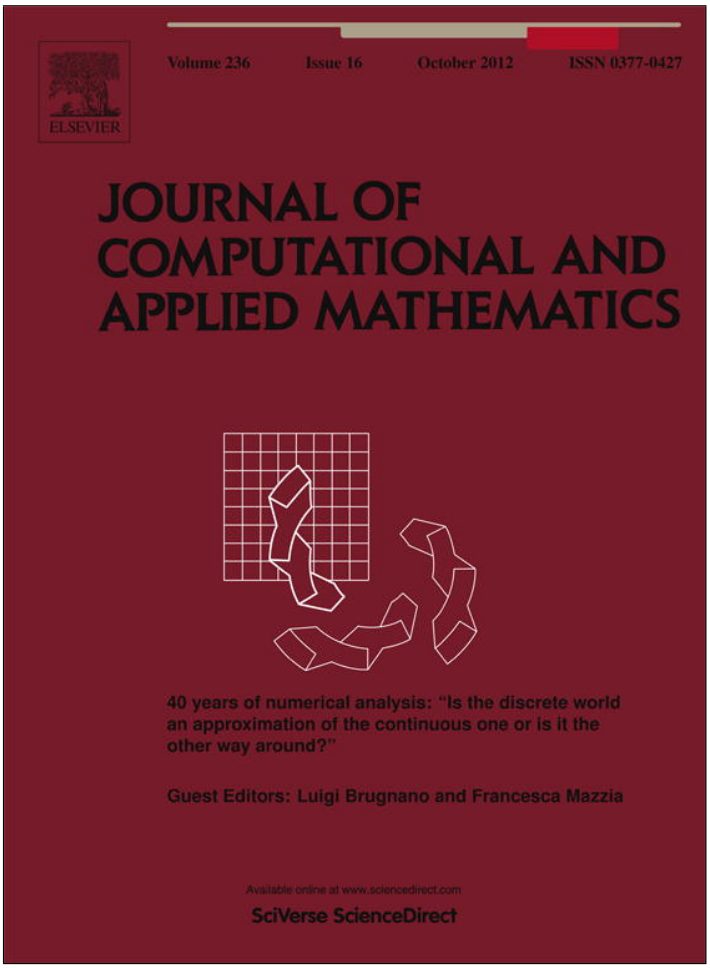

Journal of Computational and Applied Mathematics 236 (2012) 4082–4089

Contents lists available at SciVerse ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

Numerical integration of functions with a very small significant support☆

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ARTICLE INFO

Article history:
Received 6 October 2011
Received in revised form 21 December 2011

To the memory of our friend Donato Trigiante

MSC:
65D30
65R20

Keywords:
Numerical integration
Quadrature rules
Nyström interpolants

ABSTRACT

In some applications, one has to deal with the problem of integrating, over a bounded interval, a smooth function taking significant values, with respect to the machine precision or to the accuracy one wants to achieve, only in a very small part of the domain of integration. In this paper, we propose a simple and efficient numerical approach to compute or discretize integrals of this type. We also consider a class of second kind integral equations whose integral operator has the above behavior. Some numerical testing is presented.

1. Introduction

In some applications, one has to deal with the problem of integrating, over a bounded interval, a smooth function taking significant values, with respect to the machine precision or to the accuracy one wants to achieve, only in a very small part of the domain of integration. A couple of examples of this situation are given by the following integrals:

$$I_1 = \int_0^1 \frac{x}{\sqrt{1-x}} f(x)dx, \quad f(x) = \frac{e^{-(2x_1^2+x_2^2)}}{\sqrt{1+x}}, \qquad x_i = x_i(x), \tag{1}$$

$$I_2(x) = \int_\Gamma e^{-\frac{|x-y|^2}{4\epsilon}} f(y)d\Gamma_y, \quad x \in \Gamma \tag{2}$$

where I_1 is taken from [1], while in the second integral, which represents a heat potential-Gauss transform, Γ is a (open or closed) smooth curve in R^2 and $\epsilon > 0$ is a parameter whose values are (very) small.

In the applications where we have taken these two examples, it is crucial to approximate, with the needed accuracy, the integrals by means of simple quadrature rules using, if possible, a very low number of nodes. Since we are assuming that the functions $f(x)$ are very smooth, a natural approach is that of using proper Gaussian rules. However, in spite of the degree of smoothness of $f(x)$ in I_1, and of the whole integrand functions of I_2, the use of corresponding Gaussian rules would require an excessively high number of function evaluations. This is because the majorities of the quadrature nodes fall where the integrand functions take negligible values.

☆ This work was supported by Consiglio Nazionale delle Ricerche-G.N.C.S. and the Ministero dell'Istruzione, dell'Università e della Ricerca of Italy.

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doi:10.1016/j.cam.2011.12.018

In the next section we will present an alternative and more efficient approach to compute integrals of the type above. Then, in Section 3 we will use these quadrature rules to construct stable and convergent Nyström interpolants for the numerical solution of second kind integral equations, defined by integral operators whose integrands have a behavior like that described above.

2. A truncated Gaussian rule

For simplicity, we will consider the numerical evaluation of integrals of the form

$$\int_0^1 w(x)f(x)dx \tag{3}$$

where $w(x) = (1-x)^\alpha x^\beta,\ \alpha, \beta > -1$ is the Jacobi weight function, and $f(x)$ is smooth in $[0, 1]$ and decay exponentially to zero away from the endpoint $x = 0$. That is, $f(x)$ is considered significant, with respect to the machine precision or to the required accuracy, only in an interval of the type $[0, \tau)$, $0 < \tau < 1$, τ away from endpoint $x = 1$. In a more general situation the function $f(x)$ may have a two sided peak, or even more than one peak; however, by partitioning accordingly the interval of integration $(0, 1)$ we can always reduce the problem to the case considered above.

Therefore we assume that $|f(x)| \le \epsilon$, with $\epsilon > 0$ chosen by the user, in $[\tau, 1]$. The most natural approach to approximate the above integral is to use the Gauss–Jacobi rule

$$\int_0^1 w(x)f(x)dx = \sum_{i=1}^n \lambda_i f(x_i) + e_n(f) \tag{4}$$

where, here and in the following, its nodes are assumed to be ordered as follows: $0 < x_1 < \cdots < x_n < 1$. We recall that for the remainder $e_n(f)$ several estimates are well known (see, for example [2]). However, taking into account the assumption we have made on the behavior of $f(x)$, and the positivity of the coefficients λ_i (and their behavior with respect to the index i), it is equally natural to delete from the Gaussian quadrature sum all the terms referring to abscissas falling in the interval $[\tau, 1]$. That is, having defined

$$n_\tau = n_\tau(n) : x_{n_\tau} = \max\{x_i \le \tau\} \tag{5}$$

we replace (4) by

$$\int_0^1 w(x)f(x)dx = \sum_{i=1}^{n_\tau} \lambda_i f(x_i) + e_n^\tau(f) \tag{6}$$

where now we have

$$e_n^\tau(f) = \sum_{i=n_\tau+1}^{n} \lambda_i f(x_i) + e_n(f),$$

that is,

$$|e_n^\tau(f)| \le |e_n(f)| + \epsilon \sum_{i=n_\tau+1}^{n} \lambda_i < |e_n(f)| + \epsilon\mu_0$$

where we have set

$$\mu_0 = \int_0^1 w(x)dx = B(\alpha+1, \beta+1)$$

$B(x, y)$ being the well known Beta function (see [9]). Notice that in the Legendre case ($\alpha = \beta = 0$) we have $\mu_0 = 1$. Notice also that (see [9]) $n_\tau \to \infty$ as $n \to \infty$. Thus the truncated Gaussian rule has the same error behavior of the full Gaussian rule, except for an extra term of order ϵ, which can be chosen as small as we like, by taking τ sufficiently close to $x = 1$.

Of course, if the order of magnitude of the maximum of $|f(x)|$ in $[0, \tau]$ is very different from 1, one should determine first an estimate (M_f) of this value and then replace above ϵ by ϵM_f.

The truncated Gaussian rule strategy is not a new idea. We have applied it in the case of Gauss–Laguerre formulas, to approximate integrals defined on $(0, \infty)$ whose integrand functions decay exponentially at infinity (see [3,4]). It has been applied also to the Gauss–Hermite case (see [5]).

A simpler and obvious approach to approximate integral (3), alternative to the truncation one, is to write first

$$\int_0^1 w(x)f(x)dx \approx \int_0^\tau w(x)f(x)dx \tag{7}$$

Table 1
Example 1: $x_1 = 1 - 20x$, $x_2 = 1$. Relative errors.

n	GJ_n	$n_{0.5}$	$TGJ_{n,0.5}$	$GJ_{n,0.5}$	$n_{0.3}$	$TGJ_{n,0.3}$	$GJ_{n,0.3}$	$n_{0.2}$	$TGJ_{n,0.2}$	$GJ_{n,0.2}$
4	9.98E−01	2	9.98E−01	8.06E−01	1	9.98E−01	1.13E−01	1	9.98E−01	4.85E−02
8	1.64E−01	4	1.64E−01	1.02E−01	2	1.64E−01	9.95E−03	2	1.64E−01	2.59E−04
16	1.97E−02	8	1.97E−02	3.11E−04	5	1.97E−02	7.15E−07	4	1.97E−02	4.04E−09
32	3.47E−06	16	3.47E−06	1.45E−11	11	3.47E−06	1.53E−14	9	3.47E−06	4.08E−09
64	2.79E−14	32	2.79E−14	1.59E−14	23	2.79E−14	1.42E−14	18	1.03E−08	4.08E−09
128	1.14E−14	64	1.14E−14	3.74E−16	47	1.14E−14	9.35E−15	37	5.99E−09	4.08E−09
256	3.74E−15	128	3.74E−15		94	3.74E−15	1.07E−14	75	4.02E−09	4.08E−09

Table 2
Example 1: $x_1 = 1 - 40x$, $x_2 = 1$. Relative errors.

n	GJ_n	$n_{0.3}$	$TGJ_{n,0.3}$	$GJ_{n,0.3}$	$n_{0.2}$	$TGJ_{n,0.2}$	$GJ_{n,0.2}$	$n_{0.15}$	$TGJ_{n,0.15}$	$GJ_{n,0.15}$
4	1.00E+00	1	1.00E+00	4.26E−01	1	1.00E+00	5.42E−01			1.12E−01
8	3.47E−02	2	3.47E−02	5.66E−02	2	3.44E−02	1.89E−02	1	3.44E−02	1.00E−03
16	4.24E−02	5	4.24E−02	8.40E−04	4	4.24E−02	1.55E−05	3	4.24E−02	7.19E−07
32	1.06E−03	11	1.06E−03	6.81E−10	9	1.06E−03	8.01E−14	8	1.06E−03	6.85E−14
64	7.93E−10	23	7.93E−10	7.15E−14	18	7.93E−10	6.63E−14	16	7.93E−10	6.91E−14
128	9.96E−14	47	9.96E−14	8.89E−14	37	9.96E−14	7.84E−14	32	9.96E−14	7.36E−14
256	1.06E−13	94	1.06E−13	8.48E−14	75	1.06E−13	7.66E−14	64	1.06E−13	7.28E−14

hence to apply a Gauss–Jacobi Gaussian rule, with weight function x^α, to the truncated integral. Notice that under the assumption we have made on the function $f(x)$, the truncated part of the integral is bounded by $\epsilon B(\alpha + 1, \beta + 1)$.

In the following we will consider three integrals and apply to each of them the following three quadrature rules:

(i) the n-point Gauss–Jacobi rule applied to the whole interval $(0, 1)$, denoted by (GJ_n)
(ii) the truncated (n, n_τ)-point Gauss–Jacobi rule applied to the whole interval $(0, 1)$, denoted by $(TGJ_{n,\tau})$
(iii) the n-point modified Gauss–Jacobi rule applied to the interval $(0, \tau)$ $(GJ_{n,\tau})$.

Notice that the truncated rule (6), that we shall use in the next section to define a Nyström interpolant for the solution of a second kind integral equation, can also be interpreted as a rule which approximates the integral over the interval $(0, \tau)$. Indeed, the following convergence result holds.

Proposition 1. *Let $f \in C[-1, 1]$. Given any real $0 < \tau < 1$, if we define*

$$\int_0^\tau w(x) f(x) dx = \sum_{i=1}^{n_\tau} \lambda_i f(x_i) + r_n^\tau(f) \tag{8}$$

where the quadrature sum is the same we have in (6), then

$$\lim_{n\to\infty} r_n^\tau(f) = 0.$$

This result follows immediately from the well known (see, for example, [2]) convergence property of the Gaussian rules, when they are applied to bounded Riemann integrable functions. Indeed it is sufficient to consider the function

$$f_\tau(x) = \begin{cases} f(x), & 0 \le x \le \tau \\ 0, & \tau < x \le 1 \end{cases}$$

and the corresponding integral defined in $(0, 1)$.

Example 1. Here we consider integral (1) with

$$x_1 = 1 - kx, \quad k = 20, 40, \qquad x_2 = 1.$$

Notice that for $k = 20$ and $x > \tau$, $\tau = 0.5, 0.3$ the function $|f(x)|$ takes values smaller than 1.32E−71, 6.22E−23, respectively, so that in both cases all the terms ignored by the truncated rules do not give a significant contribution to the full Gaussian rule, as confirmed by the above Table 1. A similar remark applies also to the integral tail we have ignored to construct the $GJ_{n,\tau}$ rule. If instead, in this same case we take $\tau = 0.2$, where the integrand function takes the value 5.11E−09, then we expect from the $TGJ_{n,\tau}$ and $GJ_{n\tau}$ rules a maximum accuracy of order 10^{-9}.

From the results presented in Tables 1 and 2, it emerges that, according to the accuracy one wants to achieve, if one takes a proper value of the cutting point τ, rules TGJ_n and $GL_{n,\tau}$ can offer a significant function evaluation saving.

Table 3
Example 2: Relative errors.

n	$2GL_{n/2}$	$2GL_{n/2,0.4}$	$n_{0.4}$	$2TGL_{n/2,0.4}$
4	9.54E−01			
8	3.27E−03	5.37E−02	2	1.00E+00
16	2.92E−03	1.47E−04	4	7.16E−02
32	2.34E−10	3.70E−10	8	2.77E−02
64	1.86E−15	2.60E−15	14	2.11E−05
128	8.67E−16	4.96E−15	30	2.30E−13
256			118	1.25E−15

Table 4
Example 3: Relative errors.

n	GL_n	$2GL_{n/2}$	$n_{0.4}$	$2TGL_{n/2,0.4}$	$n_{0.3}$	$2TGL_{n/2,0.3}$
4	1.00E+00	1.00E+00	2	1.00E+00	2	1.00E+00
8	9.67E−01	3.14E−01	4	3.14E−01	4	3.14E−01
16	3.73E−02	4.66E−02	6	4.66E−02	6	4.66E−02
32	8.30E−03	1.50E−03	14	1.50E−03	12	1.50E−03
64	1.00E−04	7.61E−06	28	7.61E−06	24	7.61E−06
128	2.20E−08	4.63E−10	56	4.63E−10	48	4.63E−10
256	1.67E−14	7.67E−15	112	7.67E−15	94	7.67E−15

Remark 2.1. Recalling the inequalities known for the zeros of Jacobi polynomials (see [6]), it is possible to determine a priori the value of the index n_τ defined in (5). For example, in the Legendre case we have

$$x_i < \frac{1}{2}\left[1 + \cos\left(\frac{n+\frac{1}{2}-i}{n+1}\pi\right)\right], \quad i = 1, \ldots, n$$

from which it follows:

$$n_\tau = \left\lceil n + \frac{1}{2} - \frac{n+1}{\pi}\arccos(2\tau - 1)\right\rceil.$$

Example 2. Here we consider an integral of type (2), where Γ is the unit circle:

$$F(x) = \int_{-\pi}^{\pi} e^{-\frac{(x_1-\cos\theta)^2+(x_2-\sin\theta)^2}{4\epsilon}}\cos\theta d\theta$$

where we have chosen $(x_1, x_2) = (1, 0)$ and $\epsilon = 10^{-3}$.

In this case, the integrand function has a (smooth) peak at $\theta = 0$. Therefore, we split the interval of integration into two parts: $(-\pi, 0)$, $(0, \pi)$, and apply first a $n/2$-Gauss–Legendre rule to each subinterval; then, after having chosen two proper cut points, for example $\tau = \pm 0.4$, we apply the corresponding $GL_{n/2,\tau}$ and $TGL_{n/2,\tau}$ rules. The associated composite rule are denoted by $2GL_n$, $2GL_{n/2,\tau}$ and $2TGL_{n/2,\tau}$, respectively.

In Table 3 we report the relative errors given by these alternative approaches. The integer n denotes the total number of nodes used by the first two approaches, while n_τ is the total number of nodes used by the third one.

Example 3. In this last example we apply our rules to the integral

$$\int_{-1}^{1} e^{-\frac{1}{(1-x^2)^{50}}}\cos(\pi x)dx,$$

which has been taken from [7]. In this case, for $|x| > 0.4, 0.3, 0.25$ we have $|f(x)| \doteq 0, <1.87\text{E}{-}49, 8.02\text{E}{-}12$, respectively.

The results reported in Table 4 are inferior to those produced by the Gaussian rule examined in [7]. However we ought also to notice that the construction of the latter is computationally much more expensive than that of the truncated $2TGL$ one.

Finally, we remark that in all examples examined above, the $TGJ_{n,\tau}$, $TGL_{n,\tau}$ approaches have shown to be slightly more efficient than the $GL_{n,\tau}$, $2GL_{n,\tau}$ ones; this is due to the nonuniform distribution of the zeros of the Jacobi (Legendre) polynomials (see [6]). Indeed, in the latter rules we have a higher concentration of nodes not only in the "function peak" neighborhood, but also around the abscissas τ, where the integrand function is almost negligible, or even negligible if τ has been underestimated. For this reason, in the following section we will not consider the $GL_{n,\tau}$, $2GL_{n,\tau}$ approaches.

3. A Nyström interpolant for second kind integral equations

In this last section we use the truncated Gaussian rule presented in Section 2, to construct a Nyström type interpolant for the numerical solution of integral equations of the type

$$u(x) - \int_0^1 k(x,y)u(y)dy = f(x), \quad 0 \le x \le 1 \tag{9}$$

that is

$$(I - K)u = f \tag{10}$$

where $k(x,y)$ is smooth and $|k(x,y)| \le g(y)$, with $g(y)$ having one of the behaviors described in the previous section.

To test the truncated Gauss–Legendre approach, we will apply it to the following two equations:

$$u(x) - \frac{1}{4}\int_0^1 e^{-\frac{1+(x-y)^2}{(1-y)^{20}}} u(y)dy = 1, \quad 0 \le x \le 1 \tag{11}$$

$$u(x) - \frac{1}{2}\int_{-1}^1 e^{-\frac{1+(x-y)^2}{(1-y^2)^{40}}} u(y)dy = 1, \quad -1 \le x \le 1. \tag{12}$$

Notice that in these two particular examples we expect a small contribution given by Ku, hence a solution $u \approx f$. This is indeed confirmed by the numerical results we have obtained.

To each of these integral equations we will apply the complete Gauss–Legendre rule and the associated truncated version, according to the strategy described in the previous section. In both cases we will choose $\epsilon = 1\mathrm{E} - 15$, since the order of magnitude of the solution, in the uniform norm, is 1. When the order of magnitude is very different from 1, we can determine a rough approximation of $\|u\|_\infty$, let us say M_u, by solving first the integral equation using the standard Gauss–Legendre rule with a very low number of nodes. Then, before applying the truncated version, to achieve the required accuracy, we scale the equation unknown by dividing the equation by the constant M_u.

To construct our Nyström interpolant, in the case of Eq. (11) we solve first the linear system

$$u_n^\tau(x_j) - \sum_{i=1}^{n_\tau} \lambda_i k(x_j, x_i) u_n^\tau(x_i) = f(x_j), \quad j = 1, \ldots, n \tag{13}$$

for the unknown $\{u_n^\tau(x_j)\}$, and then define the continuous approximant

$$u_n^\tau(x) = f(x) + \sum_{i=1}^{n_\tau} \lambda_i k(x, x_i) u_n^\tau(x_i). \tag{14}$$

In matrix form, system (13) can be written as follows:

$$\begin{pmatrix} I - K_n^\tau & O \\ -K_n^\tau & I \end{pmatrix} \begin{pmatrix} u_n^{\tau,1} \\ u_n^{\tau,2} \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$$

where $u_n^{\tau,1}, f_1 \in R^{n_\tau}$ and $u_n^{\tau,2}, f_2 \in R^{n-n_\tau}$. Notice however that we only have to solve the system

$$(I - K_n^\tau)u_n^{\tau,1} = f_1,$$

of order n_τ.

In the case of Eq. (12), to discretize the integral operator we split the domain of integration into two parts: $(-1, 0)$, $(0, 1)$, and apply a n-point truncated Gauss–Legendre rule to each subinterval. In this particular case we take two symmetric truncation points: $\pm\tau$. Thus, denoting by

$$-1 < x_{-n} < \cdots < x_{-1} < 0 < x_1 < \cdots < x_n < 1$$

the sequence of the nodes of the $2n$-point composite Gauss–Legendre rule, the linear system we have to solve takes the form:

$$u_n^\tau(x_j) - \sum_{i=-n_\tau}^{n_\tau} \lambda_i k(x_j, x_i) u_n^\tau(x_i) = f(x_j), \quad j = -n, \ldots, n \tag{15}$$

that is,

$$\begin{pmatrix} I & -K_n^\tau & O \\ O & I - K_n^\tau & O \\ O & -K_n^\tau & I \end{pmatrix} \begin{pmatrix} u_n^{\tau,1} \\ u_n^{\tau,2} \\ u_n^{\tau,3} \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix}$$

Table 5
Relative errors at x; Eq. (11), $\tau = 0.2$.

n	n_τ	cond	$x = 0$	$x = 0.5$	$x = 0.99$
8	2	1.006	1.69E−04	1.51E−04	4.63E−05
16	5	1.005	6.04E−06	3.47E−06	1.11E−06
32	9	1.005	4.98E−09	2.85E−09	1.10E−09
64	19	1.005	7.06E−14	3.21E−14	1.33E−15

Table 6
Relative errors at x; Eq. (12), $\tau = 0.3$.

$2n$	$2n_\tau$	cond	$x = -0.9$	$x = 0.1$	$x = 0.99$
8	2	1.004	1.48E−02	3.46E−02	1.22E−02
16	4	1.087	6.19E−04	3.55E−03	9.15E−04
32	6	1.082	7.39E−05	8.28E−05	5.62E−05
64	12	1.083	1.52E−08	1.84E−06	1.39E−08
128	24	1.083	2.21E−11	1.69E−10	1.57E−11
256	50	1.083	2.19E−16	4.26E−16	2.19E−16

Table 7
Relative errors at x; Eq. (11bis), $\tau = 0.2$.

n	n_τ	cond	$x = 0$	$x = 0.5$	$x = 1$
8	2	1.41	3.61E−01	5.79E−05	6.27E−08
16	5	1.38	1.12E−02	6.52E−06	2.47E−07
32	9	1.36	3.23E−03	7.39E−08	8.88E−09
64	19	1.37	6.78E−05	2.03E−08	1.97E−09
128	38	1.37	1.69E−05	5.06E−09	4.94E−10
256	75	1.37	4.04E−06	1.21E−09	1.18E−10
512	151	1.37	8.09E−07	2.42E−10	2.37E−11

where $u_n^{\tau,1}, f_1 \in R^{n-n_\tau}$, $u_n^{\tau,2}, f_2 \in R^{2n_\tau}$ and $u_n^{\tau,3}, f_3 \in R^{n-n_\tau}$. In this case we only need to solve the system

$$(I - K_n^\tau)u_n^{\tau,2} = f_2,$$

of order $2n_\tau$.

In Tables 5 and 6 we report some results we have obtained by applying the strategy described above, with $\tau = 0.2$ in the case of (11), and $\tau = 0.3$ for Eq. (12). The reference values are those we have obtained taking $n = 512$. The choices $\tau = 0.5$ ($n_\tau = n/2$) and $\tau = 1$ ($n_\tau = n$) have given the same results. The label cond denote the 2-norm condition number of the matrix $I - K_n^\tau$.

We have also solved Eq. (11) after having replaced the coefficient 1/4 by 20, the kernel by $e^{\frac{1+|x-y|}{(1-y)^{20}}}$ and the known term $f(x) = 1$ by $f(x) = x^2$. The corresponding results are reported in the Table 7, where the equation is denoted by (11bis). The reference values have been computed taking $n = 1024$. These are: $u(0) = 1.803363487833016\text{e} - 04$, $u(0.5) = 2.500715944413509\text{e} - 01$, $u(1) = 1.000027056477329\text{e} + 00$.

As in the previous case, the choices $\tau = 0.5$ ($n_\tau = n/2$) and $\tau = 1$ ($n_\tau = n$) have given the same results.

By using standard arguments, under the assumption that $\|K\| := \|K\|_\infty < 1$ it is possible to show that the proposed Nyström method indeed allows to achieve the required accuracy, as confirmed by the numerical examples we have presented above. Unfortunately, till now we have not been able to show that this property holds by requiring only that the operator $I - K$ has a bounded inverse.

To this end, we recall that since we have assumed the kernel $k(x, y)$ to be smooth, the operator $K : C[0, 1] \to C[0, 1]$ is compact. Moreover, since $\|K\| < 1$, the operator $I - K$ has necessarily a bounded inverse. To simplify the description of the proofs we shall give next, we refer to the case where $|k(x, y)| \le g(y)$ with $g(y)$ decaying exponentially as y moves toward the endpoint 1.

We also define the following two discrete operators:

$$K_n u(x) = \sum_{i=1}^{n} \lambda_i k(x, x_i) u(x_i)$$

and

$$K_n^\tau u(x) = \sum_{i=1}^{n_\tau} \lambda_i k(x, x_i) u(x_i)$$

where n_τ has been defined in (5).

Proposition 2. *Given any real $\delta > 0$ and any truncation point $0 < \tau < 1$, for all integers n sufficiently large we have*

$$\|K_n^\tau\| < \|K_n\| < \|K\| + \delta.$$

Thus, if $\|K\| < 1$, by taking δ small enough we also have

$$\|K_n^\tau\| \le C < 1$$

which implies that the operator $I - K_n^\tau$ has a (uniformly) bounded inverse.

Proof. Recalling that

$$\|K\| = \max_{0\le x\le 1} \int_0^1 |k(x, y)|dy$$

and

$$\|K_n\| = \max_{0\le x\le 1} \sum_{i=1}^n \lambda_i |k(x, x_i)|$$

we examine the behavior of the remainder term of the Gauss–Legendre rule when this is applied to the function

$$|k(x, y)|.$$

Notice that this is Lipschitz continuous in $[0, 1] \times [0, 1]$, with respect to both variables. Because of this, there exist (see [8]) an algebraic polynomial $p_n(x, y)$ of degree n with respect to each variable, such that

$$\||k| - p_n\| \le cn^{-1}, \quad n \to \infty$$

where the uniform norm is taken with respect to both variables.

Recalling these properties we then write:

$$\begin{aligned} E_n(x) &= \int_0^1 |k(x, y)|dy - \sum_{i=1}^n \lambda_i |k(x, x_i)| \\ &= \int_0^1 [|k(x, y)| - p_n(x, y)]dy - \sum_{i=1}^n \lambda_i [|k(x, x_i)| - p_n(x, x_i)] \end{aligned} \tag{16}$$

from which we have the uniform bound

$$|E_n(x)| \le 2cn^{-1}, \quad n \to \infty$$

Notice that this implies that $\lim_{n\to\infty} \|K_n\| = \|K\|$.

Thus we have

$$\|K_n^\tau\| < \|K_n\| < \|K\| + \delta$$

where $\delta > 0$ can be made arbitrarily small, by taking n sufficiently large, so that $\|K\| + \delta < 1$. □

Proposition 2 allows us to claim that our (truncated) Nyström approach is stable. It also allows us to claim that by taking the parameter τ according to the accuracy we want to achieve (see Section 2), the Nyström interpolant will converge to the solution u, in the uniform norm, up to a given tolerance. Recall that this latter can be made arbitrarily small by taking the parameter τ close enough to the endpoint $x = 1$. Indeed, standard calculation gives the bound

$$\|u - u_n^\tau\| \le \|(I - K_n^\tau)^{-1}\| \cdot \|Ku - K_n^\tau u\| \le \frac{1}{1 - C} \|Ku - K_n^\tau u\|$$

where C is the constant defined in Proposition 2. Moreover, assuming that $ku \in C^m([0, 1] \times [0, 1])$, $m \ge 1$, and $|k(x, y)u(y)| \le \epsilon$ for $y \ge \tau$, $\forall x \in [0, 1]$, and recalling (see [8]) that for such a smooth ku there exists a polynomial $p_n(x, y)$, of degree n with respect to each variable, such that $\|ku - p_n\| = O(n^{-m})$ (the uniform norm being taken with respect to both variables), we have

$$\|Ku - K_n^\tau u\| < \epsilon + O(n^{-m}), \quad n \to \infty.$$

Recall also that

$$\frac{\|u - u_n^\tau\|}{\|u\|} \approx\le \operatorname{cond}(I - K) \frac{\|Ku - K_n^\tau u\|}{\|f\|}.$$

Remark 3.1. All the results presented in this section can be trivially extended to the case of an operator K having a Jacobi weight function.

Acknowledgment

We thank Pietro Pastore for performing most of the numerical testing reported in the paper.

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