Natural frequencies of a immersed Rayleigh-beam carrying a tip mass with rotary inertia

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Abstract - **In the present paper, the free vibration analysis of an offshore structure, having the form of a column partially immersed in a liquid, is presented. A beam without contacting with water is called the dry beam, while the one partially immersed in water is called the wet beam. The column is modelled as a Rayleigh beam. For investigating the dynamic behaviour of the structure under consideration, the effect of the concentrated mass and its eccentricity are all taken into account. The analysis of free vibration frequencies and eigenfunctions of the model, presented in this paper, enables one to obtain very accurate results. The roots of the transcendental frequency equations are obtained by means the improved conventional analytical approximate solution. The non-dimensional frequency coefficients are given in tabular form. Some numerical examples are presented and the influence of different non-dimensional parameters on frequency values is discussed.**

Keywords- Offshore tower, Free vibration, Rayleigh-column.

I. INTRODUCTION

Some structures, such as towers, piles, tall buildings, offshore platforms and onshore structure are subjected to various dynamic loads as those due to wind, waves and, not secondary in these regions classified as "seismic risk areas", to earthquakes actions. Therefore, the knowledge of the dynamic behavior of such structural systems and the ability to predict dynamic response from the modal data is of utmost interest in mechanics, ocean and coastal engineering. Moreover, since the wind and waves loads and the earthquakes actions represent the prominent sources of excitation, the calculation of natural frequencies and associated mode shapes represents a significant preliminary study to evaluate the dynamic response of offshore and onshore structures. The literature regarding the free vibration analysis of beams/columns with different boundary conditions and with/without attachments is relatively rich. In the majority of the papers, Euler-Bernoulli uniform and tapered beams were considered. For non-uniform beam (particularly the linearly tapered) beam with tip mass, the reports of Mabie et al. [1], Goel [2], Abrate [3], Craver et al. [4], Auciello et al. [5], Auciello [6-7] and Firouz-Abadi et al. [8]. In the case of beams with discontinuity of section, Auciello et al. [9-10] solve the dynamic problem the free frequencies taking into account the masses applied.

In offshore engineering, since the dynamic behaviors of the structures such as piles and towers, surrounded by water, can be predicted from a cantilever beam carrying a tip mass with reasonable accuracy, the literature concerned is plenty. For example, Uscilowska et al. [11] have presented in closed form the values of the natural vibration frequencies for a uniform tower offshore. Moreover in Auciello [12] and Wu et al [13] the free vibration analysis of variable circular cross section column, carrying a tip mass and partially immersed in a fluid, has been studied. The closed form solution can be expressed in terms of Bessel functions.

In this paper, the dynamic behavior of an offshore tower, partially immersed in liquid (water), has been considered. The tower, under consideration, consists of two span beams: for convenience, the immersed beam (in contact with water) is called the "wet" beam, and the other part (a beam without contact with water) is called the "dry" beam, which represents the special case of the "wet" beam. Finally, the analysis the influence of the various parameters is examined and some of the results are presented in tabular and graphical form.

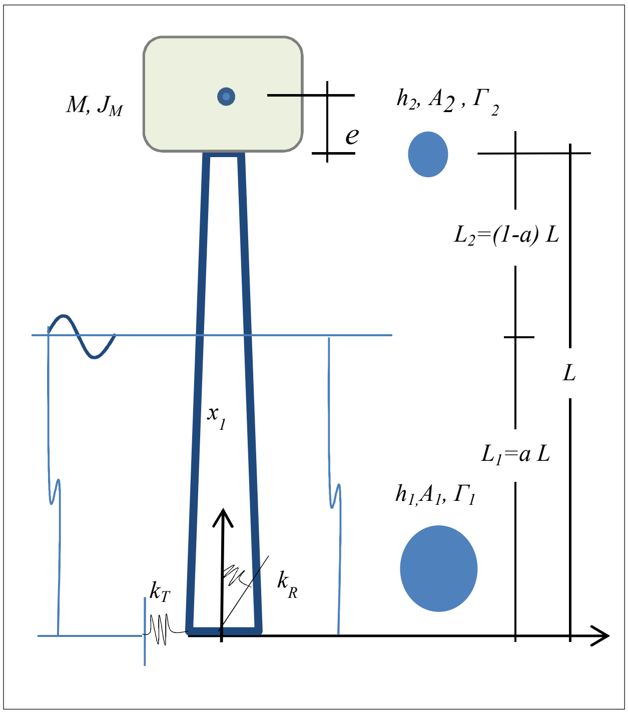

Figure 1. Offshore structure under consideration.

II. PROBLEM FORMULATION

Let us consider the tower in Figure 1, whose total span L can be divided into a partial span $L_1 = a\,L$, totally submerged, and a partial span $L_2 = (1-a)$, which is considered to be *dry*. The first span is assumed to have a tapered section, with cross sectional area A_1, moment of inertia I_1 and mass density $(\rho_w + \rho)$, where ρ_w represents the added mass density of fluid. The dry part of the beam is defined by a variable cross section with mass density ρ. The material is supposed to obey to the Hooke law, with Young modulus E, at the top the tower has an eccentric concentrated mass M with eccentricity e and rotary inertia J_M. For the immerged uniform beam the area A_1 and the moment of inertia I_1 are given by:

$$A(x) = A_1 H(x), \quad I(x) = I_1\, G(x), \quad 0 \le x \le L. \tag{1}$$

In the case of structure in which the bending effect, the axial effect are often assumed negligible and ignored.

At steady state, the system can be considered conservative and its dynamic behavior can be obtained through the *Hamilton* principle;

$$\delta \int_{t_1}^{t_2} \Phi \; dt = 0 \tag{2}$$

where $\Phi = T - U$, and T, U are respectively the kinetic and the potential energy.

The total potential energy of the system U is given by the sum of two terms: one due to bending deformation and the other due to stiffness of constraints, k_T, k_R:

$$U = \frac{1}{2} E \int_0^L I(x)\; w_{,xx}^2 \, dx + \frac{1}{2} k_T \left[w_x^2 \right]_{L_2} + \frac{1}{2} k_R \left[w_x^2 \right]_{L_2}, \tag{3}$$

where w(x,t) is the transverse displacements at the abscissa x and

$$w_{,x} = \frac{d\,w}{dx}, \qquad w_{,xx} = \frac{d^2 w}{dx^2}. \tag{4}$$

Similarly, T is the kinetic energy of the system, and if the Rayleigh beam model is assumed, the effects of the rotary inertia must be taken into account:

$$\begin{aligned} T =& \frac{1}{2}\int_0^L m(x)\, \dot{w}^2 \, dx + \frac{1}{2}\int_0^L \rho I_x \dot{w}_{,x}^2 \, dx + \\ &+ \frac{1}{2} M \left[\dot{w}^2 \right]_{x=L} + \frac{1}{2}(M\,e^2 + J_M)\left[\dot{w}_{,x}^2 \right]_{x=L} + \\ &- M\,e^2 \left[\dot{w}^2 \right]_{x=L} \left[\dot{w}_{,x}^2 \right]_{x=L} = T_1 + T_2 + T_3 \end{aligned} \tag{5}$$

where $\dot{w} = \dfrac{d\,w}{dt}$. Separating the variables, the displacement functions can be written as:

$$w(x,t) = w(x)\, e^{i\omega t}, \tag{6}$$

III. APPROXIMATE SOLUTION

In the approximate formulation the transversal displacements are assumed to be linear combination of n independent functions which satisfy the boundary equations. If functions φ_i are chosen respecting the geometrical constraints the displacements can be written; [14]:

$$w(x) = \left[\varphi_1(x) \quad \varphi_2(x) \qquad \varphi_n(x) \right] \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{bmatrix} = \boldsymbol{\varphi}^T \mathbf{q} \;,, \tag{7}$$

The elastic energy of the system can be written as

$$\begin{aligned} U =& \frac{1}{2} E\, I_1 \int_0^{L_1} G(x) \left[\left(\boldsymbol{\varphi}^T \mathbf{q} \right)_{,xx}^T \left(\boldsymbol{\varphi}^T \mathbf{q} \right)_{,xx} \right] dx + \\ &+ \frac{1}{2} E\, I_1 \int_{L_1}^{L_2} G(x) \left[\left(\boldsymbol{\varphi}^T \mathbf{q} \right)_{,xx}^T \left(\boldsymbol{\varphi}^T \mathbf{q} \right)_{,xx} \right] dx + \\ &+ \frac{1}{2} k_T \left[\left(\boldsymbol{\varphi}^T \mathbf{q} \right)^T \left(\boldsymbol{\varphi}^T \mathbf{q} \right) \right]_L + \frac{1}{2} k_R \left[\left(\boldsymbol{\varphi}^T \mathbf{q} \right)_{,x}^T \left(\boldsymbol{\varphi}^T \mathbf{q} \right)_{,x} \right], \end{aligned} \tag{8}$$

or in matrix form:

$$U = \frac{1}{2}\, \mathbf{q}^T \left\{ E\, I_1 \left(\mathbf{B}_1 + \mathbf{B}_2 \right) + k_T \mathbf{B}_T + k_R \mathbf{B}_R \right\} \mathbf{q} = \frac{1}{2}\, \mathbf{q}^T \mathbf{K}_U\, \mathbf{q}, \tag{9}$$

where

$$\begin{aligned} &\mathbf{B}_1 = \int_0^{L_1} G(x)\, \boldsymbol{\varphi}_{,xx}\, \boldsymbol{\varphi}_{,xx}^T \, dx, \qquad \mathbf{B}_2 = \int_{L_1}^{L_2} G(x)\, \boldsymbol{\varphi}_{,xx}\, \boldsymbol{\varphi}_{,xx}^T dx, \\ &\mathbf{B}_T = \left[\boldsymbol{\varphi}\, \boldsymbol{\varphi}^T \right]_{x=L}, \qquad \mathbf{B}_R = \left[\boldsymbol{\varphi}_{,x}\, \boldsymbol{\varphi}_{,x}^T \right]_{x=L}, \end{aligned} \tag{10}$$

The stiffness matrix $\mathbf{K}_U$ contains the flexural deformation energy and stiffness of constraints. Substituting (6) into (5) we get for the kinetic energy T the expression:

$$\begin{aligned} T_1 =& \frac{1}{2}\int_0^L m(x)\, \dot{w}^2 \, dx = \frac{\omega^2}{2} (\rho + \rho_w) A_1 \int_0^{L_1} H(x) \left[\left(\boldsymbol{\varphi}^T \mathbf{q} \right)^T \left(\boldsymbol{\varphi}^T \mathbf{q} \right) \right] dx + \\ &+ \frac{\omega^2}{2} \rho\, A_1 \int_{L_1}^{L_2} H(x) \left[\left(\boldsymbol{\varphi}^T \mathbf{q} \right)^T \left(\boldsymbol{\varphi}^T \mathbf{q} \right) \right] dx \end{aligned} \tag{11}$$

$$\begin{aligned} T_2 =& \frac{1}{2}\int_0^L \rho I_x \dot{w}_{,x}^2 \, dx = \frac{\omega^2}{2} (\rho + \rho_w) I_1 \int_{L_1}^{L_1} G(x) \left[\left(\boldsymbol{\varphi}^T \mathbf{q} \right)_{,x}^T \left(\boldsymbol{\varphi}^T \mathbf{q} \right)_{,x} \right] dx \\ &+ \frac{\omega^2}{2} \rho I_1 \int_{L_1}^{L_2} G(x) \left[\left(\boldsymbol{\varphi}^T \mathbf{q} \right)_{,x}^T \left(\boldsymbol{\varphi}^T \mathbf{q} \right)_{,x} \right] dx \end{aligned} \tag{12}$$

$$T_3 = \frac{\omega^2}{2} M \left[\left(\boldsymbol{\varphi}^T \mathbf{q}\right)^T \left(\boldsymbol{\varphi}^T \mathbf{q}\right) \right]_{x=L} + \frac{\omega^2}{2}(M e^2 + J_M) \left[\left(\boldsymbol{\varphi}^T \mathbf{q}\right)^T_{,x} \left(\boldsymbol{\varphi}^T \mathbf{q}\right)_{,x} \right]_{x=L} + -\omega^2 M e^2 \left[\left(\boldsymbol{\varphi}^T \mathbf{q}\right)^T \left(\boldsymbol{\varphi}^T \mathbf{q}\right) \right]_{x=L} \left[\left(\boldsymbol{\varphi}^T \mathbf{q}\right)^T_{,x} \left(\boldsymbol{\varphi}^T \mathbf{q}\right)_{,x} \right]_{x=L}. \tag{13}$$

in matrix form:

$$T_1 = \frac{\omega^2}{2}\mathbf{q}^T \left(\mathbf{M}_1 + \mathbf{M}_2\right)\mathbf{q}, \qquad T_2 = \frac{\omega^2}{2}\mathbf{q}^T \left(\tilde{\mathbf{M}}_1 + \tilde{\mathbf{M}}_2\right)\mathbf{q}, \tag{14-15}$$

$$T_3 = \frac{\omega^2}{2}\mathbf{q}^T(\mathbf{M}_3 + \mathbf{M}_4 + 2\mathbf{M}_5)\mathbf{q}, \tag{16}$$

where

$$\mathbf{M}_1 = (\rho + \rho_w) A_1 \int_0^{L_1} H(x) \boldsymbol{\varphi}\boldsymbol{\varphi}^T dx, \quad \mathbf{M}_2 = \rho A_1 \int_{L_1}^{L_2} H(x) \boldsymbol{\varphi}\boldsymbol{\varphi}^T dx, \tag{17-18}$$

$$\tilde{\mathbf{M}}_1 = (\rho + \rho_w)\Gamma_1 \int_0^{L_1} \boldsymbol{\varphi}_{,x} \boldsymbol{\varphi}^T_{,x} dx, \quad \tilde{\mathbf{M}}_2 = \rho \int_{L_1}^{L_2} I_2(x) \boldsymbol{\varphi}_{,x} \boldsymbol{\varphi}^T_{,x} dx, \tag{19-20}$$

and

$$\mathbf{M}_3 = M\left[\boldsymbol{\varphi}\,\boldsymbol{\varphi}^T\right]_{x=L}, \ \mathbf{M}_3 = (M e^2 + J_M)\left[\boldsymbol{\varphi}_{,x}\ \boldsymbol{\varphi}^T_{,x}\right]_{x=L}, \tag{21-22}$$

$$\mathbf{M}_5 = M e^2 \left[\boldsymbol{\varphi}\,\boldsymbol{\varphi}^T\right]_{x=L} \left[\boldsymbol{\varphi}_{,x}\ \boldsymbol{\varphi}^T_{,x}\right]_{x=L}. \tag{23}$$

Finally, the kinetic energy can be written as:

$$T = \frac{\omega^2}{2}\mathbf{q}^T(\mathbf{M}_1 + \mathbf{M}_2 + \tilde{\mathbf{M}}_1 + \tilde{\mathbf{M}}_2 + \mathbf{M}_3 + \mathbf{M}_4 + 2\mathbf{M}_5)\mathbf{q} = \frac{\omega^2}{2}\mathbf{q}^T \mathbf{M}\mathbf{q}, \tag{24}$$

At last, functional Φ is writer

$$\Phi = \frac{1}{2}\mathbf{q}^T \left(\mathbf{K}_U - \omega^2 \mathbf{M}\right)\mathbf{q}. \tag{25}$$

By the stationary condition of the functional in (3) leads the following eigenvalues problem:

$$\left(\mathbf{K}_U - \omega^2 \mathbf{M}\right)\mathbf{q} = 0. \tag{26}$$

The free vibration frequencies are given by calculating the roots of the characteristic polynomial

$$\det\left(\mathbf{K}_U - \omega^2 \mathbf{M}\right) = 0. \tag{27}$$

A. Non-dimensional analysis

From a numerical point of view it is convenient to introduce the non-dimensional parameters:

(9)

$$\xi = \frac{x}{L}, \quad d = \frac{e}{L}, \quad r_H = \left(\frac{A_1 L^2}{I_1}\right)^{1/2}, \ \mu = \frac{M}{m_t}, \quad k^2 = \frac{J_M}{M L^2}, \quad K_T = \frac{k_T L^3}{E I_1}, \quad K_R = \frac{k_T L}{E I_1}, \quad \lambda = \omega^2 \frac{(\rho_w + \rho) A_1 L^4}{E I_1}, \tag{28}$$

Where M is the applied tip mass and J_M is its rotary inertia. The whole mass of the beam is written as:

$$m_t = \rho A_1 \int_0^L H(x)\, dx. \tag{29}$$

As well known, the polynomial functions are chosen respecting both essential and normality conditions.

The geometric conditions are:

$$K_T w + w_{,\xi\xi\xi} = 0, \quad K_R w_{,\xi} - \frac{\partial w}{\partial \xi} = 0, \quad \text{at} \quad \xi = 0, \tag{30-31}$$

From (25) the first polynomial ϕ_1 can be obtained. After, by means of the *Gram-Schmidt* normalization, all the other requested functions can be obtained by *Mathematica* program.

IV. Free Vibration Results

In this section, the results obtained from the immersed Rayleigh beam are presented in tabular and graphical form for selected geometric and kinematic parameters. Results obtained from selected publications are also included for comparison with the Euler-Bernoulli and Rayleigh frequencies.

A. Tapered beam

Let us assume now that the variation of the cross section of the beam is given by the equations (1, 2). In the case of tapered beams, the cross section area and moment of inertia are represented by the following expressions:

$$A\ (\xi) = A_1 H(\xi), \quad I\ (\xi) = I_1\, G(\xi), \quad 0 \le \xi \le 1. \tag{31}$$

where

$$H(\xi) = [1 + \varepsilon\,\xi]^2, \quad G(\xi) = [1 + \varepsilon\,\xi]^4. \tag{32}$$

The whole mass of the beam is, (29):

$$m_t = \rho A_1 \frac{\varepsilon^2 + \varepsilon + 1}{3}. \tag{33}$$

In order to evaluate the reliability of the theory and the computer programs developed for this paper, the results of this note have been compared with those already existing in the literature; see Table I. The results have been obtained for ε= 0, i.e., [11] are reported in italics. As shown, the exact value is sensitive higher and lower, respectively, respect to the value obtained by present procedures the results are in excellent agreement.

TABLE I. COMPARISON OF FIRST THREE NATURAL FREQUENCIES OF UNIFORM EULER-BERNOULLI (E-B) BEAM; ε=0, ν=0.887, d=0, r_H=0 with K_T=K_R=0.

				Uściłowska [11]		
ε	k	μ	a	λ_1	λ_2	λ_3
0	0	1	0	*1,28589*	*4,15381*	*7,35122*
	0	2	0,5	*1,10878*	*4,05978*	*7,20006*
	$\sqrt{1/2}$	2	0,5	*0,91261*	*1,74004*	*4,89985*
			1	*0,91147*	*1,73393*	*4,84047*
				Present		
ε	k	μ	a	λ_1	λ_2	λ_3
0	0	1	0	1,28589	4,15381	7,35124
	0	2	0,5	1,10878	4,10377	7,31880
	$\sqrt{1/2}$	2	0,5	0,91261	1,74004	4,89985
			1	0,91148	1,73393	4,84047

In Table II the first three natural frequencies for a uniform Rayleigh beam are presented for two relatively extreme slenderness ratios. As expected, the Rayleigh results for the more slender beam are closer to the (E-B) results than the non-slender case. As expected, the natural frequencies of the vibration increases as the slenderness ratios →0; (E-B) beam. In Figure 2, the variation of the frequencies of the tapered Rayleigh beam and (E-B) beam is shows. It is evident from the figure that, as the slenderness ratio decreases, the Rayleigh frequencies converge the (E-B) frequencies.

TABLE II. NON-DIMENSIONAL FREQUENCIES FOR UNIFORM RAYLEIGH AS A FUNCTION OF THE LENDERNESS RATIO rH; ε=0, ν=0.887, d=0, k=0, K_T=K_R=0 and μ=2.

μ=2	*Uniform beam, a=0,5*		
	r_H=1/10	***r_H=1/30***	***(E-B)***
λ_1	1,10726	1,10861	1,10878
λ_2	3,94126	4,04573	4,05978
λ_3	6,58158	7,11601	7,20007
λ_4	8,79548	10,1409	10,3996
λ_5	10,6614	13,0321	13,5917

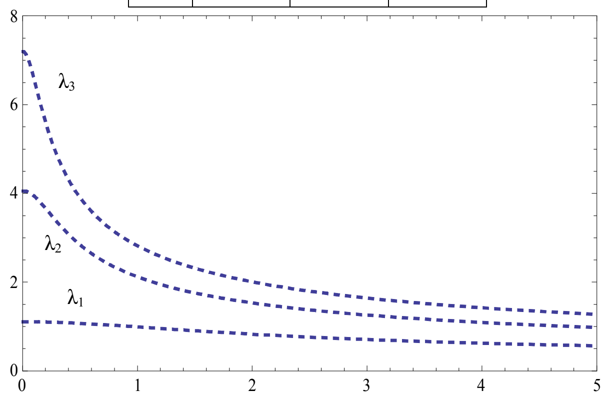

Figure 2. Natural frequencies of vibraion of uniform beam for various parameters rH; ν=0.887, d=0, k=0, KT=KR=0 and μ=2

In Table III, the first natural frequencies of tapered cantilever Rayleigh in function of taper ratio ε.

TABLE III. NON-DIMENSIONAL FREQUENCIES FOR TAPERED BEAM , ν=0.887, d=0, k=0, K_T=K_R=0 and μ=2.

a=0,5	μ=2	**r_H=1/30**	
ε	λ_1	λ_2	λ_3
0	1,10878	4,05978	7,20006
0.25	1,10971	3,91021	6,76878
0.5	1,10365	3,80816	6,51049
1	1,08222	3,66157	6,1389

Figure 3 shows the trend in the variation of the first three natural frequencies of vibration for various parameters of tip mass for $r_H=1/30$. The natural frequencies decreases when the tip mass increase.

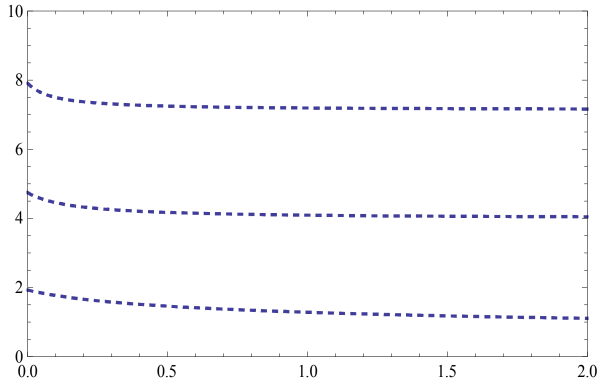

Figure 3. Natural frequencies of vibraion of non-uniform beam (rH=1/30) for various tip mass parameters μ.

V. CONCLUDING REMARKS

The approximated procedure is used the solve the free vibration problem of a immersed tapered beam based on the Rayleigh beam theory. The theory is valid for a wide range of applications the offshore structures. The column under consideration is elastically constrained at the bottom and having a mass, with rotary inertia, at its free end.

The derivation of the governing equation of motion and the algebraic manipulations are achieved by simple symbolic codes written in *Mathematica* program. A numeric algorithm is also implemented in program to compute the natural frequencies of vibration with very accurate results in comparison with the selected references in the literature available.

Also it is demonstrated in the numerical routine that, the present technique is quite simple and converges quickly to the exact solution with very minimal computational effort and resources. Furthermore, the Rayleigh theory (R-B) is proved to give very accurate results in comparison with the Euler-Bernoulli (E-B) theory. Thus, this study demonstrates the reliability and convenience of the application of the Rayleigh theory,(R-B). The natural frequencies are in excellent agreement with published results. Though for comparison purposes, the natural frequencies are kept accurate to the fourth decimal places, the precision of the natural frequencies can be increased and made as high as desired.

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