Effect of subtangential parameter on the stability and dynamic of a cantilever tapered beams subjected to followed forces

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Abstract— **This paper deals with dynamic stability of cantilever column subjected to the action of subtangential forces. The eigenfrequency equation I obtained by the generalized Rayleigh-Ritz method. A procedure is illustrated in order to study cantilever beam with variable cross-sections subjected to sub-follower forces. The analysis is based on a variational approach with Orthogonal Polynomials are chosen as trial functions polynomial trial functions. The instability regions by divergence and flutter are examined in some detail, so allowing to determine the range of applicability of the static instability criterion. In the case of applying a follower force, the static criterion is no longer valid and is necessary to use the dynamic criterion. Numerical results are tabulated for different tapered beam whit various subtangential parameters and are compared with other classical results from the bibliography, so confirming the goodness of the proposed approach.**

Keywords; tapered beams, followed forces, subtangential parameter.

I. INTRODUCTION

Nonconservative loads are present in basically every engineering field such as bioengineering (spinal cord) and civil engineering (bridges and columns) among others. Loads experienced by aircraft wings may also become partially nonconservative as their weight is a dead-load weight but the pressure exerted on the wing is a follower load.

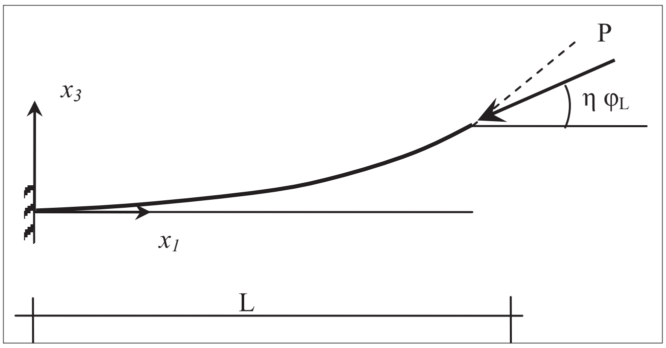

Figure 1. Cantilever beam with sub-tangential force; η=1 Beck's column.

As well known, the instability phenomena of various engineering structures depend on the boundary conditions. Moreover, it is important to consider if the applied forces admit or not a potential function, because some kind of nonconservative follower forces lead to structural behaviour which cannot be examined by using the classical static criterion. Therefore, in the presence of follower forces it is mandatory to use the dynamic criterion in order to predict the correct critical load multiplier. [1-4]

Recently, De Rosa et al. [5] and Marzani et al. [13], study the instability behaviour of beams with variable cross section subjected to sub-tangential nonconservative follower forces, and the solution is numerically attained by using a Differential Quadrature Method (DQM) procedure.

An approximed, yet general solution is proposed by Glabisz [6] for a generic structural element subjected to distributed follower forces (Leipholz) or to a concentrated tip force (Beck [6]). The problem is approximated using a power series approach, and the dynamic loss-of-stability is deduced for beams on elastic soil and for various boundary conditions.

In this paper the authors give a variational *Rayleigh-Ritz* solution to the instability problem of beams with variable cross-section subjected to subtangential follower forces. If the rotations at the ends of the beam are denoted by $\varphi_{L,}$ and φ_R , then the nonconservative applied forces will considered to be functions of $\eta\varphi_L$ and $\eta\varphi_R$ respectively. The η parameter completely defines the system of subtangential concentrated follower forces at the ends, and, consequently, completely defines the dynamic behaviour of the system.

The classical conservative Euler case is recovered at η=0, whereas if η=1 the beam is subjected to purely tangential forces (Beck problem). As η varies in the range [0,1] critical loads are reached by means of divergence or flutter, so permitting to deduce the range of applicability of the static criterion; Fig.1. Usually, a precise parameter value $\eta=\eta_c$ exists, which separate the divergence region from the flutter region, and this parameter strongly influences the frequency-force relationship. As well known, in the case of a cantilever beam with constant cross section it is $\eta_c = 0.5$; [7-12].

II. Formulation of the Problem

A. Hamilton principle

One of the most effective methods to derive the governing equations is the Energy Method. Having all of the energies in the system and applying the Hamilton Principle, the governing equations could be derived accurately:

$$\int_{t_1}^{t_2} \delta(T-U+W_c)\,dt + \int_{t_1}^{t_2} \delta W_{nc}\,dt = 0 \qquad (1)$$

where δ is the variation, t is time, T is the kinetic energy, U is the elastic energy, W_c is the work done by conservative forces and W_{nc} is the work done by non-conservative forces.

For small deflection, the slope at the free end of the rod can be approximated, moreover, cosφ≈1, and sin φ≈φ.

If the oscillations are supposed to be small, the kinetic energy, elastic energy and the work by conservative forces of the system can be written as:

$$T_2 = \frac{1}{2}\int_0^L \rho\omega^2 A(x_1) u_3{}^2 dx_1. \qquad (2)$$

$$U = \frac{1}{2}\int_0^L EI(x_1)\, u_{3,11}^2\, dx_1; \qquad W_c = \frac{1}{2}\int_0^L P u_{3,1}^2\, dx_1 \qquad (3)$$

where L, A ,ρ, EI, P and ω are the length, cross-sectional area, the mass per unit length, bending stiffness, axial force of the beam and the circular frequency.

The other, non-conservative part of the applied loads gives rise to the following virtual work:

$$\delta W_{nc} = P\sin\eta[u_{3,1}]_0^L \cong P\eta[u_{3,1}]_0^L = \\ = P\eta\left[u_{3,1}\right]_0 \delta u_3 - P\eta\left[u_{3,1}\right]_L \delta u_3 \qquad (4)$$

Since obtained an exact analytical solution to Eq. (1) is not so easy, the present study relies on the *Rayleigh-Ritz* approximation solution (R-R).

According to the R-R approximation method, the solution will be expressed as a linear combination of independent functions, and it is convenient to express the displacements of the beams as a sequence of orthogonal polynomials.

It will be:

$$u_n = q_i f_i = \mathbf{f}^T\mathbf{q} \qquad i = 1,..n \qquad (5)$$

where $\mathbf{f}$ is the eigenfunction vector, and $\mathbf{q}$ contains the lagrangian coordinates. The eigenfunctions f_i should only satisfy the essential boundary conditions, and they can be deduced following an iterative method based on the ortho-normalization method. In general the admissible functions f_i should satisfy the conditions: at least must satisfy all geometric boundary conditions; must be continuous and differentiable to the highest spatial derivative; should be a complete function; must be linearly independent.

Put $\xi = x_1/L$, the choice of trial functions $f_i \in L_2[0,1]$ is of paramount importance. In this paper they are deduced starting from a polynomial:

$$f_1(\xi) = \sum_{j=0}^{r} a_j \xi_1^j, \qquad (6)$$

$$f_h(\xi) = (\xi - B_h) f_{h-1} - C_h f_{h-2}, \quad h>1 \;\forall\; \xi\in[0,1], \qquad (7)$$

where

$$B_h = \frac{\int_0^1 \xi f_{h-1}^2\, d\xi}{\int_0^1 f_{h-1}^2\, d\xi}, \qquad C_h = \frac{\int_0^1 \xi f_{h-1} f_{h-2}\, d\xi}{\int_0^1 f_{h-1}^2\, d\xi}. \qquad (8)$$

For simplicity the following dimensionless variable are introduced:

$$A(\xi) = A_0 G(\xi), \qquad I(\xi) = I_0 H(\xi), \\ \bar{\Omega}^2 = \omega^2 \frac{\rho A_0 L^4}{E I_0}, \qquad \bar{p} = \frac{P L^2}{E I_0}, \qquad (9)$$

where A_0 and I_0 area the cross sectional area and moment of inertia at $\xi=0$, respectively.

By substitution of equations (2-4) and (5) into equation (1), the characteristic equation is finally obtained in the matrix form:

$$[\mathbf{K} + \bar{p}\,(\mathbf{B} - \eta\,\mathbf{W}) - \bar{\Omega}^2\mathbf{M}]\,\mathbf{q} = \mathbf{0}, \qquad (10)$$

with

$$K_{ij} = \int_0^1 H(\xi) f_{i,\xi\xi} f_{j,\xi\xi}\, d\xi, \qquad B_{ij} = \int_0^1 f_{i,\xi} f_{j,\xi}\, d\xi,$$

$$M_{ij} = \int_0^1 G(\xi) f_i f_j\, d\xi, \qquad W_{ij} = \left[f_i f_j\right]_{\xi=0} + \left[f_i f_j\right]_{\xi=1}$$

(11)

where M is a symmetric mass matrix, K is a symmetric matrix elastic stiffness, B is the geometric symmetric matrix due to an axial force and W is a nonsymmetric matrix due a non-conservative forces.

The free vibration frequencies can be calculated by imposing:

$$\det[\mathbf{K}+\overline{p}(\mathbf{B}-\eta\,\mathbf{W})-\overline{\Omega}^2\,\mathbf{M}]=0, \qquad (12)$$

From a computational point of view, the presence of unsymmetrical matrices leads to complex conjugate solutions, and an iterative approach seems to be the simplest solution algorithm.

The stability of the system under consideration is determined by the sign of real part σϖof the complex eigenvalue $\overline{\Omega}=\sigma\pm i\,\omega\ \ (i=\sqrt{-1})$. If σ < 0 ,the system is stable; if σ >0 and ω= 0 , the system is statically unstable, i.e., divergence type instability; if σ > 0 and ω≠0 , the system is dynamically unstable, i.e., flutter type instability; if σ = 0 , the critical distributed follower force p_c arises.

Two different cases can be faced, according to the η value. If $\eta<\eta_c$ the normalized critical load p_c corresponds to $\Omega_1=0$, and it can be deduced using the static criterion.

The condition:

$$\det[\mathbf{K}+\overline{p}(\mathbf{B}-\eta\,\mathbf{W})]=0, \qquad (13)$$

gives the solutions p_i and the critical load is $p_c=p_1$.

As η increases, a threshold value η_c is reached, beyond which the structure loses stability by flutter, and the static criterion is no longer applicable.

At $\eta>\eta_c$ the solutions p_i of equation (12) turn out to be complex, and the critical load must be calculated using eqn. (12), corresponding to the coalescence of the first two free vibration frequencies.

III. Results and Discussion

In order to illustrate some numerical examples and comparisons with other known results, let us consider now a cantilever beam with varying cross section, in which area and moment of inertia of the cross section obey to the following laws:

$$G(\xi)=\left(1+\alpha\,\xi\right)^2, \qquad H(\xi)=\left(1+\alpha\,\xi\right)^4 \qquad (14)$$

For a circular cross section, it will be:

$$I_0=a^4\frac{\pi}{4}, \qquad A_0=a^2\pi\,, \qquad (15)$$

where a is the radius of the section at $x_1=0$, and therefore:

$$\Omega_i^2=\frac{\overline{\Omega}^2}{\pi^2}, \qquad p=\frac{\overline{p}}{\pi^2} \qquad (16)$$

- Uniform cantilever beam

As a first example, the column with constant cross section has been studied, subjected to a sub-tangential load, α=0;

$$G(\xi)=1, \qquad H(\xi)=1\,. \qquad (17)$$

Using eqn. (13) the critical load is given, for different parameter value $\eta<\eta_c$. At η=0 we have two different value p_1 and p_2, whereas the difference p_1-p_2 diminishes with increasing η, and at $\eta=\eta_{cr}=0.5$ the two values coalesce.

To verify the accuracy of the numerical calculation applied, the critical force obtained for a specific parameter η of the present paper was compared with the values reported by reference.

In Table I, the critical loads p_c for $\eta<\eta_c$ are presented and compared with the results by Chen [11]. For η > 0.5 the critical load must be calculated using eqn. (12), corresponding to the coalescence of the first two free vibration frequencies. For the values of nonconservative parameter η=1, Beck column, the critical force is p_c*=2.0315*.

TABLE I. Dependence of critical force (α=0), η<0,5.

α=0	Chen [11]	Present	
η	p_1	p_1	p_2
0	0,240	0,2499	2,2499
0,20	0,337	0,3369	2,0151
0,30	------	0,4109	1,8469
0,3543	------	0,4649	1,7236
0,40	------	0,5362	1,6071
0,45	------	0,6519	1,4279
0,48	------	0,7644	1,2671
0,49	0,829	0,8291	1,1868
0,50	1	1	1

In Table II, are given the critical loads for $\eta>\eta_c$. The beam have two divergence instability forces for η≤0.3545 (DS) and divergence and flutter instability forces for 0.3543≤η≤0.5 (DFS), and only one flutter instability force for η > 0.5. At this parameter value $\eta_c=0.5$, for η>0.5 the type of instability of the column is pure flutter (FS); Fig. 2.

TABLE II. Dependence of critical force (α=0), η>0,5.

α=0	Chen [11]		Present	
η	p_C	$\Omega_1=\Omega_2$	p_C	$\Omega_1=\Omega_2$
0,51	1,627	0,732	1,6267	0,7315
0,52	-----	------	1,6274	0,7456
0,55	1,632	0,788	1,6321	0,7876
0,60	-----	-----	1,6473	0,8445
0,70	-----	-----	1,7009	0,9359
0,80	1,782	1,009	1,7815	1,0085
1,00	2,032	1,118	2,0315	1,1161
1,5	-----	-----	3,1033	1,2215
2	-----	-----	3,8272	1,1855

The comparison of the results has shown a good agreement. The variations of the first two eigenfrequency parameters Ω_1, Ω_2, with the force parameter, *p*, for nonconservativeness η=0.5, 0.6, 1, 1.5 and 2 are shown in Figure 3. When η<0.5, the first Ω_1=0 at the critical load, in which case the divergence instability occurs. For η>0.5, when *p=0* the first two natural frequencies of the beam are Ω_1=0.2499, Ω_1=0.2499. As *p* increases, the first frequency increases monotonically, while the second one decreases. At p=pc, two frequencies coincide. Any further increase of p will yield two complex conjugate frequencies. The Figure 3 is typical for all values of η larger than 0.5.

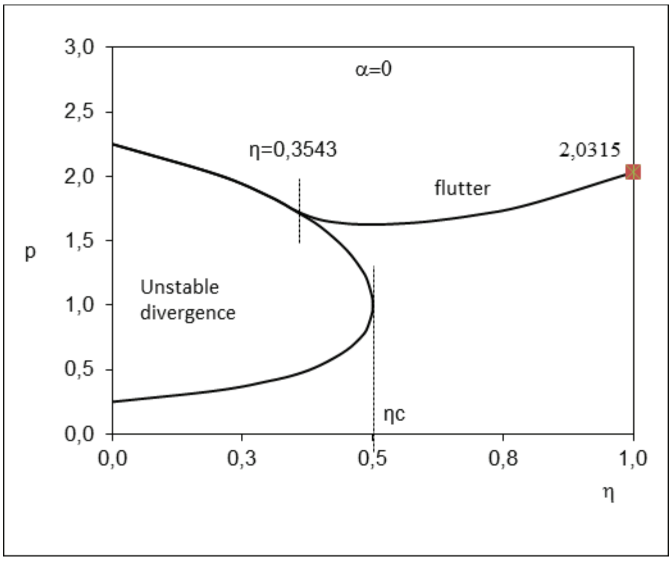

Figure 2. Effect of nonconservativeness parameter η vs *p;* uniform beam (α=0).

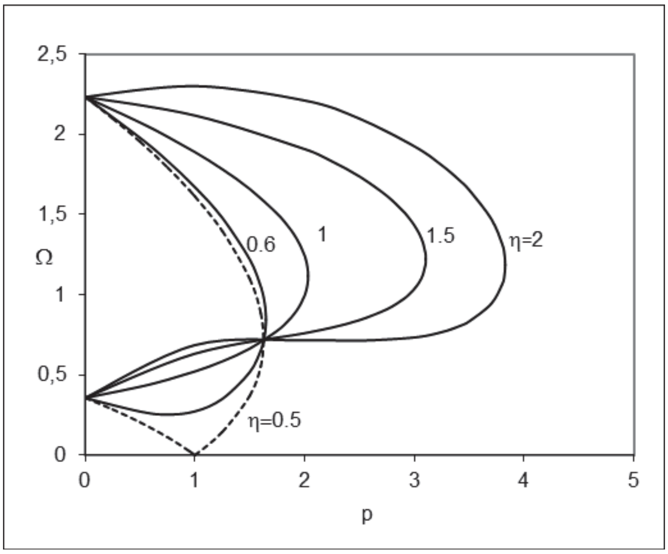

Figure 3. Non-dimensional parameters, Ω_1, Ω_2 for various follower force; uniform beam (α=0).

- *Tapered cantilever beam*

Let us consider here a linear circular tapered cantilever beam of length *L* clamped at *ξ=0*. The beam is considered subjected to a concentrated compressive sub-tangential load *p* in *x =L* Figure 1.

The cross- section of the beam varies with geometrical tapered parameter α given in (14).

For a tapered parameter α=-0.5 the beam radius at ξ =0 is twice the radius at ξ=1, the critical force are given in Table III and the results can be compared with the critical loads given by Glabisz [7] and Marzani et al. [13] . For the sake of completeness, the p_2 values are also given. A geometrical sketch of the functional relationship between axial loads *p* and subtangential parameter is given in Figure 4, as can be seen, $p_1 \rightarrow p_2$ for $\eta \rightarrow \eta_c$.

TABLE III. DEPENDENCE OF CRITICAL FORCE (α =0.5).

α=–0,5	Present		Glabisz [7]	Marzani et al [13]
η	p_1	p_2	p_1	p_1
0,00	0,1043	0,6114	-----	0,1043
0,20	0,1509	0,4960	-----	0,1509
0,30	0,2058	04007	-----	0,2058
η_c=0,3425	0,2938	0,2938	-----	-----
0,40	0,3711	-----	-----	0,3711
0,50	0,3807	-----	0,3807	-----
0,60	0,4052	-----	-----	0,4052
0,80	0,4969	-----	-----	0,4969
1	0,6592	-----	-----	0,6588
1,5	0,9487	-----	0,9484	-----
2	1,0252	-----	-----	-----

If η is higher than its critical value η_c the static criterion is no longer applicable, and the critical flutter load must be sought by applying the dynamic criterion and the complete equation (12). The critical value η_c is the threshold value between the divergence region and the flutter region.

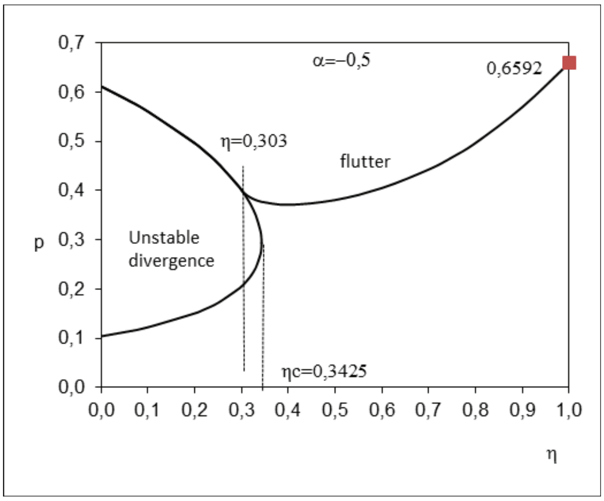

Figure 4. Effect of nonconservativeness parameter on critical force; tapered beam.(α=-0.5).

In the flutter region η≥ηc, the dynamic criterion is followed, and the axial load is found, such that the first two frequencies coalesce, $\Omega_1=\Omega_2$. The corresponding data for α=0.5, is given in the Figure 5.

In Figure 6, the eigenfrequency curves are presented for α=-0.5 and different values of the subtangential parameter, namely, η=0.3425, 0.5, 0.6, 1.0, 1.5, 2.0. While, in Figures 7 are given the eigenfrequency curves for α=0.5.

Obviously, for fixed properties of the left-hand side of the beam (ξ=0), increasing values of the tapered parameter (α) results in bigger natural frequencies as well as critical loads, as it can be seen by comparing the plot in Figures 4-7. It is interesting to note that the divergence instability range decreases for decreases tapered parameter. In fact, for α=0.5 the cantilever beam is unstable for divergence when η≤0.6014 while for α=0 (uniform beam) divergence appears only when η≤0.5.

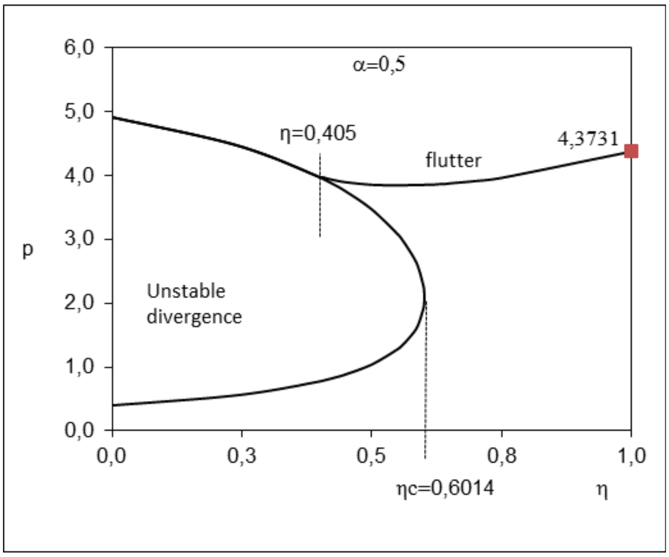

Figure 5. Effect of nonconservativeness parameter on critical force; tapered beam.(α=0.5).

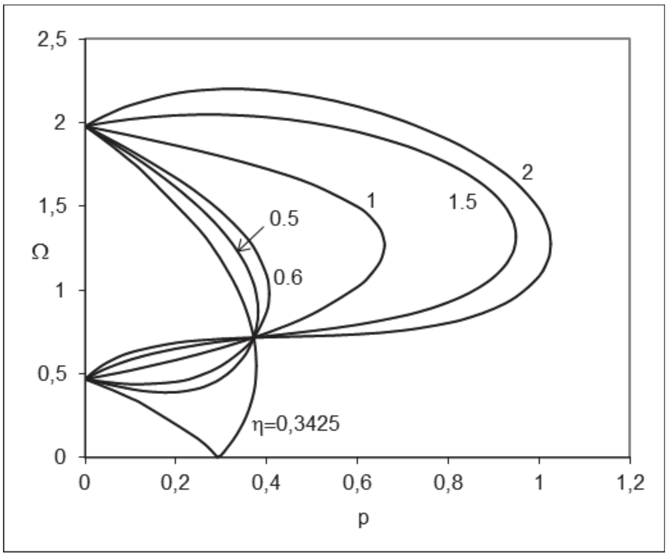

Figure 6. Non-dimensional parameters, Ω_1, Ω_2 for various follower force; (α=-0.5).

IV. COMMENTS AND CONCLUSIONS

A general approach is discussed for the analysis of tapered beams subjected to non-conservative sub-tangential loads. The analysis does not depend on the boundary conditions, and allows the determination of the critical parameters signalling the passage from divergence to flutter. The whole procedure is extremely stable, even in critical conditions. The numerical examples have been completely carried through by means of the powerful software Mathematica 6.

The proposed numerical procedure has been demonstrated to be fast, stable and accurate, presenting results in excellent agreement with the other approximate solutions.

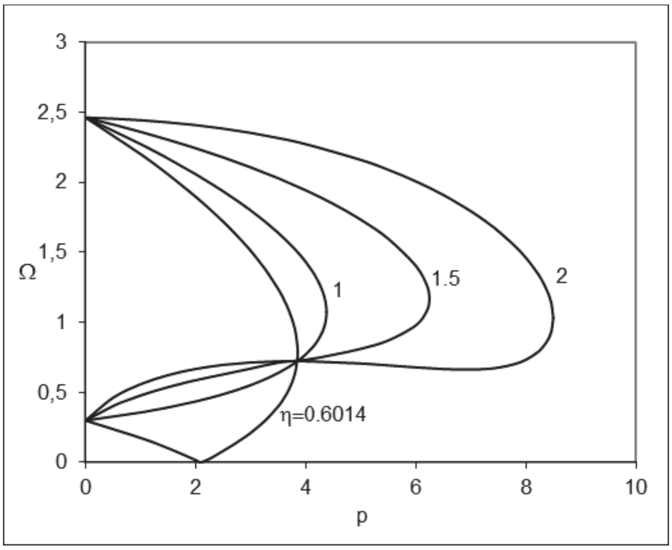

Figure 7. Non-dimensional parameters, Ω_1, Ω_2 for various follower force; (α=-0.5).

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NOMENCLATURE

a	radius of section in $x=0$
δW	virtual work
A(x); A_o	cross-sectional area of cantilever beam; cross-sectional area of in $x=0$
B; **W**	matrix in eq. (10)
E	Young's modulus of beam material
f	vector eigenfunction vector
I(x); I_o	moment of inertia; area moment of inertia in x=0
K; **M**	stiffness matrix; mass matrix
L	length of the beam
$\overline{p}$;p	dimensionless partially tangential load;, eqs. (7,14)
p_c	critical buckling load parameter
q	vector coefficients of trial function
T	kinetic energy
u_3	amplitude of the transverse deflection
V	potential energy
$\overline{\Omega}_i ; \Omega_i$	*i*th non-dimensional eigenfrequencyeq. (9)
α	thickness ratio
ξ	*x/l*; geometric parameter
η	tangential coefficient
η_c	critical tangential coefficient
ρ	mass density
ω	natural frequency, eq. (5); eq. (14)