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Two-Character Sets Arising from Gluings of Orbits

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Abstract In this paper we construct two infinite families of transitive two-character sets and hence two infinite families of symmetric strongly regular graphs. We also construct infinite families of quasi-quadrics.

Keywords Symmetric graphs · Strongly regular graphs · Two-character sets · Quasi-quadrics

Mathematics Subject Classification (2000) 51E14 · 51E20

1 Introduction

A *symmetric graph* is a (loopless, undirected) graph whose automorphism group acts transitively on ordered edges. The study of symmetric graphs goes back to work of Tutte [12], with the first significant result after that being the classification of those symmetric graphs of prime order by Chao [5].

A *strongly regular graph* $srg(v, k, \lambda, \mu)$ is a regular graph such that there are constants λ and μ with the property that every pair of adjacent vertices has λ common neighbours and every pair of non-adjacent vertices has μ common neighbours. Strongly regular graphs were introduced by Bose [2].

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A way of constructing symmetric strongly regular graphs arises from the geometry of certain subsets of finite projective spaces known as transitive two-character subsets.

A *two-character set* in the projective space $\mathrm{PG}(d, q)$ is a set $\mathcal{S}$ of n points with the property that the intersection number with any hyperplane only takes two values, $n - w_1$ and $n - w_2$. Then the positive constants w_1 and w_2 are called the *weights* of the two-character set. Embed now $\mathrm{PG}(d, q)$ as a hyperplane Π in $\mathrm{PG}(d + 1, q)$. The *linear representation graph* $\Gamma^*_d(\mathcal{S})$ is the graph having as vertices the points of $\mathrm{PG}(d + 1, q) \setminus \Pi$ and where two vertices are adjacent whenever the line defined by them meets $\mathcal{S}$. It follows that $\Gamma^*_d(\mathcal{S})$ has $v = q^{d+1}$ vertices and valency $k = (q - 1)n$. Delsarte [10] proved that this graph is strongly regular if $\mathcal{S}$ is a two-character set. If the two-character set $\mathcal{S}$ is also transitive, i.e., it has a transitive automorphism group, then the strongly regular graph $\Gamma^*_d(\mathcal{S})$ is also symmetric.

The other parameters of the graph $\Gamma^*_d(\mathcal{S})$ are $\lambda = k^2 + 3k - q(w_1 + w_2) - kq(w_1 + w_2) + q^2 w_1 w_2$ and $\mu = k^2 + k - kq(w_1 + w_2) + q^2 w_1 w_2$ by Corollary 3.7 of [4]. It is interesting to note that regarding the coordinates of the elements of $\mathcal{S}$ as columns of the generator matrix of a code $\mathcal{L}$ of length n and dimension $d + 1$, then the two-character set property of $\mathcal{S}$ translates into the fact that the code $\mathcal{L}$ has two (non-zero) weights (w_1 and w_2) [4]. Such a code is said to be a *projective two-weight code*. The weights of the code are exactly the weights of the two-character set.

Numerous authors—too many to be mentioned here—studied two-character sets, sometimes using algebra, sometimes using geometry. In this paper we construct two infinite families of transitive two-character sets and hence two infinite families of symmetric strongly regular graphs. In Corollary 1 we construct symmetric strongly regular graphs
$srg(q^4, (q^4 - q^3 - q^2 + q)/2, (q^4 - 2q^3 - q^2 + 6q)/4, (q^4 - 2q^3 - q^2 + 2q)/4)$, for any prime power $q > 3$, while in Corollary 2 we construct symmetric strongly regular graphs $srg(q^8, q^7 - q^5 - q^3 + q, q^6 - q^4 + 3q^3 + q^2 + 3q, q^6 - 2q^4 - q^3 + q^2 + q)$, for any odd prime power q.

We also give a construction of quasi-quadrics of projective spaces i.e., two-character sets having the same intersection numbers with respect to hyperplanes as a non-degenerate quadric and hence a construction of other strongly regular graphs.

Finally, we present a computational result on two-character sets arising from a particular 2-modular representation of the group $G_2(3)$.

2 Elementary Properties of the Twisted Cubic

A *twisted cubic* $\mathcal{C}$ of $\mathrm{PG}(3, q)$, $q = p^h$, p prime, where $X_1, \ldots, X_4$ are projective homogeneous coordinates, in its canonical form is the set

$$\{P(t) = (t^3, t^2, t, 1) : t \in \mathrm{GF}(q)\} \cup \{(1, 0, 0, 0)\},$$

comprising $q + 1$ points no four of which are coplanar and hence no three of which are collinear. At each point $P(t)$ of $\mathcal{C}$ there exists an *osculating plane* with equation $X_1 - 3t X_2 + 3t^2 X_3 - t^3 X_4 = 0$, that meets $\mathcal{C}$ only at $P(t)$. The set of $q + 1$ osculating planes form the *osculating developable* Γ to $\mathcal{C}$. A *chord* of $\mathcal{C}$ is a line ℓ of $\mathrm{PG}(3, q)$

joining either a pair of real points $P(t)$ and $P(t')$ of $\mathcal{C}$, possibly coincident, or a pair of complex conjugate points $P(t)$ and $P(t')$ (here t and t' are conjugate over a quadratic extension of GF(q)). We say that ℓ is a *real chord*, a *tangent* or a *imaginary chord*, respectively. When $p \neq 3$, dual to the chords of $\mathcal{C}$ are the *axes* of Γ. From [11, Lemma 21.1.9] the tangents to $\mathcal{C}$ form a $(q+1)$-span of PG$(3, q)$.

Lemma 1 ([11, Lemma 21.1.3]) *Let G_q denote the stabilizer of $\mathcal{C}$ in* PGL$(4, q)$.

1. *When $q \geq 5$, $G_q \simeq$* PGL$(2, q)$*;*
2. *$G_4 \simeq Sym(5)$;*
3. *$G_3 \simeq Sym(4) \cdot 2^3$;*
4. *$G_2 \simeq Sym(3) \cdot 2^3$.*

For our purpose we need to know the action of G_q on planes and points of PG$(3, q)$.

Lemma 2 ([11, Corollary 4, p. 234]) *Under the action of G_q there are five orbits on planes of* PG$(3, q)$*:*

1. *$\mathcal{N}_1 = \Gamma$ of size $q+1$;*
2. *$\mathcal{N}_2 = \{$planes meeting $\mathcal{C}$ at exactly two points$\}$ of size $q(q+1)$;*
3. *$\mathcal{N}_3 = \{$planes meeting $\mathcal{C}$ at exactly three points$\}$ of size $q(q^2-1)/6$;*
4. *$\mathcal{N}_4 = \{$planes not in Γ meeting $\mathcal{C}$ at exactly one point$\}$ of size $q(q^2-1)/2$;*
5. *$\mathcal{N}_5 = \{$planes disjoint from $\mathcal{C}\}$ of size $q(q^2-1)/3$.*

Lemma 3 ([11, Corollary 5 p. 235]) *Under the action of G_q there are five orbits on points of* PG$(3, q)$.

Assume $q \not\equiv 0$ (mod 3).

1. *$\mathcal{M}_1 = \mathcal{C}$ of size $q+1$;*
2. *$\mathcal{M}_2 = \{$points off $\mathcal{C}$ on a tangent$\}$ of size $q(q+1)$;*
3. *$\mathcal{M}_3 = \{$points on three osculating planes$\}$ of size $q(q^2-1)/6$;*
4. *$\mathcal{M}_4 = \{$points off $\mathcal{C}$ on exactly one osculating plane$\}$ of size $q(q^2-1)/2$;*
5. *$\mathcal{M}_5 = \{$points on no osculating plane$\}$ of size $q(q^2-1)/3$.*

Assume $q \equiv 0$ (mod 3).

1. *$\mathcal{M}_1 = \mathcal{C}$ of size $q+1$;*
2. *$\mathcal{M}_2 = \{$points on all osculating planes$\}$ of size $q+1$;*
3. *$\mathcal{M}_3 = \{$points off $\mathcal{C}$ on a tangent and one osculating plane$\}$ of size (q^2-1);*
4. *$\mathcal{M}_4 = \{$points off $\mathcal{C}$ on a real chord$\}$ of size $q(q^2-1)/2$;*
5. *$\mathcal{M}_5 = \{$points on an imaginary chord$\}$ of size $q(q^2-1)/2$.*

Remark 1 From [11, Corollary p. 242], for $q \equiv 1$ (mod 3), $\mathcal{M}_3 \cup \mathcal{M}_5 = \{$ points on a real chord$\}$ and $\mathcal{M}_4 = \{$points on an imaginary chord$\}$. For $q \equiv -1$ (mod 3), $\mathcal{M}_3 \cup \mathcal{M}_5 = \{$points on an imaginary chord$\}$ and $\mathcal{M}_4 = \{$points on a real chord$\}$.

2.1 The First Infinite Family

In this section we will show that the orbit $\mathcal{M}_4$ for all q turns out to be a two-character set.

Theorem 1 *For $q \geq 3$, the G_q-orbit $\mathcal{M}_4$ is a two-character set with $w_1 = (q^3-q^2)/2$ and $w_2 = (q^3 - q^2 - 2q)/2$.*

Proof Firstly assume that $q \not\equiv 0 \pmod 3$. Let Π be a plane of PG$(3, q)$. If $\Pi \in \mathcal{N}_1$ and $q \equiv 1 \pmod 3$ then Π meets $\mathcal{M}_4$ at $q(q-1)/2$ points. Indeed, in this case, Π does not contain imaginary chords and from Remark 1 $\mathcal{M}_4$ is the set of points on imaginary chords. The claim follows from the fact that $q(q-1)/2$ is the number of imaginary chords [11, Lemma 21.1.4]. If $q \equiv -1 \pmod 3$ then Π meets $\mathcal{M}_4$ again at $q(q-1)/2$ points. Indeed, in this case from Remark 1 $\mathcal{M}_4$ is the set of points on real chords (there are exactly q chords of $\mathcal{C}$ passing through $\Pi \cap \mathcal{C}$). Assume that $\Pi \in \mathcal{N}_2$ and let $\Pi \cap \mathcal{C} = \{P, Q\}$. Then Π contains either the tangent t_P to $\mathcal{C}$ at P or the tangent t_Q to $\mathcal{C}$ at Q. It cannot contain both because the tangents to $\mathcal{C}$ are mutually disjoint (they form a $(q+1)$-span). It follows that Π meets $\mathcal{M}_2$ at $q + (q-1)$ points. Assume that $q \equiv 1 \pmod 3$. Then $\mathcal{M}_3 \cup \mathcal{M}_5$ consists of points on real chords. There are $q(q+1)/2$ real chords of which $q-1$ on P and $q-1$ on Q distinct from PQ. It follows that Π meets $\mathcal{M}_3 \cup \mathcal{M}_5$ at the $q-1$ points of PQ and at $(q^2-3q+2)/2$ points on the remaining chords. Hence Π meets $\mathcal{M}_4$ at $q^2+q+1-2-(2q-1)-(q-1)-((q^2-3q+2)/2) = q(q-1)/2$ points. Assume that $\Pi \in \mathcal{N}_3$. Let $\Pi \cap \mathcal{C} = \{P, Q, R\}$. Then Π meets $\mathcal{C}$ at three points, $\mathcal{M}_2$ at $q-2$ points, $\mathcal{M}_3 \cup \mathcal{M}_5$ at $3(q-1)$ points on PQ, PR and QR and at further $(q^2-5q+6)/2$ points. It follows that π meets $\mathcal{M}_4$ at $q(q-1)/2$ points. Assume that $\Pi \in \mathcal{N}_4$. Then Π meets $\mathcal{C}$ at only one point, $\mathcal{M}_2$ at q points, $\mathcal{M}_3 \cup \mathcal{M}_5$ at $q(q+1)/2 - q = q(q-1)/2$ points and hence $\mathcal{M}_4$ at $q(q+1)/2$ points. Assume that $\Pi \in \mathcal{N}_5$. Then Π is skew to $\mathcal{C}$, meets $\mathcal{M}_2$ at $q+1$ points, $\mathcal{M}_3 \cup \mathcal{M}_5$ at $q(q+1)/2$ points and hence $\mathcal{M}_4$ at $q(q-1)/2$ points. When $q \equiv -1 \pmod 3$, the orbits $\mathcal{M}_4$ is exactly the set of points on real chords and hence the result follows from the previous case. Assume now that $q \equiv 0 \pmod 3$. In this case $\mathcal{M}_4$ consists of points off $\mathcal{C}$ on real chords and the result follows from the proof above. The proof is now complete.

Corollary 1 $\Gamma_3^*(\mathcal{M}_4)$ *is a symmetric strongly regular graph* srg$(q^4, (q^4-q^3-q^2+q)/2, (q^4-2q^3-q^2+6q)/2, (q^4-2q^3-q^2+2q)/2)$.

Remark 2 When $q = 3$, the group G_3 has four point-orbits of size 4, 8, 12, 16. The 12-orbit is a transitive two-character set with $w_1 = 3$ and $w_2 = 6$, the 16-orbit is a transitive two-character set with $w_1 = 4$ and $w_2 = 7$. Their union is a two-character set with $w_1 = 7$ and $w_2 = 10$. When $q = 4$, we have that $\mathcal{M}_4$ has size 30 and it is a transitive two-character set with $w_1 = 6$ and $w_2 = 10$.

3 On the Set of Tangent Lines to $Q^-(3, q)$, q odd

Let $Q^-(3, q)$ be an elliptic quadric of PG$(3, q)$, q odd, i.e., a set of $q^2 + 1$ points of PG$(3, q)$ no three of them collinear. On each point P of $Q^-(3, q)$ there are $q + 1$

tangent lines lying on a plane called *tangent plane* to $Q^-(3,q)$ at P. Hence there are $(q^2+1)(q+1)$ tangent lines to $Q^-(3,q)$. Embed $\mathrm{PG}(3,q)$ in $\mathrm{PG}(3,q^2)$ as a Baer subgeometry. Denote by T the set of points of $\mathrm{PG}(3,q^2)$, that are not in $\mathrm{PG}(3,q)$, on tangent lines to $Q^-(3,q)$ (extended over $\mathrm{GF}(q^2)$). Clearly, $|T| = (q+1)(q^2+1)(q^2-q)$.

3.1 The Second Infinite Family

Theorem 2 *The set T is a transitive two-character set with $w_1 = q^5 - q^3$ and $w_2 = q^5 - q^3 - q^2$.*

Proof The stabilizer of $Q^-(3,q)$ in $\mathrm{PGL}_4(q)$ is the group $\mathrm{PGL}_2(q^2)$. In order to show that $\mathrm{PGL}_2(q^2)$ is transitive on T it is sufficient to show that the stabilizer H in $\mathrm{PGL}_2(q^2)$ of an extended tangent line ℓ to $Q^-(3,q)$ is transitive on the set of points of ℓ not in $\mathrm{PG}(3,q)$ for $\mathrm{PGL}_2(q^2)$ acts transitively on the set of tangent lines to $Q^-(3,q)$. The group H has order $q^2(q-1)$ and acts on ℓ with a kernel of order q as $\mathrm{AGL}(1,q)$; hence H acts transitively on the set of points of ℓ not in $\mathrm{PG}(3,q)$.

A plane Π of $\mathrm{PG}(3,q^2)$ either meets $\mathrm{PG}(3,q)$ at a plane or at a line.

Assume that $\Pi \cap \mathrm{PG}(3,q)$ is a plane π. If π is a tangent plane to $Q^-(3,q)$ at a point P of $Q^-(3,q)$ then Π meets T at $(q+1)(q^2-q)$ points.

If π is a secant plane to $Q^-(3,q)$, i.e., $\pi \cap Q^-(3,q)$ ia a conic C, then Π contains the tangent lines to $Q^-(3,q)$ at the points of C and again Π meets T at $(q+1)(q^2-q)$ points.

Assume that Π meets $\mathrm{PG}(3,q)$ at a line ℓ.

If ℓ is tangent to $Q^-(3,q)$ at P then Π meets T at q^3+q^2-q points: q^2-q points of the $\mathrm{GF}(q^2)$-extension of ℓ and for any other point of $Q^-(3,q)$ there are q $\mathrm{GF}(q^2)$-extended tangent lines to $Q^-(3,q)$ meeting Π at a point.

Assume that ℓ is external to $Q^-(3,q)$. On ℓ there are exactly two tangent planes to $Q^-(3,q)$. This means that Π meets T at $(q^2-1)q = (q+1)(q^2-q)$ points.

Finally, assume that ℓ is secant to $Q^-(3,q)$ at P_1 and P_2. Then, Π meets T at $(q+1)(q^2-q)$ points. The proof is now complete.

Corollary 2 $\Gamma_3^*(T)$ *is a symmetric strongly regular graph* $\mathrm{srg}(q^8, q^7-q^5-q^3+q, q^6-q^4+3q^3+q^2+3q, q^6-2q^4-q^3+q^2+q)$.

Remark 3 When q is even, the tangent lines to $Q^-(3,q)$ form a non-singular linear complex, and the analogue of our theorem has been proved in [8].

4 A Construction of Quasi-Quadrics

In a projective space $\mathrm{PG}(k,q)$ a *quasi-quadric* is a set of points that has the same intersection numbers with respect to hyperplanes as a non-degenerate quadric in that space [9]. Clearly, non-degenerate quadrics themselves are examples of quasi-quadrics. When k is odd, quasi-quadrics have two sizes of intersections with hyperplanes and so are two-character sets (and therefore give rise to strongly regular graphs). In this section we give a construction of quasi-quadrics, arising from a non-degenerate polarity Π of $\mathrm{PG}(k,q)$, k odd.

Let $s = q$ or $s = q^2$, $q = p^h$, p prime, according as Π is symplectic, orthogonal or unitary.

Let $k = 2r - 1$. Let $X_0, \ldots, X_{2r-1}$, be homogeneous projective coordinates for $\mathrm{PG}(k, s)$, $k = 2r - 1$, $r > 2$. Let S_1 and S_2 be the subspaces with equations:

$$\begin{aligned} S_1 : X_0 &= X_1 = X_2 = \cdots = X_{r-1} = 0, \\ S_2 : X_r &= X_{r+1} = \cdots = X_{k+1} = 0. \end{aligned}$$

Assume that S_1 and S_2 are totally singular with respect to Π. Let $P := (0, 0, \ldots, 0, a_r, a_{r+1}, \ldots, a_{k+1})$ be a point of S_1, and let $P^\perp$ be the polar hyperplane of P with respect to the polarity Π. The intersection of $P^\perp$ and S_2 is a projective $(r-2)$-subspace, say S_P, and we say that S_P corresponds to P. Straightforward computations show that S_P has the following equations:

$$\begin{cases} a_r^\alpha X_0 + a_{r+1}^\alpha X_1 + \cdots + a_{k+1}^\alpha X_{r-1} = 0 \\ X_r = X_{r+1} = \cdots = X_{k+1} = 0 \end{cases}.$$

where $\alpha = 1$ if Π is symplectic or orthogonal, and $\alpha = q$ in the unitary case. Note that the subspace $\langle P, S_P \rangle$ is totally singular with respect to Π.

Take another point P' of S_1, $P' \neq P$, and suppose that its corresponding subspace $S_{P'}$ coincides with S_P. Then, the line PP' and S_P would be orthogonal to each other and hence would generate a totally singular r-dimensional projective subspace, a contradiction.

Thus, allowing P to vary over the points of S_1, the construction described above produces a family, say $\mathcal{P}_1$, of $\theta_{r-1}(s)$ distinct totally singular $(r-1)$-dimensional subspaces (here $\theta_t(s)$ denotes the number of points of a projective space $\mathrm{PG}(t, s)$). Further any two members of $\mathcal{P}_1$ meet in exactly a totally singular $(r-2)$-dimensional subspace. In a similar way, as Q varies over the points of S_2, one obtains another collection, say $\mathcal{P}_2$, of $\theta_{r-1}(s)$ distinct totally singular $(r-1)$-dimensional subspaces. This way, we get a set $\mathcal{R}$ containing at least $2\theta_{r-1}(s) + 2$ totally singular $(r-1)$-dimensional subspaces, including S_1 and S_2, but at most two of these subspaces are mutually disjoint.

Theorem 3 *The set $\mathcal{R}$ is a hyperbolic quadric if Π is orthogonal and a quasi-quadric in the symplectic and unitary cases having the same intersection numbers of a hyperbolic quadric of* $\mathrm{PG}(2r-1, s)$.

Proof Easy computations show that $|\mathcal{R}| = (s^{r-1} + 1)(s^r - 1)/(s - 1)$.

Let H be a hyperplane of $\mathrm{PG}(k, s)$. We distinguish two cases.

$\boxed{S_1 \subset H \text{ or } S_2 \subset H}$ Assume $S_1 \subset H$. The case $S_2 \subset H$ is similar. In this case the intersection between H and S_2 is an $(r-2)$-dimensional subspace S_p for some $P \in S_1$. Of course, H meets any other member of $\mathcal{P}_1$ in an $(r-2)$-dimensional subspace. It follows that H meets $\mathcal{R}$ in h_1 points where $h_1 = \theta_{r-1}(s) + \theta_{r-2}(s) + (\theta_{r-1}(s) - \theta_{r-2}(s-1) + (\theta_{r-1}(s) - 1)(\theta_{r-2}(s) - \theta_{r-3}(s) - 1) = \theta_{r-1}(s) + s^{r-2}(\theta_{r-1}(s) - 1)$.

$\boxed{S_1, S_2 \not\subset H}$ Let X_1 be the $(r-2)$-dimensional subspace $S_1 \cap H$ and X_2 the $(r-2)$-dimensional subspace $S_2 \cap H$. We have to consider two subcases.

1. *$X_2 = S_1 P$ for some $P \in X_1$*. In this case, H contains $\langle P, X_2 \rangle$, meets $(\theta_{r-2}(s) - 1)$ members of $\mathcal{P}_1$ in an $(r-2)$-dimensional subspace meeting both X_1 and X_2, and the remaining members of $\mathcal{P}_1$ in $\theta_{r-1}(s) - \theta_{r-3}(s)$ points. It follows that H meets $\mathcal{R}$ in h_1 points where $h_1 = 2\theta_{r-2}(s) + \theta_{r-1}(s) - \theta_{r-2}(s) - 1 + (\theta_{r-2}(s) - 1)(\theta_{r-2}(s) - \theta_{r-3}(s) - 1) + (\theta_{r-1}(s) - \theta_{r-2}(s))(\theta_{r-2}(s) - \theta_{r-3}(s)) = \theta_{r-1}(s) + s^{r-2}(\theta_{r-1}(s) - 1)$.
2. *X_2 does not correspond to any point of X_1*. It follows that H meets $\mathcal{R}$ in h_2 points where $h_2 = 2\theta_{r-2}(s) + \theta_{r-2}(s)(\theta_{r-2}(s) - \theta_{r-3}(s) - 1) + (\theta_{r-1}(s) - \theta_{r-2}(s) - 1)(\theta_{r-2}(s) - \theta_{r-3}(s)) = \theta_{r-2}(s) + s^{r-2}(\theta_{r-1}(s) - 1)$.

Remark 4 When q is even and Π is symplectic $\mathcal{R}$ is actually a quadric: an orthogonal polarity is also symplectic.

Remark 5 It should be noted that when $r = 3$, the chordal variety M_3^4 of the Veronese surface $\mathcal{V}$ of PG$(5, q)$ is a quasi-quadric of hyperbolic type. In [7] the geometry of conics of PG$(2, q)$ was used to determine the two intersection numbers. By means of [1], it can be proved that the point set of M_3^4 can be partitioned into $q^2 + 1$ Veronese surfaces ($\mathcal{V}$ included). It turns out that the $q^2 + q + 1$ conic planes of $\mathcal{V}$ all meet the remaining Veronese surfaces on M_3^4 at a point. Hence, $\mathcal{V}$ and each one of the Veronese surfaces on M_3^4 play the role of S_1 or S_2 in our Theorem 3.

Remark 6 It should be mentioned that when Π is unitary the set $\mathcal{R}$ can be obtained as the intersection of two Hermitian varieties in PG$(2r - 1, q^2)$, [6]

Remark 7 Of course, the strongly regular graphs arising from our quasi-quadrics and quasi-Hermitian varieties have the same parameters of those arising from the usual quadrics and Hermitian varieties, but in general we have no information about their automorphism groups: some MAGMA computations [3] for small values of k and q show that the automorphism groups of our graphs are smaller than those arising from quadrics or Hermitian varieties. In particular, in the symplectic case when $k = 5$ and $q = 3, 5, 7, 9, 11$ we found that the group of the related strongly regular graphs is $GL_3(q) \cdot 2$. In the unitary case, always when $k = 5$ and $q^2 = 9, 16, 25, 49$, we found that the group of the related strongly regular graph is $(SU_3(q) \times SU_3(q)) \cdot C_{q+1} \cdot 2$.

5 A sporadic two-character set

The group $G_2(3)$ has a 2-modular representation of 14° [13]. As a subgroup of PGL$_{14}(2)$, it has five orbits on points of PG$(13, 2)$ of sizes 378, 378, 3888, 4368 and 7371. The union of the two 378-orbits and the 3888-orbit is a two-character set with $w_1 = 2368$ and $w_2 = 2304$. (This computation was performed in Magma [3].) This gives rise to a strongly regular (16384, 4644, 1276, 1332) graph.

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