Nyström method for Fredholm integral equations of the second kind in two variables on a triangle

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Abstract

The paper deals with the approximation of the solution of the following bivariate Fredholm integral equation

$$f(\mathbf{y}) - \mu \int_{\mathrm{D}} \mathbf{K}(\mathbf{x}, \mathbf{y}) f(\mathbf{x}) \widetilde{\omega}(\mathbf{x})\, d\mathbf{x} = g(\mathbf{y}), \quad \mathbf{y} \in \mathrm{D},$$

where the domain D is a triangle. The proposed procedure, by a suitable transformation, is essentially the Nyström method based on the zeros of univariate Jacobi orthogonal polynomials. Convergence, stability and well conditioning of the method are proved. In order to illustrate the efficiency of the proposed method some numerical tests are given.

Key words: Fredholm integral equations, projection methods, Nyström method.
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1 Introduction

The present paper deals with the approximation of the solution of the Fredholm integral equation of the second kind in two variables on a bounded domain D,

$$f(\mathbf{y}) - \mu \int_{\mathrm{D}} \mathbf{K}(\mathbf{x}, \mathbf{y}) f(\mathbf{x}) \tilde{\omega}(\mathbf{x})\, d\mathbf{x} = g(\mathbf{y}), \quad \mathbf{y} \in D, \tag{1.1}$$

where $\mathbf{y} = (y_1, y_2)$, $\mathbf{x} = (x_1, x_2)$, $d\mathbf{x} = dx_1 dx_2$, $\mathbf{K}$ and g are given functions defined on D, $\tilde{\omega}$ is a weight function and f is the unknown function. This topic is of interest since various problems can be modeled by integral equations of type (1.1). In [10] the authors transform a two dimensional transport equation into an integral of type (1.1). Equations of kind (1.1) are an useful toll for modeling many problems arising in fracture mechanics, aerodynamics, two dimensional electromagnetic scattering [19], computer graphics manipulations (see for instance [2],[12]). Some of the existing papers treat the case when the domain D is a square (see for instance [1], [3], [13], [8], [14], [17], [21]) and some of the existing procedures are based on piecewise approximating polynomials [14], Monte Carlo methods [13], or discrete Galerkin method [8], Nyström methods based on cubature rules obtained as the tensor product of two univariate Gaussian rules [21],[17]. In some cases the error estimate is not given [3], [18], in others the convergence is "slow" [11]. Anyway, as far as we know, little attention was given to the stability and the convergence of the proposed procedures.

Here we consider the case D coinciding with the unit triangle T of vertices $(0,0)$, $(0,1)$, $(1,0)$ and the weight $\tilde{\omega}(x_1, x_2) = x_1^{p-1} x_2^{q-1} (x_1 + x_2)^a (1 - x_1 - x_2)^b$, with $p, q > 0$, $p + q + a > 0$, $b > -1$. Since the case we are considering includes kernels $\mathbf{K}$ having singularities on the border of T, cubature rules involving nodes on the sides (see for instance [1], [15], [9], [20]) cannot be used. For instance, when $b > 0$, $\mathbf{K}$ can be discontinuous on the side $0 \le x \le 1, 0 \le y \le 1 - x$, while in the case $a > 0$ and $p = q = 1$, it can be singular on the open sides $\{x = 0, 0 < y \le 1\}$ and/or $\{y = 0, 0 < x \le 1\}$. Hence, by using a suitable transformation, we obtain an integral equation on the square $\mathrm{Q} = [-1, 1] \times [-1, 1]$, where the new weight into the integral is the product of a pair of Jacobi weights. By this way, possible singularities of the kernel $\mathbf{K}$ on the boundary of T are moved to corresponding singularities along the border of the square and "absorbed" into the Jacobi weights. Then we propose a global approximation of the solution by means of a Nyström method based on the tensor product of two univariate Gaussian rules. Even though few results are available in the literature about the weighted polynomial approximation in two variables [5], [16], we are able to prove that the proposed method is stable and convergent, giving error estimates in weighted uniform spaces. Finally we propose some numerical experiments to test the efficiency and accuracy of our procedure and to confirm the theoretical results.

The outline of this paper is as follows. Section 2 contains notations and some basic results. The main results are stated in Section 3, where we describe the proposed method, state results about the convergence in weighted uniform spaces and prove the stability of the numerical procedure. Section 4 contains numerical tests and some computational details. All the proofs are given in Section 5.

2 Notations and basic tools

In the sequel $\mathbb{P}_m$ will denote the space of all algebraic polynomials of two variables of degree at most m in each variable separately.

Along all the paper the constant $\mathcal{C}$ will be used several times, having different meanings in different formulas. Moreover from now on we will write $\mathcal{C} \neq \mathcal{C}(a, b, \ldots)$ in order to say that $\mathcal{C}$ is a positive constant independent of the parameters $a, b, \ldots$, and $\mathcal{C} = \mathcal{C}(a, b, \ldots)$ to say that $\mathcal{C}$ depends on $a, b, \ldots$. Furthermore, if $A, B > 0$ are quantities depending on some parameters, we write $A \sim B$, if there exists a positive constant $\mathcal{C}$ independent of the parameters of A and B, such that

$$\frac{B}{\mathcal{C}} \leq A \leq \mathcal{C}B.$$

2.1 Spaces of functions

Let be $Q = [-1, 1]^2$ and $\sigma(\mathbf{z}) = \sigma(z_1, z_2) = v^{\gamma_1,\delta_1}(z_1)v^{\gamma_2,\delta_2}(z_2)$, where $\mathbf{z} = (z_1, z_2) \in (-1, 1)^2$, $\gamma_i, \delta_i \geq 0, i = 1, 2$, and $v^{\rho,\tau}(t) = (1-t)^\rho(1+t)^\tau$ is a Jacobi weight. Now, we denote by $C(A), A \subseteq Q$ the set of the continuous functions on A and by C_σ the set of functions f which are continuous on $(-1, 1)^2$, so that σf is continuous in Q and vanishing on the border ∂Q of the square, i.e.,

$$C_\sigma = \left\{ f \in C((-1, 1)^2) : \lim_{\mathbf{z} \to \partial Q} (\sigma f)(\mathbf{z}) = 0 \right\}.$$

We remark that C_σ with $\gamma_i, \delta_i > 0$ includes functions f which can be unbounded for $\mathbf{z} \to \partial Q$. If $\sigma = 1$, it is $C = C_1$. Now we set

$$\|f\|_{C_\sigma} = \|f\sigma\|_\infty = \max_{(x,y)\in Q} |f(x, y)\sigma_1(x)\sigma_2(y)|.$$

In $(C_\sigma, \|\cdot\|_{C_\sigma})$, which is a Banach space, we introduce a modulus of continuity with weight σ. To this end we denote by $f_y(x)$ the function $f(x, y)$ as a function of the only variable x and fixed y. Similarly for the notation $f_x(y)$. Recalling also the definition of the φ-modulus of continuity in [16], we set [5]

$$\omega_Q^r(f,t)_\sigma := \max\left\{\sup_{|y|\le 1}\sigma_2(y)\omega_\varphi^r(f_y,t)_{\sigma_1}, \sup_{|x|\le 1}\sigma_1(x)\omega_\varphi^r(f_x,t)_{\sigma_2}\right\}.$$

Moreover, by denoting with $E_m(f)_\sigma = \inf_{P\in\mathbb{P}_m}\|(f-P)\sigma\|$, it is not hard to prove the relations

$$E_m(f)_\sigma \le \mathcal{C}\omega_Q^r\left(f,\frac{1}{m}\right)_\sigma, \tag{2.1}$$

$$\omega_Q^r\left(f,\frac{1}{m}\right)_\sigma \le \frac{\mathcal{C}}{m^r}\sum_{k=1}^{m}(1+k)^{r-1}E_k(f)_\sigma. \tag{2.2}$$

By (2.1) and (2.2) it follows

$$f\in C_\sigma \Leftrightarrow \lim_m \omega_Q^r\left(f,\frac{1}{m}\right)_\sigma = 0 \Leftrightarrow \lim_m E_m(f)_\sigma = 0.$$

In particular, for smoother functions, by setting $\varphi(z)=\sqrt{1-z^2}$, we have

$$\begin{aligned} E_m(f)_\sigma &\le \frac{\mathcal{C}}{m^r}\max\left\{\sup_{|y|\le 1}\sigma_2(y)\|f_y\varphi^r\sigma_1\|_{[0,1]}, \sup_{|x|\le 1}\sigma_1(x)\|f_x\varphi^r\sigma_2\|_{[0,1]}\right\} \\ &=: \frac{\mathcal{C}}{m^r}M_r(f,u), \end{aligned} \tag{2.3}$$

where the positive constant $\mathcal{C}\ne\mathcal{C}(m,f)$.

2.2 Transformations

Let us denote by T the triangle defined as

$$\mathrm{T} = \{(x_1,x_2) : 0\le x_1+x_2\le 1, \quad x_1\in[0,1]\}$$

and by Q the square defined as $\mathrm{Q} := [-1,1]\times[-1,1]$. For any $\mathbf{x}=(x_1,x_2)\in\mathrm{T}$ and $\mathbf{u}=(u_1,u_2)\in\mathrm{Q}$, we consider the following transformations between the triangle T and the square Q

$$x_1 = \frac{1}{4}(1+u_1)(1+u_2) \qquad x_2 = \frac{1}{4}(1+u_1)(1-u_2) \tag{2.4}$$

or equivalently

$$u_1 = 2(x_1+x_2)-1 \qquad u_2 = \frac{x_1-x_2}{x_1+x_2}.$$

The Jacobian of the transformations is

$$J(x_1, x_2) = \frac{\partial(x_1, x_2)}{\partial(u_1, u_2)} = \frac{1}{8}(1 + u_1) = \frac{1}{8} v^{0,1}(u_1).$$

Fig. 2.1. Transformation T $\rightarrow$ Q

Trough (2.4) (see Figure 2.2) the sides $\overline{AC}$ (open in A), $\overline{CB}, \overline{AB}$ of the triangle are mapped into the sides $\overline{EF}, \overline{GF}$ and $\overline{HG}$ respectively, while the point A is exploited into the sides $\overline{HE}$. We remark that many other maps are possible, such as, for instance (see [4], [6]),

$$x_1 = \frac{1}{4}(1 + u_1)(1 - u_2) \qquad x_2 = \frac{1}{2}(1 + u_2).$$

However we will use the map in (2.4) since, as we will see, it allows us to obtain an integral equation on the square Q, with a weight function product of two Jacobi weights.

3 The numerical method

In this section we consider the two-variables Fredholm integral equation defined on the triangle T, i.e. for any $(x_1, x_2) \in$ T,

$$f(x_1, x_2) - \mu \int_{\mathrm{T}} K(x_1, x_2, y_1, y_2) f(y_1, y_2) \tilde{\omega}(y_1, y_2) \, dy_1 \, dy_2 = g(x_1, x_2),$$

where

$$\tilde{\omega}(x_1, x_2) = x_1^{p-1} x_2^{q-1} (x_1 + x_2)^a (1 - x_1 - x_2)^b, \quad p, q > 0, \; p + q + a > 0, \; b > -1.$$

For this kind of equation, by using the map in (2.4), we obtain the integral equation

$$F(\mathbf{v}) - \nu \int_{\mathrm{Q}} \mathbf{h}(\mathbf{u}, \mathbf{v}) F(\mathbf{u}) \mathbf{w}(\mathbf{u}) d\mathbf{u} = G(\mathbf{v}), \quad \mathbf{v} \in \mathrm{Q}, \tag{3.1}$$

where

$$
\begin{aligned}
F(\mathbf{v}) &= f\left(\frac{(1+v_1)(1+v_2)}{4}, \frac{(1+v_1)(1-v_2)}{4}\right), \qquad (3.2)\\
\mathbf{h}(\mathbf{u},\mathbf{v}) &= K\Bigg(\frac{(1+u_1)(1+u_2)}{4}, \frac{(1+u_1)(1-u_2)}{4},\\
&\qquad\qquad \frac{(1+v_1)(1+v_2)}{4}, \frac{(1+v_1)(1-v_2)}{4}\Bigg),\\
G(\mathbf{v}) &= g\left(\frac{(1+v_1)(1+v_2)}{4}, \frac{(1+v_1)(1-v_2)}{4}\right),\\
\nu &= \frac{\mu}{2^{2p+2q+a+b-1}}, \quad \mathbf{w}(\mathbf{u}) = w_1(u_1)w_2(u_2),\\
w_1(u_1) &= v^{b,p+q+a-1}(u_1), \quad w_2(u_2) = v^{q-1,p-1}(u_2). \qquad (3.3)
\end{aligned}
$$

Setting

$$(\mathbf{H}g)(\mathbf{v}) := \nu \int_{\mathrm{Q}} \mathbf{h}(\mathbf{u},\mathbf{v})g(\mathbf{u})\mathbf{w}(\mathbf{u})d\mathbf{u},$$

(3.1) can be written as

$$(I - \mathbf{H})F = G, \qquad (3.4)$$

where I is the identity operator. In order to study equation (3.1) in C_σ for suitable values of the parameters, we prove the following

Proposition 3.1 *Assume that the kernel* $\mathbf{h}$ *satisfy the following condition:*

$$\forall \mathbf{u} \in \mathrm{Q} \quad \sigma(\mathbf{u})\mathbf{h}_{\mathbf{u}} \in C_\sigma \qquad (3.5)$$

and that

$$\int_Q \frac{\mathbf{w}(\mathbf{u})}{\sigma^2(\mathbf{u})} d\mathbf{u} < +\infty. \qquad (3.6)$$

Then $\mathbf{H}$ *maps* C_σ *into itself and it is a compact operator.*

Proof. The compactness of $\mathbf{H}$ in C_σ is equivalent to

$$\lim_m \sup_{f \in C_\sigma} E_m(\mathbf{H}f)_\sigma = 0$$

i.e., uniformly on $f \in C_\sigma$, the function $\mathbf{H}f$ belongs to C_σ too. Recalling the definition of the space C_σ, we have to prove:

$$
\begin{aligned}
&\mathbf{H}F \in C((-1,1)^2), \qquad (3.7)\\
&\lim_{\mathbf{v}\to\partial Q} \sigma(\mathbf{v})(\mathbf{H}F)(\mathbf{v}) = 0,
\end{aligned}
$$

both of them uniformly on $F \in C_\sigma$.

Let A be a closed set of Q and let $\mathbf{v} \in A$. With $\delta = (\delta_1, \delta_2)$ we can choose δ_1, δ_2 sufficiently small so as $\mathbf{v} + \delta \in A'$, where A' is a closed subset s.t. $A \subset A' \subset (-1,1)^2$. Then we have

$$|(\mathbf{H}F)(\mathbf{v}+\delta) - (\mathbf{H}F)(\mathbf{v})| \le |\nu| \|F\sigma\| \sup_{\mathbf{u}\in Q} |\mathbf{h}_{\mathbf{u}}(\mathbf{v}+\delta) - \mathbf{h}_{\mathbf{u}}(\mathbf{v})| \, \sigma(\mathbf{u}) \int_Q \frac{\mathbf{w}(\mathbf{u})}{\sigma^2(\mathbf{u})} d\mathbf{u}$$

and the assumptions (3.5) and (3.6), (3.7) follows. Moreover,

$$|(\mathbf{H}F)(\mathbf{v})|\sigma(\mathbf{v}) \le |\nu| \|F\sigma\| \sigma(\mathbf{v}) \sup_{\mathbf{u}\in Q} |\mathbf{h}_{\mathbf{u}}(\mathbf{v})| \, \sigma(\mathbf{u}) \int_Q \frac{\mathbf{w}(\mathbf{u})}{\sigma^2(\mathbf{u})} d\mathbf{u}$$

and taking into account (3.5) and (3.6), the proof is complete. $\square$

Therefore under the assumptions (3.5) and (3.6), the operator $\mathbf{H}$ is compact in C_σ and therefore the Fredholm alternative holds true.

In order to introduce our approximation method, we recall the following cubature formula obtained as the tensor product of two univariate Gaussian rules, i.e.,

$$\int_{\mathrm{Q}} F(\mathbf{u})\mathbf{w}(\mathbf{u})d\mathbf{u} = \sum_{i=1}^{m}\sum_{j=1}^{m} \Lambda_{i,j} F(\mathbf{u}_{i,j}) + e_m(F),$$

where $\mathbf{u}_{i,j} = (\tau_i^{(1)}, \tau_j^{(2)})$, $\Lambda_{i,j} = \lambda_i^{(1)}\lambda_j^{(2)}$, and $\{(\tau_i^{(1)}, \lambda_i^{(1)})\}, \{(\tau_i^{(2)}, \lambda_i^{(2)})\}$ are the knots and the coefficients of the univariate Gaussian rule w.r.t. the weight functions w_1, w_2, respectively. We recall that $e_m(g) = 0$ for any $g \in \mathbb{P}_{2m-1}$.

Define a discrete operator $\mathbf{H}_m$ as

$$(\mathbf{H}_m F)(\mathbf{v}) = \nu \sum_{i=1}^{m}\sum_{j=1}^{m} \Lambda_{i,j} \mathbf{H}(\mathbf{v}, \mathbf{u}_{i,j}) F(\mathbf{u}_{i,j}), \quad \mathbf{v} \in \mathrm{Q},$$

and consider the following discrete operator equations

$$(I - \mathbf{H}_m)F_m = G, \quad m > m_0$$

where F_m is the unknown. Multiplying this equation by $\sigma(\mathbf{v})$ and collocating it on the grid points $\mathbf{u}_{h,\ell}, \quad h, \ell = 1, 2, \dots, m$, we have that $a_{h\ell} = F_m(\mathbf{u}_{h,\ell})\sigma(\mathbf{u}_{h,\ell})$ are the unknowns of the linear system

$$a_{h\ell} - \nu\sigma(\mathbf{u}_{h,\ell}) \sum_{i=1}^{m}\sum_{j=1}^{m} \frac{\Lambda_{i,j}}{\sigma(\mathbf{u}_{i,j})} \mathbf{H}(\mathbf{u}_{h,\ell}, \mathbf{u}_{i,j}) a_{i,j} = G(\mathbf{u}_{h,\ell})\sigma(\mathbf{u}_{h,\ell}), \quad h, \ell = 1, \dots, m. \tag{3.8}$$

If we denote by $\mathbf{a}$ the m^2-length vector obtained reordering row-by-row the matrix
$$\{a_{h,\ell}, \quad h = 1, 2, \ldots, m, \ell = 1, 2, \ldots, m\},$$
the vector $\mathbf{a}$ represents the unknown vector of the m^2 size linear system

$$\mathbf{B_m a = g},$$

where the m-blocks matrix $\mathbf{B_m}$ takes the following expression

$$\mathbf{B_m} = \begin{bmatrix} B^{(1,1)} & B^{(1,2)} & \ldots & B^{(1,m)} \\ B^{(2,1)} & B^{(2,2)} & & B^{(2,m)} \\ \vdots & & & \\ B^{(m,1)} & B^{(m,2)} & & B^{(m,m)} \end{bmatrix},$$

$$B^{(h,\ell)} = \delta_{h,\ell} I_m - \mu D_h \widetilde{H}^{(h,\ell)} \Lambda_\ell,$$

$$D_h = \text{diag}\left(\sigma(\mathbf{v}_{h,1}), \sigma(\mathbf{v}_{h,2}), \ldots, \sigma(\mathbf{v}_{h,m})\right),$$

$$\Lambda_\ell = \text{diag}\left(\frac{\Lambda_{\ell,1}}{\sigma(\mathbf{u}_{\ell,1})}, \frac{\Lambda_{\ell,2}}{\sigma(\mathbf{u}_{\ell,2})}, \ldots, \frac{\Lambda_{\ell,m}}{\sigma(\mathbf{u}_{\ell,m})}\right),$$

I_m is the identity matrix of order m and $\widetilde{H}^{(h,\ell)}$ is defined as

$$\widetilde{H}^{(h,\ell)}(i,j) = \mathbf{H}(\mathbf{v}_{h,\ell}, \mathbf{u}_{i,j}), \quad i = 1, 2, \ldots, m; \; j = 1, 2, \ldots, m$$

Setting

$$\mathbf{D} = \text{diag}(D_1, D_2, \ldots, D_m) \quad \mathbf{\Lambda} = \text{diag}(\Lambda_1, \Lambda_2, \ldots, \Lambda_m)$$

$$\widetilde{\mathbf{H}}_\mathbf{m} = \begin{bmatrix} \widetilde{H}^{(1,1)} & \widetilde{H}^{(1,2)} & \ldots & \widetilde{H}^{(1,m)} \\ \widetilde{H}^{(2,1)} & \widetilde{H}^{(2,2)} & & \widetilde{H}^{(2,m)} \\ \vdots & & & \\ \widetilde{H}^{(m,1)} & \widetilde{H}^{(m,2)} & & \widetilde{H}^{(m,m)} \end{bmatrix},$$

the matrix $\mathbf{B_m}$ takes the compact expression

$$\mathbf{B_m} = I_{m^2} - \mu \mathbf{D} \widetilde{\mathbf{H}}_\mathbf{m} \mathbf{\Lambda}.$$

By the solution $\{a^*_{h,\ell}\}_{h,\ell=1,2,\ldots,m}$ (if it exists), we are able to construct the weighted Nyström interpolant of F

$$F_m(\mathbf{v})\sigma(\mathbf{v}) = \nu \sum_{i=1}^{m}\sum_{j=1}^{m} \sigma(\mathbf{v}) \frac{\Lambda_{i,j}}{\sigma(\mathbf{u}_{i,j})} \mathbf{H}(\mathbf{v}, \mathbf{u}_{i,j}) a_{i,j}^{*} + G(\mathbf{v})\sigma(\mathbf{v})$$

that now we go to compare with the exact solution $F \in C_\sigma$.

4 Stability and convergence analysis

In this section we prove the stability and the convergence of the proposed method, i.e., that the system (3.8) is unisolvent and well-conditioned and that the Nyström interpolant converges to the exact solution in the space C_σ. To this end, the following auxiliary result is crucial.

Lemma 4.1 *Assume (3.6) and that*

$$(\forall \mathbf{v} \in \mathrm{Q}) \quad \sigma(\mathbf{v})\mathbf{h}_{\mathbf{v}} \in C_\sigma \quad \textit{and} \quad (\forall \mathbf{u} \in \mathrm{Q}) \quad \sigma(\mathbf{u})\mathbf{h}_{\mathbf{u}} \in C_\sigma. \tag{4.1}$$

Then, for any $F \in C_\sigma$, it holds true

$$\lim_{m} \|\sigma[\mathbf{H}_m(F) - \mathbf{H}(F)]\|_\infty = 0. \tag{4.2}$$

Moreover, the sequence $\{\mathbf{H}_m\}_m$ is collectively compact.

Proof. First we prove (4.2). For any fixed $\mathbf{v}$ and for any polynomial $P \in \mathbb{P}_{2m-1}$,

$$\begin{aligned}\sigma(\mathbf{v})|(\mathbf{H}_m F)(\mathbf{v}) - (\mathbf{H}F)(\mathbf{v})| &\le \left|\sigma(\mathbf{v}) \int_{\mathrm{Q}} [(\mathbf{h}_{\mathbf{v}}F)(\mathbf{u}) - P(\mathbf{u})]\mathbf{w}(\mathbf{u}) d\mathbf{u}\right| \\ &\quad + \left|\sigma(\mathbf{v}) \sum_{i=1}^{m}\sum_{j=1}^{m} \Lambda_{i,j} [\mathbf{h}_{\mathbf{v}}F - P](\mathbf{u}_{i,j})\right| \\ &=: A + B \end{aligned} \tag{4.3}$$

We have

$$A \le \mathcal{C}\sigma(\mathbf{v}) \|(\mathbf{h}_{\mathbf{v}}F - P)\sigma^2\| \int_{\mathrm{Q}} \frac{\mathbf{w}(\mathbf{u})}{\sigma^2(\mathbf{u})} d\mathbf{u} \le \mathcal{C} \sup_{\mathbf{v}\in\mathrm{Q}} \sigma(\mathbf{v}) E_{2m-1}(\mathbf{h}_{\mathbf{v}}F)_{\sigma^2}$$

and

$$B \le \mathcal{C}\sigma(\mathbf{v}) \|(\mathbf{h}_{\mathbf{v}}F - P)\sigma^2\| \sum_{i=1}^{m}\sum_{j=1}^{m} \frac{\Lambda_{i,j}}{\sigma^2(\mathbf{u}_{i,j})} \le \mathcal{C} \sup_{\mathbf{v}\in\mathrm{Q}} \sigma(\mathbf{v}) E_{2m-1}(\mathbf{h}_{\mathbf{v}}F)_{\sigma^2}.$$

Since

$$E_{2m-1}(\mathbf{h_v}F)_{\sigma^2} \leq \mathcal{C}[\|\sigma\mathbf{h_v}\|E_{m-1}(F)_\sigma + \|F\sigma\|E_{m-1}(\mathbf{h_v})_\sigma],$$

we deduce the following estimate

$$\|\sigma(\mathbf{H}_mF-\mathbf{H}F)\|_\infty \leq \mathcal{C}\left[E_{m-1}(F)_\sigma \sup_{\mathbf{v}\in \mathrm{Q}} \sigma(\mathbf{v})\|\sigma\mathbf{h_v}\| + \|F\sigma\| \sup_{\mathbf{v}\in \mathrm{Q}} \sigma(\mathbf{v})E_{m-1}(\mathbf{h_v})_\sigma\right]$$

and therefore (4.2) easily follows. Now, we prove the collective compactness of the sequence $\{\mathbf{H}_m\}_m$. To this end, it is sufficient to prove the following limit relation

$$\lim_N \sup_m \sup_{F\in C_\sigma} E_N(\mathbf{H}_mf)_\sigma = 0,$$

i.e., $\mathbf{H}_mF \in C_\sigma$ uniformly on F and m. Recalling the definition of $\mathcal{C}_\sigma$, we will prove that

(1) $\mathbf{H}_mF$ is continuous in any closed subset $A \subset \mathrm{Q}$,
(2) $\lim_{\mathbf{v}\to\partial\mathrm{Q}} \sigma(\mathbf{v})(\mathbf{H}_mF)(\mathbf{v}) = 0$

uniformly on F and m. To prove the continuity of $\mathbf{H}_mF$, following the same arguments used in the proof of the Proposition 3.1, we have

$$|(\mathbf{H}_mF)(\mathbf{v}+\delta)-(\mathbf{H}_mF)(\mathbf{v})| \leq |\nu|\|F\sigma\| \sup_{\mathbf{u}\in Q} |\mathbf{h_u}(\mathbf{v}+\delta) - \mathbf{h_u}(\mathbf{v})|\,\sigma(\mathbf{u}) \sum_{i=1}^m\sum_{j=1}^m \frac{\Lambda_{i,j}}{\sigma^2(\mathbf{u}_{i,j})}$$

and using

$$\sum_{i=1}^m\sum_{j=1}^m \frac{\Lambda_{i,j}}{\sigma^2(\mathbf{u}_{i,j})} \leq \mathcal{C}\int_Q \frac{\mathbf{w}(\mathbf{u})}{\sigma^2(\mathbf{u})}d\mathbf{u}$$

taking into account the assumption (3.6) and the second condition in (4.1), relation 1) follows. Moreover,

$$|\sigma(\mathbf{v})(\mathbf{H}_mF)(\mathbf{v})| \leq \nu\|F\sigma\| \sup_{\mathbf{u}\in \mathrm{Q}} \sigma(\mathbf{u})|\mathbf{h_u}(\mathbf{v})\sigma(\mathbf{v}) \sum_{i=1}^m\sum_{j=1}^m \frac{\Lambda_{i,j}}{\sigma^2(\mathbf{u}_{i,j})}$$

and 2) follows in view of (3.6) and the second condition in (4.1) again. □

Now, we are able to claim the following

Theorem 4.2 *Assume that the equation* (3.4) *have only one solution in* C_σ *with* σ *satisfying the assumption* (3.6). *If the kernel* h *verifies the hypotheses* (4.1) *and* $G \in C_\sigma$ *then the system (3.8) is unisolvent and well-conditioned and the Nyström interpolant* F_m *converges in* C_σ *to the exact solution* F. *Moreover, the following error estimate holds*

$$\|(F-F_m)\sigma\|_\infty \leq \mathcal{C}\left[\sup_{\mathbf{v}\in \mathrm{Q}} \|\sigma\mathbf{h_v}\|E_{m-1}(F)_\sigma + \|F\sigma\| \sup_{\mathbf{v}\in \mathrm{Q}} \sigma(\mathbf{v})E_{m-1}(\mathbf{h_v})_\sigma\right] \quad (4.4)$$

Proof. The theorem easily follows by the Lemma, which assures the strong convergence and the collective compactness of $\{\mathbf{H}\}_m$. Then, following standard arguments, it follows the uniqueness of the linear system solution and its well conditioning. Finally, the estimate (4.4) follows by

$$\|(F - F_m)\sigma\|_\infty \leq \|\sigma(\mathbf{H}_m F - \mathbf{H}F)\|_\infty$$

where the right hand side is estimated in the Lemma. $\square$

Remark 4.3 By the proposed procedure, it can be approximated the solutions F having (algebraic) singularities on the border of the rectangle Q, corresponding to functions f with singularities on the border of the triangle T. For instance, in the case $b > 0$, f can be discontinuous on the side (see Figure 2.2) $\overline{GF}$, while in the case $a > 0$ and $p = q = 1$, it can be singular on the open sides $\overline{AC}$ and/or $\overline{AB}$.

5 Numerical tests

Now we show the performance of our method by some numerical examples. Let m be the number of the Gaussian knots. The m^2- square linear systems were solved by using Gaussian elimination with partial pivoting. In the next table we set the maximum relative and absolute error attained in the grid of equally spaced points $[0.1\ 0.2\ 0.3\ 0.4\ 0.9\ 1] \times [0\ 0.1\ 0.2\ 0.3\ 0.4\ 0.5]$ About the system of linear equations, we observe that in the general case the coefficient matrix $\mathbf{B_m} \in \mathbb{R}^{m^2 \times m^2}$ is dense and unstructured. So to solve the system we used the Gaussian elimination with partial pivoting. As its is known the usual algorithm of the Gaussian elimination based on a triple loop procedure involves $m^6/3$ flops [7]. Therefore by this way also for small value of m, the required CPU time can be prohibitive in a sequential implementation. To improve computer times, parallel algorithms running on multi-core processors can be successfully used. As an example, if we implement the *block right-looking algorithm* using the Java package JAMA for basic matrix computations, and the Java message passing library MPJ, the time is reduced of almost 40% in a dual-core architecture with two concurrent processes. For instance for $m^2 = 3600$, letting b the block-size, we have the following CPU times in milliseconds

Table 3

Parallel			Sequential		
$b = 120$	$b = 180$	$b = 300$	$b = 120$	$b = 180$	$b = 300$
53880	49795	50837	88401	78647	74466

We remark that in any example the condition $\|\mathbf{H}\|_{C_\sigma \to C_\sigma} < 1$ is satisfied and therefore, any of the following equation admit an unique solution.

Example 1 As a first example we consider the following equation

$$f(y_1,y_2)-\int_{\mathrm{T}} e^{-(x_1+x_2+y_1+y_2)} f(x_1,x_2)(1-x_1-x_2)dx_1dx_2 = y_1^2+y_2^2-\frac{1}{3}\frac{(98-36e)}{e}e^{-(y_1+y_2)},$$

with the exact solution $f(y_1,y_2)=y_1^2+y_2^2$. The kernel and the parameters are

$$\mu=1, \quad K(x_1,x_2,y_1,y_2)=e^{-(x_1+x_2+y_1+y_2)}, \quad p=q=1, \quad a=0, \quad b=1.$$

By (3.2)–(3.3), we obtain

$$\nu=\frac{1}{16}, \quad \mathbf{w}(\mathbf{u})=v^{1,1}(u_1), \quad \mathbf{h}(\mathbf{u},\mathbf{v})=e^{-\frac{2+u_1+v_1}{2}},$$

$$G(\mathbf{v})=\frac{1}{8}(1+v_1)^2(1+v_2^2)-\frac{1}{3}\frac{(98-36e)}{e}e^{-\frac{1+v_1}{2}}$$

We look for the solution $f\in C_\sigma$, with $\sigma(u_1,u_2)=1$ and, according to the theoretical results, we expect a very fast convergence. The numerical results are as follows:

m	Absolute Error	Relative Error
2	$0.59e-4$	$0.29e-2$
4	$0.73e-9$	$0.37e-7$
8	$0.78e-15$	$0.39e-13$

Example 2 Let

$$f(y_1,y_2)-\int_{\mathrm{T}}[(y_1-x_1)^2+(y_2-x_2)^2]^\alpha f(x_1,x_2)x_1x_2\ dx_1\ dx_2=\sin\left(\frac{\pi}{2}(y_1+y_2)\right),$$

$$\mu=1, \quad K(x_1,x_2,y_1,y_2)=[(y_1-x_1)^2+(y_2-x_2)^2]^\alpha, \quad \alpha>0, \quad p=q=2, \quad a=b=0.$$

By (3.2)–(3.3), we obtain

$$\nu=\frac{1}{2^7}, \quad \mathbf{w}(\mathbf{u})=v^{0,3}(u_1)v^{1,1}(u_2), \quad G(\mathbf{v})=\sin\left(\frac{\pi(1+v_1)}{8}\right),$$

$$\mathbf{h}(\mathbf{u},\mathbf{v})=\left[\frac{((1+u_1)(1+u_2)-(1+v_1)(1+v_2))^2+((1+u_1)(1-u_2)-(1+v_1)(1-v_2))^2}{16}\right]^\alpha.$$

We have chosen two different values of the parameters α, corresponding to different smoothness of the kernel.

- In the case $\alpha=7/2$, in view of Proposition 3.1, the solution uniquely exists in the space C_σ, choosing $\sigma(u_1,u_2)=v^{0,\frac{1}{2}}(u_1)v^{\frac{1}{2},\frac{1}{2}}(u_2)$. By Theorem 4.2 and by (2.3) with $r=7$, neglecting the constant, we expect the machine precision with $m\geq 150$, while the numerical tests show that it is attained for $m=60$.

m	Absolute error	Relative error
2	$0.32e-2$	$0.33e-2$
4	$0.39e-5$	$0.41e-5$
8	$0.20e-8$	$0.20e-8$
16	$0.77e-11$	$0.78e-11$
30	$0.59e-13$	$0.62e-13$
60	$0.53e-15$	$0.55e-15$

- Consider now the case $\alpha = 3/2$. Let be $\sigma(u_1, u_2) = v^{0,\frac{1}{2}}(u_1)v^{\frac{1}{2},\frac{1}{2}}(u_2)$. By Theorem 4.2 and by (2.3) with $r = 3$, for $m = 40$ we can rely on $4-5$ exact digits, while the test evidences 9 significant digits.

m	Absolute Error	Relative Error
2	$0.28e-3$	$0.42e-3$
4	$0.12e-4$	$0.13e-4$
8	$0.84e-6$	$0.85e-6$
16	$0.45e-7$	$0.49e-7$
30	$0.30e-8$	$0.31e-8$
40	$0.85e-9$	$0.87e-9$

Example 3 Let

$$f(y_1, y_2) - \frac{\sqrt{2}}{5} \int_{\mathrm{T}} \left(\left| \sin\left(\frac{3 - 4y_1 - 4y_2}{2} \right) \right|^{3.2} (2x_1 + 2x_2 - 1) + \left| \cos\left(\frac{2y_2}{y_1 + y_2} \right) \right|^{3.2} \right.$$

$$\left. \times \frac{x_1 - x_2}{x_1 + x_2} \right) f(x_1, x_2) \frac{\sqrt{x_1 x_2}(1 - x_1 - x_2)^{-1/2}}{(x_1 + x_2)^2} \, dx_1 \, dx_2 = e^{(1+y_1)(1+y_2)}.$$

By (3.2)–(3.3), we obtain

$$\nu = \frac{1}{10}, \quad \mathbf{w}(\mathbf{u}) = v^{-\frac{1}{2},0}(u_1)v^{\frac{1}{2},\frac{1}{2}}(u_2), \quad G(\mathbf{v}) = e^{(1+v_1)(1+v_2)}$$

$$\mathbf{h}(\mathbf{u}, \mathbf{v}) = \left| \sin\left(\frac{1 - v_1}{2} \right) \right|^{3.2} u_1 + |\cos(1 + v_2)|^{3.2} \, u_2.$$

Let C_σ, with $\sigma(u_1, u_2) = v^{0,1/4}(u_1)v^{1/2,1/2}(u_2)$. The numerical results are as follows:

m	Absolute Error	Relative Error
4	0.17×10^{-4}	0.28×10^{-4}
8	0.56×10^{-6}	0.47×10^{-6}
16	0.23×10^{-7}	0.16×10^{-7}
32	0.32×10^{-8}	0.31×10^{-8}
50	0.44×10^{-9}	0.34×10^{-9}

We remark that with $m = 50$ we obtain 9 exact digits, while the theoretical estimate assures us only 5 correct digits.

Example 4 Finally, we consider the equation

$$f(y_1, y_2) - \int_{\mathrm{T}} e^{(x_1+x_2)(y_1+y_2)-x_1x_2} f(x_1, x_2)\ dx_1\ dx_2 = y_1^2 + y_2^2,$$

$$\mu = 1, \quad K(x_1, x_2, y_1, y_2) = e^{(x_1+x_2)(y_1+y_2)-x_1x_2}, \quad p = q = 1, \quad a = b = 0.$$

Here the exact solution is $f(y_1, y_2) = y_1^2 + y_2^2$.
By (3.2)–(3.3), we obtain

$$\nu = \frac{1}{8}, \quad \mathbf{w}(\mathbf{u}) = v^{0,1}(u_1), \quad \mathbf{h}(\mathbf{u}, \mathbf{v}) = \frac{4(1+u_1)(1+v_1) - (1+u_1)^2(1-u_2^2)}{16},$$

$$G(\mathbf{v}) = \frac{(1+v_1)^2(1+v_2^2)}{8}$$

In this case the kernel is very smooth and we can choose $\sigma(u_1, u_2) = 1$.

m	Absolute Error	Relative error
2	$0.21e-1$	$0.15e-1$
4	$0.78e-5$	$0.57e-5$
8	$0.76e-13$	$0.56e-13$
16	$0.40e-14$	$0.23e-14$

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