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L^p-convergence of Hermite and Hermite–Fejér interpolation

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Abstract

Necessary and sufficient conditions for the weighted L^p-convergence of Hermite and Hermite–Fejér interpolation of higher order based on Jacobi zeros are given, extending previous results for Lagrange interpolation. Error estimates in the weighted L^p-norm are also shown.

Keywords: Hermite and Hermite–Fejér interpolation of higher order; Orthogonal polynomials; Weighted L^p-convergence

1. Introduction and preliminary results

We start with some notation. We define in the usual way the $L^p := L^p(-1, 1)$ spaces, $1 \le p < \infty$, and, if $u(x) = v^{\gamma,\delta}(x) = (1-x)^\gamma(1+x)^\delta$, $\gamma, \delta > -\frac{1}{p}$, $1 \le p < \infty$, $|x| < 1$,

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we write $f \in L^p_u$, if $uf \in L^p$ and $\|f\|^p_{L^p_u} = \int_{-1}^1 |(fu)(t)|^p dt < \infty$. If $p = \infty$, we put $L^\infty_u = C_u = \{f \in C^0(-1,1) : \lim_{|x|\to 1}(fu)(x) = 0\}$, with the obvious modifications if $\gamma = 0$ or $\delta = 0$, and in particular we denote by $C_1 = C^0([-1,1])$ the set of all continuous functions on $[-1,1]$. The norm in C_u is defined by $\|f\|_{C_u} = \|fu\|_\infty = \max_{|x|\le 1} |(fu)(x)|$. Now we denote by $L_m(w; f; x)$, $w = v^{\alpha,\beta}$, $\alpha, \beta > -1$, the Lagrange polynomial interpolating a function $f \in C^0(-1,1)$ at the m zeros $x_k \equiv x_{m,k}$, $1 \le k \le m$, of the mth Jacobi polynomial $p_m(w)$, i.e. $L_m(w; f; x_i) = f(x_i)$, $i = 1, 2, \ldots, m$. Naturally, $L_m(w; f) \in \mathcal{P}_{m-1}$, where $\mathcal{P}_{m-1}$ is the set of all polynomials of degree at most $m-1$.

In what follows, C will stand for positive constants which can assume different values in each formula, and we shall write $C \ne C(a, b, \ldots)$ when C is independent of $a, b, \ldots$. Furthermore, $A \sim B$ will mean that, if A and B are positive quantities depending on some parameters, then there exists a positive constant C independent of these parameters such that $(A/B)^{\pm 1} \le C$.

It is well known that, with $1 \le p < \infty$, the estimate

$$\|L_m(w; f)u\|_p \le C\|f\|_\infty, \quad C \ne C(m, f), \tag{1.1}$$

holds for any continuous function on $[-1,1]$ if and only if the following condition on the weights is satisfied:

$$\frac{u}{\sqrt{w\varphi}} \in L^p, \quad \text{where } \varphi(x) = \sqrt{1-x^2}. \tag{1.2}$$

This is a special case of a theorem by P. Nevai in [17]. Obviously, under condition (1.1), the norm $r_m(f) := \|[f - L_m(w; f)]u\|_p$ can be estimated by the best approximation error; that is,

$$r_m(f) \le C E_{m-1}(f)_\infty = C \inf_{P_{m-1}\in\mathcal{P}_{m-1}} \|f - P_{m-1}\|_\infty. \tag{1.3}$$

If the function f is not continuous on $[-1,1]$ but belongs to C_u, $u = v^{\gamma,\delta}$, $\gamma, \delta > 0$, (for example $f(x) = \log(1+x)$), then, again using [17, (31)], (1.1) can be replaced by

$$\|L_m(w; f)u\|_p \le C\|fu\|_\infty, \quad C \ne C(m, f), \tag{1.4}$$

with $1 \le p < \infty$, under the necessary and sufficient conditions

$$\frac{u}{\sqrt{w\varphi}} \in L^p \ \frac{\sqrt{w\varphi}}{u} \in L^1. \tag{1.5}$$

Therefore, the estimate

$$r_m(f) \le c E_{m-1}(f)_{u,\infty} \tag{1.6}$$

holds, where

$$E_{m-1}(f)_{u,\infty} = \inf_{P_{m-1}\in\mathcal{P}_{m-1}} \|(f - P_{m-1})u\|_\infty.$$

For example, for $f(x) = \log(1+x)$, estimate (1.3) cannot be used. On the other hand, f is continuous in $(-1,1)$ and $f \in C_u$, $u = v^{0,\gamma}$, with arbitrary $\gamma > 0$, then, by (1.6), we get $r_m(f) = O(m^{-2\gamma})$.

However, in many contexts an estimate of $r_m(f)$ is required in terms of the L^p_u-norm of f (see [7]). A partial result in this direction appears in [14]:

$$r_m(f) \le \frac{C}{m} E_{m-2}(f')_{u,p}, \quad 1 < p < \infty, \ C \ne C(m, f), \tag{1.7}$$

under the sufficient hypothesis that the Fourier sums $S_m(w, f)$ are uniformly bounded in L^p_u; that is, $\sup_m \|S_m(w, f)u\|_p \le C\|fu\|_p$.

A complete "iff" result is given by the following [8].

Theorem A. *For any $f \in C^0(-1, 1)$, $u \in L^p$ and $1 < p < \infty$,*

$$\|L_m(w; f)u\|_p \sim \left(\sum_{k=1}^{m} \Delta x_k |(fu)(x_k)|^p\right)^{1/p}, \quad C \ne C(m, f), \tag{1.8}$$

with $\Delta x_k = x_{k+1} - x_k$, $x_k = x_{m,k}(w)$, uniformly w.r.t. m and f, if and only if

$$\frac{u}{\sqrt{w\varphi}} \in L^p, \qquad \frac{\sqrt{w\varphi}}{u} \in L^q, \qquad q^{-1} + p^{-1} = 1. \tag{1.9}$$

For the sake of completeness, we shall give an alternative proof of Theorem A in the Appendix, mainly to emphasize the necessity of conditions (1.9).

The sum in (1.8) is uniformly bounded with respect to m if f is continuous on $(-1, 1)$ and $fu \in L^p$. In the following, we assume that $f \in L^p_u$ and $\Omega^r_\varphi(f, t)_{u,p} t^{-1-1/p} \in L^1$, where $\Omega^r_\varphi(f, t)_{u,p}$ is the main part of the weighted $L^p r$th φ-modulus of continuity (see [5]). Now, the function f is continuous on $(-1, 1)$ (see, e.g., [6]), and since [8, pp. 281–283]

$$\left(\sum_{k=1}^{m} \Delta x_k |(fu)(x_k)|^p\right)^{1/p} \le \frac{C}{m^{1/p}} \int_0^{1/m} \frac{\Omega^r_\varphi(f, t)_{u,p}}{t^{1+1/p}} dt + c\|fu\|_p, \tag{1.10}$$

the summation in (1.8) is uniformly bounded. The following corollary holds.

Corollary A. *Let $f \in L^p_u$ and $\Omega^r_\varphi(f, t)_{u,p} t^{-1-1/p} \in L^1$, with $1 < p < \infty$. Then, with the notation and assumptions of Theorem A, we have*

$$\|[f - L_m(w; f)]u\|_p \le \frac{C}{m^{1/p}} \int_0^{1/m} \frac{\Omega^r_\varphi(f, t)_{u,p}}{t^{1+1/p}} dt, \quad C \ne C(m, f). \tag{1.11}$$

In the case when $f(x) = \log(1 + x)$, $u(x) = v^{\gamma,\gamma}(x) = (1 - x^2)^\gamma$, $\gamma > -1/p$, the error is dominated by $c/m^{2\gamma+2/p}$, as for the best L^p_u approximation.

The aim of this paper is to extend Theorem A and Corollary A to the case of Hermite interpolation. We will obtain, as a result, a close connection among the convergence of Lagrange, Hermite, and Hermite–Fejér interpolation in suitable function spaces. Moreover, the proved estimates cannot be improved, and they cover the ones appearing in the literature.

2. Main results

2.1. Hermite interpolation

Let us denote by $H_{m,r}(w; f)$, $r \ge 1$, the Hermite polynomial based on the Jacobi zeros and corresponding to a function $f \in C^{r-1}(-1, 1)$, i.e., $H^{(i)}_{m,r}(w; f; x_k) = f^{(i)}(x_k)$, $k = 1, \ldots, m$

and $i = 0, \ldots, r-1$; let us remark that $H_{m,1}(w; f) = L_m(w; f)$. A wide literature exists on the convergence of this sequence of operators $\{H_{m,r}(w)\}$. Among others, we mention [4,16,18–22].

The aim of the previous papers was an estimate of the norm $e_{m,r}(f) := \|[f - H_{m,r}(w, f)]u\|_p$ by the *unweighted* best approximation error in uniform norm, analogously to the Lagrange interpolation. Here, we want to generalize Theorem A to Hermite interpolation, i.e. we want to include u and p in the error estimation. To this purpose, we state the following theorem.

Theorem 1. *Let $u \in L^p$, $p \in (1, \infty)$, and $f \in C^{r-1}(-1, 1)$. Then the following inequality holds.*

$$\|H_{m,r}(w; f)u\|_p \le C \left[\left\|H_{m,r-1}(w; f)u\right\|_p + \left(\sum_{k=1}^{m} \Delta x_k \left| \left(f^{(r-1)} \left(\frac{\varphi}{m}\right)^{r-1} u \right)(x_k) \right|^p \right)^{1/p} \right], \tag{2.1}$$

with $C \ne C(m, f)$, if and only if

$$\frac{u}{\left(\sqrt{w\varphi}\right)^r} \in L^p, \qquad \frac{\left(\sqrt{w\varphi}\right)^r}{u} \in L^q, \qquad (p^{-1} + q^{-1} = 1). \tag{2.2}$$

Proof. Let us assume that (2.2) holds. Using an idea by Xu [22], with

$$Q(x) = \frac{H_{m,r}(w; f; x) - H_{m,r-1}(w; f; x)}{[p_m(w; x)]^{r-1}} \in \mathcal{P}_{m-1}, \tag{2.3}$$

we can write

$$H_{m,r}(w; f) = H_{m,r-1}(w; f) + Q p_m(w)^{r-1}.$$

Moreover, using the fact that for any measurable function g the relation (see [13,15])

$$\|g p_m(w)\|_p \sim \left\| \frac{g}{\sqrt{w\varphi}} \right\|_p, \tag{2.4}$$

holds, further, by (2.2) and Theorem A, with $u/\left(\sqrt{w\varphi}\right)^{r-1}$ instead of u and Q instead of f, we obtain

$$\left\| Q p_m^{r-1}(w) u \right\|_p \sim \left\| Q \frac{u}{\left(\sqrt{w\varphi}\right)^{r-1}} \right\|_p \le C \left(\sum_{k=1}^{m} \Delta x_k \left| \left(Q \frac{u}{\left(\sqrt{w\varphi}\right)^{r-1}} \right)(x_k) \right|^p \right)^{1/p} =: S.$$

Now, recalling (2.3) and (see [17])

$$\frac{1}{|p_m'(w, x_k)|} \sim \Delta x_k \sqrt{(w\varphi)(x_k)} \sim \left(\frac{\varphi}{m} \sqrt{w\varphi} \right)(x_k),$$

we get

$$|Q(x_k)| = \left|\frac{f^{(r-1)}(x_k) - H^{(r-1)}_{m,r-1}(w; f; x_k)}{(r-1)![p'_m(w; x_k)]^{r-1}}\right|$$
$$\sim \left(\sqrt{w\varphi}\right)^{r-1}(x_k)\left(\frac{\varphi}{m}\right)^{r-1}(x_k)\left|f^{(r-1)}(x_k) - H^{(r-1)}_{m,r-1}(w; f; x_k)\right|, \tag{2.5}$$

whence S is dominated by

$$\left(\sum_{k=1}^{m}\Delta x_k\left|\left(f^{(r-1)}\left(\frac{\varphi}{m}\right)^{r-1}u\right)(x_k)\right|^p\right)^{1/p}$$
$$+\left(\sum_{k=1}^{m}\Delta x_k\left|\left(H^{(r-1)}_{m,r-1}(w; f)\left(\frac{\varphi}{m}\right)^{r-1}u\right)(x_k)\right|^p\right)^{1/p} =: S_1 + S_2.$$

Now, by the Marcinkiewicz and Bernstein inequalities

$$S_2 \le C\left\|H_{m,r-1}(w, f)u\right\|_p,$$

so the first part of Theorem 1 ((2.2) ⇒ (2.1)) easily follows.

Now, we assume that, for any $f \in C^{r-1}$ and $u \in L^p$,

$$\|H_{m,r}(w; f)u\|_p \le C\left[\|H_{m,r-1}(w; f)u\|_p + \left(\sum_{k=1}^{m}\Delta x_k\left|\left(f^{(r-1)}\left(\frac{\varphi}{m}\right)^{r-1}u\right)(x_k)\right|^p\right)^{1/p}\right]. \tag{2.6}$$

Let $g_m : g_m^{(i)}(x_k) = 0,\ i = 0, \ldots, r-2,\ g_m^{(r-1)}(x_k) \ne 0,\ k = 1, 2, \ldots, m$. The function $g_m \in C^{r-1}(-1, 1)$, and $H_{m,r-1}(w; g_m) = 0$. Therefore, by (2.6),

$$\|H_{m,r}(w; g_m)u\|_p \le C\left(\sum_{k=1}^{m}\Delta x_k\left|\left(g_m^{(r-1)}\left(\frac{\varphi}{m}\right)^{r-1}u\right)(x_k)\right|^p\right)^{1/p}. \tag{2.7}$$

Letting $B_m = H_{m,r}(w; g_m)/p_m^{r-1}(w) \in \mathcal{P}_{m-1}$, by (2.4), we have

$$\|H_{m,r}(w; g_m)u\|_p = \left\|B_m p_m^{r-1}(w)u\right\|_p \ge C\left\|B_m\frac{u}{\left(\sqrt{w\varphi}\right)^{r-1}}\right\|_p$$
$$= C\left\|L_m(w; B_m)\frac{u}{\left(\sqrt{w\varphi}\right)^{r-1}}\right\|_p.$$

Thus (2.7) becomes

$$\left\|L_m(w, B_m)\frac{u}{\left(\sqrt{w\varphi}\right)^{r-1}}\right\|_p \le C\left(\sum_{k=1}^{m}\Delta x_k\left|\left(g_m^{(r-1)}\left(\frac{\varphi}{m}\right)^{r-1}u\right)(x_k)\right|^p\right)^{1/p}.$$

Therefore it suffices to show that the sum is equivalent to

$$\left(\sum_{k=1}^{m} \Delta x_k \left| \frac{(B_m u)(x_k)}{\sqrt{(w\varphi)^{r-1}(x_k)}} \right|^p\right)^{1/p}.$$

To this purpose, since, as before (see [17]),

$$\frac{1}{|p_m'(w, x_k)|} \sim \frac{\varphi(x_k)}{m} \sqrt{(\varphi w)(x_k)},$$

we have

$$|B_m(x_k)| = \frac{H_{m,r}(w; g_m)^{(r-1)}(x_k)}{(r-1)!|p_m'(w, x_k)|^{r-1}}$$
$$\sim \left| g_m^{(r-1)}(x_k) \right| \left(\frac{\varphi(x_k)}{m}\right)^{r-1} \left(\sqrt{(w\varphi)(x_k)}\right)^{r-1}$$

and

$$\Delta x_k \left| B_m(x_k) \frac{u(x_k)}{\left(\sqrt{(w\varphi)(x_k)}\right)^{r-1}} \right|^p \sim \Delta x_k \left| \left(g_m^{(r-1)} \left(\frac{\varphi}{m}\right)^{r-1} u \right)(x_k) \right|^p.$$

In conclusion, (2.6) implies that

$$\left\| L_m(w; B_m) \frac{u}{\left(\sqrt{w\varphi}\right)^{r-1}} \right\|_p \leq C \left(\sum_{k=1}^{m} \Delta x_k \left| \left(B_m \frac{u}{\left(\sqrt{w\varphi}\right)^{r-1}} \right)(x_k) \right|^p \right)^{1/p}. \tag{2.8}$$

Therefore, by Theorem A, with u replaced by $u/\left(\sqrt{w\varphi}\right)^{r-1}$, and f replaced by B_m, inequality (2.8) implies conditions (2.2). □

Obviously, Theorem 1 can be iterated. That means we obtain the following theorem, which generalizes Theorem A.

Theorem 2. *Let $u \in L^p$, $1 < p < \infty$, and $r > 1$. Then, for all $f \in C^{r-1}(-1, 1)$ the equivalence*

$$\left\| H_{m,r}(w; f)u \right\|_p \sim \left(\sum_{k=1}^{m} \Delta x_k \sum_{i=0}^{r-1} \left| \left(f^{(i)} \left(\frac{\varphi}{m}\right)^i u \right)(x_k) \right|^p \right)^{1/p} \tag{2.9}$$

holds uniformly w.r.t. m and f if and only if

$$\frac{u}{\left(\sqrt{w\varphi}\right)^r} \in L^p, \qquad \frac{\left(\sqrt{w\varphi}\right)^i}{u} \in L^q, \quad i = r, r-1, \ldots, 1. \tag{2.10}$$

$(p^{-1} + q^{-1} = 1)$.

For $f \in \mathcal{P}_{rm-1}$, formula (2.9) (with "≤") is the second Marcinkiewicz inequality with multiple nodes (proved first by Xu [22] using Fourier sums under stronger assumptions on the weights).

Our next statement generalizes Corollary A.

Corollary 1. *Let* $f^{(r-1)} \in L_u^p$ *and* $\Omega_\varphi^s(f^{(r-1)}, t)_{u\varphi^{r-1},p} t^{-1-1/p} \in L^1$, *with* $s \geq r \geq 1$ *and* $1 < p < \infty$. *Then, with the conditions and notation of Theorem 2, we have*

$$\left\| [f - H_{m,r}(w; f)]u \right\|_p \leq \frac{C}{m^{r-1+1/p}} \int_0^{1/m} \frac{\Omega_\varphi^s(f^{(r-1)}, t)_{u\varphi^{s-1},p}}{t^{1+1/p}} dt,$$

with $C \neq C(m, f)$.

Proof. We can write

$$\left(\sum_{k=1}^{m} \Delta x_k \sum_{i=0}^{r-1} \left| \left(f^{(i)} \left(\frac{\varphi}{m} \right)^i u \right) (x_k) \right|^p \right)^{\frac{1}{p}} \leq C \left(\sum_{i=0}^{r-2} \sum_{k=1}^{m} \Delta x_k \left| \left(f^{(i)} \left(\frac{\varphi}{m} \right)^i u \right) (x_k) \right|^p \right)^{\frac{1}{p}}$$
$$+ C \frac{1}{m^{r-1}} \left(\sum_{k=1}^{m} \Delta x_k \left| \left(f^{(r-1)} \varphi^{r-1} u \right) (x_k) \right|^p \right)^{\frac{1}{p}}.$$

The first sum is dominated by $C \sum_{i=0}^{r-1} \frac{\|f^{(i)} \varphi^i u\|_p}{m^i}$, using (1.10) several times, with f and u replaced by $f^{(i)}$ and $\varphi^i u$, respectively. The second sum is dominated by

$$C \left\{ \frac{\|f^{(r-1)} \varphi^{r-1} u\|_p}{m^{r-1}} + \frac{1}{m^{r-1+1/p}} \int_0^{1/m} \frac{\Omega_\varphi^s(f^{(r-1)}, t)_{u\varphi^{r-1},p}}{t^{1+1/p}} dt \right\},$$

with $s \geq r$, using again (1.10) with $f^{(r-1)}$ instead of f and $u\varphi^{r-1}$ instead of u.

If $P \in \mathcal{P}_{rm-1}$ is the best approximation polynomial of f, we have

$$\left\| \left[f - H_{m,r}(w, f) \right] u \right\|_p \leq C \left(\sum_{i=0}^{r-1} \frac{\left\| (f - P)^{(i)} \varphi^i u \right\|_p}{m^i} \right.$$
$$\left. + \frac{1}{m^{r-1+1/p}} \int_0^{1/m} \frac{\Omega_\varphi^s \left((f - P)^{(r-1)}, t \right)_{u\varphi^{r-1},p}}{t^{1+1/p}} dt \right). \quad (2.11)$$

The general term of the sum in the r.h.s. of (2.11) obeys the following inequality:

$$\frac{\left\| (f - P)^{(i)} \varphi^i u \right\|_p}{m^i} \leq C \frac{E_{rm-1-i}(f^{(i)})_{u\varphi^i,p}}{m^i} \leq C \frac{E_{(r-1)m} \left(f^{(r-1)} \right)_{u\varphi^{r-1},p}}{m^{r-1}},$$

whence

$$\sum_{i=0}^{r-1} \frac{\left\| (f - P)^{(i)} \varphi^i u \right\|}{m^i} \leq \frac{C}{m^{r-1+1/p}} \int_0^{1/m} \frac{\Omega_\varphi^s \left(f^{(r-1)}, t \right)_{u\varphi^{r-1},p}}{t^{1+1/p}} dt.$$

Moreover, the following obvious inequality holds for the second term in the r.h.s. of (2.11):

$$\frac{1}{m^{r-1+1/p}} \int_0^{1/m} \frac{\Omega_\varphi^s \left((f - P)^{(r-1)}; t \right)_{u\varphi^{r-1},p}}{t^{1+1/p}} dt$$
$$\leq \frac{1}{m^{r-1+1/p}} \int_0^{1/m} \frac{\Omega_\varphi^s \left(f^{(r-1)}, t \right)_{u\varphi^{r-1},p}}{t^{1+1/p}} dt$$
$$+ \frac{1}{m^{r-1+1/p}} \int_0^{1/m} \frac{\Omega_\varphi^s \left(P^{(r-1)}, t \right)_{u\varphi^{r-1},p}}{t^{1+1/p}} dt := A + B.$$

Now

$$\begin{aligned} B &\le \frac{C}{m^{r-1+1/p}} \int_0^{1/m} t^{s-1-1/p} dt \left\| \left[P^{(r-1)} \right]^{(s)} u\varphi^{r+s-1} \right\|_p \\ &\sim \frac{1}{m^{r-1}} \frac{1}{m^s} \left\| \left[P^{(r-1)} \right]^{(s)} u\varphi^{r+s-1} \right\|_p \\ &\le \frac{1}{m^{r-1+1/p}} \int_0^{1/m} \frac{\Omega_\varphi^s \left(f^{(r-1)}, t \right)_{u\varphi^{r-1},p}}{t^{1+1/p}} dt, \end{aligned}$$

where the last inequality follows from [5, p. 100], considering that $P^{(r-1)}$ is a polynomial approximating $f^{(r-1)}$ with the order as its best approximating polynomial in the proper metric (we shall call $P^{(r-1)}$ the *quasi best approximant*).

The proof is complete. □

2.2. *Hermite–Fejér interpolation*

As is well known, we can write

$$H_{m,r}(w, f) = F_{m,r}(w, f) + G_{m,r}(w, f), \quad r > 1,$$

where $F_{m,r}$ is the Hermite–Fejér interpolation polynomial of higher order defined, for $1 \le k \le m$, by

$$\begin{cases} F_{m,r}(w, f, x_k) = f(x_k), \\ F_{m,r}(w, f)^{(i)}(x_k) = 0, \quad i = 1, \ldots, r-1 \end{cases}$$

and

$$\begin{cases} G_{m,r}(w, f, x_k) = 0, \\ G_{m,r}(w, f)^{(i)}(x_k) = f^{(i)}(x_k) \quad i = 1, \ldots, r-1. \end{cases}$$

If $r = 1$, then $F_{m,1}(w, f) = L_m(w, f)$; now, we define $G_{m,1}(w, f) \equiv 0$. The behaviour of the polynomial $F_{m,r}(w, f)$ is given by the following.

Theorem 3. *Conditions (2.2) are equivalent to*

$$\|uF_{m,r}(w; f)\|_p \le C\|uF_{m,r-1}(w; f)\|_p, \quad C \ne C(m, f), \tag{2.12}$$

for any $f \in C^0(-1, 1)$ and $p \in (1, \infty)$.

Moreover, under assumptions (2.10), if $\Omega_\varphi(f, t)_{u,p} t^{-1-1/p} \in L^1$,

$$\|[f - F_{m,r}(w; f)]u\|_p \le \frac{C}{m^{1/p}} \int_0^{1/m} \frac{\Omega_\varphi(f, t)_{u,p}}{t^{1+1/p}} dt. \tag{2.13}$$

Proof. Proceeding as in the proof of Theorem 1, we let

$$q_{m-1} = \frac{F_{m,r}(w; f) - F_{m,r-1}(w; f)}{p_m(w)^{r-1}}.$$

Then we can write

$$F_{m,r}(w; f) = F_{m,r-1}(w; f) + q_{m-1} p_m^{r-1}(w).$$

In addition, using [17],

$$\left| p_m(w,x)\sqrt{(w\varphi)(x)} \right| \le C, \quad |x| \le 1 - \frac{c}{m^2}.$$

By (2.4) and (2.2), we get

$$\begin{aligned} \left\| q_{m-1} p_m'^{\,r-1}(w) u \right\|_p &\sim \left\| q_{m-1} \frac{u}{\left(\sqrt{w\varphi}\right)^{r-1}} \right\|_p \\ &\sim \left(\sum_{k=1}^{m} \Delta x_k \left| \left(q_{m-1} \frac{u}{\left(\sqrt{w\varphi}\right)^{r-1}} \right)(x_k) \right|^p \right)^{1/p}. \end{aligned}$$

Since

$$\begin{aligned} |q_{m-1}(x_k)| = \left| \frac{F_{m,r-1}(w; f; x_k)^{(r-1)}}{(r-1)! \left[p_m'(w,x_k) \right]^{r-1}} \right| &\sim \left(\frac{\varphi(x_k)}{m} \right)^{r-1} \left(\sqrt{w\varphi(x_k)} \right)^{r-1} \\ &\times \left| F_{m,r-1}(w, f; x_k)^{(r-1)} \right|, \end{aligned}$$

the sum is equivalent to

$$\left(\sum_{k=1}^{m} \Delta x_k \left| F_{m,r-1}(w; f; x_k)^{(r-1)} \left(\frac{\varphi(x_k)}{m} \right)^{r-1} u(x_k) \right|^p \right)^{1/p}.$$

Now, using [8, formula (2.19)] and the Bernstein theorem, the last sum is less than $C\|F_{m,r-1}(w; f)u\|_p$. So formula (2.12) follows.

Conversely, if (2.12) is true for any function $f \in C^0(-1,1)$, it will be true for a continuous function $\overline{g}_m$ such that $\overline{g}_m(x_k) \neq 0$ and $\overline{g}_m^{(i)}(x_k) = 0,\ k = 1, \ldots, m, i = 1, \ldots, r-1$; that is,

$$\|F_{m,r}(w; \overline{g}_m)u\|_p \le C\|F_{m,r-1}(w; \overline{g}_m)u\|_p.$$

By definition,

$$F_{m,r}(w; \overline{g}_m) = H_{m,r}(w; \overline{g}_m)$$

and

$$F_{m,r-1}(w; \overline{g}_m) = H_{m,r-1}(w; \overline{g}_m).$$

Therefore,

$$\left\| H_{m,r}(w; \overline{g}_m)u \right\|_p \le C \left\| H_{m,r-1}(w; \overline{g}_m)u \right\|_p;$$

that is, we have (2.1) with $f = \overline{g}_m$. Therefore, condition (2.2) follows.

To estimate the error, we observe that, if P_m is a quasi best approximant polynomial of $f \in L_u^p$, then we have

$$f - F_{m,r}(w; f) = (f - P_m) + [P_m - F_{m,r}(w; P_m)] - [F_{m,r}(w; f - P_m)].$$

Now, by iterating (2.12), we have $\|F_{m,r}(w; f)\|_p \le C\|L_m(w; f)u\|_p$, and it follows that

$$\begin{aligned} \|(f - P_m)u\|_p + \|F_{m,r}(w; f - P_m)u\|_p &\le \|(f - P_m)u\|_p + \|L_m(w; f - P_m)u\|_p \\ &\le \frac{C}{m^{1/p}} \int_0^{1/m} \frac{\Omega_\varphi(f,t)_{u,p}}{t^{1+1/p}} dt \end{aligned}$$

(see (1.11)). In order to estimate $\|[P_m - F_{m,r}(w; P_m)]u\|$, we notice that, since $H_{m,r}(w; f) = F_{m,r}(w; f) + G_{m,r}(w; f)$, with $G_{m,r}(w; f)(x_k) = 0$ and $G_{m,r}(w; f)^{(i)}(x_k) = f^{(i)}(x_k), i = 1, \ldots, r-1$, we have $P_m - F_{m,r}(w; P_m) = G_{m,r}(w; P_m)$.

Therefore, using the same arguments as in the proof of Theorem 1, we set

$$A(x) = \frac{G_{m,r}(w; P_m; x) - G_{m,r-1}(w; P_m; x)}{[p_m(w; x)]^{r-1}} \in \mathbb{P}_{m-1}$$

and

$$G_{m,r}(w; P_m; x) = G_{m,r-1}(w; P_m; x) + [p_m(w; x)]^{r-1} A(x).$$

Since

$$A(x_k) \frac{P_m^{(r-1)}(x_k) - G_{m,r-1}^{(r-1)}(w; P_m; x_k)}{(r-1)! \left[p'_m(w; x_k)\right]^{r-1}},$$

under assumptions (2.2), we easily get

$$\left\|G_{m,r}\,(w;\, P_m)\, u\right\|_p \le C \left\{ \left\|G_{m,r-1}\,(w;\, P_m)\, u\right\|_p + \left\| P_m^{(r-1)} \left(\frac{\varphi}{m}\right)^{r-1} u \right\|_p \right\}.$$

Under assumptions (2.10), we can iterate, obtaining

$$\left\|G_{m,r}\,(w;\, P_m)\, u\right\|_p \le C \left\{ \left\|G_{m,2}\,(w;\, P_m)\, u\right\|_p + \sum_{i=2}^{r-1} \left\| P_m^{(i)} \left(\frac{\varphi}{m}\right)^{i} u \right\|_p \right\}.$$

Since

$$\begin{aligned} \|G_{m,2}(w; P_m)u\|_p &= \left\| L_m \left(w; \frac{P'_m}{p'_m(w)} \right) p_m(w) u \right\|_p \\ &\sim \left\| L_m \left(w; \frac{P'_m}{p'_m(w)} \right) \frac{u}{\sqrt{w\varphi}} \right\|_p \\ &\sim \left(\sum_{k=1}^{m} \Delta x_k \left| P'_m(x_k) \frac{\varphi(x_k)}{m} u(x_k) \right|^p \right)^{1/p} \\ &\sim \left\| P'_m \left(\frac{\varphi}{m}\right) u \right\|_p, \end{aligned}$$

we get

$$\begin{aligned} \|G_{m,r}(w; P_m)u\|_p &\le C \sum_{i=1}^{r-1} \left\| P_m^{(i)} \left(\frac{\varphi}{m}\right)^{i} u \right\|_p \\ &\le rC \left\| P'_m \left(\frac{\varphi}{m}\right) u \right\|_p \\ &\le \frac{C}{m^{1/p}} \int_0^{1/m} \frac{\Omega_\varphi(f, t)_{u,p}}{t^{1+1/p}} dt. \end{aligned}$$

The proof is completed. □

Remark 1. For the sake of simplicity, we took the Jacobi zeros as interpolation points. Nevertheless, the stated theorems hold for a wider class of nodes. Among others we may use the zeros of orthonormal polynomials related to generalized Jacobi weights [1,15,17], to generalized Ditzian–Totik weights [8], to Badkov weights [2], or to the weights considered in [15]. Also, the weight u of the norm can be one of the previously mentioned weights, but, if u has some inner zeros, then the Ditzian–Totik φ-modulus has to be replaced by a modulus defined in [3] (see also [9–12]).

This means that, if

$$\mathcal{B}_u^s = \left\{ f^{(s)} \in L_u^p : \ \Omega_\varphi^k(f^{(s)}, t)_{u,p} t^{-1-1/p} \in L^1, \ s, k \geq 1 \right\},$$

we obtain the following.

For a wide class of interpolation nodes, the L_u^p-convergence of the sequence

$$\left\{ H_{m,r}(w, f), \ f \in \mathcal{B}_u^{r-1} \right\}$$

implies the L_u^p-convergence of the sequence

$$\left\{ L_m(w, f), \ f \in \mathcal{B}_u^0 \right\},$$

and this last is equivalent to the L_u^p-convergence of the sequence of Hermite–Fejér polynomials $\left\{ F_{m,r}(w, f), \ f \in \mathcal{B}_u^0 \right\}$.

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Appendix

Here, we give a new relatively simple proof for Theorem A of [8].

Proof. a. We prove that conditions (1.9) imply (1.8).

Let $A_m = [-1 + c/m^2, 1 - c/m^2]$. Then, by the Remez inequality [11], we have

$$\|L_m(w; f)u\|_p \leq \|L_m(w; f)u\|_{L^p(A_m)} = \sup_{\|g\|_q = 1} A(g),$$

$$A(g) = \int_{A_m} L_m(w; f, x) u(x) g(x) dx.$$

Hence

$$A(g) = \sum_{k=1}^{m} \frac{f(x_k)u(x_k)}{u(x_k)p_m'(w, x_k)} \int_{A_m} \frac{p_m(w, x)}{x - x_k} u(x)g(x)dx$$
$$\leq c \sum_{k=1}^{m} |(fu)(x_k)| \Delta x_k \frac{\sqrt{(w\varphi)(x_k)}}{u(x_k)} |\Gamma_m(g, x_k)|,$$

where, for arbitrary $Q \in \mathcal{P}_{lm}$ ($l \geq 1$, integer),

$$\Gamma_m(g, t) = \int_{A_m} \frac{p_m(w, x)Q(x) - p_m(w, t)Q(t)}{x - t} \frac{(gu)(x)}{Q(x)} dx \in \mathcal{P}_{2m-2}.$$

Consequently,

$$
\begin{aligned}
A(g) &\le C\left(\sum_{k=1}^{m}\Delta x_k|(fu)(x_k)|^p\right)^{1/p}\left(\sum_{k=1}^{m}\Delta x_k\left|\frac{\sqrt{w\varphi}(x_k)}{u(x_k)}\Gamma_m(w,x_k)\right|^q\right)^{1/q}\\
&=: C\left(\sum_{k=1}^{m}\Delta x_k|(fu)(x_k)|^p\right)^{1/p} B(g).
\end{aligned}
$$

By the Marcinkiewicz inequality, we get

$$
\begin{aligned}
B(g) &\le \left\|\frac{\sqrt{w\varphi}}{u}\Gamma_m(g)\right\|_{L^q(A_m)}\\
&\le \left\|\frac{\sqrt{w\varphi}}{u}\left(|H(p_m(w)ug)|+\left|p_m(w)QH\left(\frac{ug}{Q}\right)\right|\right)\right\|_{L^q(A_m)}\\
&:= J_1+J_2,
\end{aligned}
$$

where

$$
(Hf)(t)=\lim_{\varepsilon\to 0^+}\int_{|x-t|>\varepsilon}\frac{F(x)}{x-t}dt
$$

is the Hilbert transform. Now, if σ is a Jacobi weight, then $\|H(f)\sigma\|_p \le C\|f\sigma\|_p$, iff $\sigma \in L^p$ and $\sigma^{-1} \in L^q$. Therefore,

$$
J_1 \le C\left\|\frac{\sqrt{w\varphi}}{u}p_m(w)ug\right\|_q \le C\|g\|_q.
$$

Using that $0 < c(m) \le \sqrt{w\varphi} \le d(m)$ in A_m, we can choose Q such that $Q \sim \sqrt{w\varphi}$ in A_m. So

$$
\begin{aligned}
J_2 &\le \left\|\frac{\sqrt{w\varphi}}{u}p_m(w)QH\left(\frac{ug}{Q}\right)\right\|_{L^q(A_m)}\\
&\le C\left\|\frac{\sqrt{w\varphi}}{u}H\left(\frac{ug}{Q}\right)\right\|_{L^q(A_m)}\\
&\le C\left\|\frac{\sqrt{w\varphi}}{u}\frac{ug}{Q}\right\| \sim \|g\|_q.
\end{aligned}
$$

b. We prove that (1.8) implies (1.9).

Let us assume that (1.8) holds for all $f \in C^0(-1,1)$ and $u \in L^p$. Let τ_m be a piecewise linear function such that $\tau_m(x_k)=0$ if $x_k \notin [-\eta,\eta]$, $\eta < 1/2$, and $\tau_m(x_k)=|f(x_k)|\,\mathrm{sgn}\,p_m'(w;x_k)$, if $x_k \in [-\eta,\eta]$ (see [13]). Then (1.8) must hold with τ_m in place of f. Moreover, since $x-x_k<2$ and $\|up_m(w)\|_p \ge C\|u/\sqrt{w\varphi}\|_p$, we have

$$
\begin{aligned}
\|L_m(w,\tau_m)u\|_p &\ge \|L_m(w,\tau_m)u\|_{L^p\{|x|>\eta\}}\\
&\ge C\left\|\frac{p_m(w)u}{2}\right\|_{L^p\{|x|>\eta\}}\sum_{x_k\in[-\eta,\eta]}\frac{|\tau_m(x_k)u(x_k)|}{|p_m'(w,x_k)|u(x_k)}\\
&\ge C\left\|\frac{u}{\sqrt{w\varphi}}\right\|_{L^p\{|x|>\eta\}}\sum_{x_k\in[-\eta,\eta]}\frac{|(\tau_m u)(x_k)|}{|p_m'(w,x_k)u(x_k)|}.
\end{aligned}
$$

Then

$$\left\|\frac{u}{\sqrt{w\varphi}}\right\|_{L^p\{|x|>\eta\}} \sum_{x_k\in[-\eta,\eta]} \frac{\Delta^{1/p}x_k|(\tau_m u)(x_k)|}{|p_m'(w,x_k)u(x_k)\Delta^{1/p}x_k|}$$
$$\leq C\left(\sum_{x_k\in[-\eta,\eta]} \Delta x_k|(\tau_m u)(x_k)|^p\right)^{1/p},$$

and necessarily

$$\left\|\frac{u}{\sqrt{w\varphi}}\right\|_{L^p\{|x|>\eta\}} \left(\sum_{x_k\in[-\eta,\eta]}\left[\frac{1}{\Delta^{1/p}x_k|p_m'(w,x_k)u(x_k)|}\right]^q\right)^{1/p} \leq C.$$

The last sum is equivalent to

$$\left(\sum_{x_k\in[-\eta,\eta]} \Delta x_k\left|\frac{\sqrt{(w\varphi)(x_k)}}{u}\right|^q\right)^{1/p} \sim \left(\int_{-\eta}^{\eta}\left|\frac{\sqrt{(w\varphi)(x)}}{u(x)}\right|^q dx\right)^{1/q} \sim 1,$$

and therefore

$$\frac{u}{\sqrt{w\varphi}} \in L^p.$$

For the L^q-condition, we consider the function $\widetilde{\tau}_m(x)$ such that $\widetilde{\tau}_m(x_k) = 0$ for $x \in [-\eta, \eta]$ and

$$\widetilde{\tau}_m(x_k) = |f(x_k)|\mathrm{sgn}\, p_m'(w, x_k), \qquad x \notin [-\eta, \eta].$$

We have

$$\|L_m(w;\widetilde{\tau}_m)u\|_p \geq \|L_m(w;\widetilde{\tau})u\|_{L^p(-\eta,\eta)}$$
$$\geq \|p_m(w)u\|_{L^p(-\eta,\eta)} \sum_{x_k\notin[-\eta,\eta]} \frac{|(\widetilde{\tau}_m u)(x_k)|}{|p_m'(w,x_k)|u(x_k)}.$$

The L^p-norm is ~ 1, and

$$\sum_{x_k\notin[-\eta,\eta]} \frac{|(\widetilde{\tau}_m u)(x_k)|\Delta^{1/p}x_k}{\Delta^{1/p}x_k|p_m'(w,x_k)|u(x_k)} \leq c\left(\sum_{x_k\in[-\eta,\eta]} |(\widetilde{\tau}_m u)(x_k)|^p\right)^{1/p},$$

from which

$$\left(\sum_{x_k\notin[-\eta,\eta]}\left[\frac{1}{\Delta^{1/p}x_k|p_m'(w,x_k)|u(x_k)}\right]^q\right)^{1/q} \leq C.$$

The last sum is equivalent to

$$\left(\int_{|x|>\eta}\left|\frac{\sqrt{w\varphi(x)}}{u(x)}\right|^q dx\right)^{1/q},$$

from which we get the L^q-condition. □

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