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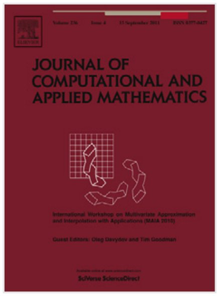

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Extended Lagrange Interpolation on the real line ☆

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Abstract

Let $\{p_m(w_\alpha)\}_m$ be the sequence of the polynomials orthonormal w.r.t. the Sonin-Markov weight $w_\alpha(x) = e^{-x^2}|x|^\alpha$. The authors study extended Lagrange interpolation processes essentially based on the zeros of $p_m(w_\alpha)p_{m+1}(w_\alpha)$, determining the conditions under which the Lebesgue constants, in some weighted uniform spaces, are optimal.

Keywords: Lagrange Interpolation, Extended Interpolation, Orthogonal Polynomials, Polynomial Approximation, Lebesgue Constants
2000 MSC: 65D05, 41A05, 41A10, 33C45

1. Introduction

Let $w_\alpha(x) = e^{-x^2}|x|^\alpha$, $\alpha > -1$, be a *Sonin-Markov* weight, and $\{p_m(w_\alpha)\}_m$ be the corresponding sequence of the orthonormal polynomials.

The Lagrange polynomial interpolating a given function f at the zeros of $Q_{2m+1} := p_{m+1}(w_\alpha)p_m(w_\alpha)$, denoted by $\mathcal{L}_{2m+1}(w_\alpha, w_\alpha, f)$, is known as the "Extended Lagrange polynomial". In the present paper we study extended processes essentially based on the zeros of Q_{2m+1} and on two special points $\pm\sqrt{2m}$. Indeed, these zeros are sufficiently far apart and, as we will show, the "good" distance is a needful ingredient in order to obtain "optimal" Lebesgue constants (i.e. behaving like $\log m$ [12, 13]) in suitable weighted uniform spaces (see Proposition 2.2). We point out that without the extra points $\pm\sqrt{2m}$ the Lebesgue constants corresponding to $\mathcal{L}_{2m+1}(w_\alpha, w_\alpha, f)$ behave algebraically (see Proposition 2.3).

A useful aspect of the "extended interpolation" is the approximation of functions by $2m$-degree interpolating polynomials by using the zeros of m and $m+1$ degree polynomials. This allows to construct "high" degree interpolation processes, overcoming the well known difficulties in computing the zeros of high

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degree orthogonal polynomials. Another application is the construction of $2m$-degree interpolating polynomials by reusing the nodes of a previously computed interpolating polynomial.

Here we introduce extended processes in order to approximate functions with an exponential growth at infinity, proving that if we neglect some of the greatest zeros, we obtain a cheap process, with "optimal" Lebesgue constants and "free" of possible overflow phenomena.

At first we will treat the approximation of continuous functions by truncated polynomials interpolating f at the zeros of Q_{2m+1} and on s "additional knots" close to 0. Successively we will consider the case of functions having a singularity in the point 0, and in this case we will drop r zeros of Q_{2m+1} in a neighborhood of 0. In both cases we state the conditions under which the weighted sequence of interpolating polynomials behave like the best polynomial approximation, except for the $\log m$ factor.

The plan of the paper is the following: the next section contains some notations and basic results about the zeros involved in the extended interpolation. In Section 3 are introduced the extended processes and their convergence is studied. Section 4 contains some numerical examples and in Section 5 the proofs of the main results are given.

2. Notations and preliminary results

In the sequel $\mathcal{C}$ will denote a generic positive constant. Moreover $\mathcal{C} \neq \mathcal{C}(a, b, ..)$ will be used to specify that the constant $\mathcal{C}$ is independent of $a, b, \ldots$. The notation $A \sim B$, where A and B are positive quantities depending on some parameters, will be used if and only if $(A/B)^{\pm 1} \leq \mathcal{C}$. Finally $[a]$ will denote the integer part of $a \in \mathbb{R}^+$.

Denote by $\mathbb{P}_m$ the space of all algebraic polynomials of degree at most m.

Consider the weight w_α and let $\{p_m(w_\alpha)\}_m$ be the corresponding sequence of orthonormal polynomials with positive leading coefficients, i.e.

$$p_m(w_\alpha, x) = \gamma_m(w_\alpha)x^m + \text{ terms of lower degree}, \quad \gamma_m(w_\alpha) > 0.$$

Let $\{x_{m,k}\}_{k=1}^{\left[\frac{m}{2}\right]}$ be the positive zeros of $p_m(w_\alpha)$ and $x_{m,-k}$ the negative ones. If m is odd, then $x_0 = 0$ is a zero of $p_m(w_\alpha)$. It is known (see for instance [6]) that the zeros lie in the range $(-\sqrt{2m}, \sqrt{2m})$, and more precisely

$$-\sqrt{2m} + \frac{\mathcal{C}}{m^{\frac{1}{6}}} < x_{m,-\left[\frac{m}{2}\right]} < \ldots < x_{m,1} < x_{m,2} < \ldots x_{m,\left[\frac{m}{2}\right]} < \sqrt{2m} - \frac{\mathcal{C}}{m^{\frac{1}{6}}}$$

where $\mathcal{C} \neq \mathcal{C}(m)$.

Since the zeros of $p_{m+1}(w_\alpha)$ interlace with those of $p_m(w_\alpha)$, the polynomial Q_{2m+1} has $2m + 1$ simple zeros $z_i,\ i = 0, \pm 1 \ldots, \pm \left[\frac{2m+1}{2}\right]$, where

$$z_{2i-1} := x_{m+1,i}, \quad i = 1, 2, \ldots, \left[\frac{m+1}{2}\right], \quad z_{2i} := x_{m,i}, \quad i = 1, 2, \ldots, \left[\frac{m}{2}\right],$$

$$z_{-i} := -z_i, \quad i = 1, 2, \ldots \left[\frac{2m+1}{2}\right] \quad z_0 := 0.$$

Now define

$$z_j = z_{j(m,\theta)} := \min\left\{z_k : z_k \geq \theta\sqrt{2m}, \quad k = 1, 2, ...\right\} \tag{1}$$

where $0 < \theta < 1$ denotes a fixed real number.

The next proposition is one of the basic tools for constructing extended Lagrange polynomials.

Proposition 2.1. *With z_j defined in (1), we have*

$$\Delta z_k = z_{k+1} - z_k \sim \frac{1}{\sqrt{m}}, \quad |k| < j, \tag{2}$$

uniformly w.r.t. $m \in \mathbb{N}$.

Note that (2) is comparable with the distance between two consecutive zeros of $p_m(w_\alpha)$ belonging to $(-\theta\sqrt{2m}, \theta\sqrt{2m})$. Indeed it is (see [6, 3, 1])

$$\Delta x_{m,k} = x_{m,k+1} - x_{m,k} \sim \frac{1}{\sqrt{m}}, \qquad x_{m,k}, x_{m,k+1} \in (-\theta\sqrt{2m}, \theta\sqrt{2m}). \tag{3}$$

Now we show how this distance seems to be a relevant ingredient in order to obtain good interpolation processes. To this end, if $w := w_0$ is the classical Hermite weight, define the space of functions

$$C(w) = \left\{f \in C^0(\mathbb{R}) : \lim_{x\to\pm\infty} f(x)w(x) = 0\right\},$$

equipped with the norm $\|f\|_{C(w)} = \|fw\|_\infty = \sup_{x\in\mathbb{R}} |f(x)|w(x)$.

Denoting by $\mathcal{X} = \left\{\xi_{m,i}, \ i = -\left[\frac{m}{2}\right], \ldots, 1, 2, \ldots, \left[\frac{m}{2}\right], \ i \neq 0, \quad m \in \mathbb{N}\right\}$, a generic indefinite triangular matrix, let $\mathcal{L}_m(\mathcal{X}, g)$ be the Lagrange polynomial interpolating $g \in C(w)$ at the elements of the m-th row of $\mathcal{X}$, i.e.

$$\mathcal{L}_m(\mathcal{X}, g; \xi_{m,i}) = g(\xi_{m,i}), \quad 1 \leq |i| \leq \left[\frac{m}{2}\right].$$

The $m-$th Lebesgue constant is defined as

$$\|\mathcal{L}_m(\mathcal{X})\| = \sup_{\|gw\|_\infty = 1} \|\mathcal{L}_m(\mathcal{X}, g)w\|_\infty, \quad m = 1, 2, \ldots$$

If the distance between the interpolation knots is too small the Lebesgue constants diverge algebrically. Indeed the following proposition holds.

Proposition 2.2. *If for m sufficiently large (say $m > m_0$), there exists $k := k(m)$ s. t.*

$$\Delta\xi_{m,k} \leq \left(\frac{\mathcal{C}}{\sqrt{m}}\right)^{\eta+1}, \quad \eta > 0, \tag{4}$$

then

$$\|\mathcal{L}_m(\mathcal{X})\| \geq \mathcal{C}(\sqrt{m})^\eta,$$

where $\mathcal{C} \neq \mathcal{C}(m)$.

Nevertheless a system of knots with the only property of a certain distance between two consecutive zeros , i.e. a distance like in (2), does not guarantee Lebesgue constants behaving like $\log m$. Indeed, denoting by $\mathcal{L}_{2m+1}(w_\alpha, w_\alpha, f)$ the Lagrange polynomial interpolating a given f at the zeros of Q_{2m+1}, i.e.

$$\mathcal{L}_{2m+1}(w_\alpha, w_\alpha, f; z_i) = f(z_i), \quad 0 \leq |i| \leq \left[\frac{m}{2}\right],$$

and working, for instance, as in the proof of [5, Theorem 3.3], the following proposition holds true.

Proposition 2.3. *For any choice of* α*, there exists a positive* τ *s.t.*

$$\|\mathcal{L}_{2m+1}(w_\alpha, w_\alpha)\| = \sup_{\|fw\|_\infty=1} \|\mathcal{L}_{2m+1}(w_\alpha, w_\alpha, f)w\|_\infty \geq \mathcal{C}m^\tau,$$

with $\mathcal{C} \neq \mathcal{C}(m)$.

Now we show how some suitable systems of knots, using the zeros of $\{Q_{2m+1}\}_m$, can be proposed in order to obtain optimal Lebesgue constants. This goal will be achieved in the next section by means of extended processes based on the zeros of Q_{2m+1} and on some additional knots.

3. Extended interpolating polynomials

As announced in the Introduction we will propose two different interpolation processes one for the case when the function f is continuous and the other if f has a singularity at the point zero.

In the first case we will use the idea from Szabados [10], what we call "method of additional points", adding $\pm\sqrt{2m}$ to the set of the interpolation knots and some extra points close to 0 (see [6]). Moreover we will consider a suitable truncation of the interpolation process (refer to [4]).

In the second case we will remove some nodes close to zero as proposed in [6] and consider the truncation again.

3.1. The case of continuous functions

Now let $L_{2m+3}(w_\alpha, w_\alpha, f)$ denote the Lagrange polynomial interpolating a given function $f \in C(w)$ at the zeros of $Q_{2m+1}(x)(2m - x^2)$, i.e we are considering the extended Lagrange interpolation with the two additional nodes $\pm\sqrt{2m}$.

If χ_j is the characteristic function of the segment (z_{-j}, z_j), where z_j is defined in (1), let us introduce the Lagrange polynomial

$$L^*_{2m+3}(w_\alpha, w_\alpha, f; x) := L_{2m+3}(w_\alpha, w_\alpha, f\chi_j; x) = \sum_{0 \leq |k| \leq j} \ell_{2m+3,k}(x) f(z_k), \quad (5)$$

with

$$\ell_{2m+3,k}(x) = \frac{Q_{2m+1}(x)(2m - x^2)}{Q'_{2m+1}(z_k)(2m - z_k^2)(x - z_k)}, \quad 0 \le |k| \le j.$$

Therefore $L^*_{2m+3}(w_\alpha, w_\alpha, f)$ is a polynomial of degree $2m+2$ such that

$$L^*_{2m+3}(w_\alpha, w_\alpha, f; \pm\sqrt{2m}) = 0 = L^*_{2m+3}(w_\alpha, w_\alpha, f; z_k) = 0, \text{ for } |k| > j.$$

We remark that the Lagrange operator $L_{2m+3}(w_\alpha, w_\alpha)$ projects $C(w)$ on $\mathbb{P}_{2m+2}$, while $L^*_{2m+3}(w_\alpha, w_\alpha)$ does not. However, setting

$$\mathcal{P}^*_{2m+2} = \left\{ q \in \mathbb{P}_{2m+2} : q(z_i) = q(\pm\sqrt{2m}) = 0, \quad |i| > j \right\} \subset \mathbb{P}_{2m+2},$$

we have that $L^*_{2m+3}(w_\alpha, w_\alpha)$ is a projector of $C(w)$ on $\mathcal{P}^*_{2m+2}$. Moreover $\bigcup_m \mathcal{P}^*_m$ is dense in $C(w)$. Indeed, with $E_m(f)_w = \inf_{P \in \mathbb{P}_m} \|(f-P)w\|_\infty$, the error of the best polynomial approximation of f in $C(w)$, and

$$\tilde{E}_{2m+2}(f)_w := \inf_{Q \in \mathcal{P}^*_{2m+2}} \|(f-Q)w\|_\infty,$$

the following result was proved in [4], in a more general context.

Lemma 3.1. *For any function $f \in C(w)$,*

$$\tilde{E}_{2m+2}(f)_w \le \mathcal{C} \left\{ E_M(f)_w + e^{-Am} \|fw\|_\infty \right\}, \tag{6}$$

where $M = \left[(m+1) \left(\frac{\theta}{1+\theta} \right)^2 \right]$ and $0 < A \ne A(m,f)$, $\mathcal{C} \ne \mathcal{C}(m,f)$.

This means that $\tilde{E}_{2m+2}(f)_w$ can be estimated by the best approximation error $E_M(f)_w$, where M is smaller than $2m+2$.

About the convergence of the proposed process the following result holds true.

Theorem 3.1. *For any function $f \in C(w)$,*

$$\|L^*_{2m+3}(w_\alpha, w_\alpha, f) w\|_\infty \le \mathcal{C} \|fw\|_\infty \log m \tag{7}$$

with $\mathcal{C} \ne \mathcal{C}(m,f)$, if and only if

$$-1 < \alpha \le 0. \tag{8}$$

Moreover

$$\|[f - L^*_{2m+3}(w_\alpha, w_\alpha, f)]w\|_\infty \le \mathcal{C} \left\{ E_M(f)_w \log m + e^{-Am} \|fw\|_\infty \right\}, \tag{9}$$

where M was defined in Lemma 3.1, $0 < A \ne A(m,f)$, $\mathcal{C} \ne \mathcal{C}(m,f)$.

Now we consider the case $\alpha > 0$, that means when (8) is not satisfied.

In this case we follow the procedure of including *additional nodes* (see [11], and the bibliography therein, and [6]). Let $t_i, i = 1, \ldots, s,\ t_i \neq 0$, be some simple knots added in the range $[z_{-1}, z_1]$ and let $B_s(x) = \prod_{i=1}^{s}(x - t_i)$. For instance take $t_i = z_{-1} + \frac{2z_1}{s+1}i,\ i = 1, 2, \ldots, s$, (providing that $t_i \neq 0$, for all i).

Denote by $L_{2m+3,s}(w_\alpha, w_\alpha, f)$ the Lagrange polynomial interpolating f at the zeros of $Q_{2m+1}(x)B_s(x)(2m - x^2)$, and define

$$L^*_{2m+3,s}(w_\alpha, w_\alpha, f) = L_{2m+3,s}(w_\alpha, w_\alpha, f\chi_j).$$

An explicit expression for this polynomial is

$$L^*_{2m+3,s}(w_\alpha, w_\alpha, f; x) = \sum_{i=1}^{s} \frac{Q_{2m+1}(x)(2m - x^2)B_s(x)}{Q_{2m+1}(t_i)(2m - t_i^2)B'_s(t_i)} \frac{f(t_i)}{(x - t_i)}$$
$$+ \sum_{0 \le |k| \le j} \frac{Q_{2m+1}(x)(2m - x^2)B_s(x)}{Q'_{2m+1}(z_k)(2m - z_k^2)B_s(z_k)} \frac{f(z_k)}{(x - z_k)} \tag{10}$$

The following result holds true.

Theorem 3.2. *If there exists an integer s such that*

$$0 \le s - \alpha - 1 \le 1, \tag{11}$$

then, for any function $f \in C(w)$, we have

$$\|L^*_{2m+3,s}(w_\alpha, w_\alpha, f)w\|_\infty \le \mathcal{C}\|fw\|_\infty \log m, \tag{12}$$

where $\mathcal{C} \neq \mathcal{C}(m, f)$. Moreover

$$\|[f - L^*_{2m+3,s}(w_\alpha, w_\alpha, f)]w\|_\infty \le \mathcal{C}\left\{E_{\bar{M}}(f)_w \log m + e^{-Am}\|fw\|_\infty\right\}, \tag{13}$$

where $\bar{M} = \left[(m + 1 + \frac{s}{2})\left(\frac{\theta}{1+\theta}\right)^2\right] \sim m$, $\mathcal{C} \neq \mathcal{C}(m, f)$, $0 < A \neq A(m, f)$.

We conclude recalling that in the Sobolev space

$$W_k(w) := \left\{f \in C(w) : \|f^{(k)}w\|_\infty < \infty\right\}, \quad k \ge 1,$$

the following estimate holds true [2]

$$E_m(f)_w \le \frac{\mathcal{C}}{\sqrt{m}^k}\|f^{(k)}w\|_\infty, \quad \mathcal{C} \neq \mathcal{C}(m, f). \tag{14}$$

Therefore by (9) and (13) it follows that the errors of the proposed interpolation processes behave like $\mathcal{O}\left(\frac{\log m}{\sqrt{m}^k}\right)$ for any $f \in W_k(w)$.

3.2. The case of functions with a singularity in zero

Now consider the case of function having an algebraic singularity in 0. Set $u(x) = e^{-x^2}|x|^\gamma$, $\gamma > 0$, and consider the space of weighted continuous functions

$$\tilde{C}(u) = \left\{ f \in C^0(\mathbb{R} - \{0\}) : \lim_{x\to\pm\infty} f(x)u(x) = 0 = \lim_{x\to 0} f(x)u(x) \right\}$$

equipped with the norm $\|fu\|_\infty = \sup_{x\in\mathbb{R}} |f(x)u(x)|$. In other words $\tilde{C}(u)$ contains functions which can grow exponentially at $\pm\infty$ and algebraically at zero. The limit condition in the definition is necessary in order for the best polynomial approximation to exist in $\tilde{C}(u)$.

We will denote $E_M(f)_u = \inf_{P\in\mathbb{P}_M} \|(f-P)u\|_\infty$.

Now let r be an integer and A_r be the monic polynomial of degree r whose zeros are the r consecutive roots of Q_{2m+1} nearest to 0, i.e.

$$A_r(x) = \begin{cases} x(x - z_{\frac{r}{2}}) \prod_{k=1}^{\frac{r}{2}-1} (x^2 - z_i^2), & r \text{ even}, \\ \\ x \prod_{k=1}^{\frac{r-1}{2}} (x^2 - z_i^2), & r \text{ odd}. \end{cases}$$

Consider the Lagrange polynomial $L_{2m+3,-r}(w_\alpha, w_\alpha, f)$ interpolating a given function $f \in \tilde{C}(u)$ at the zeros of $\dfrac{Q_{2m+1}(x)}{A_r(x)}(2m - x^2)$ and let

$$L^*_{2m+3,-r}(w_\alpha, w_\alpha, f) = L_{2m+3,-r}(w_\alpha, w_\alpha, f\chi_j).$$

Denoting by $\Gamma_{m,j}$ the set of zeros of $\frac{Q_{2m+1}(x)}{A_r(x)}$ belonging to the segment (z_{-j}, z_j), an explicit expression of $L^*_{2m+3,-r}(w_\alpha, w_\alpha, f)$ is

$$L^*_{2m+3,-r}(w_\alpha, w_\alpha, f; x) = \sum_{z_k\in\Gamma_{m,j}} \frac{Q_{2m+1}(x)(2m - x^2)A_r(z_k)}{Q'_{2m+1}(z_k)(2m - z_k^2)A_r(x)(x - z_k)} f(z_k). \tag{15}$$

Theorem 3.3. *If, for $\gamma > 0$, $\alpha > -1$, there exists an integer r such that*

$$0 \le \gamma - \alpha - r \le 1, \tag{16}$$

then, for any function $f \in \tilde{C}(u)$, we have

$$\|L^*_{2m+3,-r}(w_\alpha, w_\alpha, f)u\|_\infty \le \mathcal{C}\|fu\|_\infty \log m, \tag{17}$$

where $\mathcal{C} \ne \mathcal{C}(m, f)$. Moreover,

$$\|[f - L^*_{2m+3,-r}(w_\alpha, w_\alpha, f)]u\|_\infty \le \mathcal{C}\left\{ E_{\widehat{M}}(f)_u \log m + e^{-Am}\|fu\|_\infty \right\}, \tag{18}$$

where $\widehat{M} = \left[(m + 1 - \frac{r}{2})\left(\frac{\theta}{1+\theta}\right)^2\right] \sim m$, $\mathcal{C} \ne \mathcal{C}(m, f)$, $0 < A \ne A(m, f)$.

Finally we recall that in [7, 8] it was proved that if we consider the Sobolev-type space

$$\tilde{W}_k(u) := \left\{ f \in \tilde{C}(u) : \|f^{(k)}u\|_\infty < \infty \right\}, \quad k \geq 1,$$

then (14) holds true with u instead of w. Therefore we can conclude that if $f \in \tilde{W}_k(u)$, by (18) the error of the Lagrange interpolation process $L^*_{2m+3,-r}(w_\alpha, w_\alpha)$ goes like $\mathcal{O}\left(\frac{\log m}{\sqrt{m}^k}\right)$.

4. Numerical Examples

Now we propose some examples in order to show the performance of the interpolation processes introduced in this paper. We remark once again that all the proposed extended interpolation processes have optimal Lebesgue constants, as showed by estimates (7), (12), (17).

In order to outline the more feasibility of the proposed extended processes we compared our results with those produced by the analogous modifications of the Lagrange polynomial based on the zeros of $p_{2m+1}(\rho, x)(2m - x^2)$, where $\rho(x) = e^{-2x^2}|x|^{2\lambda}$.

To be more precise, we denote by $\{\sigma_i\}_i$ the zeros of $p_{2m+1}(\rho)$ and define

$$h = h(m, \theta) = \min_{0 \leq i \leq 2m+1} \left\{ \sigma_i \geq \theta\sqrt{2m} \right\} \tag{19}$$

where $0 < \theta < 1$ is fixed.

The Lagrange polynomial $L_{2m+3}(\rho, f)$ interpolates f at the zeros of the polynomial $p_{2m+1}(\rho, x)$ and the additional knots $\pm\sqrt{2m}$. Setting $L^*_{2m+3}(\rho, f) := L_{2m+3}(\rho, f\chi_h)$, where χ_h is the characteristic function of the segment $(-\sigma_h, \sigma_h)$, and with Theorem 3.1 in [6], combined with a more general result in [4], we have that for any $f \in C(w)$

$$-1 < \lambda \leq 0 \Longrightarrow \|L^*_{2m+3}(\rho, f)w\|_\infty \leq \mathcal{C}\|fw\|_\infty \log m, \quad \mathcal{C} \neq \mathcal{C}(m, f). \tag{20}$$

Moreover, denoting by $L^*_{2m+3,s_1}(\rho, f) := L_{2m+3,s_1}(\rho, f\chi_h)$, the Lagrange polynomial interpolating f at the zeros of $p_{2m+1}(\rho, x)(2m - x^2)$ and on s_1 additional knots "close" to zero, by Theorem 3.2 in [6] it follows

$$0 \leq s_1 - \lambda \leq 1 \Longrightarrow \|L^*_{2m+3,s_1}(\rho, f)w\|_\infty \leq \mathcal{C}\|fw\|_\infty \log m, \tag{21}$$

where $\mathcal{C} \neq \mathcal{C}(m, f)$.

Finally, denoting by $L^*_{2m+3,-r_1}(\rho, f) := L_{2m+3,-r_1}(\rho, f\chi_h)$, the Lagrange polynomial interpolating f at the zeros of $\frac{p_{2m+1}(\rho,x)}{A_{r_1}(x)}(2m - x^2)$, i.e. dropping r_1 zeros of $p_{2m+1}(\rho)$ "close" to 0, by Theorem 3.3 in [6] it can be easily deduced that for any function $f \in \tilde{C}(u)$,

$$0 \leq \gamma - r_1 - \lambda \leq 1 \Longrightarrow \|L^*_{2m+3,-r_1}(\rho, f)w\|_\infty \leq \mathcal{C}\|fw\|_\infty \log m, \tag{22}$$

where $\mathcal{C} \neq \mathcal{C}(m, f)$.

We remark that each of these interpolation processes needs the computation of the zeros of the polynomial $p_{2m+1}(\rho)$ of degree $2m+1$. On the contrary our processes, for the same choice of $2m+1$, need the computation of the zeros of $p_m(w_\alpha)$ and $p_{m+1}(w_\alpha)$. As we will see, the extended approach makes the difference, from the computational point of view, when the convergence is slow and it is necessary to construct interpolating polynomials of high degree.

We denote by D_m a set of equally spaced points belonging to the range $(-\sigma_h, \sigma_h)$, and by F_m a set of equally spaced points belonging to the range $(-z_j, z_j)$, with z_j being defined in (1).

Every table contains:

- the degree of the interpolating polynomials
- the number of knots depending on $j = j(m, \theta)$ defined in (1)
- the maximum error attained in the set F_m
- the number of knots depending on $h = h(m, \theta)$ defined in (19)
- the maximum error attained in the set D_m.

All the computations were performed in double machine precision ($eps = 2.2204 \times 10^{-16}$).

We finally remark that, in each of the examples, the intervals $(-\sigma_h, \sigma_h)$ and $(-z_j, z_j)$, which are depending on the choice of θ, were computed empirically finding the interval $I \equiv [-\eta, \eta]$ s.t. $|f(y)|w(y) \geq \mathcal{C}eps, \quad \forall y \in I$.

Example 1

$$f_1(x) = \frac{e^{\frac{x^2}{2}}}{1 + x^2}$$

We choose as interpolation weights $\rho(x) = e^{-2x^2}|x|^{-0.5}$ and $w_\alpha(x) = e^{-x^2}|x|^{-0.5}$, respectively, so that the assumptions in (20) and (8) are both fulfilled. In this case the function f_1 has an exponential growth at $\pm\infty$ and is a smooth function. Therefore we expect a fast convergence of the interpolation processes. Nevertheless the convergence depends on the quick rise of the seminorm $\|f_1^{(k)} w\|_\infty$. For instance $\|f_1^{(10)} w\|_\infty \sim 10^6$.

$E_{m,sing} = \|[f_1 - L^*_{2m+3}(\rho, f_1)]w\|_\infty$, $E_{m,ext} = \|[f_1 - L^*_{2m+3}(w_\alpha, w_\alpha, f_1)]w\|_\infty$

$2m+2$	$2h+1$	$E_{m,sing}$	$2j+1$	$E_{m,ext}$
34	33	0.12e-4	35	0.81e-4
66	65	0.11e-6	65	0.28e-5
130	105	0.23e-9	105	0.62e-8
258	155	0.13e-13	155	0.18e-12
514	277	0.93e-14	223	0.40e-14

Example 2

$$f_2(x) = e^{\frac{x}{8}}|x-1|^{\frac{11}{2}}$$

In this case $\rho(x) = e^{-2x^2}|x|^{0.5}$, $w_\alpha(x) = e^{-x^2}|x|^{0.5}$ and the choices $s = 2$, $s_1 = 1$ assure that (11) and (21) are both satisfied. Since $f_2 \in W_5(w)$, the numerical errors agree with the theoretical estimates. We remark that we obtain results comparable with the corresponding interpolation processes, by using the zeros of half degree orthogonal polynomials. In addition, by virtue of the truncation of the function f, the number of function computations is drastically reduced up to 75%, for m sufficiently large.

$$E_{m,sing} = \|[f_2 - L^*_{2m+3,s_1}(\rho, f_2)]w\|_\infty,$$
$$E_{m,ext} = \|[f_2 - L^*_{2m+3,s}(w_\alpha, w_\alpha, f_2)]w\|_\infty$$

$2m+2+s$	$2h+1+s_1$	$E_{m,sing}$	$2j+1+s$	$E_{m,ext}$
36	35	0.64e-2	35	0.23e-3
68	56	0.38e-3	55	0.71e-3
132	108	0.36e-4	81	0.50e-5
260	160	0.67e-5	117	0.28e-5
516	232	0.98e-6	167	0.45e-6
1028	330	0.59e-7	235	0.61e-7
2052	—	—	361	0.16e-8

In this case the computation of $L^*_{2052,1}(\rho, f_2)$ cannot be performed, while the extended polynomial of degree 2052 can be successfully computed.

Example 3

$$f_3(x) = \frac{\cos(x)e^{\frac{x}{8}}}{|x|^{0.1}}$$

In this case $\rho(x) = e^{-2x^2}|x|^2$, $w_\alpha(x) = e^{-x^2}|x|^2$, $u(x) = e^{-x^2}|x|^{4.2}$ and the choices $r = 2$, $r_1 = 3$ assure that (16) and (22) are both fulfilled. We observe that $f_3 \in \tilde{W}_4(u)$ and the numerical errors agree with the theoretical estimates, since we can expect at most 6 exact digits by using 1024 interpolation knots.

$$E_{m,sing} = \|[f_3 - L^*_{2m+3,-r_1}(\rho, f_3)]u\|_\infty,$$
$$E_{m,ext} = \|[f_3 - L^*_{2m+3,-r}(w_\alpha, w_\alpha, f_3)]u\|_\infty.$$

$2m+3-r$	$2h-3$	$E_{m,sing}$	$2j-2$	$E_{m,ext}$
33	31	0.14	32	0.11e-1
65	49	0.21e-1	52	0.22e-2
129	77	0.11e-2	78	0.24e-3
257	111	0.27e-3	112	0.35e-4
513	161	0.82e-5	164	0.75e-5
1025	233	0.89e-6	236	0.76e-6
2021	—	—	333	0.21e-7

Also in this case, in order to get more precise values, the computation of $L^*_{2023,-3}(\rho, f_2)$ cannot be performed, while the extended polynomial of degree 2021 can be successfully computed.

5. The Proofs

Now we collect some polynomial inequalities deduced in [6] (see also [3]).

Let $x \in [x_{m,-\left[\frac{m}{2}\right]}, x_{m,\left[\frac{m}{2}\right]}]$ and let $d = d(x)$, $1 \le |d| \le \left[\frac{m}{2}\right]$ be the index of a zero of $p_m(w_\alpha)$ closest to x. Then, for some positive constant $\mathcal{C} \ne \mathcal{C}(m, x, d)$, we have

$$p_m^2(w_\alpha, x) w_{\alpha,m}(x) \sqrt{2m - x^2 + (2m)^{\frac{1}{3}}} \sim \left(\frac{x - x_{m,d}}{x_{m,d} - x_{m,d\pm1}} \right)^2 \tag{23}$$

where $w_{\alpha,m}(x) = e^{-x^2} \left(|x| + \frac{1}{\sqrt{m}} \right)^\alpha$, by which

$$|p_m(w_\alpha, x)| \sqrt{w_\alpha(x)} \sqrt[4]{2m - x^2 + (2m)^{\frac{1}{3}}} \le \mathcal{C}, \qquad \frac{\mathcal{C}}{\sqrt{m}} \le |x| \le \sqrt{2m}. \tag{24}$$

Moreover, for $0 \le |k| \le \left[\frac{m}{2}\right]$ and setting $\Delta x_{m,k} = x_{m,k+1} - x_{m,k}$ one has

$$\frac{1}{|p'_m(w_\alpha, x_{m,k})| \sqrt{w_{\alpha,m}(x_{m,k})}} \sim \Delta x_{m,k} \sqrt[4]{2m - x_{m,k}^2 + (2m)^{\frac{1}{3}}}. \tag{25}$$

Now, setting $W_\gamma^a(x) = |x|^\gamma e^{-ax^2}$, $\gamma \ge 0$, $a > 0$, we recall the Bernstein inequality [7] (see also [4]), holding for any polynomial $P \in \mathbb{P}_m$,

$$\|P' W_\gamma^a\|_\infty \le \mathcal{C} \sqrt{m} \|P W_\gamma^a\|_\infty, \quad \mathcal{C} \ne \mathcal{C}(m, P) \tag{26}$$

and the Remez-type inequality [7]

$$\max_{x \in \mathbb{R}} |P(x) W_\gamma^a(x)| \le \mathcal{C} \max_{x \in \mathcal{A}_m} |P(x) W_\gamma^a(x)|, \tag{27}$$

with $\mathcal{A}_m = \left\{ x : \frac{\mathcal{C}}{\sqrt{m}} \le |x| \le \sqrt{2m}, \quad \mathcal{C} \ne \mathcal{C}(m) \right\}$.

Lemma 5.1. *Let $Q_{2m+1} = p_{m+1}(w_\alpha) p_m(w_\alpha)$ and $z_k, 1 \le |k| \le \left[\frac{m}{2}\right]$ be the zeros of $Q_{2m+1}(x)$. We have*

$$\frac{1}{|Q'_{2m+1}(z_k)| w_\alpha(z_k)} \le \mathcal{C} \Delta z_k \sqrt{2m - z_k^2 + (2m)^{\frac{1}{3}}}, \quad \mathcal{C} \ne \mathcal{C}(m). \tag{28}$$

Proof. Assuming that $m+1$ is odd, by (24)

$$\frac{1}{|p_m(w_\alpha, x_{m+1,k})| \sqrt{w_\alpha(x_{m+1,k})}} \le \mathcal{C} \sqrt[4]{2m - x_{m+1,k}^2 + (2m)^{\frac{1}{3}}},$$

and by using (25), it follows

$$\frac{1}{|Q'_{2m+1}(x_{m+1,k})| w_\alpha(x_{m+1,k})} \le \mathcal{C} \Delta x_{m+1,k} \sqrt{2m - x_{m+1,k}^2 + (2m)^{\frac{1}{3}}}.$$

Since an analogous estimate holds considering $(|Q'_{2m+1}(x_{m,k})|w_\alpha(x_{m,k}))^{-1}$, the Lemma follows.□

Proof of Proposition 2.1. First assume that m is odd. In view of the symmetry of the zeros w.r.t. 0, we prove the Lemma in the positive semiaxis. Therefore let $k \geq 1$. Since $x_{m+1,k+1} - x_{m,k} < x_{m+1,k+1} - x_{m+1,k}$ and taking into account (3) it follows

$$\Delta z_k \leq \frac{\mathcal{C}}{\sqrt{m}}, \qquad |z_k| \leq \theta\sqrt{2m}.$$

Now we prove the converse inequality showing that for $k < j$, $\mathcal{C} \neq \mathcal{C}(m)$,

$$x_{m+1,k+1} - x_{m,k} \geq \frac{\mathcal{C}}{\sqrt{m}}, \quad x_{m,k} - x_{m+1,k} \geq \frac{\mathcal{C}}{\sqrt{m}}$$

We prove the first inequality. The second one follows by using the same arguments.

Since $Q'_{2m+1}(x_{m+1,k+1}) > 0$ and $Q'_{2m+1}(x_{m,k}) < 0$, we have

$$0 < Q'_{2m+1}(x_{m+1,k+1}) - Q'_{2m+1}(x_{m,k}) = (x_{m+1,k+1} - x_{m,k})Q''_{2m+1}(\xi_k),$$

where $\xi_k \in (x_{m,k}, x_{m+1,k+1})$. Therefore

$$\frac{1}{x_{m+1,k+1} - x_{m,k}} \leq \frac{Q''_{2m+1}(\xi_k)}{Q'_{2m+1}(x_{m+1,k+1})}. \tag{29}$$

If $\alpha \geq 0$ (otherwise it is possible to use $w_{\alpha+1}$ instead of w_α for weighting the polynomials), by (26)-(27) and (24) it follows

$$|p'_m(w_\alpha, \xi_k)|\sqrt{w_\alpha(\xi_k)} \leq \mathcal{C}\frac{\sqrt{m}}{\sqrt[4]{2m - \xi_k^2 + (2m)^{\frac{1}{3}}}}$$

and

$$|p''_m(w_\alpha, \xi_k)|\sqrt{w_\alpha(\xi_k)} \leq \mathcal{C}\frac{m}{\sqrt[4]{2m - \xi_k^2 + (2m)^{\frac{1}{3}}}}.$$

Therefore, again by (24), we get

$$|Q''_{2m+1}(\xi_k)|w_\alpha(\xi_k) \leq \frac{m}{\sqrt{2m - \xi_k^2 + (2m)^{\frac{1}{3}}}}.$$

Using Lemma 5.1 and taking into account that $w_\alpha(x_{m+1,k+1}) \sim w_\alpha(\xi_k)$, it follows

$$\frac{|Q''_{2m+1}(\xi_k)|}{Q'_{2m+1}(x_{m+1,k+1})} \leq \mathcal{C}m\Delta x_{m+1,k+1}$$

and hence by (29) and (3), being $k < j$, it is $\dfrac{1}{x_{m+1,k+1} - x_{m,k}} \leq \mathcal{C}\sqrt{m}$.

Therefore, since the case of m even can be treated with similar arguments, the Lemma follows$\square$.

Proof of Proposition 2.2. Let be $m > m_0$ and k s.t. (4) holds true. Let g_m be the piecewise linear function defined as

$$g_m(x) = \begin{cases} \frac{x-\xi_{m,k-1}}{\xi_{m,k}-\xi_{m,k-1}}, & \xi_{m,k-1} \le x \le \xi_{m,k} \\ \\ \frac{x-\xi_{m,k+1}}{\xi_{m,k}-\xi_{m,k+1}}, & \xi_{m,k} \le x \le \xi_{m,k+1} \\ \\ 0, & \text{otherwise} \end{cases}$$

We have that $g_m(\xi_{m,k}) = 1$ and $g_m(\xi_{m,k-1}) = 0$ and hence

$$\begin{aligned} 1 &= |g_m(\xi_{m,k}) - g_m(\xi_{m,k-1})| \frac{w(\xi_{m,k})}{w(\xi_{m,k})} \\ &= |\mathcal{L}_{2m+1}(\mathcal{X}, g_m; \xi_{m,k}) - \mathcal{L}_{2m+1}(\mathcal{X}, g_m; \xi_{m,k-1})| \frac{w(\xi_{m,k})}{w(\xi_{m,k})} \\ &= \left|\mathcal{L}'_{2m+1}(\mathcal{X}, g_m; \zeta_k)\right| (\xi_{m,k} - \xi_{m,k-1}) \frac{w(\xi_{m,k})}{w(\xi_{m,k})}, \qquad \zeta_k \in (\xi_{m,k-1}, \xi_{m,k}). \end{aligned}$$

In view of (4) it is $w(\xi_{m,k}) \sim w(\zeta_{m,k})$, and

$$\mathcal{C} \le \left(\frac{1}{\sqrt{m}}\right)^{1+\eta} \left|\mathcal{L}'_{2m+1}\left(\mathcal{X}, \frac{g_m}{w}; \zeta_k\right)\right| w(\zeta_k).$$

By the Bernstein inequality (26) with $\gamma = 0$, $a = 1$, it follows

$$\mathcal{C} \le \left(\frac{1}{\sqrt{m}}\right)^{\eta} \|\mathcal{L}_{2m+1}\left(\mathcal{X}, \frac{g_m}{w}\right) w\|_\infty$$

and the Proposition follows. $\square$

The following lemmas will be useful in the proofs of the main results.

Lemma 5.2. *[6] Let $\{z_k\}_{-[\frac{m}{2}]}^{[\frac{m}{2}]}$ be the set of zeros of Q_{2m+1}. Denote by z_d a zero closest to x. Assuming $\rho, \sigma \in [0,1]$, for $x \in \mathcal{A}_m$ and for m sufficiently large, we have*

$$\sum_{k=-[\frac{m}{2}], k\ne d}^{[\frac{m}{2}]} \frac{\Delta z_k}{|x - z_k|} \frac{(2m - x^2)^\rho}{(2m - z_k^2)^\rho} \frac{|x|^\sigma}{|z_k|^\sigma} \le \mathcal{C} \log m, \quad \mathcal{C} \ne \mathcal{C}(m, x).$$

Lemma 5.3. *Let $u(x) = |x|^\gamma e^{-x^2}$, with $\gamma \ge 0$. For $x \in (z_{-[\frac{m}{2}]}, z_{[\frac{m}{2}]})$ and with $z_d, d \ne 0$, the zero of $\frac{Q_{2m+1}(x)}{x}$ closest to x, we have*

$$\frac{|Q_{2m+1}(x)|}{|Q'_{2m+1}(z_d)(x - z_d)|} \frac{u(x)}{u(z_d)} \le \mathcal{C}, \quad \mathcal{C} \ne \mathcal{C}(m, x).$$

We remark that this Lemma follows by

$$\frac{|p_m(w_\alpha,x)|\sqrt{w_\alpha(x)}}{|p'_m(w_\alpha,x_{m,d})(x-x_{m,d})|\sqrt{w_\alpha(x_{m,d})}} \sim \mathcal{C}, \quad \mathcal{C}\neq\mathcal{C}(m,x)$$

being $x_{m,d}$ a zero of $p_m(w_\alpha)$ closest to x (see [6, (8), p. 694]).

Proof of Theorem 3.1. First we prove that (8) is a sufficient condition for (7). By (27) we have

$$\|L^*_{2m+3}(w_\alpha,w_\alpha,f)w\|_\infty \le \mathcal{C}\max_{x\in\mathcal{A}_m}|L^*_{2m+3}(w_\alpha,w_\alpha,f;x)w(x)|$$

and using (5)

$$\begin{aligned}\|L^*_{2m+3}(w_\alpha,w_\alpha,f)w\|_\infty &\le \mathcal{C}\|fw\|_\infty\times\max_{x\in\mathcal{A}_m}\left\{\left|\frac{Q_{2m+1}(x)(2m-x^2)}{xQ'_{2m+1}(0)2m}\right|w(x)\right.\\ &+\left.\sum_{1\le|k|\le j}\left|\frac{Q_{2m+1}(x)(2m-x^2)}{Q'_{2m+1}(z_k)(2m-z_k^2)(x-z_k)}\right|\frac{w(x)}{w(z_k)}\right\}\\ &=: \mathcal{C}\|fw\|_\infty\max_{x\in\mathcal{A}_m}\{A_0(x)+\Sigma_m(x)\}. \qquad (30)\end{aligned}$$

By (24)

$$|Q_{2m+1}(x)|w(x)\le\mathcal{C}\frac{|x|^{-\alpha}}{\sqrt{2m-x^2+(2m)^{\frac13}}},\quad x\in\mathcal{A}_m, \qquad (31)$$

and by (23) and (25)

$$\frac{1}{|Q'_{2m+1}(z_0)|}\le\mathcal{C}m^{-\frac{\alpha}{2}},$$

and hence

$$A_0(x)\le\mathcal{C}\frac{|x|^{-\alpha-1}(2m-x^2)}{m^{\frac{\alpha}{2}+1}\sqrt{2m-x^2+(2m)^{\frac13}}}\le\frac{\mathcal{C}}{|x|^{\alpha+1}(\sqrt{m})^{\alpha+1}}\le\mathcal{C}, \qquad (32)$$

since $|x|\ge\frac{\mathcal{C}}{\sqrt{m}}$ and $\alpha+1>0$.

Consider now $\Sigma_m(x)$. In view of (28) we have, for $1\le|k|\le j$,

$$\frac{1}{|Q'_{2m+1}(z_k)|w(z_k)}\le\mathcal{C}\Delta z_k\frac{\sqrt{2m-z_k^2+(2m)^{\frac13}}}{|z_k|^{-\alpha}}.$$

Denoting by z_d the zero of $\frac{Q_{2m+1}(y)}{y}$ closest to x, we have

$$\begin{aligned}\Sigma_m(x) &\le \mathcal{C}\sum_{1\le|k|\le j,\ k\neq d}\frac{\Delta z_k}{|x-z_k|}\frac{\sqrt{2m-x^2}}{\sqrt{2m-z_k^2}}\frac{|x|^{-\alpha}}{|z_k|^{-\alpha}}\\ &+\left|\frac{Q_{2m+1}(x)(2m-x^2)}{Q'_{2m+1}(z_d)(x-z_d)(2m-z_d^2)}\right|\frac{w(x)}{w(z_d)}.\end{aligned}$$

By Lemma 5.2 under the assumption $-1 \le \alpha \le 0$, it follows

$$\sum_{1\le|k|\le j,\ k\neq d} \frac{\Delta z_k}{|x-z_k|}\frac{\sqrt{2m-x^2}}{\sqrt{2m-z_k^2}}\frac{|x|^{-\alpha}}{|z_k|^{-\alpha}} \le \mathcal{C}\log m.$$

Moreover by Lemma 5.3 with $\gamma = 0$ and taking into account that $(2m-x^2) \sim (2m-z_d^2)$, we get

$$\Sigma_m(x) \le \mathcal{C}\log m \tag{33}$$

and hence (7) follows by (32), (33) and (30).

We omit the proof of the necessity of condition (8) since, mutatis mutandis, it follows by using standard arguments, similar to those used in the proof of Theorem 3.1 in [9]. Finally (9) can be deduced immediately by (7) and (6). $\square$

Proof of Theorem 3.2 By (27) and (10) we have

$$\begin{aligned}
&\|L^*_{2m+3,s}(w_\alpha,w_\alpha,f)u\|_\infty \le \mathcal{C}\|fu\|_\infty \times \\
&\max_{\frac{\mathcal{C}}{\sqrt{m}}\le x\le\sqrt{2m}} \left\{\sum_{i=1}^{s} \frac{|Q_{2m+1}(x)(2m-x^2)B_s(x)t_i|}{|xQ_{2m+1}(t_i)(2m-t_i^2)B_s'(t_i)(x-t_i)|}\frac{u(x)}{u(t_i)}\right.\\
+ \quad &\left.\sum_{1\le|k|\le j}\frac{|Q_{2m+1}(x)(2m-x^2)B_s(x)z_k|}{|xQ'_{2m+1}(z_k)(2m-z_k^2)B_s(z_k)(x-z_k)|}\frac{u(x)}{u(z_k)}\right\}\\
=: \quad &\mathcal{C}\|fu\|_\infty \max_{\frac{\mathcal{C}}{\sqrt{m}}\le x\le\sqrt{2m}}\{C_1(x)+C_2(x)\}
\end{aligned}$$

About the sum C_2 we can prove that $C_2(x) \le \mathcal{C}\log m$, $\mathcal{C} \neq \mathcal{C}(m,x)$, by means of the same arguments used in estimating $\Sigma_m(x)$ in the proof of Theorem 3.1, simply having in mind that

$$|B_s(z_k)| \ge |z_k - t_s|^s \sim |z_k|^s \quad \text{and} \quad B_s(x) \sim B_s(z_d),$$

with z_d the zero of $\frac{Q_{2m+1}(y)}{y}$ closest to x, for any fixed x.

Hence (12) follows if we are able to prove that $C_1(x) \le \mathcal{C}$, $\mathcal{C} \neq \mathcal{C}(m,x)$. But in estimating $C_1(x)$ it is possible to proceed as in evaluating $A_0(x)$ in the proof of Theorem 3.1, taking into account that $t_i \sim \frac{1}{\sqrt{m}}$, so that

$$\left|\frac{B_s(x)}{B_s'(t_i)}\right| \le \prod_{j=1,j\neq i}^{s}\left|\frac{x-t_j}{t_i-t_j}\right| \le \mathcal{C}\left(\sqrt{m}|x|\right)^{s-1}. \quad \square$$

Proof of Theorem 3.3. By (27) and (15) we have

$$\begin{aligned}
&\|L^*_{2m+3,-r}(w_\alpha,w_\alpha,f)u\|_\infty \le \mathcal{C}\|fu\|_\infty \times\\
&\max_{\frac{\mathcal{C}}{\sqrt{m}}\le x\le\sqrt{2m}}\sum_{z_k\in\Gamma_{m,j}}\frac{|Q_{2m+1}(x)(2m-x^2)A_r(z_k)|}{|A_r(x)Q'_{2m+1}(z_k)(2m-z_k^2)(x-z_k)|}\frac{u(x)}{u(z_k)}.
\end{aligned}$$

By (31), Lemma 5.3 and taking into account that

$$|A_r(x)| \geq \mathcal{C}|x|^r, \quad |x| \geq \frac{\mathcal{C}}{\sqrt{m}}, \quad |A_r(z_k)| \leq \mathcal{C}|z_k|^r,$$

it follows that the sum is bounded by

$$\mathcal{C} \sum_{z_k \in \Gamma_{m,j}} \frac{\Delta z_k}{|x - z_k|} \left(\frac{|x|}{|z_k|} \right)^{\gamma-\alpha-r} \frac{\sqrt{2m - x^2}}{\sqrt{2m - z_k^2}}, \quad \mathcal{C} \neq \mathcal{C}(m, x),$$

and therefore (17) follows by Lemma 5.2 with $\rho = \frac{1}{2}$, $\sigma = \gamma - \alpha - r$.

Finally (18) follows immediately from (17) since (6) holds true even if w is replaced by u (see [4]). $\square$

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