Dynamic analysis of rotating tapered Rayleigh beams using two general approaches

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Abstract— **The free vibration characteristics of a rotating tapered Rayleigh beam is analysed in this study. Two approaches to the dynamic analysis of a rotating beam. The first proposed approach is the discretization "CDM" method, the structure is reduced to rigid bars, connected together by means of elastic hinges, and lower bound to the true frequencies is obtained. The second approach is a variational Rayleigh–Ritz like method, in which the vibration modes are expressed as a linear combination of orthogonal polynomials. The results are presented for Rayleigh beams with rotatory inertia effects and compared with existing solution. The parameters for the hub radius, rotational speed and taper ratio are incorporated. Numerous numerical examples are proposed for the calculation of the free frequencies of vibration to vary some parameter of taper of the beams. Good agreement with the finite element method is obtained. In the end the results are compared with those reported by other authors.**

Keywords- Rotating beam, Rayleigh-Ritz, CDM method.

I. INTRODUCTION

Rotating beam-like structures are widely used in various engineering fields such as helicopter blades, robotic manipulations, wind turbines and turbo-machinery. The free vibration frequencies and mode shapes of such structures have been a topic interest, and hence have received considerable attention. In the case of non-uniform beams, without centrifugal force, the exact solution of the motion equation is given by the well-known Bessel functions Auciello and Ercolano [1]. In all other cases, and in particular when the effects of on the transverse vibrations are taken into account, an approximate procedure is necessary. In order to investigate the effect of centrifugal force a number of methods based on the power series solution have been developed for determination of natural frequencies of rotating beams. The differential equation with variable coefficients is solved by means of the Frobenius series. Following this approach, Wright et al. [2] and Wang et al. [3], obtained the free vibrations of uniformly tapered beams according to the numbers of the terms of the series. The accuracy of the exact solution depends on the number of terms included in the Frobenius function ant it goes up with higher modes, taper, and rotation speed.

Gunda et al [4-5] used the finite element analysis method. They proposed these new hybrid-type functions to determine the free frequencies in both case, without rotation and with rotation. Some researched have also used the Dynamic Stiffness Method to (DSM) solve the natural frequencies of rotating beams. The structure is discretized with beam elements of constant section, therefore the considered stiffness is lower than the real one. Consequently, lower bound values of the frequencies are obtained. Instead, Mei [6] studied the free vibration of a rotating tapered Rayleigh beams by the method of differentiation (DTM).

In the case of the beam under axial loads, the free vibration analysis of a rotating beam can be undertaken by means of the Rayleigh-Ritz procedure with orthogonal test functions. To this aim, Boundary Characteristic Orthogonal Polynomials method (BCOP), Auciello [7], is used where the test functions are chosen in the polynomial set of those functions which respect the essential boundary conditions and then normalized by a Gram-Schmidt.

Instead, if the beam is discretized reducing it to a set of rigid bars linked together by elastic sections (elastic cells), the resulting approximate free frequencies values give a lower bound to the exact values; CDM method [10]. Consequently, it is possible to obtain a useful lower-upper approximation of the free frequencies of vibration.

In this work, flapwise bending vibration of a rotating tapered beam is studied by using the CDM method and Rayleigh-Ritz method with orthogonal polynomials method [9]. The partial differential equations are obtained through the Lagrange equations. The results of the parametric analysis have been compared with those known in literature and reported in bibliography.

II. DERIVATION OF THE MODAL EQUATIONS OF A ROTATING CANTILEVER BEAM

Consider a tapered Rayleigh beam rotating around the z1 axis with a constant speed as shown in Fig. 1. A generic point $\mathbf{p}_o$ the undeformed position is given of the vector

$$\mathbf{p}_0 = \left[r_0 + x,\ y,\ z \right]^T \tag{1}$$

If the beam now in deforms as a result of flexure and also under tension due to the centrifugal force, the position vector of the deformed point would now be given of the **P**:

$$\mathbf{p} = \left[r_0 + x + u - z\, w_{,x} - y\, v_{,x},\ y + v,\ z + w \right]^T \tag{2}$$

The velocity of a material point in deformed state is given by:

$$\mathbf{v}=\dot{\mathbf{p}}+\Omega(\mathbf{k}\times\mathbf{p})=\begin{bmatrix}\dot{u}-\Omega(y+v)-z\,\dot{w}_{,x}-y\,\dot{v}_{,x}\\ \dot{v}+\Omega(r_0+x+u)-\Omega(z\,w_{,x}+y\,v_{,x})\\ \dot{w}\end{bmatrix} \quad (3)$$

where, the derivatives with respect to the coordinates are defined with the related subscript while the time derivatives are defined with a dot. In the case of structure in which the bending effect (flapwise), the axial (stretching) effect are often assumed negligible and ignored.

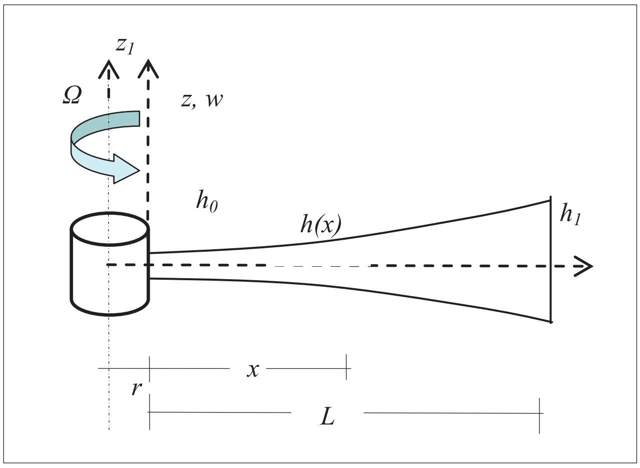

Figure 1. Non-uniform beam; parabolic thickness variation.

A. Hamilton principle

At steady state, the system can be considered conservative and its dynamic behavior can be obtained through the Hamilton principle;

$$\delta\int_{t_1}^{t_2} \mathsf{L}\, dt=0 \quad (4)$$

where $\mathsf{L}=T-V$, and T, V are respectively the kinetic and the potential energy. In the Euler-Bernoulli assumptions their explicit forms are given as follows:

$$T=\frac{1}{2}\int_0^L m(x)\,\dot{w}^2\,dx \quad (5)$$

$$V=\frac{1}{2}E\int_0^L I(x)\,w_{,xx}^2\,dx+\frac{1}{2}\int_0^L F(x)\,w_{,x}^2+\overline{F}, \quad (6)$$

where E and $I(x)$ are respectively the Young modulus and the inertia of the section referred to the x axis and $m(x)$ is the mass distribution.

The term, axial force due to centrifugal stiffening, F(x) is given as

$$F_x=\int_x^L \rho\,A(x)\,\Omega^2(r_0+x)\,dx+\overline{F}\,. \quad (7)$$

Centrifugal force acting on the beam at a distance from the origin and is due to the effect of the spin around the axis z. $\overline{F}$ is constant and related to the static inertia, in dynamic conditions its contribution is zero. Thus potential energy is made of two different terms: one due to bending deformation and the other due to centrifugal force deformation. In the hypothesis of separation of variables, the transverse displacements w(x,t) can be written as follows

$$w(x,t)=W(x)\cos\omega t, \quad (8)$$

where $W(x)$ represents the amplitude of the displacements $w(x,t)$. So, the maximum kinetic energy is

$$T=\frac{\omega^2}{2}\int_0^L \rho A(x)\,W^2(x)\,dx, \quad (9)$$

where ρ is the mass density and A(x) is the cross-sectional area. Therefore, the maximum potential energy can be written as:

$$V=\frac{1}{2}E\int_0^L I(x)\,W_{,xx}^2\,dx+\frac{1}{2}\int_0^L F(x)\,W_{,x}^2+\overline{F}. \quad (10)$$

The Lagrange's equations for free vibration of a distributed parameter are given by

$$\frac{d}{dt}\left(\frac{\partial T}{\partial\dot{q}_i}\right)-\frac{\partial T}{\partial q_i}+\frac{\partial U}{\partial q_i}=0, \qquad i=1,2....n \quad (11)$$

where n is the total number of modal coordinates. The partial derivatives of T with respect to the generalized coordinates are needed.

III. "CDM" METHOD

The beam, under consideration, is discretized reducing it to a set of rigid bars, linked together by elastic sections, "cells". In this way, the structure is reduced to a system with finite number of degrees of freedom (MDOF). The Lagragian parameters can be assumed to be the t rotations of the rigid bars, i.e. the generalized coordinates of the rigid-elastic system. All the possible configurations are functions of the following vector:

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$$\mathbf{c}=[\varphi_1,\varphi_2,\ \dots\ \varphi_t]^T \quad (12)$$

and the vertical components of the nodal displacements are given by the following expressions:

$$w_1=0,\quad w_2=-\varphi_1\,l_1\,,\quad w_i=-\varphi_i\,l_i,\quad w_{t+i}=-\varphi_t\,l_t.$$

In matrix form, being A the displacements matrix, it is possible to write:

$$\mathbf{w} = \mathbf{A}\,\mathbf{c}, \tag{13}$$

while, the nodal rotations, in the approximate form, can be written as follows:

$$w_{1,x} = 0, \quad w_{2,x} = \varphi_2\,, \quad w_{i,x} = \varphi_i, \quad w_{t+1,x} = \varphi_t, \tag{14}$$

$$\mathbf{w}_{,x} = \mathbf{R}\,\mathbf{c}\,, \tag{15}$$

Similarly, the relative rotations between the two faces of the elastic cells are given by:

$$\Delta\varphi_1 = \varphi_1, \quad \Delta\varphi_2 = \varphi_2 - \varphi_1\,, \quad \Delta\varphi_i = \varphi_i - \varphi_{i-1}, \quad \Delta\varphi_{t+1} = 0$$

or,

$$\Delta\boldsymbol{\varphi} = \mathbf{B}\,\mathbf{c}. \tag{16}$$

The rectangular matrices **A**, **R** and **B** have *t+1* rows and t columns, and each term can be calculated according to Fig.2.

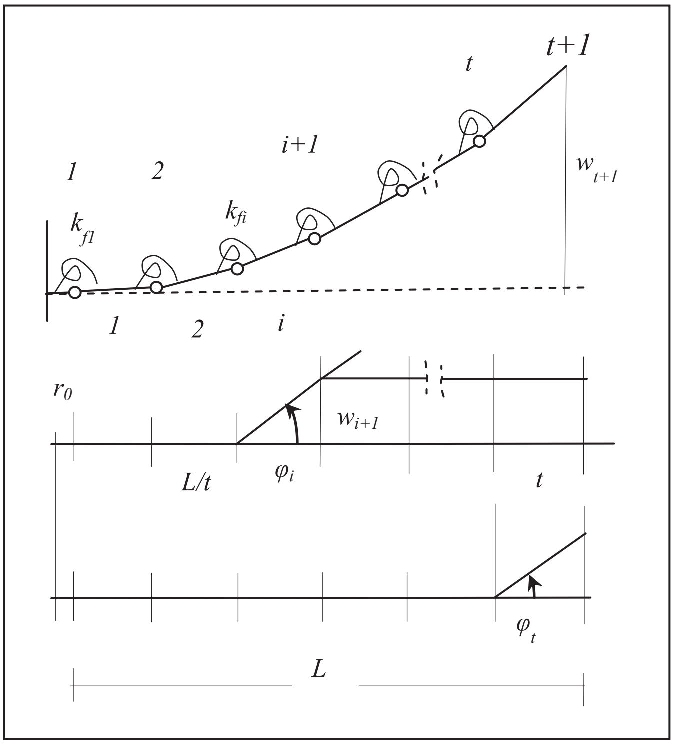

Figure 2. Discretization "CDM" method.

With the proposed method, the mass of the beam is properly concentrated at the nodal point of the rigid segments. Therefore, the mass distribution becomes:

$$\begin{aligned} m_1 &= \rho \frac{L}{2t} A_1, \\ m_i &= \rho \frac{L}{t} \frac{A_i + A_{i+1}}{2} \qquad i = 2\ldots,t \\ m_{t+1} &= \rho \frac{L}{2t} A_{t+1}, \end{aligned} \tag{17}$$

In the Rayleigh hypothesis it is necessary to consider the inertia effect of the section respect to axis y. Being Iy, the moment of inertia with respect to the axis y, it can be conveniently written:

$$\begin{aligned} \tilde{m}_1 &= \rho \frac{L}{2t} I_{y1}, \\ \tilde{m}_i &= \rho \frac{L}{t} \frac{I_{yi} + I_{y(i+1)}}{2} \qquad i = 2\ldots,t \\ \tilde{m}_{t+1} &= \rho \frac{L}{2t} I_{y(t+1)}. \end{aligned} \tag{18}$$

A. Kinetic energy

Substituting eqs (13-16) into eq. (5) one obtained the kinetic energy of rotating becomes:

$$T = \frac{1}{2}\sum_1^{t+1} m_i\,\dot{w}_i^2 + \frac{1}{2}\sum_1^{t+1} \tilde{m}_i \left[\left(\dot{w}_{,x}\right)^2 + \Omega^2 \left(w_{,x}\right)^2\right] =$$

$$= \frac{1}{2}\dot{\mathbf{c}}^T\,\mathbf{A}^T\,\mathbf{m}\,\mathbf{A}\,\dot{\mathbf{c}} + \frac{1}{2}\left[\dot{\mathbf{c}}^T\,\mathbf{R}^T\,\tilde{\mathbf{m}}\,\mathbf{R}\,\dot{\mathbf{c}} + \Omega^2\,\mathbf{c}^T\,\mathbf{R}^T\,\tilde{\mathbf{m}}\,\mathbf{R}\,\mathbf{c}\right].$$

$$T = \frac{1}{2}\dot{\mathbf{c}}^T\left[\mathbf{A}^T\,\mathbf{m}\,\mathbf{A} + \mathbf{R}^T\,\tilde{\mathbf{m}}\,\mathbf{R}\right]\dot{\mathbf{c}} + \frac{1}{2}\Omega^2\,\mathbf{c}^T\,\mathbf{R}^T\,\tilde{\mathbf{m}}\,\mathbf{R}\,\mathbf{c}. \tag{19}$$

From the preceding relationship it is underlined that kinetic energy is furnished by the sum of the really due term to the speed, ċ, anymore and the dependent share from the rotatory motion around z_1. The effect of the rotatory given of the reduction of the rigidity of the structure, eq. (19) is written:

$$T = \frac{1}{2}\dot{\mathbf{c}}^T\,\mathbf{M}\,\dot{\mathbf{c}} + \frac{1}{2}\Omega^2\,\mathbf{c}^T\,\mathbf{R}^T\,\tilde{\mathbf{m}}\,\mathbf{R}\,\mathbf{c}. \tag{20}$$

B. Strain energy

Quite often it is possible to neglect both the axial and the shear deformation effects, limiting one to the bending deformations. In such hypothesis, at each "cell", the following relation between the relative rotation $\Delta\varphi_i$ and the moment M_i can be written, as follows:

$$M_i = k_i\,\Delta\varphi_i \tag{21}$$

In the rigid-elastic formulation, relation (6) must be expressed as functions of the rotations of the rigid bars and the following relationship is easily obtained:

$$V = \frac{1}{2}\mathbf{c}^T\mathbf{B}^T\,\mathbf{k_f}\,\mathbf{B}\,\mathbf{c} + \frac{1}{2}\mathbf{c}^T\,\mathbf{F}_x\,\mathbf{c}\,, \tag{22}$$

where the cells stiffness k_i, according to the present discretization, can be written as:

$$\begin{aligned} k_1 &= 2\frac{t}{L} EI_{y1}, \\ k_i &= 2\frac{t}{L} E \frac{I_{yi}\,I_{y(i+1)}}{I_{yi} + I_{y(i+1)}} \quad (i = 2,\ldots t), \\ k_{t+1} &= 2\frac{t}{L} EI_{y(t+1)}. \end{aligned} \tag{23}$$

Substituting eqs (20-23) into eq. (11) obtained the equation of motion and the free frequencies have calculated by solving the eingevalues problem given by the following algebraic system:

$$(\mathbf{B}^T \mathbf{k_f}\, \mathbf{B} + \mathbf{F}_x - \Omega^2\, \mathbf{R}^T\, \tilde{\mathbf{m}}\, \mathbf{R})\,\mathbf{c} + \mathbf{M}\, \ddot{\mathbf{c}} = 0 \;. \tag{24}$$

IV. Rayleigh-Ritz method

The Ritz extension of the Rayleigh method is one of the convenient procedures for evaluating the modes of vibration. In the approximate formulation the transversal displacements are assumed in term of generalized coordinate q. The transversal displacements are assumed to be linear combination of n independent functions which satisfy the boundary equations. If functions ϕi are chosen respecting the geometrical constraints the displacements can be written

$$w(x,t) = \phi_i(x)\, q_i(t) = \mathbf{\Phi}^T \mathbf{q}, \tag{25}$$

where $\phi_i(x)$ represents the assumed modal functions (test function).

Substituting (26) in eq (5) the kinetic energy is

$$\begin{aligned} T &= \frac{1}{2}\int_0^L \rho A(x)\, \dot{\mathbf{q}}^T \mathbf{\Phi}\, \mathbf{\Phi}^T \dot{\mathbf{q}}\, dx + \\ &+ \frac{1}{2}\int_0^L \rho\, I\left(x\right)\left[\Omega^2\, \mathbf{q}^T \mathbf{\Phi}_{,x}\, \mathbf{\Phi}_{,x}^T\, \mathbf{q} + \dot{\mathbf{q}}^T \mathbf{\Phi}_{,x}\, \mathbf{\Phi}_{,x}^T\, \dot{\mathbf{q}}\right] dx. \end{aligned} \tag{26}$$

Therefore, the strain energy, eq (6) can be written as:

$$V = \frac{1}{2} E\int_0^L I(x)\, \mathbf{q}^T \mathbf{\Phi}_{,xx}\, \mathbf{\Phi}_{,xx}^T\, \mathbf{q}\, dx + \frac{1}{2}\int_0^L F_x\, \mathbf{q}^T \mathbf{\Phi}_{,x}\, \mathbf{\Phi}_{,x}^T\, \mathbf{q}. \tag{27}$$

The single terms of the equation of Lagrange, written in operation of the generalized coordinates q, is given by the following

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{\mathbf{q}}} = \int_0^L \rho A(x) \mathbf{\Phi}\, \mathbf{\Phi}^T\, \ddot{\mathbf{q}}\;\; dx + \int_0^L \rho\, I_y(x) \mathbf{\Phi}_{,x}\, \mathbf{\Phi}_{,x}^T\, \ddot{\mathbf{q}}\; dx = \mathbf{M}_A + \mathbf{M}_R,$$

$$\frac{\partial T}{\partial \mathbf{q}} = \Omega^2 \int_0^L \rho I_y(x) \mathbf{\Phi}_{,x}\, \mathbf{\Phi}_{,x}^T\, \mathbf{q}\; dx = \mathbf{K}_R$$

$$\frac{\partial V}{\partial \mathbf{q}} = \int_0^L EI_y(x)\;\; \mathbf{\Phi}_{,xx}\, \mathbf{\Phi}_{,xx}^T\, \mathbf{q}\; dx + \int_0^L F_x\, \mathbf{\Phi}_{,x}\, \mathbf{\Phi}_{,x}^T\; \mathbf{q}\; dx = \mathbf{K}_f + \mathbf{K}_\Omega$$

Equations of the motion is:

$$\left(\mathbf{M}_A + \mathbf{M}_R\right)\ddot{\mathbf{q}} + \left(\mathbf{K}_f + \mathbf{K}_\Omega - \mathbf{K}_R\right)\mathbf{q} = \mathbf{0} \tag{28}$$

or

$$(\mathbf{K} - \omega^2 \mathbf{M})\,\mathbf{q} = 0\,, \tag{29}$$

where:

$$\mathbf{K} = \mathbf{K}_f + \mathbf{K}_\Omega - \mathbf{K}_R, \qquad \mathbf{M} = \mathbf{M}_A + \mathbf{M}_R. \tag{30,31}$$

A. Nondimensional analysis

In order to compare the results with those reported in the literature it is useful to introduce the functions *G(x)* and *H(x)* that define, in general terms, the geometric characteristics of the structure

$$A(x) = A_0\, G(x), \qquad I(x) = I_0\, H(x), \tag{32,33}$$

where A0 and I0, are respectively the area and moment of inertia of the section at $x = 0$.

Introducing the following parameters

$$\xi = \frac{x}{L}, \qquad \delta = \frac{r_0}{L}, \qquad \gamma^2 = \rho\frac{A_0\; L^4}{E\, I_0}\Omega$$

$$\lambda^2 = \omega^2\frac{\rho A_0 L^4}{E\, I_0}, \qquad r_H = \left(\frac{A_0\, L^2}{I_0}\right)^{1/2}, \qquad \mathbf{\theta} = \frac{\mathbf{q}}{L}. \tag{33}$$

Therefore the eigenvalue problem can be placed in the following non-dimensional form

$$(\bar{K}_{ij} - \lambda_i^4 \bar{M}_{ij})q_i = 0\,. \tag{34}$$

where

$$\begin{aligned} \bar{K}_{ij} &= \int_0^1 H(\xi)\; \phi_{i,\xi\xi}\, \phi_{j,\xi\xi}\; d\xi + \\ &+ \gamma^2\left[\int_0^1 F_\xi\;\; \phi_{i,\xi}\, \phi_{j,\xi}\; d\xi - \int_0^1 H(\xi)\;\; \phi_{i,\xi}\, \phi_{j,\xi}\; d\xi\right] \end{aligned} \tag{35}$$

$$\bar{M}_{ij} = \int_0^1 G(\xi)\; \phi_i\, \phi_j\; d\xi + \int_0^1 G(\xi)\; \phi_{i,\xi}\, \phi_{j,\xi}\; d\xi\;, \tag{36}$$

As well known, the polynomial functions are chosen respecting both essential and normality conditions. The geometric conditions are:

$$W = 0, \qquad \frac{\partial W}{\partial \xi} = 0, \qquad \text{at}\;\; \xi = 0\,, \tag{37,38}$$

From (25) the first polynomial ϕ_1 can be obtained. After, by means of the Gram-Schmidt normalization, all the other requested functions can be obtained by *Mathematica* program.

V. Free vibration results

A. Tapered beam

(27)

Let us assume now that the variation of the cross section of the beam is given by the equations (23). In the case of tapered beams, the cross section area and moment of inertia are represented by the following expressions:

$$\begin{aligned} A\left(\xi\right) &= A_1\left(1 - \alpha\,\xi\right)\left(1 - \beta\,\xi\right) \\ I_y(\xi) &= I_{y1}\left(1 - \alpha\,\xi\right)^3\left(1 - \beta\,\xi\right) \end{aligned}. \tag{39}$$

where the α and β parameters define the variation of height and base of the beam cross section along its span. As already said, in the literature few papers exist which deal with the tapered Rayleigh beam theory, so that the Authors – for the sake of comparisons.

In the Table I, for α=0.5 and r_H=1/30 data, the values obtained are reported. As shown, the natural frequencies obtained by applying the CDM method are always lower bounds to the values determined by the R-R method.

TABLE I. COMPARISON OF NATURAL FREQUENCIES; TAPERED BEAMS "WEDGE BEAM".

β=0	**C.D.M.**			**R-R method**		
α=0,5	γ=0	γ=5	γ=10	γ=0	γ=5	γ=10
λ_1	3,8177	6,7332	11,4838	3,8181	6,7391	11,4978
λ_2	18,1601	21,7123	29,8974	18,1688	21,7362	29,9639
λ_3	46,2759	49,8385	59,1543	46,3265	49,9288	59,3794
λ_4	86,9716	90,4926	100,1831	87,1368	90,7623	100,8026
λ_5	139,0892	142,4541	151,9492	139,4866	143,0885	153,3487

In Figure 3, setting r_H=1/30, δ=β=0 and α=0,5, the free frequencies values of tapered beam are obtained by varying the angular speed γ. As shown, when the speed increases, the curve of fundamental frequencies tends, in asymptotic way, to the straight line of the angular speeds. In the Rayleigh beam case, for the higher modes of vibrations, the effect of taper has a relevant impact. In fact for the higher modes, the shape of cross-section beam has a relevant influence on the free frequencies results

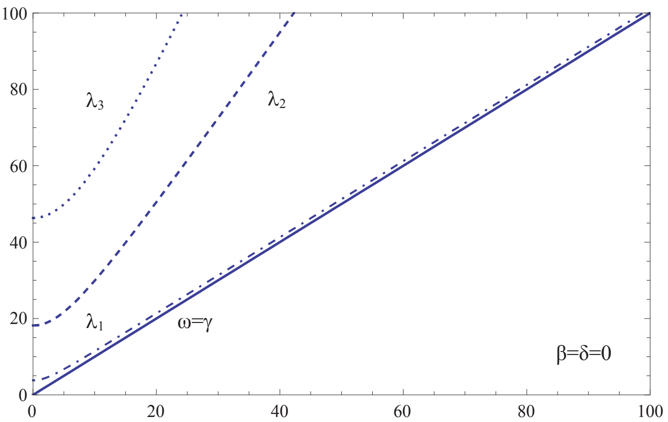

Figure 3. Natural frequencies for tapered Rayleigh beam α=0,5, and r_H=1/30.

The natural frequencies calculated by Jackson et al. [8], as reported in Tab. II,

TABLE II. TAPERED BEAMS "WEDGE BEAM"; JACKSON [8].

β=0	**Jackson [8]**		
α=0,5	γ=0	γ=5	γ=10
λ_1	3,82109	6,7356	11,4856
λ_2	18,2245	21,7911	30,0232
λ_3	46,5757	50,1876	59,6737
λ_4	87,7974	91,4413	101,5422
λ_5	140,8192	144,4462	154,7865

It can be noted as the frequencies calculated with the DTM method are strongly influenced by the angular speed ,γ . Those calculated with the proposed methods provide numerical solutions not dependent on geometric and kinematic parameters.

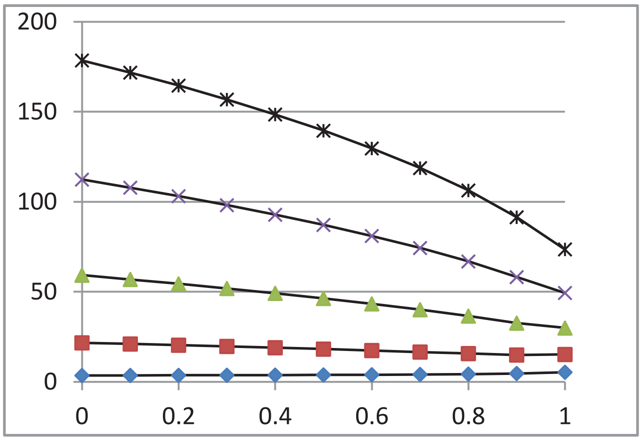

Figure 4. Variation of the nondimensional frequencies with respect to the taper ratio, α, for the five lowest modes of the beam with γ=0, β=0 and δ=0.

In the Rayleigh beam case, for the higher modes of vibrations, the effect of taper has a relevant impact. Infact for the higher modes, the shape of cross-section beam has a relevant influence on the free frequencies results. In Fig. 4, the curve of the first five fundamental frequencies are reported and for the different values of taper parameter, α.

In the case r_H=1/10 (slenderness ratio) and considering the same Rayleigh beam, a different behaviour can be observed. The resonance phenomenon occurs for γ = 62.40 (tuned angular speed), which represents the intersection between the curve relative to the fundamental frequencies, and the line of the angular speeds γ=ω, see Figure 5. If the δ parameter (named hub radius) is allowed to increase, the centrifugal force leads to an increase of the extension deformation of the beam, so that the fundamental frequencies migrate away from the line of speeds and the resonance phenomenon does not occur.

This conclusion has a practical usefulness in analyzing the rotors, where the control devices are of paramount importance.

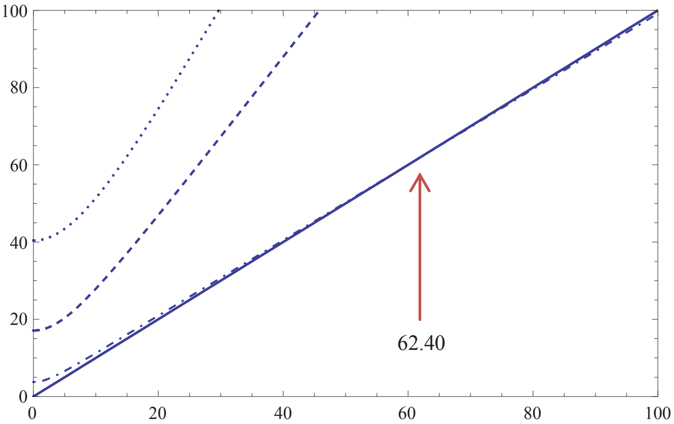

Figure 5. Natural frequencies for tapered Rayleigh beam α=0,5, and r_H=1/10.

B. Tapered beam; parabolic thickness variation.

This section is concerned with the transverse vibration of the non-uniform beam of constant breadth and depth proportional to the square of the axial co-ordinate. In particular, the geometry of the structure is given:

$$h(\xi)=h_0\left(1+(\alpha-1)\xi^2\right), \qquad b(\xi)=b_0, \tag{40}$$

the area and the inertia assume the form

$$\begin{aligned} A(\xi) &= A_0\left(1+(\alpha-1)\xi^2\right) \\ I(\xi) &= I_0\left(1+\xi^2(\alpha-1)\right)^3. \end{aligned} \tag{41}$$

In Tab III, the first five natural frequencies for rotating parabolic non-uniform Euler beam (r_H=1/1000), α=5, are presented for various angular speed, γ and hub ratios, δ.

TABLE III. COMPARISON OF FIRST FIVE NATURAL FREQUENCIES; PARABOLIC THICKESS VARIATION (EULER BEAM).

α=5		δ=0	δ=5
γ=0	λ_1	2.2608	
	λ_2	30.0559	
	λ_3	110.7650	
	λ_4	230.4959	
	λ_5	395.3289	
γ=12	λ_1	12.5806	33.4471
	λ_2	50.0796	106.4972
	λ_3	137.4859	234.4189
	λ_4	260.9709	397.4638
	λ_5	428.5189	605.8781
γ=100	λ_1	100.9850	276.5721
	λ_2	301.1600	769.3038
	λ_3	550.2041	1320.2100
	λ_4	854.6310	1968.2940
	λ_5	1223.8710	2735.9000

VI. CONCLUSIONS

In this paper the stability and dynamic analysis of rotating Rayleigh beam. Two different numerical approaches have been used and compared, namely the CDM method and the Rayleigh-Ritz method with polynomial functions. Both approaches lead to numerically stable algorithms and the numerical results are in excellent agreement. All the symbolic and numerical computations have been performed using *Mathematica.*

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