.

If $(x^1, x^2, x^3, y) \in \mathcal{F}$ is a local coordinate system adapted to Bernal–Sánchez foliation, i.e. $\mathcal{D}$ is the span of $\{\partial/\partial x^i : 1 \leqslant i \leqslant 3\}$ and y is a transverse local coordinate, then $\mathcal{T}$, λ and r are (locally) functions of y alone; hence (by (58)) $r(y) = \left(3\int \lambda^{-1}\mathcal{T}'\,\mathrm{d}y\right)^{-1}$. Whenever $\lambda \in \mathbb{R}$ (by (59)),

$$k(Z) = 1/(\lambda^{-1}\mathcal{T} + b) \tag{60}$$

for some $b \in \mathbb{R}$. If this is the case then (by (31) and $Z(\lambda) = 0$) the time function $\mathcal{T}$ is a solution to

$$\Box\mathcal{T} = \lambda^2/(\mathcal{T} + b\lambda), \tag{61}$$

if and only if $-\beta\,(d\mathcal{T})^2 + \overline{g}$ is an exact solution to field equations in free space. Simple and physically relevant splittings to which the theory developed in this paper applies (as observed by the reviewer) are standard static spacetimes (where $\mathcal{D}$ is totally geodesic) and Friedmann–Lemaitre–Robertson–Walker (FLRW) spacetimes (where $\mathcal{D}$ is totally umbilical), both captured by the GRW spacetime due to Sánchez (cf [26]) $g^f = -\pi_I^*\mathrm{d}t^2 + (f \circ \pi_I)^2\,\pi_F^* g_F$ with *base* $(I, -\mathrm{d}t^2)$ (an open interval $I \subset \mathbb{R}$), *fiber* (F, g_F) (a Riemannian manifold) and *warping* function $f = e^\theta$ ($\theta : I \to \mathbb{R}$ is a smooth function). The GRW model does not require (F, g_F) to be a space-form (and generalizes the FLRW model in this sense). It is prompted by the assumptions of homogeneity and isotropy of space, and field equations (63) below will be needed solely to control the scale factor $f(t)$. This issue will be addressed in further work.

4. Particle motion on isoparametric foliations

A celebrated result by Einstein, Infeld and Hoffmann [11] shows that geodesic motion of material particles in the gravitational field need not be postulated and is rather a consequence of the field equations. Geodesic motion actually follows from $\nabla_\sigma T^{\mu\sigma} = 0$ and the latter is implied by covariant differentiation of Einstein's equations $R^{\mu\nu} - \frac{1}{2}\rho(g)\, g^{\mu\nu} = \kappa\, T^{\mu\nu}$, as a purely geometric property of Ricci curvature $R_{\mu\nu}$.

The purpose of this section is to reset the mixed field equations (22) in the form (63) and take the covariant derivative of (63) in an attempt to discover a (mixed curvature) analogue to $\nabla_\sigma T^{\mu\sigma} = 0$. Our findings are equations (69), though limited to the case where the time function $\mathcal{T}$ is isoparametric and Bernal–Sánchez foliation is totally umbilical. Integration of (69) over a domain $\Omega \subset M$ containing a small globule of matter in uniform motion, well away from the exterior matter creating the gravitational field, followed by shrinking the globule to the limit size of a material particle, then leads to the equations of trajectories, i.e. equations (70) below.

Let us contract with $g^{\mu\nu}$ in (22) and substitute from (25), so as to derive

$$\mathrm{Scal}_{\mathcal{D}}(g) = -\rho_{\mathcal{D}}(g) + 4\Lambda - \mathfrak{a}\, T, \tag{62}$$

where $T = g^{\mu\nu}T_{\mu\nu}$ is the Laue scalar. Substituting (62) into (22) leads to field equations

$$\mathrm{Ric}_{\mathcal{D}}(g)_{\mu\nu} - \rho_{\mathcal{D}}(g)\,\omega_\mu\omega_\nu - \Lambda\, g_{\mu\nu} = \mathfrak{a}\,\{T_{\mu\nu} - (1/2)\,T\,g_{\mu\nu}\}. \tag{63}$$

Lemma 6. *Let M be a spacetime with a Bernal–Sánchez foliation. Let $\mathcal{T}$ be isoparametric and $\mathcal{F}$ be totally umbilical. Then $f = Z(r) - 3r^2$ is a basic function, i.e. $f \in \Omega^0_B(\mathcal{F})$. In particular, $(g^{\alpha\sigma} + Z^\alpha Z^\sigma)\, f_{|\sigma} = 0$.*

Indeed, (by lemma 5) $[X, Z] \in \mathfrak{X}_{\mathcal{F}}$ for every $X \in \mathfrak{X}_{\mathcal{F}}$. Next, (by theorem 3) $r \in \Omega^0_B(\mathcal{F})$; hence $X(Z(r)) = [X, Z](r) = 0$ and then $X(f) = 0$.

From now on, we work under the assumptions of lemma 6. Then (by (56) and the second equality in (57))

$$\mathrm{Ric}_{\mathcal{D}}(g)_{\mu\nu} = -f(g_{\mu\nu} + \omega_\mu\,\omega_\nu), \quad \mathrm{Scal}_{\mathcal{D}}(g) = -3f. \tag{64}$$

It should be emphasized that the first equality in (57) holds only in free space. Next (by (64))

$$\nabla_\sigma \mathrm{Ric}_{\mathcal{D}}(g)_{\mu\nu} = -(g_{\mu\nu} + \omega_\mu\,\omega_\nu)\, f_{|\sigma} - f(\omega_\nu\,\nabla_\sigma\omega_\mu + \omega_\mu\,\nabla_\sigma\omega_\nu). \tag{65}$$

Covariant derivative of (63) gives (as $\nabla_\sigma g_{\mu\nu} = 0$)

$$\nabla_\sigma \mathrm{Ric}_{\mathcal{D}}(g)_{\mu\nu} - \rho_{\mathcal{D}}(g)_{|\sigma}\omega_\mu\omega_\nu - \rho_{\mathcal{D}}(g)\{\omega_\nu\nabla_\sigma\omega_\mu + \omega_\mu\nabla_\sigma\omega_\nu\} = \mathfrak{a}\{\nabla_\sigma T_{\mu\nu} - (1/2)T_{|\sigma}g_{\mu\nu}\}$$

and substitution from (65) leads to

$$-(g_{\mu\nu} + \omega_\mu\omega_\nu)f_{|\sigma} - \rho_{\mathcal{D}}(g)_{|\sigma}\omega_\mu\omega_\nu - (f + \rho_{\mathcal{D}}(g))\{\omega_\nu\nabla_\sigma\omega_\mu + \omega_\mu\nabla_\sigma\omega_\nu\}$$
$$= \mathfrak{a}\{\nabla_\sigma T_{\mu\nu} - (1/2)T_{|\sigma}g_{\mu\nu}\}$$

or (by contraction with $g^{\alpha\mu}g^{\sigma\nu}$)

$$-(g^{\alpha\sigma} + Z^\alpha Z^\sigma)f_{|\sigma} - \rho_{\mathcal{D}}(g)_{|\sigma}Z^\alpha Z^\sigma - (f + \rho_{\mathcal{D}}(g))\{Z^\sigma\nabla_\sigma Z^\alpha + Z^\alpha\nabla_\sigma Z^\sigma\}$$
$$= \mathfrak{a}\{\nabla_\sigma T^{\alpha\sigma} - (1/2)T_{|\sigma}g^{\alpha\sigma}\}. \tag{66}$$

Using lemma 6 and the identity

$$Z^\sigma\nabla_\sigma Z^\alpha + Z^\alpha\nabla_\sigma Z^\sigma = -3rZ^\alpha$$

(itself a consequence of (49)), equation (66) becomes

$$-\rho_{\mathcal{D}}(g)_{|\sigma}\, Z^\alpha Z^\sigma + 3r(f + \rho_{\mathcal{D}}(g))Z^\alpha = \mathfrak{a}\,\{\nabla_\sigma T^{\alpha\sigma} - (1/2)\,T_{|\sigma}\, g^{\alpha\sigma}\}. \tag{67}$$

Substituting the second relation of (64) into (62) leads to

$$\rho_{\mathcal{D}}(g) = 4\Lambda - \mathfrak{a}T + 3f;$$

hence

$$\rho_{\mathcal{D}}(g)_{|\sigma} = -\mathfrak{a}T_{|\sigma} + 3f_{|\sigma}$$

and (67) becomes

$$-hZ^{\alpha} = \mathfrak{a}\,\{\nabla_{\sigma}T^{\alpha\sigma} - (1/2)\,T_{|\sigma}\,g^{\alpha\sigma}\}, \tag{68}$$

where $h = 3\,\{Z(f) - 4rf\} - 12\Lambda r - \mathfrak{a}\,\{Z(T) - 3rT\}$. Contraction with ω_{α} in (68) gives

$$h = \mathfrak{a}\{\omega_{\alpha}\nabla_{\sigma}T^{\alpha\sigma} - (1/2)Z(T)\};$$

hence, (by substitution into (68)) one derives

$$\left(\delta^{\alpha}_{\mu} + Z^{\alpha}\omega_{\mu}\right)\nabla_{\sigma}T^{\mu\sigma} - (1/2)\,\{g^{\alpha\sigma}\,T_{|\sigma} + Z^{\alpha}\,Z(T)\} = 0. \tag{69}$$

This is the analogue to the conservation law $\nabla_{\sigma}T^{\alpha\sigma} = 0$ one seeks for. It will be used to derive the equations of motion of a point particle in a gravitational field controlled by equations (63). We adopt physical arguments and relegate a more rigorous treatment (similar to that by Infeld and Plebanski, [17]) to further work.

Theorem 5. *Let M be a spacetime with a totally umbilical Bernal–Sánchez foliation $\mathcal{F}$ and an isoparametric time function $\mathcal{T}$. Then trajectories of point particles in a gravitational field obeying mixed field equations are*

$$2\,\varphi^{\alpha}_{\mu}\left(\frac{\mathrm{d}u^{\mu}}{\mathrm{d}s} + \Gamma^{\mu}_{\sigma\rho}u^{\sigma}u^{\rho}\right) = \frac{\mathrm{d}}{\mathrm{d}s}[\varphi^{\alpha 0}(u^{0})^{-1}u^{\mu}u_{\mu}] - u^{\mu}u_{\mu}[\varphi^{\alpha\sigma}{}_{|\sigma} + \varphi^{\alpha\sigma}\,(\log\sqrt{-\mathfrak{g}})_{|\sigma}], \tag{70}$$

where $\varphi^{\alpha}_{\mu} \equiv \delta^{\alpha}_{\mu} + Z^{\alpha}\omega_{\mu}$ and $u^{\mu} \equiv \mathrm{d}x^{\mu}/\mathrm{d}s$.

Let φ be the $(1, 1)$-tensor field on M locally given by $\varphi^{\alpha}_{\mu} = \delta^{\alpha}_{\mu} + Z^{\alpha}\omega_{\mu}$. Then $\varphi^{\alpha}_{\mu}Z^{\mu} = 0$ (in particular, φ has rank 3). Equations (69) become

$$\varphi^{\alpha}_{\mu}\,\nabla_{\sigma}T^{\mu\sigma} = (1/2)\,\varphi^{\alpha\sigma}T_{|\sigma}\,. \tag{71}$$

Covariant derivative may be written as

$$\nabla_{\sigma}T^{\mu\sigma} = T^{\mu\sigma}{}_{|\sigma} + \Gamma^{\mu}_{\rho\sigma}T^{\rho\sigma} + \Gamma^{\sigma}_{\sigma\rho}T^{\mu\rho};$$

hence (by substitution of contracted Christoffel symbols $\Gamma^{\sigma}_{\sigma\rho} = (\log\sqrt{-\mathfrak{g}})_{|\rho}$) equations (71) become

$$\varphi^{\alpha}_{\mu}\left(T^{\mu\sigma}{}_{|\sigma} + \frac{1}{2\mathfrak{g}}\mathfrak{g}_{|\rho}T^{\mu\rho} + \Gamma^{\mu}_{\rho\sigma}T^{\rho\sigma}\right) = (1/2)\varphi^{\alpha\sigma}T_{\sigma}.$$

Multiplication with $\sqrt{-\mathfrak{g}}$ then leads to

$$\varphi^{\alpha}_{\mu}\,\{(\sqrt{-\mathfrak{g}}\,T^{\mu\sigma})_{|\sigma} + \sqrt{-\mathfrak{g}}\,\Gamma^{\mu}_{\nu\sigma}T^{\nu\sigma}\} = (1/2)\,\sqrt{-\mathfrak{g}}\,\varphi^{\alpha\sigma}T_{|\sigma}\,. \tag{72}$$

To prove theorem 5, one considers a local coordinate system (U, x^{α}) and a domain $\Omega \subset U$ where there is but one globule of matter described by the density field ρ_0 and the velocity vector u^{α} with $u^{0} \neq 0$. One further assumes that $\rho_0 \neq 0$ in globule's (small) volume and $\rho_0 = 0$ on its surface. Also, u^{α} is assumed to be nearly constant over globule's volume. We wish to study globule's motion, as a particle with velocity $\mathrm{d}x^{\alpha}/\mathrm{d}s = u^{\alpha}$. The stress–energy tensor within the globule is assumed to be

$$T^{\mu\nu} = \rho_0\,u^{\mu}\,u^{\nu}\,. \tag{73}$$

As is well known, the meaning of (73) is that globule's matter is incoherent and subject but to gravitational interaction as yielded by field equations (63). Let Ω be the region in U comprised between the hypersurfaces $x^0 = a$ and $x^0 = b$ with $a < b$. Globule's motion generates a world-tube in Ω intersecting $x^0 = a$ and $x^0 = b$ in two spatial regions Σ_1 and Σ_2, respectively. The boundary of the world tube is $\Sigma \equiv \Sigma_1 \cup \Sigma_2 \cup \Sigma_3$ (where Σ_3 is the 'lateral surface'). Exterior matter responsible for the ambient gravitational field is assumed to be far from the tube. Let us integrate (72) over Ω, so as to obtain

$$\mathfrak{J} + \mathfrak{K} = \frac{1}{2}\int_\Omega \varphi^{\alpha\sigma}\, T_{|\sigma}\, \sqrt{-\mathfrak{g}}\, \mathrm{d}^4x, \tag{74}$$

$$\mathfrak{J} \equiv \int_\Omega \varphi^\alpha_\mu (\sqrt{-\mathfrak{g}} T^{\mu\sigma})_{|\sigma}\, \mathrm{d}^4x, \quad \mathfrak{K} \equiv \int_\Omega \varphi^\alpha_\mu \sqrt{-\mathfrak{g}} \Gamma^\mu_{\nu\sigma} T^{\nu\sigma}\, \mathrm{d}^4x.$$

To compute $\mathfrak{J}$, one integrates by parts

$$\mathfrak{J} = \int_\Omega \left\{\varphi^\alpha_\mu \sqrt{-\mathfrak{g}} T^{\mu\sigma}\right\}_{|\sigma} \mathrm{d}^4x - \mathfrak{L} = \sum_{i=1}^{3} \int_{\Sigma_i} \varphi^\alpha_\mu \sqrt{-\mathfrak{g}} T^{\mu\sigma} N_\sigma\, \mathrm{d}\Sigma - \mathfrak{L},$$

where N_σ is the outward unit normal on Σ and $\mathrm{d}\Sigma$ is the 'surface' measure. Also, $\mathfrak{L}$ is the 'volume' integral

$$\mathfrak{L} \equiv \int_\Omega (Z^\alpha \omega_\mu)_{|\sigma} \sqrt{-\mathfrak{g}} T^{\mu\sigma}\, \mathrm{d}^4x.$$

Yet $\rho_0 = 0$ on Σ_3 and the unit normals on Σ_1 and Σ_2 are respectively $(-1, 0, 0, 0)$ and $(1, 0, 0, 0)$; hence

$$\mathfrak{J} = \sum_{i=1}^{2} (-1)^i \int_{\Sigma_i} \rho_0 u^\mu \varphi^\alpha_\mu\, \sqrt{-\mathfrak{g}}\, u^0\, \mathrm{d}^3x - \mathfrak{L}. \tag{75}$$

The special relativistic interpretation of $\sqrt{-\mathfrak{g}}\, \mathrm{d}^4x$ is $\sqrt{-\mathfrak{g}}\, \mathrm{d}^4x = c\, \mathrm{d}\tau\, \mathrm{d}V$, where $\mathrm{d}\tau$ and $\mathrm{d}V$ are the proper time interval and the proper volume element. On the other hand, on an arbitrary Lorentzian manifold M, we may consider a coordinate system which is locally Minkowskian at a given point and such that the line element $\mathrm{d}s$ and proper time $\mathrm{d}\tau$ are related by $\mathrm{d}s = c\, \mathrm{d}\tau$. With respect to this coordinate system, $\sqrt{-\mathfrak{g}}\, \mathrm{d}^4x = \mathrm{d}V\ \mathrm{d}s$, where $\mathrm{d}V$ is globule's volume element at rest. World tube is thin, i.e. ω_μ and Z^α may be thought of as nearly constant on its s-sections. Next,

$$\sqrt{-\mathfrak{g}}\, u^0\, \mathrm{d}^3x = \sqrt{-\mathfrak{g}}\, \frac{\mathrm{d}x^0}{\mathrm{d}s}\, \mathrm{d}^3x = \sqrt{-\mathfrak{g}}\, \frac{\mathrm{d}^4x}{\mathrm{d}s} = \mathrm{d}V;$$

hence (as $\varphi^\alpha_\mu\, \mathrm{d}x^\mu/\mathrm{d}s$ is nearly constant over globule's volume) (75) reads

$$\begin{aligned}\mathfrak{J} &= \sum_{i=1}^{2} (-1)^i \int_{\Sigma_i} \rho_0 \frac{\mathrm{d}x^\mu}{\mathrm{d}s} \varphi^\alpha_\mu \mathrm{d}V - \mathfrak{L} \\ &= -\, m_0 \left\{\varphi^\alpha_\mu \frac{\mathrm{d}x^\mu}{\mathrm{d}s}\right\}_{x^0=a} + m_0 \left\{\varphi^\alpha_\mu \frac{\mathrm{d}x^\mu}{\mathrm{d}s}\right\}_{x^0=b} - \mathfrak{L},\end{aligned}$$

where

$$m_0 = \int_{\Sigma_i} \rho_0\, \mathrm{d}V$$

is globule's proper mass. Let L be the path described in spacetime by globule's center in motion. To make the adopted model precise, one shrinks the globule to the limit of a material point, with finite rest mass m_0, and whose world line is L. Hence the first two terms in $\mathfrak{J}$ may be expressed as a line integral

$$\mathfrak{J} = m_0 \int_{L_{ab}} \frac{\mathrm{d}}{\mathrm{d}s} \left[\varphi^\alpha_\mu \frac{\mathrm{d}x^\mu}{\mathrm{d}s}\right] \mathrm{d}s - \mathfrak{L}, \tag{76}$$

where L_{ab} denotes the portion of L comprised between $x^0 = a$ and $x^0 = b$. To compute $\mathfrak{L}$, one lets once again globule's volume become arbitrarily small and integrates over globule's proper volume, i.e.

$$\mathfrak{L} = \int_\Omega (Z^\alpha \omega_\mu)_{|\sigma} T^{\mu\sigma} \, ds \, dV = m_0 \int_{L_{ab}} (Z^\alpha \omega_\mu)_{|\sigma} \frac{dx^\mu}{ds} \frac{dx^\sigma}{ds} \, ds. \tag{77}$$

Similarly

$$\mathfrak{K} = \int_\Omega \varphi^\alpha_\mu \Gamma^\mu_{\nu\sigma} T^{\nu\sigma} \, ds \, dV = m_0 \int_{L_{ab}} \varphi^\alpha_\mu \Gamma^\mu_{\nu\sigma} \frac{dx^\nu}{ds} \frac{dx^\sigma}{ds} \, ds. \tag{78}$$

Then (by (76)–(78)) the left-hand side of (74) is

$$\mathfrak{J} + \mathfrak{K} = m_0 \int_{L_{ab}} \left\{ \frac{d}{ds} \left[\varphi^\alpha_\mu \frac{dx^\mu}{ds} \right] - (Z^\alpha \omega_\mu)_{|\sigma} \frac{dx^\mu}{ds} \frac{dx^\sigma}{ds} + \varphi^\alpha_\mu \Gamma^\mu_{\nu\sigma} \frac{dx^\nu}{ds} \frac{dx^\sigma}{ds} \right\} ds$$
$$= m_0 \int_{L_{ab}} \varphi^\alpha_\mu \left(\frac{d^2 x^\mu}{ds^2} + \Gamma^\mu_{\nu\sigma} \frac{dx^\nu}{ds} \frac{dx^\sigma}{ds} \right) ds.$$

As to the right-hand side of (74),

$$\int_\Omega \varphi^{\alpha\sigma} T_{|\sigma} \sqrt{-\mathfrak{g}} \, d^4x = \int_\Omega \{ (\sqrt{-\mathfrak{g}} \varphi^{\alpha\sigma} T)_{|\sigma} - (\sqrt{-\mathfrak{g}} \varphi^{\alpha\sigma})_{|\sigma} T \} d^4x$$
$$= \sum_{i=1}^{3} \int_{\Sigma_i} \sqrt{-\mathfrak{g}} T \varphi^{\alpha\sigma} N_\sigma \, d\Sigma - \mathfrak{M} \tag{79}$$

(by Green's lemma), where $\mathfrak{M}$ is

$$\mathfrak{M} \equiv \int_\Omega (\sqrt{-\mathfrak{g}} \varphi^{\alpha\sigma})_{|\sigma} T \, d^4x.$$

Hence the integral (79) is

$$\sum_{i=1}^{2} (-1)^i \int_{\Sigma_i} \sqrt{-\mathfrak{g}} T \varphi^{\alpha 0} d^3x - \mathfrak{M}$$
$$= \sum_{i=1}^{2} (-1)^i \int_{\Sigma_i} \varphi^{\alpha 0} (u^0)^{-1} \rho_0 u^\mu u_\mu dV - \mathfrak{M}$$
$$= -m_0 \{ \varphi^{\alpha 0} (u^0)^{-1} u^\mu u_\mu \}_{x^0 = a} + m_0 \{ \varphi^{\alpha 0} (u^0)^{-1} u^\mu u_\mu \}_{x^0 = b} - \mathfrak{M}$$
$$= m_0 \int_{L_{ab}} \frac{d}{ds} \{ \varphi^{\alpha 0} (u^0)^{-1} u^\mu u_\mu \} ds - \mathfrak{M},$$

where $u_\mu = g_{\mu\nu} u^\nu$. Next one computes $\mathfrak{M}$ as

$$\mathfrak{M} = \int_\Omega T \{ \varphi^{\alpha\sigma}{}_{|\sigma} + \varphi^{\alpha\sigma} (\log \sqrt{-\mathfrak{g}})_{|\sigma} \} dV \, ds$$
$$= m_0 \int_{L_{ab}} u^\mu u_\mu \{ \varphi^{\alpha\sigma}{}_{|\sigma} + \varphi^{\alpha\sigma} (\log \sqrt{-\mathfrak{g}})_{|\sigma} \} ds.$$

Finally (by substitution into (79))

$$\int_\Omega \varphi^{\alpha\sigma} T_{|\sigma} \sqrt{-\mathfrak{g}} d^4x = m_0 \int_{L_{ab}} \left\{ \frac{d}{ds} [\varphi^{\alpha 0} (u^0)^{-1} u^\mu u_\mu] - u^\mu u_\mu [\varphi^{\alpha\sigma}{}_{|\sigma} + \varphi^{\alpha\sigma} (\log \sqrt{-\mathfrak{g}})_{|\sigma}] \right\} ds,$$

and (74) may be written as

$$\int_{L_{ab}} \left\{ 2\varphi^\alpha_\mu \left(\frac{d^2 x^\mu}{ds^2} + \Gamma^\mu_{\nu\sigma} \frac{dx^\nu}{ds} \frac{dx^\sigma}{ds} \right) - \frac{d}{ds} [\varphi^{\alpha 0} (u^0)^{-1} u^\mu u_\mu] \right.$$
$$\left. + u^\mu u_\mu [\varphi^{\alpha\sigma}{}_{|\sigma} + \varphi^{\alpha\sigma} (\log \sqrt{-\mathfrak{g}})_{|\sigma}] \right\} ds = 0;$$

hence (as $a, b \in \mathbb{R}$ are arbitrary) equations (70) must hold. If $C^\mu(t)$ is a reparametrization of a solution $x^\mu(s)$ to (70) under a parameter change $t = \psi(s)$, then $C^\mu(t)$ is an allowed trajectory if and only if

$$\left[2\varphi^\alpha_\mu \frac{dC^\mu}{dt}\frac{dC^0}{dt} - \varphi^{\alpha 0} g_{\mu\nu}\frac{dC^\mu}{dt}\frac{dC^\nu}{dt}\right]\psi'' = 0;$$

hence affine parameter transformations $\psi(s) = as + b$ leave (70) invariant.

It should be emphasized that shrinking the size of the moving volume and assuming that the mass remains finite yields infinite density ρ_0; hence arguments above may be set on a rigorous mathematical basis only through the use of Schwartz distribution theory (as in [17]). The treatment in [17] not only implies that trajectories $x(s)$ of material particles are geodesics, but also proves (73) as the statement that the stress–energy tensor is proportional to the tensorial distribution $(dx/ds) \otimes (dx/ds)$ supported on x. The arguments in [11] were used by Souriau (cf [28]) to derive the equation of a string and were generalized, from curves (*a posteriori* geodesics) to higher dimensional submanifolds, by Sternberg (cf [29]) relating the subject matter to harmonic maps theory (with potential applications to the physical theory of branes).

Any geodesic $x^\mu(s)$ satisfying the first-order ODE system

$$\varphi^{\alpha\rho}\Gamma^0_{\rho\sigma}u^0u^\sigma - \varphi^{\alpha 0}\Gamma^0_{\rho\sigma}u^\rho u^\sigma + \varphi^{0\rho}\Gamma^\alpha_{\rho\sigma}u^0u^\sigma - \varphi^{\sigma\rho}\Gamma^\alpha_{\sigma\rho}(u^0)^2 \\ = r[2Z^\alpha(u^0)^2 - 2Z^\alpha Z^0 u^0\omega_\sigma u^\sigma - Z^0u^0u^\alpha] \tag{80}$$

is an allowed trajectory. Indeed, if $x^\mu(s)$ is a geodesic of (M, g), then $u^\mu u_\mu = \mathcal{E}_0 \in \mathbb{R}$; hence $x^\mu(s)$ satisfies (70) if and only if

$$\frac{d}{ds}[\varphi^{\alpha 0}(u^0)^{-1}] = \varphi^{\alpha\sigma}{}_{|\sigma} + \varphi^{\alpha\sigma}\Gamma^\rho_{\rho\sigma}\,. \tag{81}$$

Equation (81) is equivalent to

$$\varphi^{\alpha 0}{}_{|\sigma}u^0u^\sigma - \varphi^{\alpha 0}\frac{du^0}{ds} = \left[\varphi^{\alpha\sigma}{}_{|\sigma} + \varphi^{\alpha\sigma}\Gamma^\rho_{\rho\sigma}\right](u^0)^2$$

and one may use the identities

$$\varphi^{\alpha 0}{}_{|\sigma} = -\varphi^{\alpha\rho}\Gamma^0_{\rho\sigma} - \varphi^{0\rho}\Gamma^\alpha_{\rho\sigma} - r\left(Z^0\delta^\alpha_\sigma + Z^\alpha\delta^0_\sigma + 2Z^0Z^\alpha\omega_\sigma\right),$$

$$\varphi^{\alpha\sigma}{}_{|\sigma} = -g^{\alpha\rho}\Gamma^\sigma_{\sigma\rho} - g^{\sigma\rho}\Gamma^\alpha_{\sigma\rho} - \Gamma^\alpha_{\rho\sigma}Z^\rho Z^\sigma - \Gamma^\rho_{\rho\sigma}Z^\alpha Z^\sigma - 3rZ^\alpha,$$

$$\frac{du^0}{ds} = -\Gamma^0_{\rho\sigma}u^\rho u^\sigma$$

to derive (80).

5. Mixed field equations in the classical limit

Let $M = \mathbb{R}^4$ be the Minkowski space with the special relativity coordinates $x^0 = ct$, $x^1 = x$, $x^2 = y$, $x^3 = z$, where c is the speed of light. Let g_0 be the Minkowski metric $g_0 = -c^2\,dt^2 + dx^2 + dy^2 + dz^2$ and let $\mathcal{F}_0$ be the foliation of M whose leaves are $\mathbb{R}^4/\mathcal{F}_0 = \{\{ct\} \times \mathbb{R}^3 : t \in \mathbb{R}\}$. The (integrable) Pfaffian system $\mathcal{D} = T(\mathcal{F}_0)$ is spacelike with respect to g_0. Let $\gamma \in \mathcal{M}(\mathcal{D}, Z)$ where $Z = \partial/\partial t$. To linearize (63), we set

$$g_{\mu\nu} = (g_0)_{\mu\nu} + \epsilon\gamma_{\mu\nu} \tag{82}$$

(so that $g \in \mathcal{M}(\mathcal{D}, Z)$) and compute $\mathrm{Ric}_{\mathcal{D}}(g)_{\mu\nu}$ by dropping the terms of order $O(\epsilon^2)$ or higher. One has

$$Z^0 = 1, \quad Z^i = 0, \quad \gamma_{0i} = 0, \quad g_{0i} = 0, \quad 1 \leqslant i \leqslant 3. \tag{83}$$

Using the first-order inverse $g^{\mu\nu} = (g_0)^{\mu\nu} - \epsilon\gamma^{\mu\nu}$ to (82), we obtain

$$\Gamma_{\alpha\beta\sigma} = \epsilon\ \{\alpha\beta, \sigma\}, \quad \Gamma^{\alpha}_{\mu\nu} = \epsilon\ (g_0)^{\alpha\sigma}\ \{\mu\nu, \sigma\}, \tag{84}$$

where $\{\alpha\beta, \sigma\} = \frac{1}{2}(\gamma_{\alpha\sigma|\beta} + \gamma_{\beta\sigma|\alpha} - \gamma_{\alpha\beta|\sigma})$. Moreover, (by (83)–(84)) $A^{\alpha}_{\beta} = \nabla_{\beta} Z^{\alpha} = Z^{\alpha}{}_{|\beta} + \Gamma^{\alpha}_{\beta\sigma} Z^{\sigma} = \Gamma^{\alpha}_{\beta 0}$ or

$$A^{\alpha}_{\beta} = \epsilon\ (g_0)^{\alpha\sigma}\ \{\beta 0, \sigma\}. \tag{85}$$

Consequently, trace $(A) = A^{\alpha}_{\alpha}$ is given by

$$\text{trace}\ (A) = \frac{\epsilon}{2}\ (g_0)^{\alpha\sigma}\ \gamma_{\alpha\sigma|0}. \tag{86}$$

Also (by (82) and (85))

$$\nabla_{\mu}\omega_{\nu} = g_{\nu\alpha}\nabla_{\mu} Z^{\alpha} = g_{\nu\alpha} A^{\alpha}_{\mu} = \epsilon\ (g_0)_{\nu\alpha}\ (g_0)^{\alpha\sigma}\ \{\mu 0, \sigma\}$$

or

$$\nabla_{\mu}\omega_{\nu} = \epsilon\{\mu 0, \nu\}. \tag{87}$$

Therefore (by (86)–(87))

$$\text{trace}\ (A)\ \nabla_{\mu}\omega_{\nu} = O(\epsilon^2). \tag{88}$$

Next (by (84)–(85))

$$\begin{aligned}\nabla_{\sigma} A^{\alpha}_{\mu} &= A^{\alpha}_{\mu|\sigma} + A^{\rho}_{\mu}\Gamma^{\alpha}_{\sigma\rho} - \Gamma^{\rho}_{\sigma\mu} A^{\alpha}_{\rho} \\ &= \epsilon\ (g_0)^{\alpha\beta}\ \{\mu 0, \beta\}_{|\sigma} + \epsilon\ (g_0)^{\rho\beta}\ \{\mu 0, \beta\}\Gamma^{\alpha}_{\sigma\rho} - \epsilon\ (g_0)^{\alpha\beta}\ \{\rho 0, \beta\}\Gamma^{\rho}_{\sigma\mu}\end{aligned}$$

or

$$\nabla_{\sigma} A^{\alpha}_{\mu} = \epsilon\ (g_0)^{\alpha\beta}\ \{\mu 0, \beta\}_{|\sigma}. \tag{89}$$

It follows that (by (82) and (89))

$$Z^{\sigma}\ \nabla_{\sigma}\nabla_{\mu}\omega_{\nu} = \epsilon\ \{\mu 0, \nu\}_{|0}. \tag{90}$$

Again by (85) and (87),

$$A^{\sigma}_{\mu}\ \nabla_{\sigma}\omega_{\nu} = O(\epsilon^2), \tag{91}$$

$$\omega_{\nu}\ \nabla_{\sigma} A^{\sigma}_{\mu} = \epsilon\ \omega_{\nu}\ (g_0)^{\sigma\beta}\ \{\mu 0, \beta\}_{|\sigma}, \tag{92}$$

$$\nabla_{\sigma}\omega_{\mu} \cdot \nabla^{\sigma}\omega_{\nu} = O(\epsilon^2). \tag{93}$$

Finally (similar to (90))

$$\omega_{\mu}\ \nabla_{\sigma}\nabla^{\sigma}\omega_{\nu} = \epsilon\ \omega_{\mu}\ (g_0)^{\sigma\rho}\ \{\rho 0, \nu\}_{|\sigma}. \tag{94}$$

Dropping the terms of order $O(\epsilon^2)$, one has (by (88)–(94))

$$\text{Ric}_{\mathcal{D}}(g)_{\mu\nu} = \epsilon\ \{\mu 0, \nu\}_{|0} + \epsilon\ (g_0)^{\sigma\rho}\ (\omega_{\nu}\{\mu 0, \rho\}_{|\sigma} - \omega_{\mu}\{\rho 0, \nu\}_{|\sigma}). \tag{95}$$

To compute $\rho_{\mathcal{D}}(g) = -Z^{\mu} Z^{\nu} R_{\mu\nu}$, one starts from the second identity in (84)

$$\Gamma^{\alpha}_{\mu\nu|\beta} = \epsilon\ (g_0)^{\alpha\rho}\ \{\mu\nu, \rho\}_{|\beta};$$

hence

$$\begin{aligned}R^{\alpha}_{\beta\mu\nu} &= \Gamma^{\alpha}_{\mu\nu|\beta} - \Gamma^{\alpha}_{\beta\nu|\mu} + \Gamma^{\sigma}_{\mu\nu}\Gamma^{\alpha}_{\beta\sigma} - \Gamma^{\sigma}_{\beta\nu}\Gamma^{\alpha}_{\mu\sigma} \\ &= \epsilon (g_0)^{\alpha\rho}(\{\mu\nu, \rho\}_{|\beta} - \{\beta\nu, \rho\}_{|\mu}) + O(\epsilon^2)\end{aligned}$$

so that (by contracting α and β)

$$R_{\mu\nu} = \epsilon (g_0)^{\alpha\rho} (\{\mu\nu, \rho\}_{|\alpha} - \{\alpha\nu, \rho\}_{|\mu}). \tag{96}$$

By (83), $\rho_{\mathcal{D}}(g) = -R_{00}$; hence (by (96) with $\mu = \nu = 0$)

$$\rho_{\mathcal{D}}(g) = \epsilon\, (g_0)^{\alpha\beta} (\{\alpha 0, \beta\}_{|0} - \{00, \beta\}_{|\alpha}). \tag{97}$$

Finally, (by (95) and (97)) the linearized field equations (63) (with $\Lambda = 0$) are

$$\epsilon\, \{\mu 0, \nu\}_{|0} + \epsilon\, (g_0)^{\alpha\rho} [\omega_\nu \{\mu 0, \rho\}_{|\alpha} - \omega_\mu \{\rho 0, \nu\}_{|\alpha} + \omega_\mu \omega_\nu (\{00, \rho\}_{|\alpha} - \{\alpha 0, \rho\}_{|0})]$$
$$= \mathfrak{a}(T_{\mu\nu} - (1/2) T g_{\mu\nu}). \tag{98}$$

We assume from now on that the perturbation matrix $[\gamma_{\mu\nu}]$ is static, i.e. $\gamma_{\mu\nu|0} = 0$. Then $\{\mu 0, \nu\}_{|0} = 0$ and equations (98) simplify accordingly (by $(g_0)^{\alpha\rho} = \epsilon^\alpha \delta^{\alpha\rho}$ with $\epsilon_0 = -1 = -\epsilon_i$)

$$\epsilon\, \epsilon^\alpha [\omega_\nu \{\mu 0, \alpha\}_{|\alpha} - \omega_\mu \{\alpha 0, \nu\}_{|\alpha} + \omega_\mu \omega_\nu \{00, \alpha\}_{|\alpha}] = \mathfrak{a} \left(T_{\mu\nu} - \tfrac{1}{2} T\, g_{\mu\nu}\right). \tag{99}$$

Next, by taking into account

$$\{\mu 0, \alpha\}_{|\alpha} = \tfrac{1}{2} (\gamma_{0\alpha|\mu|\alpha} - \gamma_{\mu 0|\alpha|\alpha}), \quad \{\alpha 0, \nu\}_{|\alpha} = \tfrac{1}{2} (\gamma_{0\nu|\alpha|\alpha} - \gamma_{\alpha 0|\nu|\alpha}),$$

$$\{00, \alpha\}_{|\alpha} = -\tfrac{1}{2} \gamma_{00|\alpha|\alpha},$$

$$\epsilon^\alpha (\omega_\nu \gamma_{0\alpha|\mu|\alpha} + \omega_\mu \gamma_{\alpha 0|\nu|\alpha}) = \epsilon^i (\omega_\nu \gamma_{0i|\mu|i} + \omega_\mu \gamma_{i0|\nu|i}) = 0,$$

the field equations (99) may be written as

$$\frac{\epsilon}{2} [\omega_\nu\, \Box \gamma_{\mu 0} + \omega_\mu\, \Box \gamma_{0\nu} + \omega_\mu \omega_\nu\, \Box \gamma_{00}] = \mathfrak{a}(T_{\mu\nu} - (1/2)\, T\, g_{\mu\nu}), \tag{100}$$

where $\Box f = -\epsilon^\alpha\, f_{|\alpha|\alpha}$ for any $f \in C^2(M)$. Since $\omega_\alpha = g_{\alpha\beta} Z^\beta = g_{\alpha 0}$, one has

$$\omega_0 = -1 + \epsilon \gamma_{00}\,, \quad \omega_i = 0, \quad 1 \leqslant i \leqslant 3. \tag{101}$$

By (83) and (101) in free space (i.e. $T_{\mu\nu} = 0$), the only nontrivial component of (100) is obtained for $\mu = \nu = 0$, i.e. $\omega_0(\omega_0 + 2)\, \Box \gamma_{00} = 0$ and (again by (101)) $\omega_0(\omega_0 + 2) = -1 + O(\epsilon^2)$. We may conclude that

Theorem 6. *The linearized mixed field equations in free space are*

$$\Delta \gamma_{00} \equiv -\sum_{i=1}^{3} \gamma_{00|i|i} = 0 \tag{102}$$

so that

$$\gamma_{00} = -\frac{\kappa \mathrm{M}}{r}, \quad r = (x^2 + y^2 + z^2)^{1/2},$$

where M *is the mass of the body at* $r = 0$ *and* κ *is the gravitational constant. Consequently, each Lorentz metric*

$$g_\epsilon = -c^2 \left(1 + \frac{\epsilon \kappa \mathrm{M}}{r}\right) \mathrm{d}t^2 + g_{ij}\, \mathrm{d}x^i \otimes \mathrm{d}x^j\,, \tag{103}$$

$$g_{ij} = \delta_{ij} + \epsilon \gamma_{ij}, \quad \gamma_{i0} = 0, \quad \gamma_{ij|0} = 0,$$

is a solution to the linearized mixed field equations in free space.

As to the approximate solution (103), the manifold is $M = \mathbb{R}^4 \setminus \{(ct, 0, 0, 0) : t \in \mathbb{R}\}$, M is foliated by the foliation $\mathcal{F}$ whose leaf space is $M/\mathcal{F} = \{L \setminus \{r = 0\} : L \in M/\mathcal{F}_0\}$, and

$$\beta = 1 + \frac{\epsilon\kappa \mathrm{M}}{r}, \quad \mathcal{T} = ct.$$

If $X = X^\mu \partial_\mu \in \mathfrak{X}(M)$, then $X \in \mathfrak{X}_\mathcal{F} \Longleftrightarrow X^0 = 0$. Thus (by (29) and (84))

$$\alpha(X, Y) = \{X(Y(\mathcal{T})) - (\nabla_X Y)(\mathcal{T})\}\pi Z$$

$$= \{X(Y^\sigma) + \epsilon(g_0)^{\sigma\nu} X^\alpha Y^\mu \{\alpha\mu, \nu\}\}\, \mathcal{T}_{|\sigma}\pi Z$$

(as $\mathcal{T}_{|0} = 1$, $\mathcal{T}_{|i} = 0$ and $Y^0 = 0$)

$$= \epsilon(g_0)^{0\nu} X^\alpha Y^\mu \{\alpha\mu, \nu\}\pi Z = \epsilon(g_0)^{0\nu} X^i Y^j \{ij, \nu\}\pi Z = -\epsilon X^i Y^j \{ij, 0\}\pi Z = 0$$

because $\gamma_{i0} = 0$ and $\gamma_{ij|0} = 0$. We may conclude that

Corollary 2. *$\mathcal{F}$ is totally geodesic in $(\mathbb{R}^4 \setminus (\mathbb{R} \times \{\mathbf{0}\}),\ g_\epsilon)$ to first order in ϵ.*

Next, one examines the linearized field equations in the presence of matter, in order to pinpoint the classical gravitational potential. Equation (100) for $\mu = \nu = 0$ (respectively for $\mu = i$) is

$$-\frac{\epsilon}{2}\Box\gamma_{00} = \mathfrak{a}\,[T_{00} + (1/2)\,T\,(1 - \epsilon\,\gamma_{00})]\,, \tag{104}$$

$$T_{i\nu} = (1/2)\,T\,g_{i\nu}\,. \tag{105}$$

Equivalence of matter and radiation under $E = mc^2$ implies that energy density along with mass density acts as a source of gravitational potential; hence density ρ_0 must be the time–time component of the energy–momentum tensor, i.e. $T^{00} = \rho_0$. By (105),

$$T = g^{\mu\nu}T_{\mu\nu} = g^{00}T_{00} + g^{i0}T_{i0} + g^{\mu j}T_{\mu j}$$

$$= g^{00}\rho_0 + (1/2)T g^{\mu j} g_{\mu j} = g^{00}\rho_0 + (3/2)T,$$

i.e. the Laue scalar is

$$T = 2\left(1 - \epsilon\,\gamma^{00}\right)\rho_0\,. \tag{106}$$

Substitution from (106) into (104) yields (by dropping $O(\epsilon^2)$)

$$-\frac{1}{2\mathfrak{a}}\frac{\epsilon}{2 - \epsilon(\gamma_{00} + \gamma^{00})}\Box\gamma_{00} = \rho_0$$

or (as $\epsilon/(2 - \epsilon(\gamma_{00} + \gamma^{00})) = \epsilon/2 + O(\epsilon^2)$)

$$-\frac{\epsilon}{4\mathfrak{a}}\Box\gamma_{00} = \rho_0\,. \tag{107}$$

A parallel of (107) to the classical gravitational equation for nonempty space $\sum_{i=1}^3 \phi_{|i|i} = 4\pi\rho_0\kappa$ (Poisson equation) prompts the choice of the gravitational potential

$$\phi = \frac{\pi\kappa\epsilon}{\mathfrak{a}}\gamma_{00}\,. \tag{108}$$

To determine the coupling constant $\mathfrak{a}$, we examine motion equations (70) in the weak field and low velocity limit. Let $C(t) = (ct, C^1(t), C^2(t), C^3(t))$, $|t| < \delta$, be a timelike curve and let $v^i = \mathrm{d}C^i/\mathrm{d}t$ be the velocity vector along $C(t)$. Moreover, let us set

$$s = \psi(t) = \int_0^t [-g_{C(\tau)}(\dot{C}(\tau), \dot{C}(\tau))]^{1/2}\mathrm{d}\tau,$$

and $x^\mu(s) = C^\mu(\psi^{-1}(s))$, where $C^0(t) = ct$. In particular, $\mathcal{E}_0 = -1$, i.e.

$$g_{\mu\nu}(x(s))\frac{dx^\mu}{ds}(s)\frac{dx^\nu}{ds}(s) = -1.$$

One has (by (82))

$$g_{C(t)}(\dot{C}(t), \dot{C}(t)) = -c^2 + \|\mathbf{v}(t)\|^2 + \epsilon\gamma_{\mu\nu}(C(t))\frac{dC^\mu}{dt}(t)\frac{dC^\nu}{dt}(t)$$
$$= c^2\left[-1 + \eta^2 + \epsilon\gamma_{00}(C(t)) + 2\epsilon\gamma_{0j}(C(t))\frac{v^j(t)}{c} + \epsilon\gamma_{ij}(C(t))\frac{v^i(t)}{c}\frac{v^j(t)}{c}\right]$$

where $\mathbf{v} = (v^1, v^2, v^3)$, $\|\mathbf{v}\| = (\sum_{i=1}^3 (v^i)^2)^{1/2}$ and $\eta = \|\mathbf{v}\|/c$. The last two terms are respectively of orders $O(\epsilon\,\eta)$ and $O(\epsilon\,\eta^2)$; hence (as $\epsilon \ll 1$ and $\eta \ll 1$)

$$\psi'(t) \approx c\,[1 - \epsilon\,\gamma_{00}(C(t))]^{1/2}\ . \tag{109}$$

Then $C^\mu(t) = x^\mu(\psi(t))$ yields (by (109))

$$\frac{dx^\alpha}{ds}(\psi(t)) = \frac{1}{c\,[1 - \epsilon\,\gamma_{00}(C(t))]^{1/2}}\frac{dC^\alpha}{dt}(t). \tag{110}$$

Differentiation with respect to t in (110) yields

$$c^2\,[1 - \epsilon\gamma_{00}(C(t))]\,\frac{d^2x^\alpha}{ds^2}(\psi(t)) = \frac{d^2C^\alpha}{dt^2}(t) + \frac{\epsilon}{2\,[1 - \epsilon\gamma_{00}(C(t))]}\gamma_{00|\sigma}(C(t))\frac{dC^\alpha}{dt}(t)\frac{dC^\sigma}{dt}(t)$$

and

$$\epsilon/(1 - \epsilon\gamma_{00}) = \epsilon + O(\epsilon^2), \qquad \gamma_{00|0} = 0,$$
$$\epsilon\gamma_{00|\sigma}\frac{dC^\sigma}{dt} = \epsilon\gamma_{00|i}v^i = O(\epsilon\eta),$$

imply

$$\frac{d^2x^\alpha}{ds^2}(\psi(t)) = \frac{1}{c^2\,[1 - \epsilon\,\gamma_{00}(C(t))]}\frac{d^2C^\alpha}{dt^2}(t). \tag{111}$$

By (110) for $t = \psi^{-1}(s)$ and the second identity in (84),

$$\Gamma^\mu_{\sigma\rho}(x(s))\frac{dx^\sigma}{ds}(s)\frac{dx^\rho}{ds}(s) = \frac{\epsilon}{c^2\,[1 - \epsilon\gamma_{00}(C(t))]}\,(g_0)^{\mu\nu}\,\{\sigma\rho,\nu\}(C(t))\frac{dC^\sigma}{dt}\frac{dC^\rho}{dt}$$
$$= \epsilon\,(g_0)^{\mu\nu}\left[\{00,\nu\} + 2\{0i,\nu\}\frac{v^i}{c} + \{ij,\nu\}\frac{v^i}{c}\frac{v^j}{c}\right]$$
$$= \Gamma^\mu_{00} + O(\epsilon\eta) + O(\epsilon\eta^2),$$

i.e.

$$\Gamma^\mu_{\sigma\rho}\frac{dx^\sigma}{ds}\frac{dx^\rho}{ds} = \Gamma^\mu_{00}\,. \tag{112}$$

Next (by (111)–(112) and $\omega_i = 0$)

$$\varphi^\alpha_\mu\left(\frac{d^2x^\mu}{ds^2} + \Gamma^\mu_{\sigma\rho}\frac{dC^\sigma}{ds}\frac{dC^\rho}{ds}\right) = \left(\delta^\alpha_\mu + Z^\alpha\omega_\mu\right)\left[\frac{1}{c^2\,[1 - \epsilon\gamma_{00}(C(t))]}\frac{d^2C^\mu}{dt^2} + \Gamma^\mu_{00}(C(t))\right]$$
$$= \frac{1}{c^2\,(1 - \epsilon\gamma_{00})}\frac{d^2C^\alpha}{dt^2} + \Gamma^\alpha_{00}$$
$$+ Z^\alpha\left[\frac{\omega_0}{c^2(1 - \epsilon\gamma_{00})}\frac{d^2C^0}{dt^2} + \omega_0\Gamma^0_{00}\right],$$

and (101) together with

$$1/(1 - \epsilon\gamma_{00}) = 1 + \epsilon\gamma_{00} + O(\epsilon^2),$$

$$\Gamma^0_{00} = \epsilon\,(g_0)^{0\sigma}\,\{00,\sigma\} = \epsilon\epsilon^0\delta^{0\sigma}\{00,\sigma\} = -\epsilon\{00,0\} = 0,$$

allows one to conclude that

$$\varphi^\alpha_\mu\left(\frac{d^2x^\mu}{ds^2} + \Gamma^\mu_{\sigma\rho}\frac{dC^\sigma}{ds}\frac{dC^\rho}{ds}\right) = \frac{1}{c^2}\left[(1+\epsilon\,\gamma_{00})\,\frac{d^2C^\alpha}{dt^2} + c^2\Gamma^\alpha_{00}\right]. \tag{113}$$

Next (by (110)–(111))

$$\frac{d}{ds}[\varphi^{\alpha 0}(u^0)^{-1}u^\mu u_\mu] = -\frac{d}{ds}\left[\varphi^{\alpha 0}\left(\frac{dx^0}{ds}\right)^{-1}\right]$$
$$= -\varphi^{\alpha 0}{}_{|\sigma}\frac{dx^\sigma}{ds}\left(\frac{dx^0}{ds}\right)^{-1} + \varphi^{\alpha 0}\left(\frac{dx^0}{ds}\right)^{-2}\frac{d^2x^0}{ds^2},$$

i.e.

$$\frac{d}{ds}[\varphi^{\alpha 0}(u^0)^{-1}u^\mu u_\mu] = -\varphi^{\alpha 0}{}_{|\sigma}\,\frac{1}{c}\,\frac{dC^\sigma}{dt}\,. \tag{114}$$

Also (by (83)–(84))

$$\varphi^{\alpha 0}{}_{|\sigma} = (g^{\alpha 0} + Z^\alpha Z^0)_{|\sigma} = g^{\alpha 0}{}_{|\sigma} = -g^{\alpha\mu}\Gamma^0_{\mu\sigma} - g^{0\mu}\Gamma^\alpha_{\sigma\mu}$$
$$= -\epsilon\epsilon^\alpha\delta^{\alpha\mu}\epsilon^0\delta^{0\nu}\{\mu\sigma,\nu\} - \epsilon\epsilon^0\delta^{0\mu}\epsilon^\alpha\delta^{\alpha\nu}\{\sigma\mu,\nu\},$$

i.e.

$$\varphi^{\alpha 0}{}_{|\sigma} = \epsilon\,\epsilon^\alpha\,[\{\alpha\sigma,0\} + \{\sigma 0,\alpha\}] = \epsilon\,\epsilon^\alpha\,\gamma_{\alpha 0|\sigma}\,. \tag{115}$$

Similarly,

$$\varphi^{\alpha\sigma}{}_{|\sigma} = -\epsilon\epsilon^\alpha\epsilon^\sigma\,[\{\sigma\alpha,\sigma\} + \{\sigma\sigma,\alpha\}] = -\epsilon\,\epsilon^\alpha\,\epsilon^\sigma\,\gamma_{\alpha\sigma|\sigma}\,, \tag{116}$$

$$\varphi^{\alpha\sigma}\Gamma^\rho_{\rho\sigma} = \epsilon\,\epsilon^\rho[\epsilon^\alpha\,\{\rho\alpha,\rho\} + Z^\alpha Z^\sigma\,\{\rho\sigma,\rho\}]$$
$$= \frac{\epsilon}{2}\epsilon^\rho\left(\epsilon^\alpha\gamma_{\rho\rho|\alpha} + Z^\alpha Z^\sigma\gamma_{\rho\rho|\sigma}\right). \tag{117}$$

Thus (by (115)–(117) and $\gamma_{\mu\nu|0} = 0$, $\epsilon^i = 1$ and $Z^i = 0$)

$$\frac{d}{ds}[\varphi^{\alpha 0}(u^0)^{-1}u^\mu u_\mu] - u^\mu u_\mu\left[\varphi^{\alpha\sigma}{}_{|\sigma} + \varphi^{\alpha\sigma}\Gamma^\rho_{\rho\sigma}\right] = \epsilon\epsilon^\alpha\sum_{i=1}^{3}\left(\frac{1}{2}\gamma_{ii|\alpha} - \gamma_{\alpha i|i}\right) + O(\epsilon\eta). \tag{118}$$

Let us substitute (113) and (118) into (70), so as to yield the linearized motion equations

$$2\,(1+\epsilon\gamma_{00})\,\frac{d^2C^\alpha}{dt^2} - \epsilon\,\epsilon^\alpha\,c^2\,\gamma_{00|\alpha} = \epsilon\,\epsilon^\alpha\,c^2\sum_{i=1}^{3}\left(\frac{1}{2}\,\gamma_{ii|\alpha} - \gamma_{\alpha i|i}\right). \tag{119}$$

This is identically satisfied for $\alpha = 0$, while for $\alpha = j$ (dividing by $2(1+\epsilon\gamma_{00})$ and approximating $\epsilon/(1+\epsilon\gamma_{00})$ to order $O(\epsilon)$),

$$\frac{d^2\mathbf{r}}{dt^2} = \nabla\phi + \mathbf{F}(\mathbf{r}(t)), \tag{120}$$

where $\mathbf{r}(t) = C^j(t)e_j$ and

$$\phi = (\epsilon/2)\,c^2\,\gamma_{00}\,,\quad \mathbf{F} = (\epsilon/4)\,c^2\sum_{i,j=1}^{3}(\gamma_{ii|j} - 2\,\gamma_{ji|i})e_j\,, \tag{121}$$

and $\{e_j : 1 \leqslant j \leqslant 3\} \subset \mathbb{R}^3$ is the canonical linear basis. Finally, we may parallel (108) and (121) to conclude that $\mathfrak{a} = 2\pi\kappa/c^2$.

6. Conclusions and final comments

Legitimate by Bernal and Sánchez's recent results (cf [3–5]) on the existence of a smooth version of Geroch's splitting (cf [14]) on a globally hyperbolic spacetime M, we proposed the new action S given by (2). This is imitative of Einstein–Hilbert's functional (cf Weinberg, [33], p 364) except that the scalar curvature is replaced by the mixed scalar curvature $\rho_{\mathcal{D}}(g)$ (cf Rovenski [23]). Here, $\mathcal{D} = T(\mathcal{F})$ is the tangent bundle to the foliation $\mathcal{F}$ by level hypersurfaces of the time function $\mathcal{T}$ and the metric tensor g is varied in the manifold $\mathcal{M}(\mathcal{D}, Z)$ of all Lorentzian metrics such that $\mathcal{D}$ possesses a common orthogonal complement spanned by the timelike vector field $Z \in \mathfrak{X}(M)$. We follow the ideas in [23–25] (first demonstrated within Riemannian geometry) in an attempt to embody the infinitesimal decomposition $T(M) = \mathcal{D} \oplus \mathbb{R}Z$ into the gravitational field equations. A physical meaning may be attached to S within quantum physics (cf [13]). While the quantization of the gravitational field appears, aside from the weak field case, as a tantalizing problem (cf [8]), we may, as a matter of principle, consider two spacelike hypersurfaces $\mathcal{T}^{-1}(t_a)$ $(a = 1, 2)$ and the formal Feynman propagator between two geometries $\mathfrak{G}_a$ at $\mathcal{T}^{-1}(t_a)$,

$$K[\mathfrak{G}_2\,,\ \mathfrak{G}_1] = \sum_\Gamma \exp\left\{\frac{\mathrm{i}}{\hbar} S[\Gamma]\right\} = \sum_\Gamma \exp\left\{\frac{\mathrm{i}c^2}{4\pi\kappa\hbar}\int_\Gamma \rho_{\mathcal{D}}(g)\sqrt{-\mathfrak{g}}\,\mathrm{d}^4x\right\}. \tag{122}$$

Here, one calculates the action S along any path, i.e. along any sequence of geometries starting from $\mathfrak{G}_1$ at $\mathcal{T}^{-1}(t_1)$ and ending on $\mathfrak{G}_2$ at $\mathcal{T}^{-1}(t_2)$. As $\hbar \to 0$, to reach the classical physics limit, $\exp\{(\mathrm{i}/\hbar)\,S(\Gamma)\}$ (the probability amplitude for the system to go from $\mathfrak{G}_1$ to $\mathfrak{G}_2$ along Γ) oscillates rapidly and systematic phasing (preventing $K[\mathfrak{G}_2\,,\ \mathfrak{G}_1]$ from vanishing) requires $\delta S[\Gamma] = 0$. That is, in the classical limit $\hbar \to 0$, only paths for which

$$\delta S = 0 \tag{123}$$

contribute to (122), yielding the physical interpretation sought for (though other geometries, required to satisfy $(c^2/4\pi\kappa)\int \rho_{\mathcal{D}}(g)\,\sqrt{-\mathfrak{g}}\,\mathrm{d}^4x \lesssim \hbar$, should be included in (122) as well). An inspection of the Euler–Lagrange equations associated with the variational principle (123) leads to a new kind of Ricci curvature $\mathrm{Ric}_{\mathcal{D}}(g)$ and scalar curvature $\mathrm{Scal}_{\mathcal{D}}(g)$, whose properties need to be further investigated. $\mathrm{Scal}_{\mathcal{D}}(g)$ is indeed the contraction of $\mathrm{Ric}_{\mathcal{D}}(g)$, yet $\mathrm{Scal}_{\mathcal{D}}(g) = -\rho_{\mathcal{D}}(g)$ only when g is already a solution to mixed field equations in free space. Also, $\mathrm{Ric}_{\mathcal{D}}(g)$ and the partial Ricci curvature in [23] appear as logically distinct concepts. Our discussion of the geometry of a Bernal–Sánchez foliation is imitative of that in [31, pp 104–16] and leads to results close to those in the Riemannian case e.g. that mean curvature $\mathbf{k}(Z)$ is a basic function on $(M, \mathcal{F})$. The geometric analysis in section 3 suffices however for determining an exact solution, of the form $-\beta\,(\mathrm{d}\mathcal{T})^2 + \overline{g}$, to mixed field equations in free space. The equations of motion for a material particle in the gravitational field obeying (63) are derived as a consequence of the conservation law (69), itself a consequence of the mixed field equations alone, and besides presenting independent interest are exploited in the weak field and low velocity limit, together with the very field equations in the classical limit, to compute the coupling constant $\mathfrak{a} = 2\pi\kappa/c^2$ appearing in (2). This is perhaps our most important result in section 5. In comparison with Einstein's relativity (where the field equations imply geodesic motion reducing, in the classical limit, to Newton's law of motion in a central force field), our linearized motion equations $\mathrm{d}^2\mathbf{r}/\mathrm{d}t^2 = \nabla\phi + \mathbf{F}$ contain the additional force field $\mathbf{F} = (\epsilon/4)\,c^2\sum_{i,j=1}^3\left(\gamma_{ii|j} - 2\,\gamma_{ji|i}\right)e_j$ (and $\mathbf{F}(\mathbf{r}(t)) = 0$ in the case of geodesic motion). An interesting question, raised by the reviewer, is whether our calculations in sections 2 and 3 may be related to the spacelike energy (due to Gil-Medrano and Hurtado, [15]) of the *reference frame* Z. If $A_Z = -\nabla Z$ and $P_Z X = X + g(X, Z)X$ for every $X \in \mathfrak{X}(M)$, then

$$\tilde{B}_\Omega(Z) = \frac{1}{2}\int_\Omega g^*\left(A_Z \circ P_Z, A_Z \circ P_Z\right)\, \mathrm{dvol}(g)$$

is the *spacelike energy* of Z (cf also [9], p 412). A reference frame Z is *spatially harmonic* if it is a critical point of $\tilde{B}_\Omega$ for any relatively compact domain $\Omega \subset\subset M$. Let $M = I \times_f F$ be the GRW model, i.e. $I \times F$ together with the warped product metric g^f in section 3 (cf again [26]). Then $Z = \partial/\partial t$ is a *comoving* reference frame on $I \times_f F$ (cf [1]). By a result in [15], $Z = \partial/\partial t$ is spatially harmonic (cf also example 8.29 in [9], p 435). The resulting situation appears similar to Kaluza–Klein-type theories where gravity is coupled with scalar fields (cf e.g. [16, 32, 10]) and the additional scalar fields follow to be harmonic. Further investigation of these issues (on a globally hyperbolic spacetime carrying a Bernal–Sánchez foliation) is relegated to further work.

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