Flapwise bending vibration of rotating Euler-Bernoulli beam with non-uniform tapers

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Abstract— **In this work the dynamic behavior of beams with variable section rotating around an axis is analyses. The natural frequency of the flapwise bending vibration is investigate for the rotating beam. The model used is respectful of the Euler-Bernoulli hypotheses and obtains the natural frequencies and modes according to the rotation speed. Using the Boundary Characteristic Orthogonal Polynomial method is proposed to solve the natural frequency of rotating tapered beam at high angular velocity. Numerous numerical examples are proposed for the calculation of the free frequencies of vibration to vary some parameter of taper of the beams. In the end the results are compared with those reported by other authors.**

Keywords: Hamilton principle, vibration beams, centrifugal force, Rayleigh-Ritz method.

I. INTRODUCTION

Rotating beam-like structures are widely used in various engineering fields, such helicopter blades, robotic manipulators, wind turbines and turbo-machinery. The free vibration frequencies and mode shapes of such structures have been a topic of interest, and hence have received considerable attention [1-3]. A rotating beam differs from a non-rotating beam in having additional centrifugal stiffness and Coriolis effects on its dynamics. These previous studies usually have been based on Euler-Bernoulli beam theory and various approximate solution techniques have been used to obtain the dynamic characteristics of such rotating beams. In order to investigate the effect of centrifugal force Yoo and al. [4] used a modal formulation to obtain the natural frequencies.

A number of methods based on the power series solution have been developed for determination of natural frequencies of rotating tapered beams. The differential equation with variable coefficients is solved by means of the Frobenius series. Following this approach, Wright et al. [2] and Wang et al. [6] obtained the free vibrations of uniformly tapered beams according to the numbrs of the terms of the series. The accuracy of the exact solution depends on the number of terms included in the Frobenius function ant it goes up with higher modes, taper, and rotation speed.

Some researched have also used the Dynamic Stiffness Method (DSM), to solve the natural frequencies of rotating beams. Benerjee et al. [7] studied the free vibration frequencies for tapered beams with various boundary conditions. The structure is discretized with beam elements of constant section, therefore the considered stiffness is lower than the real one. Ozgumus and al. [8] obtained the free vibrations of rotating beams by the method of differentiation (DTM), [15]. Gunda et al [10] used the linear combination of terms of the functions derived from the exact solution of the governing static differential equation of a stiff-string and that of a non-rotating beam. They proposed these new hybrid-type functions to determine the free frequencies in both case, without rotation and with rotation.

In this work, flapwise bending vibration of a rotating tapered Bernoulli-Euler beam is studied by using the approximate Ritz method. The partial differential equations are obtained through the Hamilton energy principle written in the test functions space. Further, minimizing the Rayleigh quotient, the frequencies equation is also obtained. Worth mentioning that the exposed procedure gives upper bounds values of the free frequencies. The results of the parametric analysis have been compared with those known in literature and reported in bibliography. The purpose of this paper is to perform the modal analysis of rotating cantilever beams based on the modeling method.

II. MATHEMATICAL FORMULATIONS

Consider a tapered Euler-Bernoulli beam rotating around the z_1 axis with a constant speed, Ω, as shown in Fig. 1. It is assumed that, the cross profile of the beam is symmetric. This implies that the locus of the centroids and shear centers coincide along the span of the beam hence nullifying any bending and torsion coupling effects. The material proprieties of the beam are isotropic and homogeneous,

At steady state, the system can be considered conservative and its dynamic behavior can be obtained through the *Hamilton* principle;

$$\delta\int_{t_1}^{t_2} \mathsf{L}\, dt = 0 \qquad (1)$$

where $\mathsf{L} = T - V$, and T, V are respectively the kinetic and the potential energy. In the Euler-Bernoulli assumptions their explicit forms are given as follows:

$$T = \frac{1}{2}\int_0^L m(x)\,\dot{w}^2\, dx \qquad (2)$$

$$V = \frac{1}{2} E \int_0^L I(x)\; w_{,xx}^2 \, dx + \frac{1}{2} \int_0^L F(x)\; w_{,x}^2 + \overline{F}, \qquad (3)$$

where *E* and *I(x)* are respectively the Young modulus and the inertia of the section referred to the *x* axis and *m(x)* is the mass distribution. The derivatives with respect to the coordinates are defined with the related subscript while the time derivatives are defined with a dot.

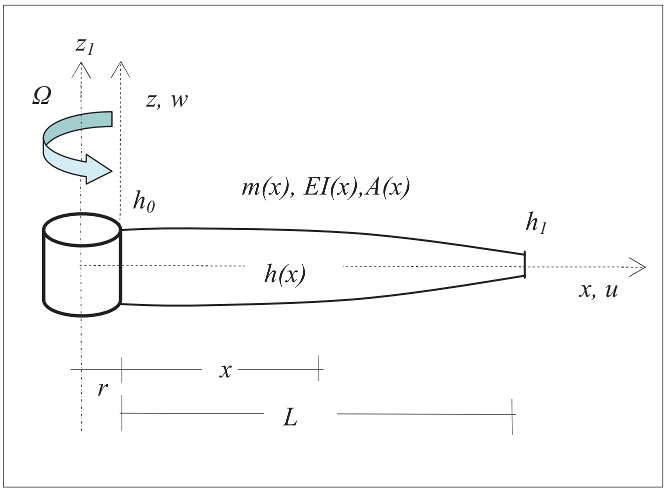


Figure 1. Configuration of a rotating, non-uniform, cantilever beam.

The term

$$F(x) = \frac{1}{2} \int_x^L m(x)\;\; \Omega^2\, (r + x)\, dx + \overline{F} \qquad (4)$$

is the centrifugal force acting on the beam at a distance from the origin and is due to the effect of the spin around the axis z. The term $\overline{F}$ is constant and related to the static inertia, in dynamic conditions its contribution is zero.

Thus potential energy is made of two different terms: one due to bending deformation and the other due to centrifugal force deformation.

In the hypothesis of separation of variables, the transverse displacements *w(x,t)* can be written as follows

$$w(x,t) = W(x) \cos \omega t, \qquad (5)$$

where *W(x)* represents the amplitude of the displacements *w(x,t)*. So, the maximum kinetic energy is

$$T_{\max} = \frac{\omega^2}{2} \int_0^L \rho A(x)\; W^2(x)\, dx, \qquad (6)$$

where ρ is the mass density and A(x) is the cross-sectional area. Therefore, the maximum potential energy can be written as:

$$V_{\max} = \frac{1}{2} E \int_0^L I(x)\; W_{,xx}^2 \, dx + \frac{1}{2} \int_0^L F(x)\; W_{,x}^2 + \overline{F}. \qquad (7)$$

Equating T_{max} with V_{max}, by means of the *Rayleigh* quotient, the natural frequencies can be obtained by the following

$$\omega^2 = \frac{V_{\max}}{T_{\max}}. \qquad (8)$$

Because the maximum amplitude for the displacements has been assumed, the frequencies obtained in (8) are higher than the exact ones. From a theoretical point of view this means that a constraint has been added with the obvious consequence of an increase of the global stiffness of the system. Obviously upper bound values will be obtained.

In the approximate formulation the transversal displacements are assumed to be linear combination of n independent functions which satisfy the boundary equations. If functions φi are chosen respecting the geometrical constraints the displacements can be written

$$W(x) = \sum_{J=1}^{n} q_j\, \varphi_j(x) = \boldsymbol{\varphi}^T \mathbf{q} \;, \qquad j = 1, \ldots.n \;, \qquad (9)$$

where $\varphi_i(x)$, are orthogonal functions and q is the generalized displacements vector. Substituting (9) in eq. (8), the *Rayleigh* ratio has the following form

$$R(\omega) = \frac{E \int_0^L I(x)[(\boldsymbol{\varphi}_{,xx}^T \mathbf{q})^T \boldsymbol{\varphi}_{,xx}^T \mathbf{q}]\, dx + \int_x^L F(x)\, [(\boldsymbol{\varphi}_{,x}^T \mathbf{q})^T \boldsymbol{\varphi}_{,x}^T \mathbf{q}] + F}{\int_0^L \rho\, A(x)[(\boldsymbol{\varphi}^T \mathbf{q})^T \boldsymbol{\varphi}^T \mathbf{q}]\, dx}.$$

Imposing the stationary conditions (1) with respect to the q_i , the homogeneous system in the unknown q_j is obtained

$$\frac{\partial R[\omega]}{\partial q_j} = 0, \quad j = 1 \ldots.n, \qquad (10)$$

and consequently the eigenvalue problem :

$$(K_{ij} - \omega_i^2 M_{ij}) q_i = 0, \qquad (11)$$

where K_{ij} and M_{ij} are given as

$$K_{ij} = E \int_0^L I(x)\, \varphi_{i,xx}\, \varphi_{j,xx}\, dx + \int_0^L F(x)\, \varphi_{i,x}\, \varphi_{j,x}\, dx \qquad (12)$$

$$M_{ij} = \int_0^L \rho\, A(x)\;\, \varphi_i\, \varphi_j\, dx\,. \qquad (13)$$

III. ADIMENSIONAL ANALYSIS

In order to compare the results with those reported in the literature it is useful to introduce the functions *G(x)* and *H(x)* that define, in general terms, the geometric characteristics of the structure

$$A(x) = A_0\, G(x), \qquad I(x) = I_0\, H(x), \qquad (14,15)$$

where A_0 and I_0, are respectively the area and moment of inertia of the section at $x = 0$.

Applying the following non-dimensional parameters

$$\xi = \frac{x}{L}, \quad \delta = \frac{r}{L}, \quad \gamma^2 = \rho \frac{A_0\, \Omega\, L^4}{E\, I_0}, \quad \lambda_i^2 = \omega_i^2 \frac{\rho A_0 L^4}{E\, I_0}, \quad (16)$$

the terms in (12) and (13) can be rewritten as

$$\bar{K}_{ij} = \int_0^1 H(\xi)\, \varphi_{i,\xi\xi}\, \varphi_{j,\xi\xi}\, d\xi + \gamma^2 \int_0^1 F(\xi) \varphi_{i,\xi}\, \varphi_{j,\xi}\, d\xi \quad (17)$$

$$\bar{M}_{ij} = \int_0^1 G(\xi)\, \varphi_i\, \varphi_j\, d\xi\,. \quad (18)$$

Therefore the eigenvalue problem can be placed in the following non-dimensional form

$$(\bar{K}_{ij} - \lambda_i^4 \bar{M}_{ij}) q_i = 0\,. \quad (19)$$

As well known, the polynomial functions are chosen respecting both essential and normality conditions. The geometric conditions are:

Cantilever beam

$$W = 0, \qquad \frac{\partial W}{\partial \xi} = 0, \qquad \text{at} \quad \xi = 0\,, \quad (20)$$

The first polynomial φ_1 can be obtained. After, by means of the Gram-Schmidt normalization, all the other requested functions can be obtained by *Mathematica* program.

As shown before, the weak formulation of the problem contains implicitly the so called natural conditions. Obviously all natural and essential conditions can be considered in the test functions but in this case these functions will be polynomials of higher degree. As consequence, the computer time requested for the integrals in (17-18) would increase without having an appreciable higher precision.

IV. FREE VIBRATION RESULTS

Geometry of the structure is given through the introduction of two functions *G (ξ)* and *H (ξ)* which supply the tapering laws:

$$A(\xi) = A_0\, G(\xi)$$
$$I(\xi) = I_0\, H(\xi). \quad (21)$$

If the tapering is defined by a linear variation of both the height and the thickness of the beam the geometric functions, are

$$h(\xi) = h_0 (1 - \alpha \xi), \qquad b(\xi) = b_0 (1 - \beta \xi), \quad (22)$$

Consequently the area and the inertia assume the form

$$A(\xi) = A_0 (1 - \alpha \xi)(1 - \beta \xi) = A_0\, G(\xi)$$
$$I(\xi) = I_0 (1 - \alpha \xi)^3 (1 - \beta \xi) = I_0\, H(\xi). \quad (23)$$

To varying of α and β, the various geometric conditions reported in bibliography can be compared. In particular, for α=β, the geometric distribution is given by

$$A(\xi) = A_0 (1 - \alpha \xi)^n \quad (16)$$
$$I(\xi) = I_0 (1 - \alpha \xi)^{n+2}, \quad (24)$$

where α is the tapering coefficient and *n* is respectively, *1* or *2*.

TABLE I. COMPARISON OF FIRST FIVE NATURAL FREQUENCIES; UNIFORM BEAM.

α=0				α=0			
δ=0	γ=5	λ_1	6.44950	δ=1	γ=5	λ_1	8.9403
		λ_2	25.4461			λ_2	29.3528
		λ_3	65.2050			λ_3	69.7607
		λ_4	124.5660			λ_4	129.5800
		λ_5	203.6220			λ_5	208.9110
	γ=12	λ_1	13.17020		γ=12	λ_1	19,72150
		λ_2	37.6031			λ_2	51,0701
		λ_3	79.6145			λ_3	98,5268
		λ_4	140.5340			λ_4	163,7240
		λ_5	220.5360			λ_5	246,7160
δ=2	γ=5	λ_1	10.8616	δ=5	γ=5	λ_1	15.2012
		λ_2	32.7642			λ_2	41.2249
		λ_3	73.9844			λ_3	85.1837
		λ_4	134.3660			λ_4	147.5860
		λ_5	214.0410			λ_5	228.5820
	γ=12	λ_1	24.54910		γ=12	λ_1	35.2082
		λ_2	6.4464			λ_2	84.9141
		λ_3	113.7890			λ_3	149.7830
		λ_4	183.3260			λ_4	230.2920
		λ_5	269.7410			λ_5	326.9570

Uniform beam

Assuming α = 0, the simplest case of the constant section beam is recovered. Free frequencies are obtained using respectively 8 polynomial functions. With N = 8 the procedure supplies frequencies in full agreement with the exact solution. Comparing the values obtained a really small difference on the first frequency can be observed. The results, for cantilever beam, reported in Tab. I and compared with the free vibration calculated by Wang and Wereley [6], Hodges and Rutkowsky [1]. As usual, the differences gradually increase for the higher frequencies. This depends from the rotational parameters too, in particular from the angular speed that is related to γ. For γ=12, δ=0 values are exactly identical to those calculated in [6] and [1]. For δ=5 and γ=12, the differences on the first five frequencies are under the 2%.

The flapwise bending natural frequencies variation are shown in Figure 2. The lower three natural frequencies are plotted for three case of hub radius ratio, δ. The dimensionless natural frequencies increase as the angular speed (γ) increase, and the increasing rates becomes larger as the hub radius ratio (δ) becomes larger. This results from the centrifugal inertia force which increases as the angular speed and the hub radius increase.

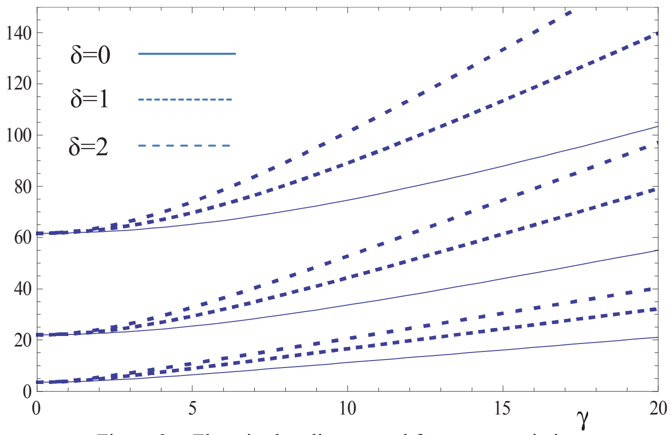
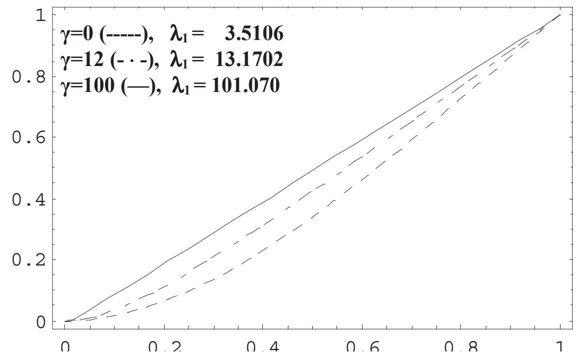


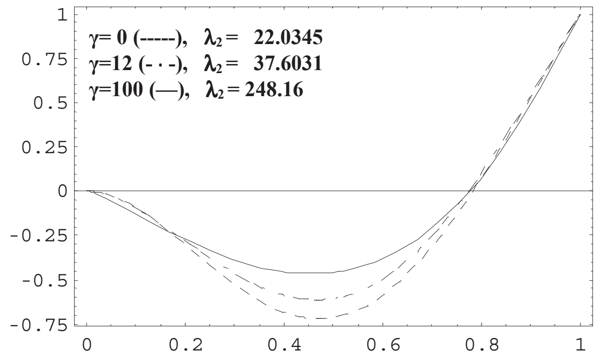
Figure 2. Flapwise bending natural frequency variation.

In order to show the relevance of the rotary displacements, the first three modes, for δ=0, are reported in Fig. 3, respectively for γ = 0, γ=12 and γ=100. As can be seen the dashed lines represent the mode shapes of the beam with no rotational motion while the dot-dashed line and the solid line represent those of the beam with rotational motion. Noticeable difference exists between the three sets of lines. If speed increases, the effect of centrifugal force becomes more evident. In this case, an increase of the positive tension and a reduction of the vibration period can be observed. The tension term will increase and consequently become dominant at very high rotation speed.

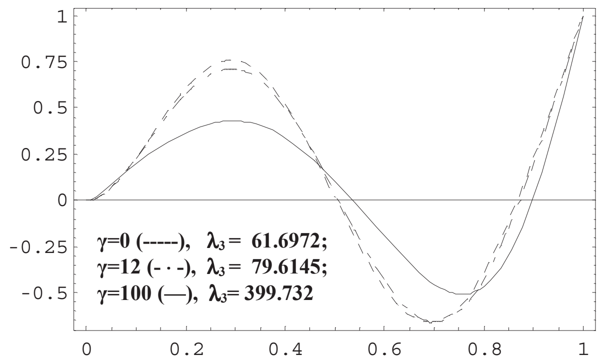
Noticeable difference exists between the third sets of lines. Information about mode shapes variation (e.g. position of nodal points) may be utilized usefully for the control of a rotating beam.



(a) First mode shapes



(b) Second mode shapes



(c) Third mode shapes

Figure 3. Flapwise bending mode shape variation due to rotation; uniform beam.

- *Double Tapered beam* $\alpha = \beta$

This is the case of a beam that tapers linearly in one plane according to the relation in (23). For α =0.5 and δ=0 variation of the natural frequencies with respect to the rotational speed parameter γ have been reported. The results have been compared with those obtained using the Dynamic Stiffness Method (DSM); Banerjee et al. [7].

In Tab.II, assuming α = 0.5, free frequencies were reported for both n = *1* "wedge beam" and n = *2* "cone beam".

The values are then compared with the solution proposed in [7].

TABLE II. COMPARISON OF FIRST FIVE NATURAL FREQUENCIES.

δ=0	α=0.5				
		n=*1*		n=*2*	
		Present	[7]	Present	[7]
γ=1	λ_1	3.98662	3.98662	4.76405	4.76405
	λ_2	18.4740	18.4740	19.6803	19.6803
	λ_3	47.4173	47.4173	48.7073	48.7073
	λ_4	90.6039	90.6039	91.9409	91.9409
	λ_5	148.1560	148.1560	149.5180	149.5180
γ=4	λ_1	5.87876	5.87877	6.47262	6.47262
	λ_2	20.6852	20.6851	21.5749	21.5749
	λ_3	49.6456	49.6456	50.5939	50.5938
	λ_4	92.8730	92.8730	93.8415	93.8415
	λ_5	150.4540	150.4540	151.4310	151.4310
γ=10	λ_1	11.5015	11.5015	11.9415	11.9415
	λ_2	30.1827	30.1827	30.0299	30.0299
	λ_3	60.5639	60.5639	60.0399	60.0399
	λ_4	104.6120	104.6120	103.8100	103.8100
	λ_5	162.6770	162.6770	161.7010	161.7010
γ=100	λ_1	101.3890		101.793	
	λ_2	232.9540		220.319	
	λ_3	368.1380		344.948	
	λ_4	509.3910		475.396	
	λ_5	659.1540		613.529	

The Tab.III provides frequencies to varying of both angular velocity of rotation γ and hub radium ratio (δ).

TABLE III. COMPARISON OF FIRST FIVE NATURAL FREQUENCIES; TAPERED BEAM.

α=1		δ=0		δ=5	
		n=*1*	n=*2*	n=*1*	n=*2*
γ=0	λ_1	5.31510	8,71930		
	λ_2	15.2072	21,1457		
	λ_3	30.0198	38.4538		
	λ_4	49.7633	60.6801		
	λ_5	74.4400	87.8340		
γ=12	λ_1	12.5846	14,7417	33.3232	36,9264
	λ_2	27.0213	29.8678	62.5629	62,9212
	λ_3	44.3132	48.4782	92.9544	90,3908
	λ_4	65.5339	71.4349	125.3550	120,4300
	λ_5	91.1435	99.0363	160.3440	153,6610
γ=100	λ_1	102.4330	104.243	321.4320	349,7010
	λ_2	208.0900	194.746	595.2460	580,6361
	λ_3	308.6180	278.455	865.5160	804,7162
	λ_4	408.4140	360.997	1134.9400	1026.7510
	λ_5	508.5510	443.885	1404.2000	1248.1020

- Beam with parabolic thickness variation

This section is concerned with the transverse vibration of the non-uniform beam shown in Fig. 4, a beam of constant breadth and depth proportional to the square of the axial co-ordinate.

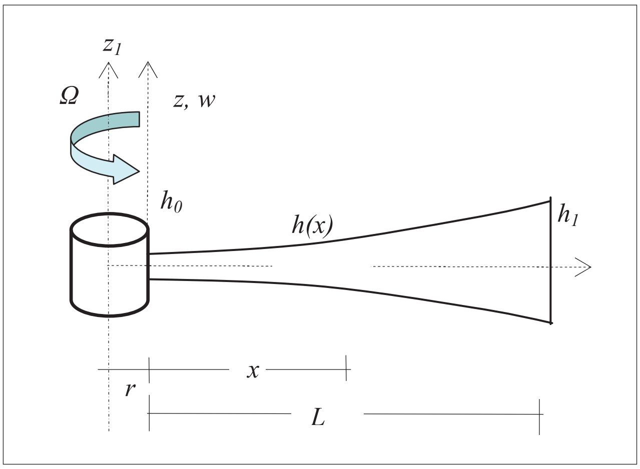


Figure 4. Non-uniform beam; parabolic thickness variation.

In particular, the geometry of the structure is given:

$$h(\xi)=h_0\left(1+(\alpha-1)\xi^2\right), \qquad b(\xi)=b_0, \tag{25}$$

the area and the inertia assume the form

$$A(\xi)=A_0\left(1+(\alpha-1)\xi^2\right)$$
$$I(\xi)=I_0\left(1+\xi^2(\alpha-1)\right)^3. \tag{26}$$

TABLE IV. COMPARISON OF FIRST FIVE NATURAL FREQUENCIES; PARABOLIC THICKESS VARIATION.

α=5		δ=0	δ=5
γ=0	λ_1	2.2608	
	λ_2	30.0559	
	λ_3	110.7650	
	λ_4	230.4959	
	λ_5	395.3289	
γ=12	λ_1	12.5806	33.4471
	λ_2	50.0796	106.4972
	λ_3	137.4859	234.4189
	λ_4	260.9709	397.4638
	λ_5	428.5189	605.8781
γ=100	λ_1	100.9850	276.5721
	λ_2	301.1600	769.3038
	λ_3	550.2041	1320.2100
	λ_4	854.6310	1968.2940
	λ_5	1223.8710	2735.9000

In Tab IV, the first five natural frequencies for rotating parabolic non-uniform beam, α=5, are presented for various angular speed, γ and hub ratios, δ.

If δ = 0 as γ increases the first natural frequency of the beam tends to the angular velocity (ω ≡ γ), causing the phenomenon of resonance, which is usually referred to as “angular speed turner”. An increase of parameter δ results in an increase in the fundamental frequency of the beam for which the phenomenon of “turner angular speed” doesn’t occur.

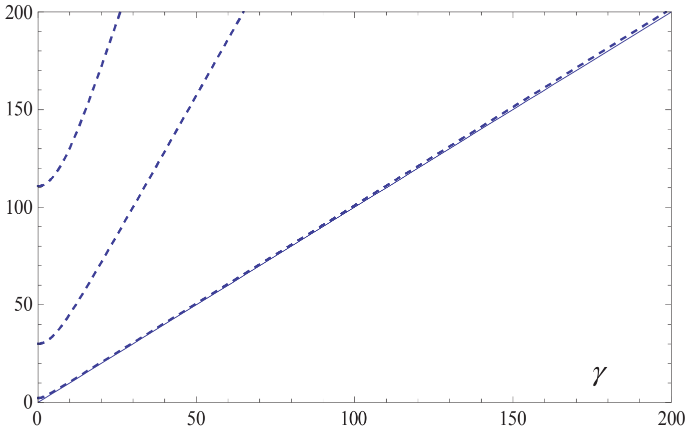


Figure 5. Angular speed and flapwise bending; α=5, r=0.

The effect of the parameter α, has a deep impact even on the free frequencies results. For values of γ≠0, the centrifugal force influences the dynamic behavior of beam: the free frequencies increase to increase, for increasing values of the rotation, γ. From a practical point of view, greater values of the angular velocity lead to greater centrifugal forces, and in turn to stiffener beams. By increasing the extensional deformation, one gets increasing values of the natural frequencies of vibration.

V. CONCLUDING REMARKS

The Hamilton principle is used to solve the free vibration problem of a rotating non-uniform beam based on the Euler beam theory. The effects of angular speed, γ and hub radius, δ, parameters are discussed in detail. The natural frequencies increase for increasing angular speed. That is due to the increase of the beam stiffness, and it is due to the increase of centrifugal force. In particular, the effect is evident on higher mode shapes. The advantage of the procedure used is the generality of polynomial functions which only need to satisfy the essential conditions. The numerical examples have been completely carried through by means of the powerful symbolic software.

Also, it is demonstrated in the numerical routines that, the use of the Boundary Characteristic Orthogonal Polynomial technique is quite simple and converges quickly to the exact solution with very minimal computation effort and resources. This investigation is also intended to form the foundation for the application of the Euler theory to other rotating beam problems.

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