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Partial Differential Equations — *Stability of solutions of evolution equations*, by ALBERTO CIALDEA and FLAVIA LANZARA, communicated on 8 March 2013.

Dedicated to the Memory of Gaetano Fichera

ABSTRACT. — While the choice of a norm in the space where an evolution problem is posed is ineffective as far as the smoothness properties of the solution with respect to the space variables are concerned, the asymptotic behavior of this solution when $t \to +\infty$ is greatly effected by a change of the norm in the space. We illustrate this consideration by studying existence, uniqueness and asymptotic behavior for the solution of a simple but very significant evolution problem.

KEY WORDS: Parabolic potential, integral equation, asymptotic behavior of solutions.

MATHEMATICS SUBJECT CLASSIFICATION: 35K10, 35B40.

1. INTRODUCTION

In 1995 Gaetano Fichera gave a talk concerning the asymptotic behaviour of solutions of evolution problems in the VIII International Conference on waves and stability in continuous media (Palermo, October 9–14, 1995). Only the transparencies were published on the Proceedings of the Conference ([2]).

Later Gaetano Fichera started to prepare a paper containing all the proofs and details. Unfortunately his sudden death prevented him to finish it. Only the first part of the paper was in a definitive form and it is published in [3].

The present paper is the continuation of [3] and hinges on the non organized and incomplete part of the notes left by Fichera. We have completed them in order to prove all the results he presented in Palermo talk.

Let us consider an evolution problem

$$\begin{cases} \dfrac{d\mathscr{U}}{dt} = A(t, \mathscr{U}) \\ \mathscr{U}(0) = \mathscr{U}_0, \end{cases} \tag{1.1}$$

where $A(t, \mathscr{U})$ is an operator defined on $\mathscr{U}$, which for each $t \in \overline{\mathbb{R}}^+$ maps $\mathscr{U}$ into itself. Here $\mathscr{U}$ is a function class of functions defined for each $t \geq 0$ and with values in a topological vector space S.

Remarking that there exist infinitely many topologies compatible with the linear structure of an infinite dimensional function space, Fichera posed a basic question (see [3]):

Let us suppose that system (1.1) *is connected with some physical phenomenon; then the subtle problem arises:* is the topology to be introduced in the vector space S uniquely determined by the physical problems under investigation?

The answer to this question is in general negative, how it can be shown by simple examples. Since the mathematical model of the physical phenomenon given by the Evolution Problem (1.1) *depends on the topology introduced in S, we have to face a very serious methodological difficulty due to the fact that the asymptotics of a* trajectory *of* (1.1) *depends on the relevant topology introduced by the mathematicians in S. This, from the point of view of Physics, is quite inadmissible.*

After proving an existence and uniqueness theorem (Section 2), in Section 3 we study an example showing how the asymptotic behavior of the solution when $t \to +\infty$ is greatly effected by a change of the norm in the space, despite the fact that the regularity of solutions with respect to the space variables does not change.

2. Existence and uniqueness theorem for an evolution problem

Suppose that the weight function $w \in C^0(\mathbb{R})$, $w(x) \geq 1$ and satisfies the following conditions:

$i)_w$ $\forall \gamma > 0$, there exists $a_\gamma(T)$ such that

$$\int_{-\infty}^{+\infty} \frac{e^{-(x-\xi)^2/4(t-\tau)}}{[w(\xi)]^\gamma} d\xi \leq \sqrt{t-\tau}\,\frac{a_\gamma(T)}{[w(x)]^\gamma}, \quad 0 \leq \tau < t \leq T; \tag{2.1}$$

$ii)_w$ $\forall \gamma > 0$, there exists $b_\gamma(T)$ such that

$$\int_{-\infty}^{+\infty} \frac{|\xi - x|}{2\sqrt{t-\tau}} \frac{e^{-(x-\xi)^2/4(t-\tau)}}{[w(\xi)]^\gamma} d\xi \leq \sqrt{t-\tau}\,\frac{b_\gamma(T)}{[w(x)]^\gamma}, \quad 0 \leq \tau < t \leq T; \tag{2.2}$$

$iii)_w$ there exists $\alpha_0 \geq 0$ such that

$$\int_{-\infty}^{+\infty} \frac{dx}{[w(x)]^\alpha} < +\infty, \quad \forall \alpha > \alpha_0. \tag{2.3}$$

For example $w(x) = 1 + x^2$ and $w(x) = e^{|x|}$ satisfy conditions $i)_w$, $ii)_w$, $iii)_w$ (see Subsections 2.1 and 2.2 below).

Let T be a positive real number. We denote by S_T the strip

$$S_T = \{(x,t) : x \in \mathbb{R}, 0 < t \leq T\}.$$

By S_∞ we denote the half-plane $t > 0$ of the (x,t)-plane. Let $\gamma > 0$. In the following we denote by F_γ and $F_\gamma(T)$ the spaces F_γ^∞ and $F_\gamma^\infty(T)$, respectively (see [3, Section 2]), that is the Banach spaces of real valued measurable functions such that

$$U(x) \in F_\gamma \quad \Leftrightarrow \quad \|U\|_\gamma = \operatorname*{ess\,sup}_{x \in \mathbb{R}} |U(x)[w(x)]^\gamma| < +\infty;$$

$$u(x,t) \in F_\gamma(T) \quad \Leftrightarrow \quad \begin{cases} u(\cdot,t) \in F_\gamma, \quad \forall t \in (0,T); \\ \|u\|_\gamma = \operatorname*{ess\,sup}_{S_T} |u(x,t)[w(x)]^\gamma| < +\infty. \end{cases}$$

We consider also

– L^p_γ the space of real valued functions $U(x)$, measurable in $\mathbb{R}$, such that

$$\|U\|_{\gamma,p} = \Big(\int_{\mathbb{R}} |U(x)|^p [w(x)]^\gamma \, dx\Big)^{1/p} < +\infty;$$

– $L^p_\gamma(T)$ the space of real valued functions $u(x,t)$ such that $u(\cdot,t) \in L^p_\gamma$ for all $0 < t < T$ and

$$\|u\|_{\gamma,p} = \operatorname*{ess\,sup}_{(0,T)} \Big(\int_{\mathbb{R}} |u(x,t)|^p [w(x)]^\gamma \, dx\Big)^{1/p} < +\infty.$$

Given $\gamma > 0$, we denote by $\mathscr{U}_\gamma(T)$ the class $\mathscr{U}^\infty_\gamma(T)$ defined in [3, Section 2] of all functions defined in S_∞ such that $u, u_x, u_{xx}, u_t \in F_\gamma(T) \cap C^\infty(S_\infty)$.

Let $c(x,t)$ be a real valued function such that

$i)_c$ $c(x,t) \in C^\infty(\overline{S}_\infty)$;
$ii)_c$ $\forall T > 0$, $\forall h = 0,1,2$ $\exists c_h(T) > 0 : |c_{x^h}(x,t)| \le c_h(T)$ $\forall (x,t) \in S_T$.

The evolution problem we are going to consider is the following

Given $c(x,t)$ satisfying $i)_c$, $ii)_c$ find $u(x,t)$ belonging to $\mathscr{U}_\gamma(T)$ for any $T > 0$, such that

$$\begin{cases} u_t(x,t) = u_{xx}(x,t) + c(x,t)u(x,t) & (x,t) \in S_\infty \quad (2.4) \\ u(x,0) = \varphi(x) & x \in \mathbb{R}, \quad (2.5) \end{cases}$$

when $\varphi \in C^\infty(\mathbb{R})$ and

$$\varphi^{(h)}(x) \in F_\gamma, \quad h = 0, \dots, 4. \tag{2.6}$$

Theorem 2.1. *Set*

$$v(x,t) = u(x,t) - \varphi(x); \tag{2.7}$$

$$\mathscr{K}g(x,t) = \frac{-1}{2\sqrt{\pi}} \int_0^t \frac{d\tau}{(t-\tau)^{1/2}} \int_{-\infty}^{+\infty} g(\xi,\tau) \mathrm{e}^{-(x-\xi)^2/4(t-\tau)} \, d\xi.$$

For any $T > 0$, problem (2.4)–(2.5) *is equivalent to the following integral equation*

$$v + \mathscr{K}(cv) = \mathscr{K}f, \quad v \in F_\gamma(T) \tag{2.8}$$

where

$$f(x,t) = -\varphi''(x) - c(x,t)\varphi(x) \in F_\gamma(T) \cap C^\infty(S_\infty). \tag{2.9}$$

Proof. It is evident that under the hypothesis (2.6) $u(x,t) \in \mathscr{U}_\gamma(T)$ if and only if $v(x,t) \in \mathscr{U}_\gamma(T)$.

A function u is solution of (2.4)–(2.5) if and only if v in (2.7) is solution of the problem: *Find $v(x,t)$ belonging to $\mathscr{U}_\gamma(T)$ for any $T > 0$, such that*

$$\begin{cases} v_{xx}(x,t) - v_t(x,t) + c(x,t)v(x,t) = f(x,t), \quad (x,t) \in S_\infty & (2.10) \\ v(x,0) = 0. & (2.11) \end{cases}$$

If $v \in \mathscr{U}_\gamma(T)$, for any $T > 0$, is solution of (2.10)–(2.11) then, for [3, Theorem 2.1], the function v is solution of the integral equation (2.8).

This result can be inverted in the following sense: if $f \in F_\gamma(T) \cap C^\infty(S_\infty)$ and v is a solution of (2.8) which belongs to $F_\gamma(T)$, then from (2.8) we deduce that

$$\iint_{S_T} c(x,t)\varphi(x,t)[\mathscr{K}(f - cv)(x,t) - v(x,t)]\,dx\,dt = 0, \tag{2.12}$$
$$\forall \varphi \in \mathring{C}^\infty(S_T - \partial S_T).$$

If (see [3, (2.3)])

$$\mathscr{K}^*\psi(x,t) = -\int_t^T d\tau \int_{-\infty}^{+\infty} \psi(\xi,\tau)\Gamma(\xi,\tau;x,t)\,d\xi$$

then (2.12) is equivalent to

$$\begin{aligned}
&\iint_{S_T} (f(x,t) - c(x,t)v(x,t))(\varphi(x,t) + \mathscr{K}^*(c\varphi)(x,t))\,dx\,dt \\
&= \iint_{S_T} (f(x,t) - c(x,t)v(x,t))\mathscr{K}^*(\varphi_{xx} + \varphi_t + c\varphi)(x,t)\,dx\,dt \\
&= \iint_{S_T} \mathscr{K}(f - cv)(x,t)(\varphi_{xx}(x,t) + \varphi_t(x,t) + c(x,t)\varphi(x,t))\,dx\,dt \\
&= \iint_{S_T} v(x,t)(\varphi_{xx}(x,t) + \varphi_t(x,t) + c(x,t)\varphi(x,t))\,dx\,dt \\
&= \iint_{S_T} \varphi(x,t)(v_{xx}(x,t) - v_t(x,t) + c(x,t)v(x,t))\,dx\,dt = 0
\end{aligned}$$

and in conseguence v is a solution of (2.10), (2.11) in the weak sense. From the classical regularization theory we deduce that $v \in F_\gamma(T) \cap C^\infty(S_\infty)$. Moreover

$$\frac{\partial^2 v_x}{\partial x^2} - \frac{\partial v_x}{\partial t} + c(x,t)v_x(x,t) = -c_x(x,t)v(x,t) + f_x(x,t); \quad v_x(x,0) = 0$$

$$\frac{\partial^2 v_{xx}}{\partial x^2} - \frac{\partial v_{xx}}{\partial t} + c(x,t)v_{xx}(x,t) = -2c_x(x,t)v_x(x,t) - c_{xx}(x,t)v(x,t) + f_{xx}(x,t)$$

$$v_{xx}(x,0) = 0.$$

This implies $v_x \in F_\gamma(T) \cap C^\infty(S_\infty)$ and, in consequence, $v_{xx} \in F_\gamma(T) \cap C^\infty(S_\infty)$ *i.e.* $v \in \mathscr{U}_\gamma(T)$. □

We have shown the perfect equivalence between problem (2.4)–(2.5) in $\mathscr{U}_\gamma(T)$ and the integral equation (2.8) in $F_\gamma(T)$, for any $T > 0$.

Lemma 2.1. *If $f(x,t) \in F_\gamma(T)$ then $(\mathscr{K}f)(x,t) \in F_\gamma(T)$.*

Proof. If $f \in F_\gamma(T)$ then

$$|\mathscr{K}f(x,t)| \le \frac{1}{2\sqrt{\pi}} \int_0^t \frac{d\tau}{(t-\tau)^{1/2}} \int_{-\infty}^{+\infty} |f(\xi,\tau)|[w(\xi)]^\gamma \frac{e^{-(x-\xi)^2/4(t-\tau)}}{[w(\xi)]^\gamma} d\xi$$

$$\le \frac{\|f\|_\gamma}{2\sqrt{\pi}} \int_0^t \frac{d\tau}{(t-\tau)^{1/2}} \int_{-\infty}^{+\infty} \frac{e^{-(x-\xi)^2/4(t-\tau)}}{[w(\xi)]^\gamma} d\xi.$$

From (2.1) we get

$$|\mathscr{K}f(x,t)| \le \frac{\|f\|_\gamma}{2\sqrt{\pi}} \frac{ta_\gamma(T)}{[w(x)]^\gamma} \tag{2.13}$$

which implies

$$\|\mathscr{K}f\|_\gamma \le \frac{Ta_\gamma(T)}{2\sqrt{\pi}} \|f\|_\gamma.$$ □

Lemma 2.2. *If $v(x,t) \in F_\gamma(T)$ then $(\mathscr{K}cv)(x,t) \in F_\gamma(T)$ and $(\mathscr{K}cv)_x(x,t) \in F_\gamma(T)$.*

Proof. Lemma 2.1 and the hypotheses on $c(x,t)$ imply that $(\mathscr{K}cv)(x,t)$ belongs to $F_\gamma(T)$. We have

$$(\mathscr{K}cv)_x(x,t) = \frac{1}{2\sqrt{\pi}} \int_0^t \frac{d\tau}{(t-\tau)^{1/2}} \int_{-\infty}^{+\infty} c(\xi,\tau)v(\xi,\tau) \frac{\xi - x}{2(t-\tau)} e^{-(x-\xi)^2/4(t-\tau)} d\xi.$$

From (2.2) it follows

$$|(\mathscr{K}cv)_x(x,t)| \le \frac{c_0(T)}{\sqrt{\pi}} \frac{t^{1/2} b_\gamma(T)}{[w(x)]^\gamma} \|v\|_\gamma \tag{2.14}$$

and

$$\|(\mathscr{K}cv)_x\|_\gamma \le \frac{c_0(T)}{\sqrt{\pi}} T^{1/2} b_\gamma(T)\|v\|_\gamma. \qquad \square$$

LEMMA 2.3. *If* $v \in F_\gamma(T)$ *then,* $\forall p \ge 1$, $\forall \mu < \gamma - \alpha_0$, $\mathscr{K}(cv) \in L^p_\mu(T)$ *and* $(\mathscr{K}(cv))_x \in L^p_\mu(T)$.

PROOF. From (2.13) and (2.14)

$$\Big(\int_{\mathbb{R}} |(\mathscr{K}cv)(x,t)|^p [w(x)]^\mu \, dx\Big)^{1/p} \le \frac{c_0(T) T a_\gamma(T)}{2\sqrt{\pi}} \|v\|_\gamma \Big(\int_{\mathbb{R}} [w(x)]^{\mu - \gamma p} \, dx\Big)^{1/p};$$

$$\Big(\int_{\mathbb{R}} |(\mathscr{K}cv)_x(x,t)|^p [w(x)]^\mu \, dx\Big)^{1/p} \le \frac{c_0(T) T^{1/2} b_\gamma(T)}{\sqrt{\pi}} \|v\|_\gamma \Big(\int_{\mathbb{R}} [w(x)]^{\mu - \gamma p} \, dx\Big)^{1/p}.$$

The integrals in the right hand sides are finite for $\mu < p\gamma - \alpha_0$, for all $p \ge 1$. This implies our assertion. $\square$

THEOREM 2.2. *Assume* $T > 0$. *For any* $\psi \in F_\gamma(T)$, *the equation*

$$v + \mathscr{K}(cv) = \psi \tag{2.15}$$

has one and only one solution in $F_\gamma(T)$. *We have*

$$v = \sum_{s=0}^{+\infty} (-1)^s (\mathscr{K}c)^s \psi.$$

The above series and the series obtained by differentiating with respect to x converge in the norm of $F_\gamma(T)$.

PROOF. For a fixed $\varrho > 0$, we define the space $F^{(\varrho)}_\gamma(T)$ formed by all functions $v(x,t) \in F_\gamma(T)$, equipped with the norm

$$|||v|||_\gamma = \operatorname*{ess\,sup}_{S_T} |e^{-\varrho t}[w(x)]^\gamma (|v(x,t)| + |v_x(x,t)|).$$

We have

$$e^{-\varrho T}(\|v\|_\gamma + \|v_x\|_\gamma) \le |||v|||_\gamma \le \|v\|_\gamma + \|v_x\|_\gamma.$$

For any $t \in [0,T]$, keeping in mind (2.1) and (2.2), we find

$$|\mathscr{K}cv(x,t)| \le \frac{c_0(T)}{2\sqrt{\pi}} |||v|||_\gamma \int_0^t \frac{e^{\varrho\tau}}{(t-\tau)^{1/2}} d\tau \int_{\mathbb{R}} \frac{e^{-(x-\xi)^2/4(t-\tau)}}{[w(\xi)]^\gamma} d\xi$$

$$\le \frac{c_0(T)}{2\sqrt{\pi}} |||v|||_\gamma \frac{a_\gamma(T)}{[w(x)]^\gamma} \Big(\frac{e^{\varrho t}}{\varrho} - \frac{1}{\varrho}\Big);$$

$$
\begin{aligned}
|(\mathscr{K}cv)_x(x,t)| &\le \frac{c_0(T)}{2\sqrt{\pi}}|||v|||_\gamma \int_0^t \frac{e^{\varrho\tau}}{(t-\tau)}d\tau \int_{\mathbb{R}} \frac{|\xi-x|}{2\sqrt{t-\tau}} \frac{e^{-(x-\xi)^2/4(t-\tau)}}{[w(\xi)]^\gamma}d\xi \\
&\le \frac{c_0(T)}{2\sqrt{\pi}}|||v|||_\gamma \frac{b_\gamma(T)}{[w(x)]^\gamma}\int_0^t \frac{e^{\varrho\tau}}{\sqrt{t-\tau}}d\tau \\
&= \frac{c_0(T)}{2\sqrt{\pi}}|||v|||_\gamma \frac{b_\gamma(T)}{[w(x)]^\gamma}\frac{e^{\varrho t}\sqrt{\pi}}{\sqrt{\varrho}}\operatorname{erf}(\sqrt{\varrho t}).
\end{aligned}
$$

Hence

$$
\operatorname*{ess\,sup}_{S_T}[e^{-\varrho t}[w(x)]^\gamma|\mathscr{K}cv(x,t)|] \le \frac{c_0(T)}{2\sqrt{\pi}}\frac{a_\gamma(T)}{\varrho}|||v|||_\gamma;
$$

$$
\operatorname*{ess\,sup}_{S_T}[e^{-\varrho t}[w(x)]^\gamma|(\mathscr{K}cv)_x(x,t)|] \le \frac{c_0(T)}{2}\frac{b_\gamma(T)}{\sqrt{\varrho}}|||v|||_\gamma.
$$

We deduce that

$$
|||\mathscr{K}cv|||_\gamma \le \delta_T|||v|||_\gamma, \quad \delta_T = \frac{c_0(T)}{2}\Big(\frac{a_\gamma(T)}{\varrho\sqrt{\pi}} + \frac{b_\gamma(T)}{\sqrt{\varrho}}\Big).
$$

If we assume $\varrho > \big(\frac{c_0(T)(a_\gamma(T)+\sqrt{\pi}b_\gamma(T))}{2\sqrt{\pi}} + 1\big)^2$ then $\delta_T \in (0,1)$ and $Tv = \mathscr{K}cv + \psi$ is a contraction map on the space $F_\gamma^{(\varrho)}(T)$. Hence equation (2.8) has a unique solution in the space $F_\gamma^{(\varrho)}(T)$ given by

$$
v(x,t) = \sum_{s=0}^{+\infty}(-1)^s((\mathscr{K}c)^s\psi)(x,t). \tag{2.16}
$$

The series (2.16) converges in the norm of $F_\gamma^{(\varrho)}(T)$. This completes the proof. □

THEOREM 2.3. *Let γ be a fixed positive real number. Assume that φ satisfies conditions* (2.6) *and c satisfies $i)_c$, $ii)_c$. There exists one and only one solution of the problem* (2.4), (2.5) *such that*

- **i)** $u(x,t) \in C^\infty(\overline{S}_\infty)$;
- **ii)** $\forall T > 0$: $u(x,t) \in F_\gamma(T)$; $u_x(x,t) \in F_\gamma(T)$;
- **iii)** $\forall p \ge 1$ *and* $\forall \mu < \gamma - \alpha_0$: $u(x,t) \in L_\mu^p(T)$, $u_x(x,t) \in L_\mu^p(T)$.

We have

$$
u(x,t) = H_t\varphi(x) = \varphi(x) + (\mathscr{K}f)(x,t) + \sum_{s=1}^{+\infty}(-1)^s(\mathscr{K}c)^s\mathscr{K}f(x,t) \tag{2.17}
$$

where $f(x,t)$ is defined in (2.9).

The above series and the series obtained by differentiating with respect to x converge in the norms of $F_\gamma(T)$, $L^p_\mu(T)$ and, in particular, they are totally convergent in $\overline{S}_T$.

The series obtained by differentiating either twice with respect to x or once with respect to t are totally convergent in every compact subset of S_T.

Proof. Problem (2.4), (2.5) is equivalent to the integral equation (2.15) with $\psi = \mathscr{K} f \in F_\gamma$ (Lemma 2.1). Existence and uniqueness follows from Theorem 2.2. For Theorem 2.2 and (2.7) we deduce (2.17) and the convergence of the series on the right hand side, together with the series obtained by differentiating with respect to x, in the norm of $F_\gamma(T)$. We deduce **i)** and **ii)**.

From (2.13), (2.14) and $ii)_c$ we have

$$|\mathscr{K}(c\psi)(x,t)| \le \frac{c_0(T)a_\gamma(T)}{2\sqrt{\pi}} \frac{t\|\psi\|_\gamma}{[w(x)]^\gamma}; \quad |(\mathscr{K}c\psi)_x(x,t)| \le \frac{c_0(T)b_\gamma(T)}{\sqrt{\pi}} \frac{t^{1/2}\|\psi\|_\gamma}{[w(x)]^\gamma}.$$

We prove by induction that, for $s \ge 1$

$$|(\mathscr{K}c)^s\psi(x,t)| \le \frac{(d(T)t)^s}{s!} \frac{1}{[w(x)]^\gamma}\|\psi\|_\gamma, \quad d(T) = \frac{c_0(T)a_\gamma(T)}{2\sqrt{\pi}} \tag{2.18}$$

Indeed, for (2.18) and (2.1)

$$\begin{aligned}
|(\mathscr{K}c)^{s+1}\psi(x,t)| &= \frac{1}{2\sqrt{\pi}}\left|\int_0^t \frac{d\tau}{\sqrt{t-\tau}}\int_{\mathbb{R}} c(\xi,\tau)[(\mathscr{K}c)^s\psi](\xi,\tau)\mathrm{e}^{-(x-\xi)^2/4(t-\tau)}\,d\xi\right| \\
&\le |||\psi\|_\gamma \frac{c_0(T)}{2\sqrt{\pi}} d(T)^s \int_0^t \frac{\tau^s}{s!}\frac{d\tau}{\sqrt{t-\tau}}\int_{\mathbb{R}} \frac{\mathrm{e}^{-(x-\xi)^2/4(t-\tau)}}{[w(x)]^\gamma}\,d\xi \\
&\le \|\psi\|_\gamma\, d(T)^{s+1}\frac{1}{[w(x)]^\gamma}\int_0^t \frac{\tau^s}{s!}\,d\tau = \|\psi\|_\gamma \frac{[d(T)t]^{s+1}}{(s+1)!}\frac{1}{[w(x)]^\gamma}.
\end{aligned}$$

Hence

$$\|(\mathscr{K}c)^s\psi\|_{\mu,p} \le \frac{(d(T)T)^s}{s!}\left(\int_{\mathbb{R}}[w(x)]^{\mu-\gamma p}\,dx\right)^{1/p}\|\psi\|_\gamma$$

and, $\forall p \ge 1$ and $\forall \mu < \gamma - \alpha_0$, there exists a constant $A_{p,\mu,\gamma}$ such that

$$\sum_{s=0}^{+\infty}\|(\mathscr{K}c)^s\psi\|_{\mu,p} \le A_{p,\mu,\gamma}\|\psi\|_\gamma$$

i.e. the convergence in $L^p_\mu(T)$.

Moreover, keeping in mind (2.18) and (2.2), for any $s \ge 1$,

$$
\begin{aligned}
(2.19)\quad & |((\mathscr{K}c)^{s+1}\psi(x,t))_x| \\
&= \frac{1}{2\sqrt{\pi}}\left|\int_0^t \frac{d\tau}{\sqrt{t-\tau}}\int_{\mathbb{R}} c(\xi,\tau)[(\mathscr{K}c)^s\psi](\xi,\tau)\frac{\xi-x}{2(t-\tau)}\mathrm{e}^{-(x-\xi)^2/4(t-\tau)}\,d\xi\right| \\
&\le \|\psi\|_\gamma \frac{c_0(T)}{2\sqrt{\pi}}\frac{(c_0(T)a_\gamma(T))^s}{2^s\pi^{s/2}}\int_0^t \frac{\tau^s}{s!}\frac{d\tau}{\sqrt{t-\tau}} \\
&\quad\times \int_{\mathbb{R}} \frac{|\xi-x|}{2(t-\tau)}\frac{\mathrm{e}^{-(x-\xi)^2/4(t-\tau)}}{[w(x)]^\gamma}\,d\xi \\
&\le \|\psi\|_\gamma \frac{(c_0(T)a_\gamma(T))^s c_0(T)b_\gamma(T)}{2^{s+1}\pi^{(s+1)/2}}\frac{1}{s!}\int_0^t \frac{\tau^s}{\sqrt{t-\tau}}d\tau\frac{1}{[w(x)]^\gamma} \\
&= \|\psi\|_\gamma \frac{(c_0(T))^{s+1}(a_\gamma(T))^s b_\gamma(T)}{2^{s+1}\pi^{s/2}}\frac{t^{s+1/2}}{s!}\frac{\Gamma(1+s)}{\Gamma(3/2+s)}\frac{1}{[w(x)]^\gamma} \\
&\le \|\psi\|_\gamma \frac{2}{\sqrt{\pi}}\frac{(c_0(T))^{s+1}(a_\gamma(T))^s b_\gamma(T)}{2^{s+1}\pi^{s/2}}\frac{t^{s+1/2}}{s!}\frac{1}{[w(x)]^\gamma}.
\end{aligned}
$$

The last inequality follows from

$$\frac{\Gamma(1+s)}{\Gamma(3/2+s)} = \frac{2}{\sqrt{\pi}}\frac{(2s)!!}{(2s+1)!!} \le \frac{2}{\sqrt{\pi}}.$$

Then

$$\|((\mathscr{K}c)^s\psi)_x\|_{\mu,p} \le \|\psi\|_\gamma \frac{(c_0(T))^{s+1}(a_\gamma(T))^s b_\gamma(T)}{2^s\pi^{(s+1)/2}}\frac{T^{s+1/2}}{s!}\left(\int_{\mathbb{R}}[w(x)]^{\mu-\gamma p}\,dx\right)^{1/p}$$

and, $\forall p \ge 1$ and $\forall \mu < \gamma - \alpha_0$,

$$\sum_{s=0}^{+\infty}\|((\mathscr{K}c)^s\psi)_x\|_{\mu,p} \le B_{p,\mu,\gamma}\|\psi\|_\gamma.$$

This shows that also u_x belongs to $L^p_\mu(T)$ and then **iii)** holds. Moreover the series in (2.17) and the series obtained by differentiating with respect to x converge in the norm of $L^p_\mu(T)$.

For $i)_c$, $ii)_c$, (2.14) and [3, Theorems 2.3 and 2.4]:

$$
\begin{aligned}
|\mathscr{K}(c\psi)(x,t))_{xx}| &\le \|(c\psi)_x\|_{L^\infty(S_T)}\frac{2}{\sqrt{\pi}}\left(\frac{4}{\sqrt{\mathrm{e}}}-1\right)\sqrt{t} \\
&\le \left[c_1(T)\frac{Ta_\gamma(T)}{2\sqrt{\pi}} + c_0(T)\frac{T^{1/2}b_\gamma(T)}{\sqrt{\pi}}\right]\|f\|_\gamma\frac{2}{\sqrt{\pi}}\left(\frac{4}{\sqrt{\mathrm{e}}}-1\right)\sqrt{T}.
\end{aligned}
$$

For any $s \geq 1$,

$$|((\mathscr{K}c)^{s+1}\psi(x,t))_{xx}| = |(\mathscr{K}(c((\mathscr{K}c)^s\psi))(x,t))_{xx}|$$
$$\leq \|((c(\mathscr{K}c)^s\psi))_x\|_{L^\infty(S_T)} \frac{2}{\sqrt{\pi}}\Big(\frac{4}{\sqrt{e}} - 1\Big)\sqrt{t}.$$

Keeping in mind $ii)_c$, (2.18) and (2.19)

$$\begin{aligned}
&|((c(\mathscr{K}c)^s\psi))_x(x,t)| \\
&\quad \leq |c_x((\mathscr{K}c)^s\psi)| + |c((\mathscr{K}c)^s\psi)_x| \\
&\quad \leq \Big(c_1(T)\frac{[d(T)t]^s}{s!} + c_0(T)\frac{(c_0(T))^s(a_\gamma(T))^{s-1}b_\gamma(T)}{2^{s-1}\pi^{s/2}}\frac{t^{s-1/2}}{s!}\Big)\|\psi\|_\gamma.
\end{aligned}$$

We deduce the convergence of the series

$$\sum_{s=1}^{+\infty} |((\mathscr{K}c)^{s+1}\psi)_{xx}(x,t)|.$$

The convergence of the series obtained by differentiating with respect to t follows from the relation

$$((\mathscr{K}c)^{s+1}\psi(x,t))_t = ((\mathscr{K}c)^{s+1}\psi(x,t))_{xx} - c(x,t)(\mathscr{K}c)^s\psi(x,t), \quad (x,t) \in S_T. \quad \square$$

2.1. The weight $w(x) = (1 + x^2)$

The function $w(x) = (1 + x^2)$ obviously satisfies condition (2.3) with $\alpha_0 = 1/2$. We show that it satisfies the hypotheses (2.1) and (2.2).

LEMMA 2.4. *For any $\gamma \geq 0$ there exists $a_\gamma(T)$ such that*

$$\int_{\mathbb{R}} \frac{e^{-(x-\xi)^2/4(t-\tau)}}{(1+\xi^2)^\gamma} d\xi \leq \sqrt{t-\tau}\frac{a_\gamma(T)}{(1+x^2)^\gamma}, \quad 0 \leq \tau < t \leq T.$$

PROOF. Setting $\sigma = (\xi - x)/(2\sqrt{t-\tau})$,

$$\begin{aligned}
\int_{\mathbb{R}} \frac{e^{-(x-\xi)^2/4(t-\tau)}}{(1+\xi^2)^\gamma} d\xi = 2\sqrt{t-\tau}\Big(&\int_{|x|/4\sqrt{T}}^{+\infty} \frac{e^{-\sigma^2}\,d\sigma}{(1+(x-2\sigma\sqrt{t-\tau})^2)^\gamma} \\
&+ \int_{-|x|/4\sqrt{T}}^{|x|/4\sqrt{T}} \frac{e^{-\sigma^2}\,d\sigma}{(1+(x+2\sigma\sqrt{t-\tau})^2)^\gamma} \\
&+ \int_{|x|/4\sqrt{T}}^{+\infty} \frac{e^{-\sigma^2}\,d\sigma}{(1+(x+2\sigma\sqrt{t-\tau})^2)^\gamma}\Big).
\end{aligned} \tag{2.20}$$

We have (see [1, 7.1.13])

$$\int_{|x|/4\sqrt{T}}^{+\infty} \frac{e^{-\sigma^2}\,d\sigma}{(1+(x-2\sigma\sqrt{t-\tau})^2)^\gamma} \le \int_{|x|/4\sqrt{T}}^{+\infty} e^{-\sigma^2}\,d\sigma = \frac{\sqrt{\pi}}{2}\operatorname{erfc}\frac{|x|}{4\sqrt{T}}$$

$$\le \frac{\sqrt{\pi}}{2}\frac{e^{-x^2/16T}}{\frac{|x|}{4\sqrt{T}}+\sqrt{\frac{x^2}{16T}+\frac{4}{\pi}}} \le c_T\frac{e^{-x^2/16T}}{\sqrt{x^2+1}}.$$

Since the function $\psi(y) = e^{-y/16T}(1+y)^{\gamma-1/2}$ is bounded in $[0,\infty)$ then

$$\int_{|x|/4\sqrt{T}}^{+\infty} \frac{e^{-\sigma^2}\,d\sigma}{(1+(x-2\sigma\sqrt{t-\tau})^2)^\gamma} \le \frac{C_T}{(1+x^2)^\gamma}.$$

For $|\sigma| \le \frac{|x|}{4\sqrt{T}}$ we get

$$|x+2\sigma\sqrt{t-\tau}| \ge |x| - 2|\sigma|\sqrt{t-\tau} \ge |x|/2. \tag{2.21}$$

We deduce that

$$\int_{-|x|/4\sqrt{T}}^{|x|/4\sqrt{T}} \frac{e^{-\sigma^2}\,d\sigma}{(1+(x+2\sigma\sqrt{t-\tau})^2)^\gamma} \le \frac{4^\gamma}{(4+x^2)^\gamma}\int_{\mathbb{R}} e^{-\sigma^2}\,d\sigma \le \frac{4^\gamma\sqrt{\pi}}{(1+x^2)^\gamma}.$$

Concerning the last integral in (2.20)

$$\int_{|x|/4\sqrt{T}}^{+\infty} \frac{e^{-\sigma^2}\,d\sigma}{(1+(x+2\sigma\sqrt{t-\tau})^2)^\gamma} \le \frac{1}{(1+x^2)^\gamma}\int_{|x|/4\sqrt{T}}^{+\infty} e^{-\sigma^2}\,d\sigma \le \frac{\sqrt{\pi}}{2(1+x^2)^\gamma}.$$

This completes the proof. □

LEMMA 2.5. *For any $\gamma \ge 0$ there exists $b_\gamma(T)$ such that*

$$\int_{\mathbb{R}} \frac{|\xi-x|}{2\sqrt{t-\tau}}\frac{e^{-(x-\xi)^2/4(t-\tau)}}{(1+\xi^2)^\gamma}d\xi \le \sqrt{t-\tau}\frac{b_\gamma(T)}{(1+x^2)^\gamma}, \quad 0 \le \tau < t \le T.$$

PROOF. Setting $\sigma = (\xi - x)/(2\sqrt{t-\tau})$,

$$\begin{aligned} &\int_{\mathbb{R}} \frac{|\xi-x|}{2\sqrt{t-\tau}}\frac{e^{-(x-\xi)^2/4(t-\tau)}}{(1+\xi^2)^\gamma}d\xi \\ &= 2\sqrt{t-\tau}\Big(\int_{|x|/4\sqrt{T}}^{+\infty} \frac{\sigma e^{-\sigma^2}\,d\sigma}{(1+(x-2\sigma\sqrt{t-\tau})^2)^\gamma} \\ &\quad + \int_{-|x|/4\sqrt{T}}^{|x|/4\sqrt{T}} \frac{|\sigma| e^{-\sigma^2}d\sigma}{(1+(x+2\sigma\sqrt{t-\tau})^2)^\gamma} \\ &\quad + \int_{|x|/4\sqrt{T}}^{+\infty} \frac{\sigma e^{-\sigma^2}d\sigma}{(1+(x+2\sigma\sqrt{t-\tau})^2)^\gamma}\Big). \end{aligned} \tag{2.22}$$

We have

$$\int_{|x|/4\sqrt{T}}^{+\infty} \frac{\sigma e^{-\sigma^2}\, d\sigma}{(1+(x-2\sigma\sqrt{t-\tau})^2)^\gamma} \le \int_{|x|/4\sqrt{T}}^{+\infty} \sigma e^{-\sigma^2}\, d\sigma = \frac{e^{-x^2/16T}}{2}.$$

Since the function $\psi(y) = e^{-y/16T}(1+y)^\gamma$ is bounded in $[0,\infty)$ then

$$\int_{|x|/4\sqrt{T}}^{+\infty} \frac{\sigma e^{-\sigma^2}\, d\sigma}{(1+(x-2\sigma\sqrt{t-\tau})^2)^\gamma} \le \frac{C_T}{(1+x^2)^\gamma}.$$

Due to (2.21) we deduce that

$$\int_{-|x|/4\sqrt{T}}^{|x|/4\sqrt{T}} \frac{|\sigma| e^{-\sigma^2}\, d\sigma}{(1+(x+2\sigma\sqrt{t-\tau})^2)^\gamma} \le \frac{4^\gamma}{(4+x^2)^\gamma}\int_{\mathbb{R}} |\sigma| e^{-\sigma^2}\, d\sigma \le \frac{4^\gamma}{(1+x^2)^\gamma}.$$

For the last integral in (2.22)

$$\int_{|x|/4\sqrt{T}}^{+\infty} \frac{\sigma e^{-\sigma^2}\, d\sigma}{(1+(x+2\sigma\sqrt{t-\tau})^2)^\gamma} \le \frac{1}{(1+x^2)^\gamma}\int_{|x|/4\sqrt{T}}^{+\infty} \sigma e^{-\sigma^2}\, d\sigma \le \frac{1}{2(1+x^2)^\gamma}.$$

The last three inequalities and (2.22) lead to our assertion. □

2.2. *The function* $w(x) = e^{|x|}$

In this subsection we show that the function $e^{|x|}$ satisfies (2.1) and (2.2). (2.3) is obviously satisfied with $\alpha_0 = 0$.

LEMMA 2.6. *For any* $\gamma \ge 0$,

$$\int_{-\infty}^{+\infty} e^{-(x-\xi)^2/4(t-\tau)} e^{-\gamma|\xi|}\, d\xi \le \frac{3}{2}\sqrt{\pi}\sqrt{t-\tau}\, e^{T\gamma^2} e^{-\gamma|x|}, \quad 0 \le \tau < t \le T.$$

PROOF. Set $\alpha = 2\sqrt{t-\tau}$. We have

$$\int_{-\infty}^{+\infty} e^{-(x-\xi)^2/\alpha^2} e^{-\gamma|\xi|}\, d\xi = \alpha \int_{-\infty}^{+\infty} e^{-\sigma^2} e^{-\gamma|x+\alpha\sigma|}\, d\sigma$$

$$= \alpha e^{\gamma x}\int_{-\infty}^{-x/\alpha} e^{-\sigma^2+\alpha\gamma\sigma}\, d\sigma + \alpha e^{-\gamma x}\int_{-x/\alpha}^{+\infty} e^{-\sigma^2-\alpha\gamma\sigma}\, d\sigma.$$

If $x < 0$ then

$$e^{\gamma x}\int_{-\infty}^{-x/\alpha} e^{-\sigma^2+\alpha\gamma\sigma}\, d\sigma = e^{-\gamma|x|} e^{\alpha^2\gamma^2/4}\int_{-\infty}^{|x|/\alpha} e^{-(\sigma-\alpha\gamma/2)^2}\, d\sigma;$$

$$e^{-\gamma x}\int_{-x/\alpha}^{+\infty} e^{-\sigma^2-\alpha\gamma\sigma}\, d\sigma = e^{\gamma|x|} e^{\alpha^2\gamma^2/4}\int_{|x|/\alpha}^{+\infty} e^{-(\sigma+\alpha\gamma/2)^2}\, d\sigma.$$

If $x > 0$ then

$$e^{-\gamma x} \int_{-x/\alpha}^{+\infty} e^{-\sigma^2 - \alpha\gamma\sigma}\, d\sigma = e^{-\gamma|x|} e^{\alpha^2\gamma^2/4} \int_{-\infty}^{|x|/\alpha} e^{-(\sigma - \alpha\gamma/2)^2}\, d\sigma;$$

$$e^{\gamma x} \int_{-\infty}^{-x/\alpha} e^{-\sigma^2 + \alpha\gamma\sigma}\, d\sigma = e^{\gamma|x|} e^{\alpha^2\gamma^2/4} \int_{|x|/\alpha}^{+\infty} e^{-(\sigma + \alpha\gamma/2)^2}\, d\sigma.$$

Hence

$$\int_{-\infty}^{+\infty} e^{-(x-\xi)^2/\alpha^2} e^{-\gamma|\xi|}\, d\xi$$
$$= \alpha e^{\alpha^2\gamma^2/4} \Big(e^{-\gamma|x|} \int_{-\infty}^{|x|/\alpha} e^{-(\sigma - \alpha\gamma/2)^2}\, d\sigma + e^{\gamma|x|} \int_{|x|/\alpha}^{+\infty} e^{-(\sigma + \alpha\gamma/2)^2}\, d\sigma \Big).$$

The inequalities

$$\int_{-\infty}^{|x|/\alpha} e^{-(\sigma - \alpha\gamma/2)^2}\, d\sigma \le \int_{-\infty}^{+\infty} e^{-(\sigma - \alpha\gamma/2)^2}\, d\sigma \le \sqrt{\pi};$$

$$\int_{|x|/\alpha}^{+\infty} e^{-(\sigma + \alpha\gamma/2)^2}\, d\sigma = \int_{|x|/\alpha + \alpha\gamma/2}^{+\infty} e^{-\tau^2}\, d\tau = \frac{\sqrt{\pi}}{2} \operatorname{erfc}\Big(\frac{|x|}{\alpha} + \frac{\alpha\gamma}{2} \Big)$$
$$\le \frac{\sqrt{\pi}}{2} e^{-(|x|/\alpha + \alpha\gamma/2)^2} \le \frac{\sqrt{\pi}}{2} e^{-2\gamma|x|}$$

complete the proof. □

LEMMA 2.7. *For any $\gamma \ge 0$ there exists $b_\gamma(T)$ such that*

$$\int_{\mathbb{R}} \frac{|\xi - x|}{2\sqrt{t-\tau}} e^{-(x-\xi)^2/4(t-\tau)} e^{-\gamma|\xi|}\, d\xi \le \sqrt{t-\tau}\, b_\gamma(T) e^{-\gamma|x|}, \quad 0 \le \tau < t \le T. \tag{2.23}$$

PROOF. Set $\alpha = 2\sqrt{t-\tau}$. We have

$$\int_{\mathbb{R}} \frac{|\xi - x|}{2\sqrt{t-\tau}} e^{-(x-\xi)^2/4(t-\tau)} e^{-\gamma|\xi|}\, d\xi$$
$$= \alpha \int_{-\infty}^{+\infty} |\sigma| e^{-\sigma^2} e^{-\gamma|x + \alpha\sigma|}\, d\sigma$$
$$= \alpha e^{\gamma x} \int_{-\infty}^{-x/\alpha} |\sigma| e^{-\sigma^2 + \alpha\gamma\sigma}\, d\sigma + \alpha e^{-\gamma x} \int_{-x/\alpha}^{+\infty} |\sigma| e^{-\sigma^2 - \alpha\gamma\sigma}\, d\sigma.$$

If $x < 0$ then

$$\mathrm{e}^{\gamma x}\int_{-\infty}^{-x/\alpha}|\sigma|\mathrm{e}^{-\sigma^2+\alpha\gamma\sigma}\,d\sigma = \mathrm{e}^{-\gamma|x|}\mathrm{e}^{\alpha^2\gamma^2/4}\int_{-\infty}^{|x|/\alpha}|\sigma|\mathrm{e}^{-(\sigma-\alpha\gamma/2)^2}\,d\sigma;$$

$$\mathrm{e}^{-\gamma x}\int_{-x/\alpha}^{+\infty}|\sigma|\mathrm{e}^{-\sigma^2-\alpha\gamma\sigma}\,d\sigma = \mathrm{e}^{\gamma|x|}\mathrm{e}^{\alpha^2\gamma^2/4}\int_{|x|/\alpha}^{+\infty}\sigma\mathrm{e}^{-(\sigma+\alpha\gamma/2)^2}\,d\sigma.$$

In the case $x > 0$

$$\mathrm{e}^{\gamma x}\int_{-\infty}^{-x/\alpha}|\sigma|\mathrm{e}^{-\sigma^2+\alpha\gamma\sigma}\,d\sigma = \mathrm{e}^{\gamma|x|}\mathrm{e}^{\alpha^2\gamma^2/4}\int_{|x|/\alpha}^{+\infty}\sigma\mathrm{e}^{-(\sigma+\alpha\gamma/2)^2}\,d\sigma;$$

$$\mathrm{e}^{-\gamma x}\int_{-x/\alpha}^{+\infty}|\sigma|\mathrm{e}^{-\sigma^2-\alpha\gamma\sigma}\,d\sigma = \mathrm{e}^{-\gamma|x|}\mathrm{e}^{\alpha^2\gamma^2/4}\int_{-\infty}^{|x|/\alpha}|\sigma|\mathrm{e}^{-(\sigma-\alpha\gamma/2)^2}\,d\sigma.$$

We deduce that

$$\int_{\mathbb{R}}\frac{|\xi-x|}{\alpha}\mathrm{e}^{-(x-\xi)^2/\alpha^2}\mathrm{e}^{-\gamma|\xi|}\,d\xi$$
$$= \alpha\mathrm{e}^{\alpha^2\gamma^2/4}\Big(\mathrm{e}^{-\gamma|x|}\int_{-\infty}^{|x|/\alpha}|\sigma|\mathrm{e}^{-(\sigma-\alpha\gamma/2)^2}\,d\sigma + \mathrm{e}^{\gamma|x|}\int_{|x|/\alpha}^{+\infty}\sigma\mathrm{e}^{-(\sigma+\alpha\gamma/2)^2}\,d\sigma\Big).$$

From the inequalities

$$\begin{aligned}\int_{|x|/\alpha}^{+\infty}\sigma\mathrm{e}^{-(\sigma+\alpha\gamma/2)^2}\,d\sigma &= \int_{|x|/\alpha+\alpha\gamma/2}^{+\infty}\Big(\tau-\frac{\alpha\gamma}{2}\Big)\mathrm{e}^{-\tau^2}\,d\tau\\ &\le \int_{|x|/\alpha+\alpha\gamma/2}^{+\infty}\tau\mathrm{e}^{-\tau^2}\,d\tau = \frac{1}{2}\mathrm{e}^{-(\alpha\beta/2+|x|/\alpha)^2} \le \frac{1}{2}\mathrm{e}^{-2\gamma|x|};\\ \int_{-\infty}^{|x|/\alpha}|\sigma|\mathrm{e}^{-(\sigma-\alpha\gamma/2)^2}\,d\sigma &= \int_{-\infty}^{|x|/\alpha+\alpha\gamma/2}\Big|\tau+\frac{\alpha\gamma}{2}\Big|\mathrm{e}^{-\tau^2}\,d\tau \le \int_{-\infty}^{+\infty}\Big|\tau+\frac{\alpha\gamma}{2}\Big|\mathrm{e}^{-\tau^2}\,d\tau\\ &= \int_{-\infty}^{-\alpha\gamma/2}\Big(-\tau-\frac{\alpha\gamma}{2}\Big)\mathrm{e}^{-\tau^2}\,d\tau + \int_{-\alpha\gamma/2}^{+\infty}\Big(\tau+\frac{\alpha\gamma}{2}\Big)\mathrm{e}^{-\tau^2}\,d\tau\\ &= \mathrm{e}^{-\alpha^2\gamma^2/4} + \frac{\alpha\gamma}{2}\sqrt{\pi}\,\mathrm{erf}\Big(\frac{\alpha\gamma}{2}\Big)\end{aligned}$$

we deduce that

$$\int_{\mathbb{R}}\frac{|\xi-x|}{\alpha}\mathrm{e}^{-(x-\xi)^2/\alpha^2}\mathrm{e}^{-\gamma|\xi|}\,d\xi \le \alpha\mathrm{e}^{\alpha^2\gamma^2/4}\mathrm{e}^{-\gamma|x|}\Big(\frac{1}{2}+\mathrm{e}^{-\alpha^2\gamma^2/4}+\frac{\alpha\gamma}{2}\sqrt{\pi}\,\mathrm{erf}\Big(\frac{\alpha\gamma}{2}\Big)\Big).$$

Then (2.23) is proved with $b_\gamma(T) = 2 + \mathrm{e}^{T\gamma^2}(1+2\gamma\sqrt{T\pi})$. □

3. Examples

While, as far as the smoothness properties of the solution with respect to the space variables are concerned, the choice of a norm is ineffective[1], the asymptotic behavior of this solution when $t \to +\infty$ is greatly effected by a change of the norm in the space. It is true that the norm which is introduced in the space S is suggested by physical considerations, however, there is not only one norm (e.g. one topology) which suits the physical problem and, on the other hand, physical experiments are not able to determine what the norm should be. We illustrate this consideration by a simple but very significant example.

For fixed $p \geq 1$, $t > 0$, $a \geq 0$, $b \geq 0$ and $a + b > 0$, consider the space $\mathscr{H}_\mu^p(t)$ of real valued function $u(\cdot, t) \in C^1(\mathbb{R})$, $u(\cdot, t), u_x(\cdot, t) \in L_\mu^p(\mathbb{R})$ equipped with the norm $\|u(\cdot, t)\|_{p,\mu} = aM(t) + bN(t)$ where

$$M(t) = \Big(\int_{-\infty}^{+\infty} |u(x,t)|^p (1 + x^2)^\mu \, dx\Big)^{1/p};$$

$$N(t) = \Big(\int_{-\infty}^{+\infty} |u_x(x,t)|^p (1 + x^2)^\mu \, dx\Big)^{1/p}.$$

The function $u(x,t) = (t+1)^h (1 + x^2)^{-\gamma} \mathrm{e}^{-(x^2/(x^2+1))(t+1)^k}$ solves the problem (2.4), (2.5) with $\varphi(x) = (1 + x^2)^{-\gamma} \mathrm{e}^{-x^2/(x^2+1)}$ and

$$\begin{aligned} c(x,t) = {} & (h(t+1)^{-1} - k(t+1)^{k-1}) + (k(t+1)^{k-1} - 2\gamma - 4\gamma^2)(1 + x^2)^{-1} \\ & + 2[2\gamma(1+\gamma) - 3(t+1)^k - 4\gamma(t+1)^k](1 + x^2)^{-2} \\ & + 4[2(1+\gamma)(t+1)^k - (t+1)^{2k}](1 + x^2)^{-3} + 4(t+1)^{2k}(1 + x^2)^{-4}. \end{aligned}$$

The function $\varphi \in C^\infty(\mathbb{R})$ and satisfies conditions (2.6); for any fixed $h, k \in \mathbb{Z}$ and $t \in [0, T]$, the function c satisfies $i)_c$ and $ii)_c$.

In the following we assume $p \geq 1$, $\gamma > 1/2$ and $\mu < \gamma - 1/2$. We obtain

$$\begin{aligned} \text{(3.1)} \qquad M(t) & = (t+1)^h \Big(2 \int_0^{+\infty} \frac{\mathrm{e}^{-(px^2/(x^2+1))(t+1)^k}}{(1 + x^2)^{p\gamma - \mu}} \, dx\Big)^{1/p} \\ & = (t+1)^h \Big(\int_0^1 \mathrm{e}^{-p(t+1)^k s} s^{-1/2} (1 - s)^{p\gamma - \mu - 3/2} \, ds\Big)^{1/p} \\ & = (t+1)^{h - k/2p} \Big(\int_0^{(t+1)^k} \Big(1 - \frac{\sigma}{(t+1)^k}\Big)^{\gamma p - \mu - 3/2} \mathrm{e}^{-\sigma p} \sigma^{-1/2} \, d\sigma\Big)^{1/p}. \end{aligned}$$

[1] These smoothness properties in general depend on the smoothness of the data, no matter how a norm in the vector space has been introduced.

Define

$$f_t(\sigma) = \begin{cases} \left(1 - \frac{\sigma}{(t+1)^k}\right)^{\gamma p - \mu - 3/2} \mathrm{e}^{-\sigma p}\sigma^{-1/2} & 0 \leq \sigma \leq (t+1)^k \\ 0 & (t+1)^k < \sigma < +\infty \end{cases}$$

We have, for any $k > 0$ and $\gamma \geq \mu + 3/2$

$$\lim_{t\to+\infty} f_t(\sigma) = \mathrm{e}^{-\sigma p}\sigma^{-1/2}; \quad |f_t(\sigma)| \leq \mathrm{e}^{-\sigma p}\sigma^{-1/2} \quad (\sigma > 0).$$

Then

$$\lim_{t\to\infty} \int_0^{(t+1)^k} f_t(\sigma)\, d\sigma = \int_0^{+\infty} \mathrm{e}^{-\sigma p}\sigma^{-1/2}\, d\sigma = \frac{\sqrt{\pi}}{\sqrt{p}}.$$

We deduce that

$$\lim_{t\to+\infty} M(t) = \begin{cases} +\infty & h > \dfrac{k}{2p} \\ \left(\dfrac{\pi}{p}\right)^{1/2p} & h = \dfrac{k}{2p} \\ 0 & h < \dfrac{k}{2p} \end{cases}$$

From (3.1), for $\mu < \gamma - 1/2$, if $k = 0$

$$\lim_{t\to+\infty} M(t) = \begin{cases} +\infty & h > 0 \\ \left(\displaystyle\int_0^1 \mathrm{e}^{-ps} s^{1/2}(1-s)^{p\gamma - \mu - 3/2}\, ds\right)^{1/p} & h = 0; \\ 0 & h < 0 \end{cases}$$

if $k < 0$ we deduce that

$$\lim_{t\to+\infty} M(t) = \begin{cases} +\infty & h > 0 \\ f_{p,\mu,\gamma} & h = 0 \\ 0 & h < 0 \end{cases}$$

where

$$f_{p,\mu,\gamma} = \left(\int_0^1 s^{1/2}(1-s)^{p\gamma-\mu-3/2}\, ds\right)^{1/p} = \left(\frac{\sqrt{\pi}}{2}\frac{\Gamma\left(p\gamma - \mu - \frac{1}{2}\right)}{\Gamma(p\gamma - \mu + 1)}\right)^{1/p}.$$

Since

$$u_x(x,t) = -\frac{2x}{1+x^2}\left(\gamma + \frac{(t+1)^k}{1+x^2}\right)u(x,t)$$

one finds

$$N(t) = (t+1)^h \Big(2\int_0^{+\infty} \frac{2^p|x|^p}{(1+x^2)^p}\Big(\gamma + \frac{(t+1)^k}{1+x^2}\Big)^p |u(x,t)|^p\,dx\Big)^{1/p}$$

$$= 2(t+1)^h \Big(\int_0^1 e^{-sp(t+1)^k}(\gamma + (t+1)^k(1-s))^p$$

$$\times (1-s)^{\gamma p-\mu-3/2+p/2} s^{(p-1)/2}\,ds\Big)^{1/p}$$

$$= (t+1)^{h-k/2p+k/2}\Big(\int_0^{+\infty} g_t(\sigma)\,d\sigma\Big)^{1/p}$$

where

$$g_t(\sigma) = \begin{cases} 2\Big(1-\frac{\sigma}{(t+1)^k}\Big)^{\gamma p-\mu-3/2+p/2}\Big(1+\frac{\gamma}{(t+1)^k}-\frac{\sigma}{(t+1)^k}\Big)^p e^{-\sigma p}\sigma^{(p-1)/2} & 0\le\sigma\le(t+1)^k \\ 0 & \sigma > (t+1)^k. \end{cases}$$

For $k>0$ and $\mu \le \gamma - 1$

$$\lim_{t\to+\infty} g_t(\sigma) = 2e^{-\sigma p}\sigma^{(p-1)/2}; \quad |g_t(\sigma)| \le 2(1+\gamma)^p e^{-\sigma p}\sigma^{(p-1)/2} \quad (\sigma>0).$$

Then

$$\lim_{t\to\infty}\int_0^{(t+1)^k} g_t(\sigma)\,d\sigma = 2\int_0^{+\infty} e^{-\sigma p}\sigma^{(p-1)/2}\,d\sigma = 2\Gamma\Big(\frac{p+1}{2}\Big)p^{-(p+1)/2}.$$

We deduce that

$$\lim_{t\to+\infty} N(t) = \begin{cases} +\infty & h > \frac{k}{2p}-\frac{k}{2} \\ \Gamma\Big(\frac{p+1}{2}\Big)^{1/p} p^{-(p+1)/2p} & h = \frac{k}{2p}-\frac{k}{2} \\ 0 & h < \frac{k}{2p}-\frac{k}{2} \end{cases}$$

For $k=0$

$$\lim_{t\to+\infty} N(t) = \begin{cases} +\infty & h>0 \\ c_{p,\mu,\gamma} & h=0 \\ 0 & h<0 \end{cases}$$

$$c_{p,\mu,\gamma} = 2\Big(\int_0^1 e^{-sp}(\gamma+(1-s))^p(1-s)^{\gamma p-\mu-3/2+p/2}s^{(p-1)/2}\,ds\Big)^{1/p}.$$

For $k < 0$

$$\lim_{t\to+\infty} N(t) = \begin{cases} +\infty & h > \dfrac{k}{2p} - \dfrac{k}{2} \\ d_{p,\mu,\gamma} & h = \dfrac{k}{2p} - \dfrac{k}{2} \\ 0 & h < \dfrac{k}{2p} - \dfrac{k}{2} \end{cases}$$

where

$$d_{p,\mu,\gamma} = 2\gamma\Big(\frac{\Gamma\big(\frac{p+1}{2}\big)\Gamma\big(\gamma p + \frac{p}{2} - \mu - \frac{1}{2}\big)}{\Gamma(p(\gamma+1) - \mu)}\Big)^{1/p}.$$

Summarizing, we obtain that, for $k > 0$, $\gamma > 3/2$ and $\mu \le \gamma - 3/2$

$$\lim_{t\to+\infty} \|u(\cdot,t)\|_{p,\mu} = \begin{cases} a\cdot\infty + b\cdot\infty & h > \dfrac{k}{2p} \\ a\cdot\Big(\dfrac{\pi}{p}\Big)^{1/2p} + b\cdot\infty & h = \dfrac{k}{2p} \\ a\cdot 0 + b\cdot\infty & \dfrac{k}{2p} - \dfrac{k}{2} < h < \dfrac{k}{2p} \\ a\cdot 0 + \Gamma\Big(\dfrac{p+1}{2}\Big)^{1/p} p^{-(p+1)/2p} & h = \dfrac{k}{2p} - \dfrac{k}{2} \\ a\cdot 0 + b\cdot 0 & h < \dfrac{k}{2p} - \dfrac{k}{2} \end{cases}$$

For $\gamma > 1/2$ and $\mu < \gamma - 1/2$, if $k = 0$

$$\lim_{t\to+\infty} \|u(\cdot,t)\|_{p,\mu} = \begin{cases} a\cdot\infty + b\cdot\infty & h > 0 \\ a\cdot\Big(\dfrac{\pi}{p}\Big)^{1/2p} + b\cdot c_{p,\mu,\gamma} & h = 0; \\ a\cdot 0 + b\cdot 0 & h < 0 \end{cases}$$

if $k < 0$ then

$$\lim_{t\to+\infty} \|u(\cdot,t)\|_{p,\mu} = \begin{cases} a\cdot\infty + b\cdot\infty & h > 0 \\ a\cdot f_{p,\mu,\gamma} + b\cdot d_{p,\mu,\gamma} & h = 0 \\ a\cdot 0 + b\cdot 0 & h < 0 \end{cases}$$

In addition it is easy to show that

$$\sup_{x\in\mathbb{R}} |u(x,t)| = (t+1)^h; \quad \lim_{t\to+\infty} \sup_{x\in\mathbb{R}} |u(x,t)| = \begin{cases} 0 & h < 0 \\ 1 & h = 0 \\ +\infty & h > 0. \end{cases}$$

For example, consider $p = 2$, $\gamma > 3/2$ and $\mu < \gamma - 3/2$. If $k > 0$, $h = k/(2p) = k/4 > 0$ we have

$$\lim_{t \to +\infty} \Big(\int_{-\infty}^{+\infty} |u(x,t)|^2 (1 + x^2)^{\mu} \, dx \Big)^{1/2} = \Big(\frac{\pi}{2} \Big)^{1/4}.$$

If $k > 0$ and $0 < h < k/4$,

$$\lim_{t \to +\infty} \Big(\int_{-\infty}^{+\infty} |u(x,t)|^2 (1 + x^2)^{\mu} \, dx \Big)^{1/2} = 0.$$

If $k < 0$, $h = -k/4 > 0$

$$\lim_{t \to +\infty} \Big(\int_{-\infty}^{+\infty} |u_x(x,t)|^2 (1 + x^2)^{\mu} \, dx \Big)^{1/2} = 2\gamma \Big(\frac{\sqrt{\pi}}{2} \frac{\Gamma\big(2\gamma - \mu + \frac{1}{2}\big)}{\Gamma(2\gamma - \mu + 2)} \Big)^{1/p}.$$

In all of the cases we have that

$$\lim_{t \to +\infty} \sup_{x \in \mathbb{R}} |u(x,t)| = +\infty.$$

In the last transparency [2] Gaetano Fichera wrote *What of these norms has a physical meaning?* and he suggested that *Asymptotic Theory of PDE should be investigated considering the norm in the function space, where $u(x,t)$ is valued, like a* datum*; in other words by considering* stability *even with respect to the norm.*

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