Mediterr. J. Math. 10 (2013), 1–15
DOI 10.1007/s00009-012-0210-8
1660-5446/13/010001-15, *published online* June 24, 2012

Mediterranean Journal of Mathematics

A Weighted Generalization of the Classical Kantorovich Operator. II: Saturation

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Dedicated to Professor Francesco Altomare on his 60th birthday

Abstract. In [1], we have introduced a new weighted type of modification of the classical Kantorovich operator. The advantage of this operator is that there is no restriction on the parameters of the weight, and the class of functions is wider than in the earlier version of the weighted operator (cf. the monograph of Ditzian and Totik [3]). Direct and converse theorems and a Voronovskaya-type relation were proved. Here we solve the saturation problem of the operator (Theorem 2.1). We follow the method developed in [3], but the details are much more involved. A surprising fact emerges in determining the trivial class of saturation (Theorem 3.1).

Mathematics Subject Classification (2010). Primary 41A10; Secondary: 41A25.

Keywords. Kantorovich operator, weighted L^p approximation, saturation.

1. Introduction

Since the Bernstein polynomials are not defined for $f \in L^p([0,1])$, $1 \le p < \infty$, the Kantorovich polynomials

$$K_n(f;x) = \sum_{k=0}^{n} p_{n,k}(x)(n+1)\int_{I_k} f(u)\,du, \tag{1.1}$$

$$I_k := \left[\frac{k}{n+1}, \frac{k+1}{n+1}\right], \quad p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k},\ x \in [0,1],$$

The work of Giuseppe Mastroianni was supported by University of Basilicata (local funds). Research of József Szabados was supported by OTKA No. T049196.
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were introduced and studied deeply (see, e.g., [3, p. 115 ff.]). The case of weighted approximation by K_n operator with the Jacobi weight

$$w(x) = x^{\alpha}(1-x)^{\beta}, \quad \alpha, \beta > -1,$$

was also investigated (see [3, p. 159]) under the restrictions

$$-\frac{1}{p} < \alpha, \ \beta < 1 - \frac{1}{p} \tag{1.2}$$

on the weight parameters. Here the left side inequalities for the weight parameters are necessary. In order to remove the right hand side inequalities, we constructed a weighted generalization of the classical Kantorovich operator

$$B_n^*(f;x) = \sum_{k=0}^{n} \frac{\int_{I_k} (wf)(t)\,dt}{\int_{I_k} w(t)\,dt} p_{n,k}(x), \quad x \in [0,1], \tag{1.3}$$

where

$$-\frac{1}{p} < \alpha, \ \beta, \quad 1 \le p \le \infty$$

and

$$f \in L_w^p := \begin{cases} \{f \mid wf \in L^p(0,1)\}, & \text{if } 1 \le p < \infty, \\ \{f \mid f \in C(0,1), \ \lim_{x(1-x)\to 0}(wf)(x) = 0\}, & \text{if } p = \infty. \end{cases}$$

The norm in L_w^p is defined as

$$\|wf\|_p = \begin{cases} \left(\int_0^1 |(wf)(x)|^p\,dx\right)^{1/p}, & \text{if } 1 \le p < \infty, \\ \sup_{0\le x\le 1} |(wf)(x)|, & \text{if } p = \infty. \end{cases}$$

In order to formulate the results of [1], we need the notion of weighted modulus of smoothness:

$$\begin{aligned} \omega_\varphi^2(f;t)_{w,p} = \sup_{0<h\le t} & \|w\Delta_{h\varphi}^2\|_{L^p([Ch^2,1-Ch^2])} \\ & + \|w\overleftarrow{\Delta}_h^2 f\|_{L^p([1-Ch^2,1])} + \|w\overrightarrow{\Delta}_h^2 f\|_{L^p([0,Ch^2])}, \end{aligned} \tag{1.4}$$

where

$$\begin{aligned} \overrightarrow{\Delta}_h^2 f(x) &= f(x) - 2f(x+h) + f(x+2h), \\ \overleftarrow{\Delta}_h^2 f(x) &= f(x) - 2f(x-h) + f(x-2h), \end{aligned}$$

$$\Delta_{h\varphi}^2 f(x) = f\left(x + h\frac{\varphi(x)}{2}\right) - 2f(x) + f\left(x - h\frac{\varphi(x)}{2}\right)$$

and

$$\varphi(x) = \sqrt{x(1-x)}.$$

(cf. [3, p. 218]).

Also, for $f \in L_w^p$ define

$$E_0(f)_{w,p} := \inf_{c\in\mathbf{R}} \|w(f-c)\|_p$$

to be the best approximation of f in weighted L_w^p spaces by constants.

In [1], we proved the following approximation results.

Theorem 1.1. *If $f \in L_w^p$, $1 \le p \le \infty$, then*

$$\|w[f - B_n^*(f)]\|_p \le C\left[\omega_\varphi^2\left(f; \frac{1}{\sqrt{n}}\right)_{w,p} + \frac{E_0(f)_{w,p}}{n}\right]. \tag{1.5}$$

Theorem 1.2. *If $f \in L_w^p$, $1 \le p \le \infty$, then*

$$\|w[f - B_n^*(f)]\|_p = O(n^{-\gamma/2}) \;\Leftrightarrow\; \omega_\varphi^2(f; h)_{w,p} = O(h^\gamma), \quad 0 < \gamma < 2.$$

Theorem 1.3. *For $f'' \in C[0,1]$ we have*

$$\lim_{n\to\infty} n[B_n^*(f;x) - f(x)] = \begin{cases} \frac{\alpha+1}{\alpha+2} f'(0), & \text{if } x = 0, \\ \frac{1-2x}{2} f'(x) + \frac{x(1-x)}{2} f''(x), & \text{if } 0 < x < 1, \\ -\frac{\beta+1}{\beta+2} f'(1), & \text{if } x = 1. \end{cases} \tag{1.6}$$

2. The saturation class

Theorem 1.2 excludes the case $\gamma = 2$, which means the saturation. Here we consider this situation.

Theorem 2.1. *Let $1 < p < \infty$, and*

$$-1 < -\frac{1}{p} < \alpha, \quad \beta \ne 1 - \frac{1}{p}.$$

Then

$$\|w[f - B_n^*(f)]\|_p = O\left(\frac{1}{n}\right)$$

if and only if f is differentiable, $f' \in \mathrm{AC}_{\mathrm{loc}}$ and $\|w\varphi^2 f''\|_p < \infty$.

Remark 2.2. The case when α and/or β is equal to $1 - \frac{1}{p}$ is excluded, and remains unsolved. This situation is similar to the unsettled case $p = 1$ for the weighted ordinary Kantorovich polynomials (cf. [3], p. 176).

For the proof we need a few lemmas.

Lemma 2.3. *We have*

$$\frac{\Gamma(n+\alpha)}{n^\alpha \Gamma(n)} = 1 + O\left(\frac{1}{n}\right), \quad n \in \mathbf{N}^+,\ -1 < \alpha \in \mathbf{R}$$

where the constant in the remainder on the right hand side depends on α.

Proof. Using the Stirling formula with remainder

$$\Gamma(x) = \sqrt{2\pi} x^{x-\frac{1}{2}} e^{-x+\frac{\theta_x}{12x}}, \quad 0 < \theta_x < 1, \quad x > 0$$

(cf. [7], Section 12.33, p. 253), we obtain

$$\begin{aligned}\frac{\Gamma(n+\alpha)}{n^\alpha\Gamma(n)} =&\frac{(n+\alpha)^{n+\alpha-\frac{1}{2}}e^{-n-\alpha+\frac{\theta_{n+\alpha}}{12(n+\alpha)}}}{n^\alpha n^{n-\frac{1}{2}}e^{-n+\frac{\theta_n}{12n}}}\\ =&\left(1+\frac{\alpha}{n}\right)^{n+\alpha-\frac{1}{2}}e^{-\alpha+O\left(\frac{1}{n}\right)} = e^{-\alpha}\left(1+\frac{\alpha}{n}\right)^n\left(1+O\left(\frac{1}{n}\right)\right)\\ =&e^{-\alpha+n\log\left(1+\frac{\alpha}{n}\right)}\left(1+O\left(\frac{1}{n}\right)\right) = e^{O\left(\frac{1}{n}\right)}\left(1+O\left(\frac{1}{n}\right)\right)\\ =&1+O\left(\frac{1}{n}\right).\end{aligned}$$

□

Lemma 2.4. *For any* $g \in C^2(0,1)$ *with compact support in* $(0,1)$ *and* $wf \in L_1(0,1)$ *we have*

$$n\left|\int_0^1 w(x)[B_n^*(f,x)-f(x)]g(x)\,dx\right| \le c_g\|wf\|_1$$

where c_g *depends only on* g.

Remark 2.5. This remarkable inequality was proved by Ditzian and May [2], Lemma 5.3, in case of the unweighted original Kantorovich operator. The weighted case is stated in Ditzian and Totik [3], (10.6.2). There is no proof given, just a reference to [2] (in fact, the L_p norm on the right hand side could be replaced by L_1 norm). We will see below that the proof in the weighted case (for our operator) is quite involved, which is probably the case for the original weighted Kantorovich operator as well.

Proof. We have

$$\int_0^1 w(x)B_n^*(f,x)g(x)\,dx = \int_0^1 w(x)\sum_{k=0}^n \frac{\int_{I_k} w(t)f(t)\,dt}{\int_{I_k} w(t)\,dt}p_{nk}(x)g(x)\,dx.$$

Here let

$$g(x) = g(t) + g'(t)(x-t) + \frac{1}{2}g'(\xi)(x-t)^2, \quad \xi \in (t,x).$$

Then

$$\begin{aligned}\int_0^1 w(x)B_n^*(f,x)g(x)\,dx =& \sum_{k=0}^n \frac{\int_{I_k} w(t)f(t)g(t)\,dt}{\int_{I_k} w(t)\,dt}\int_0^1 w(x)p_{nk}(x)\,dx\\ &+ O(\|g'\|_C)\sum_{k=0}^n \frac{\int_{I_k} w(t)|f(t)|\left|\int_0^1 w(x)p_{nk}(x)(x-t)\,dx\right|\,dt}{\int_{I_k} w(t)\,dt}\\ &+ O(\|g''\|_C)\sum_{k=0}^n \frac{\int_{I_k} w(t)|f(t)|\int_0^1 w(x)p_{nk}(x)(x-t)^2\,dx\,dt}{\int_{I_k} w(t)\,dt}\\ =:& A_1+A_2+A_3.\end{aligned}$$

Estimation of A_1. By assumption, assume that $g(x)$ is supported in the interval $[a,b] \subset (0,1)$. Then we get

$$A_1 = \sum_{an\le k\le bn} \frac{\int_{I_k} w(t)f(t)g(t)\,dt}{\int_{I_k} w(t)\,dt} \int_0^1 w(x)p_{nk}(x)\,dx.$$

Using the beta integral

$$\int_0^1 x^\alpha (1-x)^\beta\,dx = \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)} \tag{2.1}$$

we obtain by Lemma 2.3

$$\begin{aligned}\int_0^1 w(x)p_{nk}(x)\,dx &= \binom{n}{k}\int_0^1 x^{k+\alpha}(1-x)^{n-k+\beta}\,dx\\ &= \binom{n}{k}\frac{\Gamma(k+\alpha+1)\Gamma(n-k+\beta+1)}{\Gamma(n+\alpha+\beta+2)}\\ &= \frac{(n+1)^{\alpha+\beta+1}\Gamma(n+1)}{\Gamma(n+\alpha+\beta+2)}\frac{\Gamma(k+\alpha+1)}{(k+1)^\alpha\Gamma(k+1)}\\ &\quad\times\frac{\Gamma(n-k+\beta+1)}{(n-k+1)^\beta\Gamma(n-k+1)}\frac{(k+1)^\alpha(n-k+1)^\beta}{(n+1)^{\alpha+\beta+1}}\\ &= \left(1+O\left(\frac{1}{n}\right)\right)\frac{(k+1)^\alpha(n-k+1)^\beta}{(n+1)^{\alpha+\beta+1}}, \quad an\le k\le bn.\end{aligned}$$

On the other hand,

$$\int_{I_k} w(x)\,dx = \frac{1}{n+1}\left(\frac{k+\theta}{n+1}\right)^\alpha\left(\frac{n-k+1-\theta}{n+1}\right)^\beta, \quad 0<\theta<1. \tag{2.2}$$

Thus

$$\begin{aligned}\frac{\int_0^1 w(x)p_{nk}(x)\,dx}{\int_{I_k} w(x)\,dx} &= \left(1+O\left(\frac{1}{n}\right)\right)\left(\frac{k+1}{k+\theta}\right)\left(\frac{n-k+1}{n-k-\theta+1}\right)^\beta\\ &= 1+O\left(\frac{1}{n}\right),\end{aligned}$$

and

$$\begin{aligned}A_1 &= \sum_{an\le k\le bn}\left(1+O\left(\frac{1}{n}\right)\right)\int_{I_k} w(t)f(t)g(t)\,dt\\ &= \int_a^b w(t)f(t)g(t)\,dt + O\left(\frac{1}{n}\right)\int_a^b w(t)|f(t)g(t)|\,dt\\ &= \int_0^1 w(t)f(t)g(t)\,dt + O\left(\frac{\|g\|_C}{n}\right)\|wf\|_1,\end{aligned}$$

i.e.,

$$\int_0^1 w(x)[B_n^*(f,x)-f(x)]g(x)\,dx = O\left(\frac{\|g\|_C}{n}\right)\|wf\|_1 + A_2 + A_3. \tag{2.3}$$

Estimation of A_2. Using again (2.1),

$$\begin{aligned}
&\left|\int_0^1 w(x)\, p_{nk}(x)(x-t)\,dx\right| \\
&\quad=\binom{n}{k}\left|\frac{\Gamma(k+\alpha+2)\Gamma(n-k+\beta+1)}{\Gamma(n+\alpha+\beta+3)}-t\frac{\Gamma(k+\alpha+1)\Gamma(n-k+\beta+1)}{\Gamma(n+\alpha+\beta+2)}\right| \\
&\quad=\binom{n}{k}\frac{\Gamma(k+\alpha+1)\Gamma(n-k+\beta+1)}{\Gamma(n+\alpha+\beta+2)}\left|\frac{k+\alpha+1}{n+\alpha+\beta+2}-t\right| \\
&\quad\le\frac{c}{n}\frac{(n+1)^{\alpha+\beta+1}\Gamma(n+1)}{\Gamma(n+\alpha+\beta+2}\frac{\Gamma(k+\alpha+1)}{(k+1)^{\alpha}\Gamma(k+1)} \\
&\qquad\times\frac{\Gamma(n-k+\beta+1)}{(n-k+1)^{\beta}\Gamma(n-k+1)}\frac{(k+1)^{\alpha}(n-k+1)^{\beta}}{(n+1)^{\alpha+\beta+1}} \\
&\quad\le c\frac{(k+1)^{\alpha}(n-k+1)^{\beta}}{n^{\alpha+\beta+2}},\quad t\in I_k,\quad k=0,1,\ldots,n.
\end{aligned}$$

Thus by (2.2) and by $\int_0^{\frac{1}{n+1}} w(x)\,dx \ge \frac{c}{n^{\alpha+1}}$,

$$\frac{\left|\int_0^1 w(x)p_{nk}(x)(x-t)\,dx\right|}{\int_{I_k} w(x)\,dx}\le\frac{c}{n},\quad t\in I_k,\quad k=0,1,\ldots,n. \tag{2.4}$$

Hence

$$A_2=O\left(\frac{\|g'\|_C}{n}\right)\sum_{k=0}^{n}\int_{I_k} w(x)|f(x)|\,dx=O\left(\frac{1}{n}\right)\|g'\|_C\|wf\|_{L_1}. \tag{2.5}$$

Estimation of A_3. We have, again by (2.1),

$$\begin{aligned}
\int_0^1 w(x)p_{nk}(x)(x-t)^2\,dx&=\binom{n}{k}\left[\frac{\Gamma(k+\alpha+3)\Gamma(n-k+\beta+1)}{\Gamma(n+\alpha+\beta+4)}\right. \\
&\quad\left.-2t\frac{\Gamma(k+\alpha+2)\Gamma(n-k+\beta+1)}{\Gamma(n+\alpha+\beta+3)}+t^2\frac{\Gamma(k+\alpha+1)\Gamma(n-k+\beta+1)}{\Gamma(n+\alpha+\beta+2)}\right] \\
&=\binom{n}{k}\frac{\Gamma(k+\alpha+1)\Gamma(n-k+\beta+1)}{\Gamma(n+\alpha+\beta+2)}\left[\frac{(k+\alpha+1)(k+\alpha+2)}{(n+\alpha+\beta+2)(n+\alpha+\beta+3)}\right. \\
&\quad\left.-2t\frac{k+\alpha+1}{n+\alpha+\beta+2}+t^2\right].
\end{aligned}$$

Here

$$\begin{aligned}
[\ldots]&\le t\left|t-\frac{k+\alpha+1}{n+\alpha+\beta+2}\right|+\frac{k+\alpha+1}{n+\alpha+\beta+2}\left|t-\frac{k+\alpha+2}{n+\alpha+\beta+3}\right| \\
&\le\frac{c(k+1)}{n^2}\le\frac{c}{n},\quad k=0,1,\ldots,n,
\end{aligned}$$

and the rest follows like in case of A_2, yielding

$$A_3=O\left(\frac{1}{n}\right)\|g''\|_C\|wf\|_{L_1}. \tag{2.6}$$

Collecting the estimates of (2.3), (2.5) and (2.6), we obtain the statement of the lemma. □

Lemma 2.6. *We have*

$$B_n^*(t-x,x)=\frac{1-2x}{2n}+O\left(\frac{1}{n^2\varphi(x)^2}\right),\quad 0<x<1 \tag{2.7}$$

and

$$B_n^*((t-x)^2,x)=\frac{\varphi(x)^2}{n}+O\left(\frac{\varphi(x)}{n^{3/2}}\right),\quad 0\le x\le 1. \tag{2.8}$$

Proof. Using the reproduction property

$$\sum_{k=0}^{n}\frac{k}{n}\binom{n}{k}x^k(1-x)^{n-k}=x$$

of the Bernstein polynomials we get

$$\begin{aligned}B_n^*(t-x,x)&=\sum_{k=0}^{n}\frac{\int_{I_k}(t-x)w(t)\,dt}{\int_{I_k}w(t)\,dt}p_{nk}(x)\\&=\sum_{k=0}^{n}\frac{\int_{I_k}\left(t-\frac{k}{n}\right)w(t)\,dt}{\int_{I_k}w(t)\,dt}p_{nk}(x)\end{aligned} \tag{2.9}$$

Here, using the integral mean value theorem

$$\begin{aligned}\frac{\int_{I_k}\left(t-\frac{k}{n}\right)w(t)\,dt}{\int_{I_k}w(t)\,dt}&=\frac{w(\xi_k)}{w(\eta_k)}\frac{\int_{I_k}\left(t-\frac{k}{n+1}\right)\,dt}{\int_{I_k}dt}-\frac{k}{n(n+1)}\\&=\left(\frac{\xi_k}{\eta_k}\right)^{\alpha}\left(\frac{1-\xi_k}{1-\eta_k}\right)^{\beta}\frac{1}{2(n+1)}-\frac{k}{n(n+1)}\\&=\frac{1}{n+1}\left[1+O\left(\frac{1}{k}\right)\right]^{\alpha}\left[1+O\left(\frac{1}{n-k}\right)\right]^{\beta}\left(\frac{1}{2}-\frac{k}{n}\right)\\&=\left[\frac{1}{n+1}+O\left(\frac{1}{k(n-k)}\right)\right]\left(\frac{1}{2}-\frac{k}{n}\right),\\&\quad\xi_k,\eta_k\in I_k,\ k=1,\dots,n-1.\end{aligned}$$

Substituting this into (2.9), we obtain

$$\begin{aligned}B_n^*(t-x,x)&=\frac{\int_{I_0}tw(t)\,dt}{\int_{I_0}w(t)\,dt}(1-x)^n\\&+\sum_{k=1}^{n-1}\left[\frac{1}{n+1}+O\left(\frac{1}{k(n-k)}\right)\right]\left(\frac{1}{2}-\frac{k}{n}\right)p_{nk}(x)\\&+\frac{\int_{I_n}(t-1)w(t)\,dt}{\int_{I_n}w(t)\,dt}x^n=\frac{1}{n+1}\sum_{k=0}^{n}\left(\frac{1}{2}-\frac{k}{n}\right)p_{nk}(x)\\&+O\left(\sum_{k=1}^{n-1}\frac{p_{nk}(x)}{k(n-k)}+\frac{x^n+(1-x)^n}{n}\right)=\frac{1-2x}{2(n+1)}\end{aligned}$$

$$+O\left(\frac{1}{n^2\varphi(x)^2}\sum_{k=0}^{n+2}\binom{n+2}{k+1}x^{k+1}(1-x)^{n-k+1}+\frac{x^n+(1-x)^n}{n}\right)$$

$$=\frac{1-2x}{2(n+1)}+O\left(\frac{1}{n^2\varphi(x)^2}+\frac{x^n+(1-x)^n}{n}\right).$$

Here evidently

$$\frac{(1-x)^n+x^n}{n}<\frac{1}{n^2\varphi(x)^2},\quad 0<x<1,$$

and (2.7) is proved.

We now prove (2.8). We have

$$\begin{aligned}B_n^*((t-x)^2,x)&=\sum_{k=0}^{n}\frac{\int_{I_k}(t-x)^2w(t)\,dt}{\int_{I_k}w(t)\,dt}p_{nk}(x)\\&=\sum_{k=0}^{n}\frac{\int_{I_k}\left(t-\frac{k}{n}\right)^2w(t)\,dt}{\int_{I_k}w(t)\,dt}p_{nk}(x)\\&\quad+2\sum_{k=0}^{n}\left(\frac{k}{n}-x\right)\frac{\int_{I_k}\left(t-\frac{k}{n}\right)w(t)\,dt}{\int_{I_k}w(t)\,dt}p_{nk}(x)\\&\quad+\sum_{k=0}^{n}\left(x-\frac{k}{n}\right)^2p_{nk}(x).\end{aligned}\tag{2.10}$$

Here the first sum is evidently $O(1/n^2)$, and the last sum is $\frac{\varphi(x)^2}{n}$. For estimating the remaining middle sum we use

$$\frac{\int_{I_k}\left(t-\frac{k}{n}\right)w(t)\,dt}{\int_{I_k}w(t)\,dt}=O\left(\frac{1}{n}\right),\quad k=0,\dots,n$$

to get

$$\begin{aligned}&2\sum_{k=0}^{n}\left(\frac{k}{n}-x\right)\frac{\int_{I_k}\left(t-\frac{k}{n}\right)w(t)\,dt}{\int_{I_k}w(t)\,dt}p_{nk}(x)=O\left(\frac{1}{n}\right)\sum_{k=0}^{n}\left|\frac{k}{n}-x\right|p_{nk}(x)\\&=O\left(\frac{1}{n}\right)\left(\sum_{k=0}^{n}\left(\frac{k}{n}-x\right)^2p_{nk}(x)\right)^{1/2}\\&=O\left(\frac{\varphi(x)}{n^{3/2}}\right)\end{aligned}$$

which completes the proof of (2.8). □

Lemma 2.7. *For any $g\in C^2(0,1)$ with compact support in $(0,1)$ and $wf\in L_1(0,1)$ we have*

$$\begin{aligned}&\lim_{n\to\infty}2n\int_0^1 w(x)[B_n^*(f,x)-f(x)]g(x)\,dx\\&\qquad=\int_0^1 f(x)[-(w(x)(1-2x)g(x))'+(w(x)\varphi(x)^2g(x))'']\,dx.\end{aligned}$$

Proof. Assume first that $f(x) \in C^2[0,1]$, and expand f into a quadratic Taylor series:

$$f(t) = f(x) + f'(x)(t-x) + \frac{1}{2}f''(x)(t-x)^2 + R(t,x), \tag{2.11}$$

where

$$R(t,x) := \frac{1}{2}[f''(\xi) - f''(x)](t-x)^2, \quad \xi \in (t,x).$$

Applying the operator B_n^* and using Lemma 2.6 we get

$$\begin{aligned} B_n^*(f,x) - f(x) =& f'(x)B_n^*(t-x,x) + \frac{f''(x)}{2}B_n^*((t-x)^2,x) + B_n^*(R(t,x),x) \\ =& f'(x)\left[\frac{1-2x}{2n} + O\left(\frac{1}{n^2}\right)\right] + \frac{f''(x)}{2}\left[\frac{\varphi(x)^2}{n} + O\left(\frac{1}{n^{3/2}}\right)\right] \\ &+ B_n^*(R(t,x),x), \end{aligned}$$

where $x \in [a,b] \subset (0,1)$ ($[a,b]$ is the support of g). Hence

$$\begin{aligned} |2n[B_n^*(f,x) - f(x)] - & [f'(x)(1-2x) + f''(x)\varphi(x)^2]| \\ & \le n|B_n^*(R(t,x),x)| + O\left(\frac{1}{\sqrt{n}}\right), \ x \in [a,b]. \end{aligned} \tag{2.12}$$

Because of the continuity of f'', to every $\varepsilon > 0$ there exists a $\delta > 0$ such that $|f''(\xi) - f''(x)| < \varepsilon$ whenever $|\xi - x| < \delta$. Thus

$$\begin{aligned} n|B_n^*(R(t,x),x)| \le & \frac{n\varepsilon}{2} \sum_{|k-nx| \le n\delta/2} \frac{\int_{I_k}(t-x)^2 w(t)\,dt}{\int_{I_k} w(t)\,dt} p_{nk}(x) \\ & + \|f''\|_C \sum_{|k-nx| > n\delta/2} p_{nk}(x) \end{aligned}$$

since in the first sum

$$|\xi - x| \le |t-x| \le \left|t - \frac{k}{n}\right| + \left|\frac{k}{n} - x\right| \le \left(\frac{1}{n} + \frac{\delta}{2}\right) \le \delta, \quad t \in I_k$$

provided that $n \ge 2/\delta$. Hence using (2.8)

$$\begin{aligned} n|B_n^*(R(t,x),x)| \le & \frac{n\varepsilon}{2} B_n^*((t-x)^2,x) + 16n\frac{\|f''\|_C}{n^4\delta^4} \\ & \times \sum_{|k-nx| > n\delta} (k-nx)^4 p_{nk}(x) \\ \le & O(\varepsilon) + O\left(\frac{1}{n\delta^4}\right) = O(\varepsilon), \ x \in [a,b] \end{aligned}$$

provided that $n \ge \frac{1}{\varepsilon\delta^4}$. This together with (2.12) shows that

$$\lim_{n\to\infty} 2n[B_n^*(f,x) - f(x)] = f'(x)(1-2x) + f''(x)\varphi(x)^2$$

uniformly in $x \in [a,b]$. Multiplying by $w(x)g(x)$ and integrating over $[a,b]$ we obtain by integration by parts

$$\begin{aligned}\lim_{n\to\infty} 2n \int_a^b w(x)[B_n^*(f,x) - f(x)]g(x)\,dx \\ &= \int_0^1 w(x)[f'(x)(1-2x) + f''(x)\varphi(x)^2]g(x)\,dx \\ &= \int_0^1 f(x)[-(w(x)(1-2x)g(x))' + (w(x)\varphi(x)^2 g(x))'']\,dx\end{aligned}$$

which proves the lemma for $f \in C^2[0,1]$. In other words, we have proved that the sequence of linear functionals

$$a_n(f) := 2n \int_0^1 w(x)[B_n^*(f,x) - f(x)]g(x)\,dx, \quad n = 1,2,\dots \tag{2.13}$$

converges to the linear functional

$$a(f) := \int_0^1 f(x)[-(w(x)(1-2x)g(x))' + (w(x)\varphi(x)^2 g(x))'']\,dx$$

in the space of twice continuously differentiable functions, which is dense in the weighted $L_1(0,1)$ space. Moreover the functionals (2.13) are bounded by Lemma 2.4. Hence the convergence in the whole weighted space $L_1(0,1)$. □

Proof of Theorem 2.1. If

$$f' \in \mathrm{AC}_{\mathrm{loc}} \quad \text{and} \quad \|w\varphi^2 f''\|_p < \infty, \tag{2.14}$$

then $\omega_\varphi^2(f;t)_{w,p} = O(t^2)$ (this follows from (6.1.7) and (6.1.1) of [3]), whence by Theorem 1.1

$$\|w[f - B_n^*(f)]\|_p = O\left(\frac{1}{n}\right). \tag{2.15}$$

This proves the direct statement.

In order to prove the converse, assume (2.15). Then, by Lemma 2.7, it follows that f is differentiable, f' is locally absolutely continuous, and f satisfies the linear second order differential equation

$$(1-2x)f'(x) + \varphi(x)^2 f''(x) = \frac{h(x)}{w(x)} \quad \text{a.e.} \tag{2.16}$$

where $h \in L_p(0,1)$ (for details, see [3], pp. 177-178). By symmetry, it will be sufficient to prove the boundedness of the norm in (2.14) in the interval $[0,1/2]$.

Case 1: $-1/p < \alpha < 1-1/p$. Then the situation is exactly the same as in case of weighted ordinary Kantorovich polynomials, and the proof on pp. 177-178 of [3] applies; we omit the repetition of the arguments there.

Case 2: $\alpha > 1-1/p$. Then we have to modify the arguments of the cited proof. The solution of the differential equation (2.16) can be given in the form

$$f(x) = \int_x^{1/2} \frac{1}{\varphi(u)^2} \int_u^1 \frac{h(\tau)}{w(\tau)}\,d\tau\,du + c_1 + c_2 \log\frac{x}{1-x},$$

where c_1, c_2 are constants. Hence

$$f''(x) = \frac{1-2x}{\varphi(x)^4}\int_x^1 \frac{h(u)}{w(u)}\,du + \frac{h(x)}{\varphi(x)^2 w(x)} - \frac{c_2(1-2x)}{\varphi(x)^4},$$

and thus

$$w(x)\varphi(x)^2 f''(x) = h(x) + \frac{(1-2x)w(x)}{\varphi(x)^2}\int_x^1 \frac{h(u)}{w(u)}\,du - \frac{c_2(1-2x)w(x)}{\varphi(x)^2}.$$

We have to show that the L_p norm of this function in $[0,1/2]$ is finite. We get

$$w(x)\varphi(x)^2|f''(x)| \le |h(x)| + 2^{\beta+1}x^{\alpha-1}\int_x^1 \frac{|h(u)|}{u^\alpha}\,du + 2c_2x^{\alpha-1},\ 0<x<1/2.$$

Here the first and third terms are in $L_p(0,1)$. For the second term we apply Hardy's inequality to get

$$\int_0^1 x^{(\alpha-1)p}\left(\int_x^1 \frac{|h(u)|}{u^\alpha}\,du\right)^p dx \le c(p,\alpha)\int_0^1 |h(x)|^p\,dx < \infty$$

(see [4], Theorem 330). Hence Case 2, and thus the theorem is proved. □

Remark 2.8. It is interesting to observe that while in Case 1 the log-function was eliminated (see the above quoted proof from [3]), in Case 2 it was part of the function representing the saturation class. This peculiar role of the log-function will be seen in the next section as well.

3. The trivial class

When an operator is saturated with order $O(\varepsilon_n)$, then the class of functions which are approximated with order $o(\varepsilon_n)$ are called the *trivial class* of saturation. This class, in general, consists of some simple functions (constants, linear functions), and actually the operator reproduces them. Surprisingly, as we will see, this is not the case with our weighted Kantorovich operator, at least for some values of the parameters.

To begin with, we state the following result.

Theorem 3.1. *Let* $1 \le p \le \infty$, $\gamma := \min(\alpha,\beta) > 1 - 1/p$ *and*

$$\psi(x) = \log\frac{x}{1-x}.$$

Then

$$\|w[\psi - B_n^*(\psi)]\|_p = O\left(\frac{\log n}{n^{\gamma+1/p}}\right) = o\left(\frac{1}{n}\right).$$

Remark 3.2. We have seen in the previous section that this log-function plays an important role in connection with Kantorovich type operators. Theorem 3.1 says that for some values of the parameters this nontrivial function can be better approximated by our operator than elements of the saturation class.

Proof. We consider the case $1 \le p < \infty$; for $p = \infty$ the proof can be easily established from the arguments below.

First we estimate the L_p norm on the intervals $[0, 1/n]$ and $[1-1/n, 1]$. By symmetry, it is sufficient to consider $[0, 1/n]$. Since $x \log^p \frac{1}{x}$ increases for small x, we get

$$\begin{aligned}\int_0^{1/n} w(x)^p \psi(x)^p \, dx \le & c \int_0^{1/n} x^{\alpha p} \log^p x \, dx \\ & \le \frac{\log^p n}{n} \int_0^{1/n} x^{\alpha p - 1} \, dx \le \frac{c \log^p n}{n^{\alpha p + 1}}.\end{aligned}$$

Similarly, since $|B_n^*(\psi, x)| \le c \log n$,

$$\int_0^{1/n} w(x)^p |B_n^*(\psi, x)|^p \, dx \le \frac{c \log^p n}{n^{\alpha p+1}}.$$

Next we have to estimate the L_p norm on the interval $[1/n, 1/2]$ (the case of interval $[1/2, 1-1/n]$ is similar). We consider the quadratic Taylor expansion

$$\begin{aligned}\psi(t) =& \psi(x) + \psi'(x)(x-t) + \frac{1}{2}\psi''(x)(x-t)^2 + R(x,t) \\ =& \psi(x) + \frac{1}{\varphi(x)^2}(x-t) - \frac{1-2x}{2\varphi(x)^4}(x-t)^2 + R(x,t)\end{aligned}$$

where

$$R(x,t) = \frac{1}{2}[\psi''(\xi) - \psi''(x)](x-t)^2, \quad \xi \in (x,t), \tag{3.1}$$

and use Lemma 2.6 when applying the operator B_n^*:

$$\begin{aligned}B_n^*(\psi, x) - \psi(x) =& \frac{1}{\varphi(x)^2} B_n^*(x-t, x) - \frac{1-2x}{2\varphi(x)^4} B_n^*((x-t)^2, x) + B_n^*(R, x) \\ =& \frac{1}{\varphi(x)^2}\left[\frac{1-2x}{2n} + O\left(\frac{1}{n^2\varphi(x)^2}\right)\right] \\ & - \frac{1-2x}{2\varphi(x)^4}\left[\frac{\varphi(x)^2}{n} + O\left(\frac{\varphi(x)}{n^{3/2}}\right)\right] + B_n^*(R,x) \\ =& O\left(\frac{1}{(nx)^{3/2}} + |B_n^*(R,x)|\right), \quad \frac{1}{n} \le x \le \frac{1}{2}.\end{aligned}$$

We obtain from (3.1)

$$\begin{aligned}|R(x,t)| \le & \left(\left|\frac{1}{x^2} - \frac{1}{t^2}\right| + \left|\frac{1}{(1-x)^2} - \frac{1}{(1-t)^2}\right|\right)(x-t)^2 \\ =& c\left(\frac{1}{x^2\varphi(t)^2} + \frac{1}{x\varphi(t)^4}\right)|x-t|^3 \\ :=& R_1(x,t) + R_2(x,t), \quad \frac{1}{n} \le x \le \frac{1}{2}.\end{aligned}$$

Here

$$\frac{\int_{I_k} R_1(x,t)w(t)\,dt}{\int_{I_k} w(t)\,dt} \le \frac{c}{x^2}\frac{(n+2)(n+1)}{(k+1)(n-k+1)}\left(\left|x-\frac{k}{n+2}\right|^3+\frac{1}{n^3}\right)$$
$$\left(t\in I_k,\ 1\le k\le n,\ \frac{1}{n}\le x\le\frac{1}{2}\right).$$

The latter inequality extends easily to $k=0$; namely then

$$R_1(x,t)\le\frac{c}{x^2t}(x-t)^3\le\frac{cx}{t},\quad t\in I_0,\ \frac{1}{n}\le x\le\frac{1}{2}.$$

Thus

$$\begin{aligned}B_n^*(R_1,x)\le&\frac{c}{x^2}\left[\sum_{k=0}^{n}\left|x-\frac{k}{n+2}\right|^3\binom{n+2}{k+1}x^k(1-x)^{n-k}+\frac{1}{n^3x}\right]\\ \le&\frac{c}{x^3}\sum_{k=0}^{n+2}\left|x-\frac{k}{n+2}\right|^3p_{n+2,k+1}(x)+\frac{c}{(nx)^3},\quad\frac{1}{n}\le x\le\frac{1}{2}.\end{aligned}$$

Here by Cauchy–Schwarz inequality

$$\begin{aligned}&\sum_{k=0}^{n+2}\left|x-\frac{k}{n+2}\right|^3p_{n+2,k+1}(x)\\ &\le\left\{\sum_{k=0}^{n+2}\left(x-\frac{k}{n+2}\right)^2p_{n+2,k+1}(x)\sum_{k=0}^{n+2}\left(x-\frac{k}{n+2}\right)^4p_{n+2,k+1}(x)\right\}^{1/2}\\ &\le c\left(\frac{x}{n}\right)^{3/2},\quad\frac{1}{n}\le x\le\frac{1}{2}\end{aligned}$$

(cf. [5], p. 16). Thus

$$B_n^*(R_1,x)\le\frac{c}{(nx)^{3/2}},\quad\frac{1}{n}\le x\le\frac{1}{2},$$

whence

$$\int_{1/n}^{1/2}w(x)^p|B_n^*(R_1,x)|^p\,dx\le\frac{c}{n^{3p/2}}\int_{1/n}^{1}x^{(\alpha-3/2)p}\,dx\le c\frac{\log^p n}{n^{\alpha p+1}}.$$

Using similar arguments, the same estimate can be obtained for $R_2(x,t)$. Collecting all the above estimates we obtain the statement of the theorem. □

Finally, we formulate the statement about the trivial class of saturation.

Theorem 3.3. *Let $1<p<\infty$. We have*

$$\|w[f-B_n^*(f)]\|_p=o\left(\frac{1}{n}\right)$$

if and only if (i) *$f(x)$ is constant, when $-1/p<\gamma=\min(\alpha,\beta)<1-1/p$;* (ii) *$f(x)=c_1+c_2\psi(x)$, when $\gamma>1-1/p$ (where c_1, c_2 are arbitrary constants).*

Proof. The direct statements follow from Theorem 3.1 and the fact that constants are reproduced by B_n^*.

The converse statement in (a) follows from the corresponding part of the proof of Theorem 2.1. Namely, suppose e.g. that $-1/p < \alpha < 1 - 1/p$. Then in the quoted proof the function $h(x)$ is identically zero, and also $c_2 = 0$. Similarly, in (b) again $h(x)$ is identically zero, and $f(x) = c_1 + c_2\psi(x)$, for which

$$f'(x) = \frac{c_2}{\varphi(x)^2} \in \mathrm{AC}_{\mathrm{loc}}, \quad f''(x) = -\frac{c_2(1-2x)}{\varphi(x)^4},$$

and

$$\|w\varphi^2 f''\|_p^p \le c_2^p \int_0^1 x^{(\alpha-1)p}(1-x)^{(\beta-1)p}\, dx < \infty. \qquad \square$$

Acknowledgments

The authors thank the referee for careful reading and suggestions concerning the original manuscript.

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Received: November 23, 2011.
Revised: February 23, 2012.
Accepted: March 7, 2012.