Free vibration analysis of rotating Rayleigh beams at high angular velocity: variational approach.

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Abstract— **This paper presents a dynamic model for the vibration of rotating Rayleigh beam. The governing differential equations of motion of the beam in free vibration are derived using Lagrange's equations and include the effect of an arbitrary hub radius. Three linear partial differential equations are derived. Two of the linear differential equations are coupled through the stretch and chordwise deformation. The other equation is an uncoupled one for the flapwise deformation. A method based on the Rayleigh-Ritz solution is proposed to solve the natural frequency of very slender rotating beam at high angular velocity. The parameters for the hub radius, rotational speed, tapered ratio, rotary inertia and slenderness ratio are incorporated into the equation of motion. Finally the resonance frequency of rotating bema is evaluated. The non-dimensional frequency coefficients are given in tabular form. Some numerical examples are presented and the influence of different non-dimensional parameters on frequency values is discussed.**

Keywords- Lagrange's equation, dynamics, rotating Rayleigh beam

I. INTRODUCTION

There are many engineering example which can be idealized as rotating beams, such as helicopter blades, turbine blades, satellite booms, aircraft rotary wings, etc. Rotating beams differ from non-rotating beams in having additional centrifugal force and Coriolis effects on their dynamics. The stretching causes the increment of the bending stiffness of the structures, which naturally results in the variation of natural frequencies and mode shapes. Vibration in many cases greatly affects the nature of engineering designs. Vibrational properties of engineering structures are often limiting factors in their performance. Consequently, considerable attention has been paid to free vibration analysis involving the study of natural frequencies and mode shapes of such structures. Identifying such structural properties is essential to the analysis of structural dynamics and the suppression of unwanted vibrations. Centrifugally stiffened rotating beams involve variable coefficients in the governing equations. The variable coefficient differential equations in general cannot be solved by using ordinary trigonometric or hyperbolic functions. Standard approximated approaches such as Rayleigh-Ritz, Galerkin and Finite Element have been used in solving free vibration problems of such structures. Power series approaches are also applied in obtaining solutions of rotating beam structures.

Due to the wide range of applications and the specific geometric feature of beams, in which one dimension is much larger than the other two, various beam models have been employed to simulate the structural dynamics of aircraft wings, helicopter blades, spacecraft antennas, robot arms, towers and for many other industrial applications. Numerous methods such as experimental, analytical and numerical methods have been developed and used to analyze the structural dynamics of beam-like structures. In this respect, the modal analysis is a well-known practical technique for investigation of the dynamic response and vibrations of beams. The modal approach gives the solution in a series in terms of natural mode shapes and the corresponding generalized coordinates. Subsequently, a first need is determine the natural mode shapes and frequencies of free vibrations, analytically or numerically for using such techniques. Indeed transverse free vibrations of non-uniform beams have been studied by numerous researchers in both aeronautical and mechanical engineering fields either analytically or numerically. Added to this, several analytical solutions, most of which are applied for linearly tapered beams, have been represented in exact procedure with Bessel functions [1], terms of orthogonal polynomials [2] in approximation method, power series by Frobenius method [3], differential stiffness method [4] and finite element analysis [5]. On the other hand, a wide range of approximate and numerical solutions such as Rayleigh-Ritz, Galerkin, finite difference, finite element and spectral finite element methods have been used to obtain the natural vibration characteristics of variable-section beams [6-9].

In the present study, the equations of motion of rotating Rayleigh beam are derived by the Lagrange's equation. In order to capture all inertia effect and coupling between extensional and flexural deformation, the consistent linearization of the fully geometrically non-linear beam theory. The problem with many discrete degrees of freedom is studied through the adoption of orthogonal polynomial functions satisfying the essential conditions only. Finally, numerical examples have been completely carried through by means of the powerful symbolic software*; Mathematica* [14].

II. DYNAMIC MODEL

Considered a tapered Rayleigh beam length L rigidly mounted on the periphery of rigid hub with radius r rotating about its axis fixed in space at a constant angular velocity Ω. Figure 1 show the deformation of the neutral axis of the beam. The origin of the coordinate system is chosen to be the intersection of the centroid axes of the hub and the undeformed beam. A generic point P_o the undeformed position is given of the vector:

$$\mathbf{r}_0 = \left[r + x_1,\ x_2,\ x_3 \right]^T . \tag{1}$$

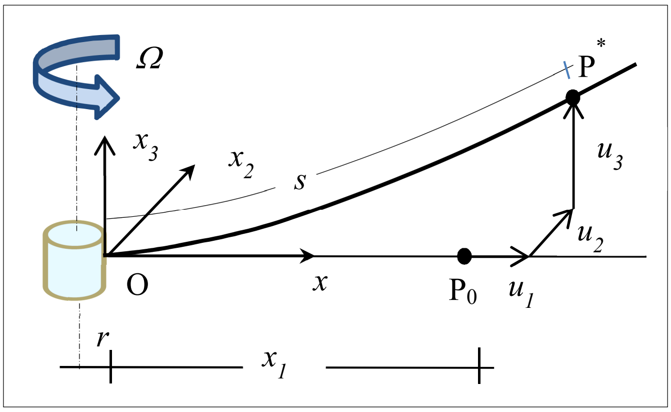

Figure 1. Deformed of the blade neutral axis

If the beam now in deforms as a result of flexure and also under tension due to the centrifugal force, the position vector of the deformed point would now be given of the $\mathbf{r}$:

$$\mathbf{r} = \left[r + x_1 + u_1 - x_3\, u_{3,1} - x_2\, u_{2,1},\ x_2 + u_2,\ x_3 + u_3 \right]^T \tag{2}$$

The velocity of a material point in deformed state is given by:

$$\mathbf{v} = \dot{\mathbf{r}} + (\Omega \mathbf{e}_3 \times \mathbf{r}) = \\ = \begin{bmatrix} \dot{u}_1 - \Omega(x_2 + u_2) - x_3\, \dot{u}_{3,1} - x_2\, \dot{u}_{2,1} \\ \dot{u}_2 + \Omega(r + x_1 + u_1) - \Omega(x_3\, u_{3,1} + x_2\, u_{2,1}) \\ \dot{u}_3 \end{bmatrix}, \tag{3}$$

where the time derivatives are defined with a dot, while *(,1)* represents the partial derivative with respect to the integral domain variable x_1.

From a geometrical point of view the length, s, is a function of Cartesian coordinates and is given by the following relationship [11]:

$$s = u_1 + \frac{1}{2}\int_0^{x_1} \left[\left(u_{2,\tau}\right)^2 + \left(u_{3,\tau}\right)^2 \right] d\tau, \\ \Downarrow \\ u_1 = s - \frac{1}{2}\int_0^{x_1} \left[\left(u_{2,\tau}\right)^2 + \left(u_{3,\tau}\right)^2 \right] d\tau \tag{4}$$

with τ, is dummy variable. The governing differential equations of motion of the rotating tapered beam in free vibration are derived by applying Lagrange's equation which requires the expression for kinetic and strain energies.

The kinetic energy of the system is given by

$$T = \frac{1}{2}\int_V m\left(\mathbf{v}\bullet\mathbf{v}\right) dV = \frac{1}{2}\int_0^L \frac{\rho}{A}\int_A \left[v_1^2 + v_2^2 + v_3^2 \right] dA\, dx_1 . \tag{5}$$

By substituting Eq. (3) in Eq. (5) the kinetic energy becomes; $T=T_1+T_2+T_3$ where

$$T_1 = \frac{1}{2}\int_V \rho(\dot{u}_1 - \Omega(x_2 + u_2) - x_3\, \dot{u}_{3,1} - x_2\, \dot{u}_{2,1})^2\ dV, \tag{6}$$

$$T_2 = \frac{1}{2}\int_V \rho\left[\dot{u}_2 + \Omega(r + x_1 + u_1 - x_3\, u_{3,1} - x_2\, u_{2,1}) \right]^2 dV, \tag{7}$$

$$T_3 = \frac{1}{2}\int_V (\dot{u}_3)^2\, dV. \tag{8}$$

and

$$T_1 = \frac{1}{2}\int_0^L \rho A(\dot{u}_1^2 + \Omega^2 u_2^2 - 2\Omega u_2\, \dot{u}_1)\, dx_1 + \\ + \frac{1}{2}\int_0^L \rho \begin{bmatrix} I_2\, \dot{u}_{3,1}^2 + I_3 \dot{u}_{2,1}^2 + I_3\, \Omega^2 + \\ + 2 I_{23}\, \dot{u}_{2,1}\, \dot{u}_{3,1} + 2\Omega I_{23}\, \dot{u}_{3,1} + 2\Omega I_3\, \dot{u}_{2,1} \end{bmatrix}, \tag{9}$$

$$T_2 = \frac{1}{2}\int_0^L \rho A \begin{bmatrix} \dot{u}_2^2 + \Omega^2 (r + x_1 + u_1)^2 + \\ + 2\Omega\, \dot{u}_2 (r + x_1 + u_1) \end{bmatrix} dx_1 + \\ + \frac{1}{2}\int_0^L \rho \Omega^2 \left[I_2\, u_{3,1}^2 + I_3 u_{2,1}^2 + 2 I_{23}\, u_{2,1}\, u_{3,1} \right] dx_1, \tag{10}$$

$$T_3 = \frac{1}{2}\int_V (\dot{u}_3)^2\, dV, \tag{11}$$

where A is the cross-sectional area of the beam, I_2, I_3 and I_{23} are the second area moments of inertia and the second area products of inertia of the cross-section respectively.

The stain energy U of the rotating *Rayleigh* beam is defined [8]

$$U = \frac{1}{2}\int_0^L \begin{bmatrix} EA(s_{,1})^2 + EI_3 (u_{2,11})^2 + \\ + EI_2 (u_{3,11})^2 + EI_{23} (u_{2,11})(u_{3,11})\, dx_1 \end{bmatrix}, \tag{12}$$

E is Young's modulus of the beam. For x_2, x_3 principal axes of inertia $I_{23} = 0$, therefore, I_2 and I_3 are the principal second area moments of the cross-section. In the present study, s, u_2 and u_3 are approximated by spatial functions and the corresponding coordinates.

By employing the Rayleigh-Ritz method the variables are approximated as follows:

$$s(x_1,t) = \Phi_{1i}\, q_{1i} = \mathbf{\Phi}_1^T\, \mathbf{q}_1, \qquad i = 1,2.....n_1 \\ u_2(x_1,t) = \Phi_{2j}\, q_{2j} = \mathbf{\Phi}_2^T\, \mathbf{q}_2, \qquad j = 1,2.....n_2 \\ u_3(x_1,t) = \Phi_{3k}\, q_{3k} = \mathbf{\Phi}_3^T\, \mathbf{q}_3. \qquad k = 1,2.....n_3 \tag{13}$$

where n_1, n_2 and n_3 are the number of the generalized coordinates $\mathbf{q}_1$, $\mathbf{q}_2$ and $\mathbf{q}_3$; and Φ_{1i}, Φ_{2j}, and Φ_{3k} are the functions for s, u_2 and u_3. It has been already mentioned that the shape functions must obey only the geometric boundary conditions, so that it will be possible to write:

$$\begin{aligned} &\Phi_{11}(x_1) = a_j\, x_1^j, \\ &\Phi_{21}(x_1) = b_j\, x_1^j, \\ &\Phi_{31}(x_1) = c_j\, x_1^j, \qquad j = 0,1,2 \end{aligned} \tag{14}$$

where the geometric conditions must be imposed on the vertical displacements and rotations, respectively. The coefficients a_j, b_j and c_j can be determined imposing the boundary conditions, whereas the higher-order functions can be sought by means of the *Gram–Schmidt* iterative procedure [13]. The geometric boundary conditions at the ends of the beam can be written as follows:

$$x_1 = 0 \rightarrow \begin{cases} s(0) = 0, & s_{,1}(0) = 0 \\ u_2(0) = 0, & u_{2,1}(0) = 0 \\ u_3(0) = 0, & u_{3,1}(0) = 0 \end{cases} \tag{15}$$

The Lagrange's equations for free vibration of a distributed parameter are given by

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial U}{\partial q_i} = 0, \qquad i = 1,2....n \tag{16}$$

where n is the total number of modal coordinates. The partial derivatives of T with respect to the generalized coordinates are needed. Substituting (11) in terms of kinetic energy and strain, derivatives are obtained with respect to generalized coordinates given in Appendix A.

By substituting the partial derivatives into eq. (16), the linearized equations of motion can be obtained as follows:

$$\mathbf{M}^{11}\ddot{\mathbf{q}}_1 - 2\Omega\mathbf{M}^{12}\dot{\mathbf{q}}_2 + \left(\mathbf{K}^s - \Omega^2\mathbf{M}^{11}\right)\mathbf{q}_1 - \Omega^2(r\,\mathbf{P}_1 + \mathbf{Q}_1) = \mathbf{0} \tag{17}$$

$$(\mathbf{M}^{R2} + \mathbf{M}^{22})\ddot{\mathbf{q}}_2 + 2\,\Omega\,\mathbf{M}^{21}\dot{\mathbf{q}}_1 + \left[\mathbf{K}^{B2} + \Omega^2\left(\mathbf{M}^{\rho 2} - \mathbf{M}^{22} - \mathbf{M}^{R2}\right)\right]\mathbf{q}_2 = \mathbf{0} \tag{18}$$

$$\left(\mathbf{M}^{R3} + \mathbf{M}^{33}\right)\ddot{\mathbf{q}}_3 + \left(\mathbf{K}^{B3} + \Omega^2(\mathbf{M}^{\rho 3} - \mathbf{M}^{R3})\right)\mathbf{q}_3 = \mathbf{0}, \tag{19}$$

where

$$\mathbf{M}^{11} = \int_0^L \rho\, A(x_1)\, \mathbf{\Phi}_1 \mathbf{\Phi}_1^T\, dx_1, \quad \mathbf{M}^{12} = \int_0^L \rho\, A(x_1)\, \mathbf{\Phi}_1 \mathbf{\Phi}_2^T\, dx_1,$$

$$\mathbf{M}^{22} = \int_0^L \rho\, A(x_1)\, \mathbf{\Phi}_2 \mathbf{\Phi}_2^T\, dx_1, \quad \mathbf{M}^{21} = \int_0^L \rho\, A(x_1)\, \mathbf{\Phi}_2 \mathbf{\Phi}_1^T\, dx_1,$$

$$\mathbf{M}^{33} = \int_0^L \rho\, A(x_1)\, \mathbf{\Phi}_3 \mathbf{\Phi}_3^T\, dx_1$$

$$\mathbf{M}^{R2} = \int_0^L \rho\, I_3(x_1)\, \mathbf{\Phi}_{2,1} \mathbf{\Phi}_{2,1}^T\, dx_1,$$

$$\mathbf{M}^{\rho 2} = \int_0^L \rho\, A(x_1)\left[r\left(L - x_1\right) + \frac{1}{2}\left(L^2 - x_1^2\right)\right]\mathbf{\Phi}_{2,1} \mathbf{\Phi}_{2,1}^T\, dx_1,$$

$$\mathbf{M}^{R3} = \int_0^L \rho\, I_2(x_1)\, \mathbf{\Phi}_{3,1} \mathbf{\Phi}_{3,1}^T\, dx_1,$$

$$\mathbf{M}^{\rho 3} = \int_0^L \rho A(x_1)\left[r\left(L - x_1\right) + \frac{1}{2}\left(L^2 - x_1^2\right)\right]\mathbf{\Phi}_{3,1} \mathbf{\Phi}_{3,1}^T\, dx_1,$$

and

$$\mathbf{K}^S = \int_0^L EA(x_1)\, \mathbf{\Phi}_{1,1} \mathbf{\Phi}_{1,1}^T\, dx_1, \quad \mathbf{K}^{B2} = \int_0^L EI_3(x_1)\, \mathbf{\Phi}_{2,11} \mathbf{\Phi}_{2,11}^T\, dx_1,$$

$$\mathbf{K}^{B3} = \int_0^L EI_2(x_1)\, \mathbf{\Phi}_{3,11} \mathbf{\Phi}_{3,11}^T\, dx_1,$$

$$\mathbf{P}_1 = \int_0^L \rho\, A(x_1)\, \mathbf{\Phi}_1\, dx_1, \qquad \mathbf{Q}_1 = \int_0^L \rho\, A(x_1)\, x_1 \mathbf{\Phi}_1\, dx_1.$$

Equation (17) is coupled with equation (18) through gyroscopic coupling terms.

III. DIMENSIONLESS TRASFORMATION

In order to compare the results with those reported in the literature it is useful to introduce the functions *G(x)* and *H(x)* which define, in general terms, the geometric characteristics of the structure

(15)

$$\begin{aligned} A(x_1) = A_0\, G(x_1), \quad I_2(x_1) = I_{02}\, H_2(x_1), \\ I_3(x_1) = I_{03}\, H_3(x_1) \end{aligned} \tag{20}$$

where A_0, I_{o2}, and I_{03}, are respectively the area and moment of inertia of the section at $x_1 = 0$.

It is convenient to express the previous formulae in terms of the non-dimensional parameters:

$$\begin{aligned} &\xi = \frac{x_1}{L}, \quad \delta = \frac{r}{L}, \quad \gamma_f^2 = \rho\frac{A_0\,\Omega\,L^4}{E\,I_{02}}, \quad \gamma_c^2 = \rho\frac{A_0\,\Omega\,L^4}{E\,I_{03}}, \\ &r_{H2} = \left(\frac{A_0\,L^2}{I_{02}}\right)^{1/2}, \qquad r_{H3} = \left(\frac{A_0\,L^2}{I_{03}}\right)^{1/2}, \end{aligned} \tag{21}$$

$$T_2 = \left(\frac{\rho A_0 \, L^4}{EI_{02}} \right)^{1/2}, \qquad T_3 = \left(\frac{\rho A_0 \, L^4}{EI_{02}} \right)^{1/2},$$

$$\tau = \frac{t}{T}, \quad \lambda_i = \omega_i \sqrt{\frac{\rho A_0 \, L^4}{EI}}, \quad \boldsymbol{\theta}_2 = \frac{\mathbf{q}_2}{L}, \quad \boldsymbol{\theta}_3 = \frac{\mathbf{q}_3}{L}.$$

Substituting the dimensionless variables and parameters defined in eq. (21) into Eqs. (17-19), the dimensionless equations of motion can be written as

$$\bar{\mathbf{M}}^{11} \ddot{\boldsymbol{\theta}}_1 - 2\Omega \mathbf{M}^{12} \dot{\boldsymbol{\theta}}_2 + \left(r_{H3}^2 \, \mathbf{K}^s - \gamma_c^2 \, \mathbf{M}^{11} \right) \boldsymbol{\theta}_1 = \mathbf{0} \tag{22}$$

$$\begin{aligned} &\left(\frac{\bar{\mathbf{M}}^{R2}}{r_{H3}^2} + \bar{\mathbf{M}}^{22} \right) \ddot{\boldsymbol{\theta}}_2 + 2\gamma_c \, \bar{\mathbf{M}}^{21} \dot{\boldsymbol{\theta}}_2 + \\ &\qquad + \left[\bar{\mathbf{K}}^{B2} + \gamma_c^2 \left(\bar{\mathbf{M}}^{\rho 2} - \bar{\mathbf{M}}^{22} - \frac{\bar{\mathbf{M}}^{R2}}{r_{H3}^2} \right) \right] \boldsymbol{\theta}_2 = \mathbf{0} \end{aligned} \tag{23}$$

$$\left(\frac{\bar{\mathbf{M}}^{R3}}{r_{H2}^2} + \bar{\mathbf{M}}^{33} \right) \ddot{\boldsymbol{\theta}}_3 + \left[\bar{\mathbf{K}}^{B3} + \gamma_f^2 \left(\bar{\mathbf{M}}^{\rho 3} - \frac{\bar{\mathbf{M}}^{R3}}{r_{H2}^2} \right) \right] \boldsymbol{\theta}_3 = \mathbf{0} \tag{24}$$

where

$$\bar{\mathbf{M}}^{11} = \int_0^1 G(\xi) \boldsymbol{\Psi}_1 \boldsymbol{\Psi}_1^T \, d\xi, \; \bar{\mathbf{M}}^{12} = \int_0^1 G(\xi) \boldsymbol{\Psi}_1 \boldsymbol{\Psi}_2^T \, d\xi,$$

$$\bar{\mathbf{M}}^{21} = \int_0^1 G(\xi) \boldsymbol{\Psi}_2 \boldsymbol{\Psi}_1^T \, d\xi$$

$$\bar{\mathbf{M}}^{22} = \int_0^1 G(\xi) \; \boldsymbol{\Psi}_2 \boldsymbol{\Psi}_2^T \, d\xi, \quad \bar{\mathbf{M}}^{R2} = \int_0^1 H_3(\xi) \; \boldsymbol{\Psi}_{2,1} \boldsymbol{\Psi}_{2,1}^T d\xi,$$

$$\mathbf{M}^{\rho 2} = \int_0^1 G(\xi) \left[\delta(1-\xi) + \frac{1}{2}(1-\xi^2) \right] \boldsymbol{\Psi}_{2,\xi} \boldsymbol{\Psi}_{2,\xi}^T \, d\xi,$$

$$\bar{\mathbf{M}}^{R3} = \int_0^1 H_2(\xi) \; \boldsymbol{\Psi}_{3,\xi} \boldsymbol{\Psi}_{3,\xi}^T \; d\xi,$$

$$\bar{\mathbf{M}}^{\rho 3} = \int_0^1 G(\xi) \left[\delta(1-\xi) + \frac{1}{2}(1-\xi^2) \right] \boldsymbol{\Psi}_{3,\xi} \boldsymbol{\Psi}_{3,\xi}^T \; d\xi,$$

and

$$\bar{\mathbf{K}}^{S} = \int_0^1 G(\xi) \; \boldsymbol{\Psi}_{1,\xi} \boldsymbol{\Psi}_{1,\xi}^T \; d\xi, \quad \bar{\mathbf{K}}^{B2} = \int_0^1 H_3(\xi) \; \boldsymbol{\Psi}_{2,\xi\xi} \boldsymbol{\Psi}_{2,\xi\xi}^T \; d\xi,$$

$$\bar{\mathbf{K}}^{B3} = \int_0^1 H_2(\xi) \; \boldsymbol{\Psi}_{3,\xi\xi} \boldsymbol{\Psi}_{3,\xi\xi}^T \; d\xi$$

The flapwise bending vibration of the rotating beam is governed by equation (24) which is not coupled with equations (22-23). From equation (24), an eigenvalue problem for the flapwise bending vibration of a rotating cantilever beam can be formulated by assuming that the u_3,*s* are harmonic function of *t*. The variables, γ_f and δ represent the angular speed ratio and the hub radius ratio, respectively.

The coupling terms are often assumed negligible and ignored. This assumption is usually reasonable since the first stretching natural frequency is far separated from the first bending natural frequency. When the coupling effect between stretching and bending is ignored and when gyroscopic coupling terms are negligible, the equation of motion in chordwise bending vibration as

$$\left(\frac{\bar{\mathbf{M}}^{R2}}{r_{H3}^2} + \bar{\mathbf{M}}^{22} \right) \ddot{\boldsymbol{\theta}}_2 + \left[\bar{\mathbf{K}}^{B2} + \gamma_c^2 \left(\bar{\mathbf{M}}^{\rho 2} - \bar{\mathbf{M}}^{22} - \frac{\bar{\mathbf{M}}^{R2}}{r_{H3}^2} \right) \right] \boldsymbol{\theta}_2 = \mathbf{0}. \tag{25}$$

IV. NUMERICAL RESULTS

In order to obtain accurate numerical results, several assumed modes are used to construct the matrices defined in Eqs. (24). Any compact set of functions which satisfy the essential boundary condition of the Rayleigh beam can be used as the test functions; [2].

The normalized modes of a non-rotating cantilever beam, the orthogonal polynomial can be used as test functions in the numerical calculation. The span-wise variations of the cross sectional are and the second moments of area of the beams are defined by:

$$\begin{aligned} A(\xi) &= A_0 (1 - \alpha \, \xi)(1 - \beta \, \xi) = A_0 \, G(\xi), \\ I(\xi) &= I_0 (1 - \alpha \, \xi)^3 (1 - \beta \, \xi) = I_0 \, H(\xi). \end{aligned} \tag{26}$$

The flapwise bending vibration is governed by eq. (24). Assuming $\alpha=0,5$,$\beta=0$, the free frequencies are obtained using respectively *14* polynomial functions. In Table 1, the first five natural frequencies (λ_i) of the rotating tapered Rayleigh beam are given for various values of parameter r_{H2}. Results obtained are also included for comparison with Jackson et al. [6]. The full agreement with the solution in [6] is quite evident, small discrepancies can be noticed only for higher frequencies.

When Table 1 is examined, it is noticed that the rotational speed parameter has an increasing effect on the natural frequencies, which is due to the stiffening effect of the centrifugal force that is proportional to the square of the rotational speed. The full agreement with the frequencies obtained in [6] is quite evident, small discrepancies can be noticed only for higher frequencies.

In Fig.2 where the variation of the first three natural frequencies with respect to the hub radius parameter, δ and the rotational speed parameter, γ_f is plotted, the effect of the hub radius parameter, δ, is investigated for $\alpha=0.5$, $r_{H2}=1/30$.

TABLE I. THE FIRST FIVE NON-DIMENSIONAL FREQUENCIES OF ROTATING TAPERED BEAMS (α=0,5) WHIT R_{H2}=1/30; JACKSON ET AL. [6].

$\delta=\beta=0$	Present			[6]		
α=0,5	γ_f=0	γ_f=5	γ_f=10	γ_f=0	γ_f=5	γ_f=10
λ_1	3,8180	6,7391	11,4978	3,8211	6,7356	11,4856
λ_2	18,1688	21,7362	29,9639	18,2245	21,7911	30,0232
λ_3	46,3265	49,9288	59,3794	46,5757	50,1876	59,6737
λ_4	87,1368	90,7623	100,8026	87,7974	91,4413	101,5422
λ_5	139,4866	143,0885	153,3487	140,8192	144,4462	154,7865

The non-dimensional natural frequencies, λ_i, increase as the speed velocity, γ_f, increases and the rate of increase becomes larger for increasing hub radius, δ. Moreover, it is noticed that the effect of the hub radius, δ. As it can be seen from the results, the non-dimensional frequencies decrease as the hub radius ratio.

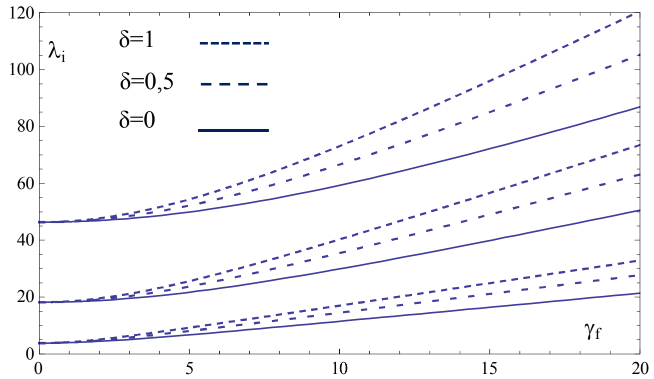

Figure 2. The effect of the hub radius ratio on the first three natural frequencies; α=0,5, r_{H2}=1/30

Resonance will occur when the angular speed of the rotating beam equals one of the natural frequencies of the beam.

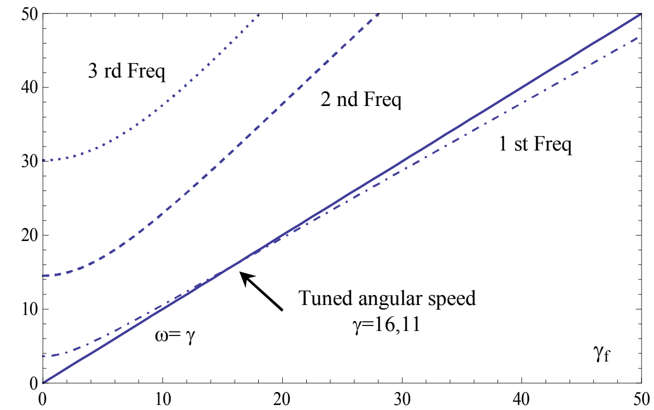

Figure 3. Tuned angular speeds; α=0,5, β=0.0 and r_{H2}=1/5.

The angular speed causing the resonance is called tuner angular speed.

In Fig. 3 the trajectories of lower three natural frequencies (for $\delta=0$, r_{H2}=1/10, $\beta=0$ and $\alpha=0,5$) and the straight line of $\omega=\gamma$ are plotted. The tuned angular speed occurs at γ=16,11 for $\delta=0$, but it does not exist for $\delta=0.5$ or δ= 1.

V. CONCLUSION

In the present study, the equation of motion for the vibration analysis of rotating *Rayleigh* beams are derived using the Lagrange's equation. The equation of motion are transformed into dimensionless forms in which the dimensionless parameter are identified. The effect of the rotational speed, slenderness ratio and hub radius ratio on the natural frequencies are investigated with the following results:

- the free vibration increases as rotational speed increases because of the stiffening effect of the centrifugal force induced from rotation, this effect is more significant on higher modes than on lower modes. The rate of increase of the natural frequencies increases as the hub radius ratio increases. Natural frequencies increase with an increasing slenderness ratio of a beam, and frequencies of a Rayleigh beam are lower than those of an Euler-Bernoulli beam;
- the tuned angular speed exists, at which resonance may occur. For a rotating Rayleigh beam, the hub radius ratio, tapered ratio and the slenderness ratio are dominant factors in affecting the tuned angular speed.

The advantage of the procedure used is the generality of polynomial functions which only need to satisfy the essential conditions.

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Appendix A:

Taking into account the (9-11) we obtain the derivatives $\frac{\partial \mathrm{T}}{\partial \mathrm{q_i}}$, neglecting higher order non-linear terms, one has,

$$\frac{\partial T}{\partial \mathbf{q}_1} = \Omega^2 \int_0^L \rho A \left[\mathbf{\Phi}_1 \mathbf{\Phi}_1^T \mathbf{q}_1 + (r + x_1) \mathbf{\Phi}_1 \right] dx_1 + \Omega \int_0^L \rho A \ \mathbf{\Phi}_1 \mathbf{\Phi}_2^T \dot{\mathbf{q}}_2 \, dx_1 \tag{A.1}$$

$$\begin{aligned} \frac{\partial T}{\partial \mathbf{q}_2} &= \Omega^2 \int_0^L \rho A \left\{ \mathbf{\Phi}_2 \mathbf{\Phi}_2^T - \left[r(L - x_1) + \frac{1}{2}(L^2 - x_1^2) - \frac{I_3}{A} \right] \mathbf{\Phi}_{2,1} \mathbf{\Phi}_{2,1}^T \right\} \mathbf{q}_2 \, dx_1 + \\ &-\Omega \int_0^L \rho A \mathbf{\Phi}_2 \mathbf{\Phi}_1^T \dot{\mathbf{q}}_1 dx_1 + \Omega^2 \int_0^L \rho I_{23} \mathbf{\Phi}_{2,1} \mathbf{\Phi}_{3,1}^T \mathbf{q}_3 dx_1 \end{aligned} \tag{A.2}$$

$$\frac{\partial T}{\partial \mathbf{q}_3} = -\Omega^2 \int_0^L \rho A \left[r(L - x_1) + \frac{1}{2}(L^2 - x_1^2) - \frac{I_2}{A} \right] \mathbf{\Phi}_{3,1} \mathbf{\Phi}_{3,1}^T \mathbf{q}_3 \, dx_1 + \Omega^2 \int_0^L \rho I_{23} \mathbf{\Phi}_{3,1} \mathbf{\Phi}_{2,1}^T \mathbf{q}_2 dx_1 \,. \tag{A.3}$$

In the same way we have:

$$\frac{d}{dt} \frac{\partial T}{\partial \dot{\mathbf{q}}_1} = \int_0^L \rho A \mathbf{\Phi}_1 \mathbf{\Phi}_1^T \ddot{\mathbf{q}}_1 \, dx_1 - \int_0^L \rho A \Omega \mathbf{\Phi}_1 \mathbf{\Phi}_2^T \dot{\mathbf{q}}_2 \, dx_1 \tag{A.4}$$

$$\begin{aligned} \frac{d}{dt} \frac{\partial T}{\partial \dot{\mathbf{q}}_2} &= \int_0^L \rho A \mathbf{\Phi}_2 \mathbf{\Phi}_2^T \ddot{\mathbf{q}}_2 \, dx_1 + \int_0^L \rho I_3 \mathbf{\Phi}_{2,1} \mathbf{\Phi}_{2,1}^T \ddot{\mathbf{q}}_2 \, dx_1 + \Omega \int_0^L \rho A \mathbf{\Phi}_2 \mathbf{\Phi}_1^T \dot{\mathbf{q}}_1 \, dx_1 + \\ &+ \int_0^L \rho I_{23} \mathbf{\Phi}_{2,1} \mathbf{\Phi}_{3,1}^T \ddot{\mathbf{q}}_3 \, dx_1 \end{aligned} \tag{A.5}$$

$$\frac{d}{dt} \frac{\partial T}{\partial \dot{\mathbf{q}}_3} = \int_0^L \rho A \mathbf{\Phi}_3 \mathbf{\Phi}_3^T \ddot{\mathbf{q}}_3 \, dx_1 + \int_0^L \rho I_2 \mathbf{\Phi}_{3,1} \mathbf{\Phi}_{3,1}^T \ddot{\mathbf{q}}_3 \, dx_1 + \int_0^L \rho I_{23} \mathbf{\Phi}_{3,1} \mathbf{\Phi}_{2,1}^T \ddot{\mathbf{q}}_2 \, dx_1 \,. \tag{A.6}$$

Using eq. (10), the partial derivatives of U with respect to the $\mathbf{q}_i$ can be obtained as

$$\frac{\partial U}{\partial \mathbf{q}_1} = \int_0^L EA \mathbf{\Phi}_{1,1} \mathbf{\Phi}_{1,1}^T \mathbf{q}_1 \, dx_1, \quad \frac{\partial U}{\partial \mathbf{q}_2} = \int_0^L EI_3 \mathbf{\Phi}_{2,11} \mathbf{\Phi}_{2,11}^T \mathbf{q}_2 \, dx_1 + \int_0^L EI_{23} \mathbf{\Phi}_{2,11} \mathbf{\Phi}_{3,11}^T \mathbf{q}_3 \, dx_1, \tag{A.7, 8}$$

$$\frac{\partial U}{\partial \mathbf{q}_3} = \int_0^L EI_2 \mathbf{\Phi}_{3,11} \mathbf{\Phi}_{3,11}^T \mathbf{q}_3 \, dx_1 + \int_0^L EI_{23} \mathbf{\Phi}_{3,11} \mathbf{\Phi}_{2,11}^T \mathbf{q}_2 \, dx_1 \,.$$