Use of Compensation Theorem for the Robustness Assessment of Electromagnetic Devices Optimal Design

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Structured Abstract:

- Purpose:

This paper aims to propose the Compensation Theorem, well known in the analysis of linear electric networks, to compute sensitivity of the performance functions used in the robust design or tolerance analysis of electromagnetic devices.

- Design/methodology/approach:

The Compensation Theorem is first illustrated in the case of a simple field analysis problem, then, using numerical simulations performed with a finite elements code based on an integral formulation, the effectiveness of Compensation approach for assessing impact of small modification of material properties is shown.

- Findings :

The complexity of additional computations to assess the effect of small variations involved in sensitivity analysis can be reduced.

- Research limitations/implications:

The method can be applied only to linear systems; in addition, although compensation applies to any variations, the reduction of computational complexity is achieved only for small variations, giving localized effects.

- Practical implications:

The method proposed in the paper can speed up the computations of sensitivity arrays in the robust design and tolerance analysis of electromagnetic device, when numerical methods are applied.

- Originality/value:

The use of Compensation Theorem in field computations is not new, but its adoption in the sensitivity computation is new to the best knowledge of authors.

Keywords: Numerical Computation, Robust Design, Thermonuclear Fusion, Tolerance Analysis

Article Classification: Research paper

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INTRODUCTION

The tolerance analysis is a widely adopted tool in the analysis of electromagnetic devices, whose main aim is to assess the impact of some independent uncertain parameters (e.g. geometrical dimensions or materials properties) on a device performance as, for example, the electromagnetic field in a volume of interest. Tolerance analysis can be performed either at the end of the design process on the final configuration to assess the robustness of the final performance, but also during the design phase, to look for robust solutions; in the latter case the process is called "robust design" (Cioffi, 2004; Cioffi, 2006).

Tolerance analysis can be either performed using statistical approach, keeping under control average value and standard deviation of device performance function, or using worst case approaches, thus looking for satisfactory performance also in the most adverse conditions. The latter approach is particularly suited for life-support devices (e.g. pace-makers) or for very expensive devices, built in a single prototype (e.g. magnets for plasma confinement and control in Tokamak devices).

Both approaches take great advantage from the knowledge of the performance "sensitivity" with respect to design parameters variations.

In order to define the "performance sensitivity", let's consider a *performance function* $\mathcal{F}$ of the device to be designed. In general the function $\mathcal{F}$ depends on the nominal value $\underline{p}_0$ of a set of design parameters vector $\underline{p}$, and on the uncertainties $\Delta\underline{p}$ on the parameters. The performance function $\mathcal{F}$ is in general rather involved, but, for the sake of the following analysis, it is profitable to refer to its polynomial expansion with respect to design parameters:

$$\mathsf{F}(\underline{p}_0+\Delta\underline{p}) = \mathsf{F}(\underline{p}_0) + \sum_i \left.\frac{\partial \mathsf{F}}{\partial p_i}\right|_{\underline{p}_0} \Delta p_i + \frac{1}{2}\sum_{i,j} \left.\frac{\partial^2 \mathsf{F}}{\partial p_i \partial p_j}\right|_{\underline{p}_0} \Delta p_i \Delta p_j + \dots \qquad (1)$$

The function $\mathcal{F}$ is considered scalar and deterministic here. Of course, the performance function and its derivatives may well depend on other variables, such as the space coordinates **x** and the time *t*, but this is not reported in (1) for the sake of readability. The term $\mathcal{F}(\underline{p}_0)$ is the actual performance and then does not depend on tolerances, while in the first order term each summation elements depends proportionally on the uncertainty on a single variable. Further terms describe mutual interactions, with increasing order.

Advantage can be taken from the hypothesis of small relative values of tolerances. In this case it is possible to neglect higher order terms and assume that the second term in (1) is linear with respect to each uncertainty Δp_i. As a consequence, the performance variation can be expressed as:

$$\mathsf{F}(\underline{p}_0+\Delta\underline{p}) - \mathsf{F}(\underline{p}_0) \cong \sum_i \left.\frac{\partial \mathsf{F}}{\partial p_i}\right|_{\underline{p}_0} \Delta p_i \qquad (2)$$

The partial derivative of $\mathcal{F}$ with respect to p_i is called the *performance sensitivity* with respect to the i-th parameter, and represents a first order approximation to the effect of parametric variations on the device performance.

The Compensation Theorem (CT), well known in circuit theory, provides a powerful tool to compute the effect of a parametric variation in linear electric circuits. The advantage is that the source in the "compensation circuit" is localized in the perturbed branch only. The theorem can be easily extended to the static or quasi-static electromagnetic fields analysis in the linear hypothesis and, in addition, assuming that the parameter variation regards just the local conductibility and not the magnetic or dielectric properties. Thanks to its capability to effectively analyze the effect of parameter variation, the CT is well suited to perform the sensitivity analysis of linear electromagnetic devices.

In the following, in order to assess the effectiveness of the approach, an example of tolerance analysis for thermonuclear fusion machines of the Tokamak type will be presented. The magnetic field map of a Tokamak must comply with tight requirements to guarantee the requested plasma confinement and a suitable stability. The discrepancy between the nominal and the actual field map is known as "error field" (Park, 2008). The assessment of error fields impact on Tokamak performance is a very complex task; fortunately, in the case of linear behavior, the superposition of a high number of elementary sources, able to model the actual system, can be effectively applied (Knaster, 2011). On the other hand, the analysis of error fields due to currents induced in Tokamak conducting structures does require quite relevant computational burden and, although efficient formulations are able to reduce the computation burden (Albanese, 1998), the analysis of the full geometry is still beyond possibilities of present computers.

In the paper, the extension of CT to assess the impact of material properties variation in electromagnetic devices performance is discussed and, for exemplification purposes, the computation of error field in Tokamaks due to lack of periodicity in passive structures is presented.

MATHEMATICAL FORMULATION

For the sake of exposition, in the following the mathematical formulation of the method will be sketched with regards to stationary conduction case, but the kernel of the method can be extended to more general linear electromagnetic systems.

The method starts from the classical Compensation Theorem, well known in the frame of circuit theory. The CT can be applied to a wide class of linear networks. A formulation suited for the following applications, restricted to well-posed networks composed of bipolar elements, but highlighting the basic properties is reported here, using a quite classical statement in order to simplify the reading:

In a linear a-dynamic circuit composed of bipolar elements, the current or voltage variation along any branch, due to a resistance (or conductance) variation in any other branch, can be evaluated using a topologically equivalent circuit, with all original sources switched off, but with a single suitable "compensation" voltage (current) source connected in series (in parallel) to the varied resistance (conductance). The compensation source is characterized by an electromotive force $-I_0\Delta R$ *(a current amplitude* $-V_0\Delta G$*), where* I_0 *(*V_0*) is the initial current (voltage) in the varied branch, and* ΔR *(*ΔG*) is the resistance (conductance) variation.*

Limiting to variations of resistive elements only, which is suited for a number of relevant applications, quite a similar statement of the CT can be given also for linear dynamical circuits in periodic regimes (using the Steinmetz transform for any of the harmonics present in the sources periodic waveform), or for more general source waveforms.

The extension of CT to electromagnetic fields has been proposed in many areas, e.g. in Eddy Current Testing applications (see for example Morozov, 2006; Albanese, 1999 and references therein).

In this paper its use for robust design and tolerance assessment of electromagnetic devices is proposed, in particular to ease sensitivities computation.

Let's then consider a current map $\mathbf{J_0}(\mathbf{x})$ in a linear, a-dynamic, non dispersive (in space and time) domain $\mathscr{D}$, characterized by a conductivity map $\sigma(\mathbf{x})$, where $\mathbf{x}$ is the coordinate vector. $\mathbf{J}_0$ and the corresponding electric field $\mathbf{E}_0(\mathbf{x}) = \mathbf{J_0}(\mathbf{x})/\sigma(\mathbf{x})$ can be generated by possible impressed current densities or electric fields, or even by suitable boundary conditions on the domain boundary $\partial\mathscr{D}$.

Due to a change $\Delta\sigma(\mathbf{x})$ in the conductivity map, the current density distribution, and the electric field as well, will change all over $\mathscr{D}$. Such a variation $\Delta\mathbf{J}(\mathbf{x})$ can be calculated, thanks to the CT, by examining a "compensation" problem defined on $\mathscr{D}$, but with only a "compensation" impressed electric field $-\mathbf{E_0}\Delta\sigma$ and, in addition, with homogeneous boundary conditions. Of course, similar considerations apply for a resistivity map change.

In fig. 1 the main aspects of the CT are summarized: a variation is considered, making conductivity vanish in a brick-shaped region in the centre of a plate with initially uniform conductivity. The system is excited by a voltage drop on opposite sides of the plate.

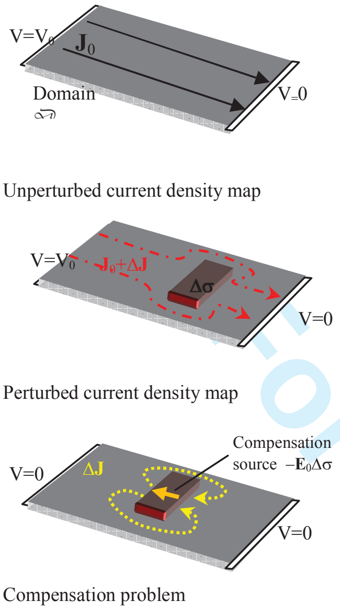


Figure 1. The CT applied to the current density map computation in a conducting plate with a brick-shaped conductivity perturbation.

A number of quite effective consequences can be drawn:

- the “compensation system” is forced just by the compensation sources (localized only in the region $\Delta\mathscr{D}_C$ where the conductivity actually changes); in addition all the boundary conditions are confirmed but with vanishing values;
- the conclusions can be applied also to any linear effect of the field, as for example, the current across an oriented surface, the voltage between any two points, the magnetic field in the surrounding space (thanks to the assumed linearity, implying of course, a linear behavior also of the magnetic materials);
- the modeling based on assumption of the conductivity (resistivity) change is able to represent also a vanishing conductivity $\sigma+\Delta\sigma$ (resistivity $\rho+\Delta\rho$) in the modified system; in this way an actual hole (vanishing conductivity) in $\mathscr{D}$ can be effectively represented in the modified system configuration.
- any changes in the system is proportional to both the initial electric field $\mathbf{E}_0$ and conductivity change $\Delta\sigma$; as a consequence, the sensitivity with respect to both conductivity changes and initial electric field $\mathbf{E}_0$ are directly available;
- also in the field case, the CT can be extended to dynamical systems.

The use of CT can simplify a number of very cumbersome, repetitive calculations as, for example, parametric analysis on the conductivity, because it just requires updating the single source in the compensation system, linearly depending on the conductivity adjustment.

Due to its characteristics, CT could represent an effective tool in the analysis of passive conductive components in nuclear devices, whose geometry is often characterized by holes due to the presence of cooling pipes, bolts, insulated wires, etc. (Fresa, 2005).

As an example of the simplifications introduced by CT, let's consider a current density map $\mathbf{J}(\mathbf{x})$, which can be regarded as a modification of a simpler map $\mathbf{J}_0(\mathbf{x})$ ($\mathbf{J}_0$ can be simpler thanks, e.g., to some symmetry, lost in the modification); and let's assume that the modification can be considered "small" (in the sense of the relative amplitude), or "local" (in the sense that it is significant only in a relatively small part of $\mathscr{D}$). Of course the application of the compensation theorem is not limited to the cases of small modification; however, in these cases, a reduced computational burden is required, since in the computation of **J** a detailed numerical analysis can be limited just to the region where significant field change is expected, while vanishing boundary conditions or coarse details can be used to reduce computational load in the remaining part.

A more general advantage of CT can be recognized when using numerical approaches (e.g. Finite or Boundary elements) for field problems. In these cases, a matrix formulation $\underline{\mathbf{A}}\underline{u}=\underline{b}$ is usually involved. With reference to the above introduced problem in the domain $\mathscr{D}$, $\underline{u}$ can be the unknowns array (e.g., the components of **J** in the nodes of a mesh), $\underline{b}$ can be the array of known terms, and $\underline{\mathbf{A}}$ is the coefficient matrix, depending on the conductivity map. When a nominal solution $\underline{\mathbf{A}}_0\,\underline{u}_0 = \underline{b}_0$ is available, a change in the conductivity produces a modification in the coefficient matrix and the numerical model becomes:

$$(\underline{\mathbf{A}}_0 + \Delta\underline{\mathbf{A}})\,(\underline{u}_0 + \Delta\underline{u}) = \underline{b}_0, \tag{3}$$

where $\Delta\underline{\mathbf{A}}$ and $\Delta\underline{u}$ are the impact of the conductivity change on the coefficient matrix and on the solution, respectively. Taking into account that $\underline{\mathbf{A}}_0\,\underline{u}_0 = \underline{b}_0$, we get from (3):

$$(\underline{\mathbf{A}}_0 + \Delta\underline{\mathbf{A}})\,\Delta\underline{u} = -\Delta\underline{\mathbf{A}}\,\underline{u}_0, \tag{4}$$

which represent the "matrix" statement of CT. Note that from (3) and (4) it emerges that the property holds for any linear system, that the relevant matrix is the "modified" matrix $\underline{\mathbf{A}}_0 + \Delta\underline{\mathbf{A}}$, that the only forcing term is the "compensation source", and finally that also boundary conditions vanish in the compensation problem.

In several cases, depending on the pattern of $\Delta\underline{\mathbf{A}}$, advantages can be taken from the knowledge of the inverse of $\underline{\mathbf{A}}_0$. This is the case, for example, of a sparse pattern with a concentration of non vanishing elements in a limited sub-matrix. In this case only the modified submatrix needs to be inverted (Woodbury Algorithm, see Albanese 1998)

Numerical solver

In the following, the application of the CT to compute the effect of conductance variations in the Vacuum Vessel (VV) of Tokamaks will be discussed. The nominal current density map $\mathbf{J}_0$ induced in the conducting VV by time varying external fields was computed using a 3D integral code (CARIDDI) designed for the solution of the Maxwell equations in the magneto-quasi-stationary limit. In the mathematical formulation (Albanese, 1998), the solenoidality of the current density field **J**, and the continuity of its normal component, are guaranteed by adopting as unknown the electric vector potential **T**, in such a way that $\nabla\times\mathbf{T} = \mathbf{J}$. The vector potential is expanded in terms of edge elements basis functions and the uniqueness of **T** is imposed directly in the discrete approximation by retaining only the degrees of freedom associated with the edges of the co-tree in a tree–co-tree decomposition of the mesh.

The method is tightly related to the classical loop currents method for the solution of the electrical networks. From a physical point of view, the unknown associated to an edge, I_k, is the line integral of **T** along the edge; consequently, the total current flowing across a single facet of the mesh is a linear combination of the unknowns associated to the edges of the facet. This formulation presents several advantages:

- only the conductive regions need to be modeled;

- boundary conditions at the interface surfaces between electrically insulating media are simply imposed by keeping out the unknowns associated to the boundary facets;
- the imposition of the current across a boundary surface is realized by prescribing the right values to the degrees of freedom associated to facets modeling the surface (Current electrodes);
- the use of the edge elements allows the automatic treatment of interfaces between media of different conductivity.

The numerical model obtained by imposing the Ohm's law in the conductive regions using a weak formulation provides a set of equations quite similar to the time domain model of LR networks:

$$\underline{\underline{L}}d\underline{I}/dt+\underline{\underline{R}}\underline{I} = \underline{V} \tag{5}$$

where:

- the elements of the "resistance" diagonal matrix are defined as:

$$R_{i,j} = \int_{V_c} \eta \mathbf{w}_i \cdot \mathbf{w}_j \, dV ,$$

  and depend on conductors geometry as well as material properties;
- the elements of the "inductance" full matrix, describing long-range relationships among degrees of freedom, are defined as:

$$L_{i,j} = \frac{\mu_0}{4\pi} \int_{V_c} \int_{V_c} \frac{\mathbf{w}_i(\mathbf{r}') \cdot \mathbf{w}_j(\mathbf{r})}{|\mathbf{r}-\mathbf{r}'|} dV' \, dV$$

  and depend only on the conductors geometry and relative positions. The inductance matrix describes the long-distance interactions between the unknown;
- finally, the elements of the "voltages" array are defined as:

$$V_i = -\int_{V_s} \frac{\partial \mathbf{A}_s}{\partial t} \cdot \mathbf{w}_i \, dV \, ; \; \mathbf{A}_s = \frac{\mu_0}{4\pi} \int_{V_s} \frac{\mathbf{J}_s(\mathbf{r}',t)}{|\mathbf{r}-\mathbf{r}'|} dV'$$

  $\mathbf{J}_s$ being the source (impressed) current density in the domain V_s.

The "perturbed" model corresponding to conductivity variations in a sub-volume of the mesh (and, of course, the "compensation" model) can be derived from the original one by simply updating the $R_{i,j}$ coefficients associated to the "varying" mesh elements. On the other side, if geometrical variations are involved, also the blocks of $\underline{\underline{L}}$ describing interactions between the unknowns in the modified region must be recomputed.

Note that in both cases, sensitivities computations imply variations of only a little part in the reference domain. As a consequence, the adoption of CT, eventually coupled with quick algorithms for matrix inversion suited for these conditions, such as the Woodbury algorithm (Albanese, 1998), can help reduce significantly the computational burden.

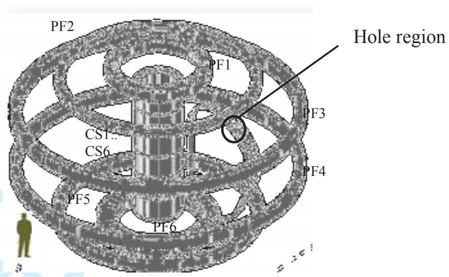
EXAMPLE OF APPLICATION

As an example of application of CT for the tolerance assessment of electromagnetic devices, the effect of variations in the size of the VV apertures - designed to let gas or particles flow in the plasma chamber - on the quality of the magnetic field in the plasma region is considered. Possible variations in the passive structures as VV have an important impact on the plasma stability (Portone, 2005).

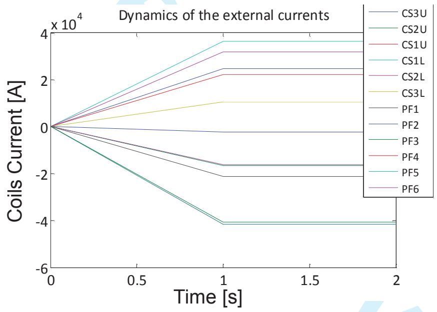
Since the effectiveness of magnetic confinement in Tokamaks is sensitive mainly to *localized* perturbations to field maps, as long as VV characteristics are equivalent on all the sections of the vessel along the toroidal direction, the field produced by eddy currents induced in VV, including the effect of apertures, has no localized perturbations. In addition, thanks to the "periodicity" of the structure, the eddy current maps can be computed using a model limited to just one of the sectors, imposing suitable periodicity condition on boundaries.

Anyway, if in one of the sectors one of the apertures differs from others (e.g. for manufacturing tolerances, of for assembly needs), than a low-periodicity disturbance appears. Such a situation can be well analyzed using CT. For the sake of exposition, a symmetric VV, with no ports, is used for the nominal configuration, and "apertures perturbation" are described as "holes" in the VV.

In fig. 2a a sketch of the considered geometry, including the active coils, inducing eddy currents in the VV, is reported; in addition, quite simple waveforms of the coil currents have been chosen (fig. 2b), able to excite in a simple but effective way the main modes of the structure. The rise times of the currents are 1s, in the same range of the longest time constants of the structure.



(a)



(b)

Figure 2. A sketch of the analyzed structure (a): active coils are dark grey, while passive structures (just half of a single periodicity module) are light grey; (b) waveforms of the inducing currents (colors in on-line version).

The eddy current map deformation due to a 10 cm × 10 cm hole (see the highlighted region in Fig. 2a) has been computed by both solving the full perturbed model, and applying the CT. Only 1/18 of the full VV (i.e. a 20° sector) has been modeled, using 817 brick elements (corresponding to 882 degrees of freedom), while inducing currents are provided by active coils, not requiring mesh. Fig. 3 shows a detail of current density map in the hole region, at t=1s, for the unperturbed case (a), for perturbed case (b), and in the case of CT (c). Note that compensation sources are represented by an imposed current density in the bricks corresponding to the hole. Due to the symmetry properties of the structure, just the right half of the hole region is reported in Fig. 3a and 3b.

Fig. 4 reports the waveforms of toroidal current component for unperturbed, perturbed and compensation cases in a sub domain located in the symmetry plane of the VV sector at φ=0°, in the upper part of the vessel (as shown in Fig. 2a). Note that the perturbed case is obtained with a fairly good approximation by superposition of unperturbed and compensation currents.

Care must be taken when computing the perturbed magnetic field in the plasma chamber using the compensation current. As a matter of fact, due to the automatic treatment of the geometrical periodicity, the adopted compensation source automatically corresponds to a set of 18-fold periodic holes.

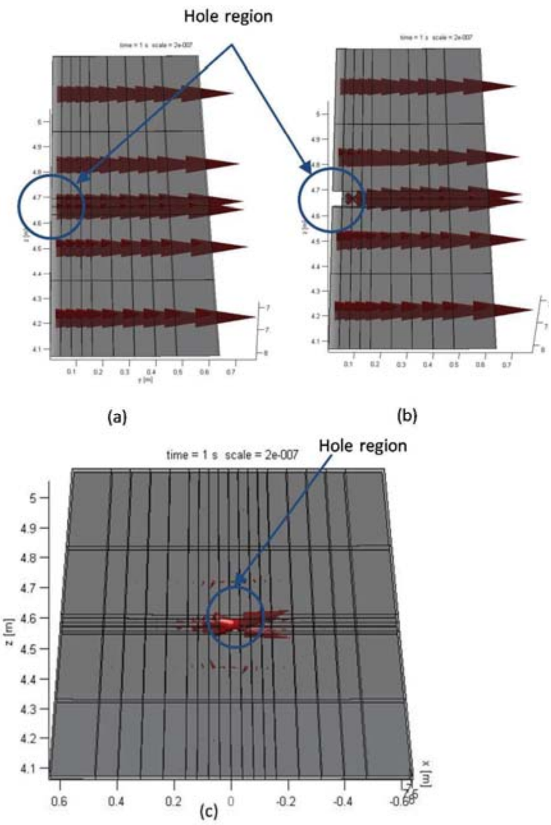


Figure 3. Current density maps in the hole region for the unperturbed case (a), for the perturbed case (b), and for the compensation case (c)

In some cases, as that here illustrated, due to the limited dimension of the hole, just local effect appears and therefore a perturbation in a single sector can be approximately evaluated, considering the perturbation vanishing all over the structure, except for the local region.

CONCLUSIONS

The use of compensation theorem appears a promising approach for the tolerance and robustness assessment of conducting structures in the performance analysis of highly demanding electromagnetic devices, when modification from nominal design may introduce perturbations that, although small and as such hard to evaluate, impact on the device performance.

The analysis of small, non-symmetric perturbation in a Tokamak has been used to exemplify the effectiveness of the compensation scheme.

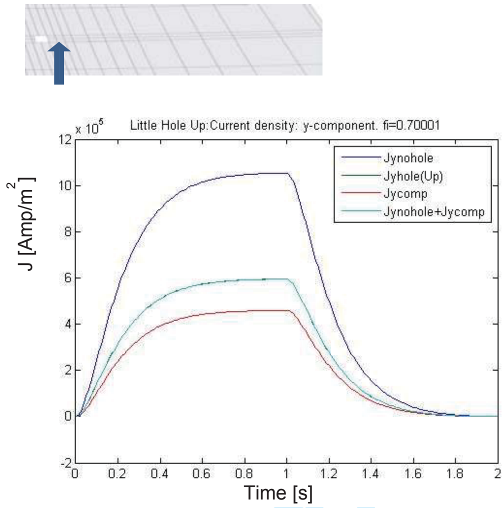


Figure 4. Waveform of y-component of current density in a point near the hole (colors in on-line version).

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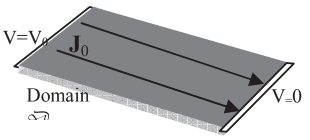
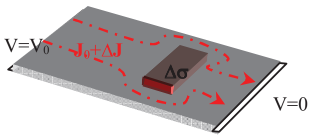
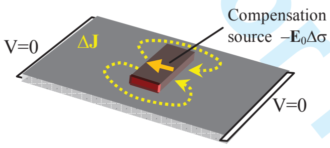


(a) Unperturbed current density map

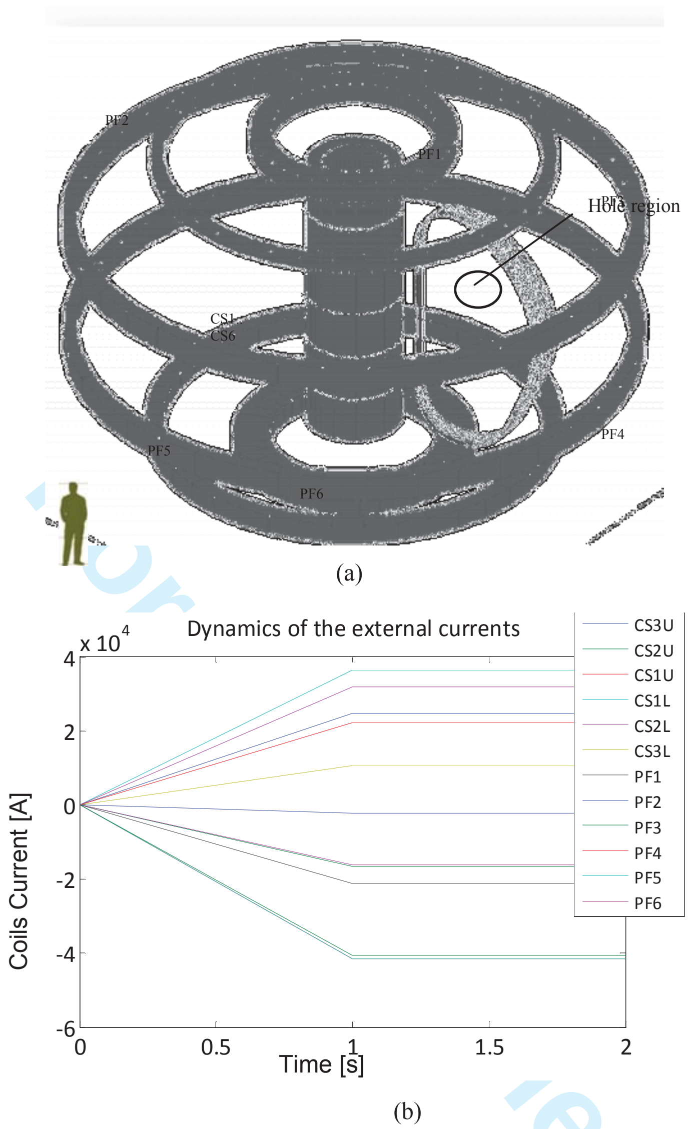


(b) Perturbed current density map



(c) Compensation problem

Figure 1. The CT applied to the current density map computation in a conducting plate with a brick-shaped conductivity perturbation.



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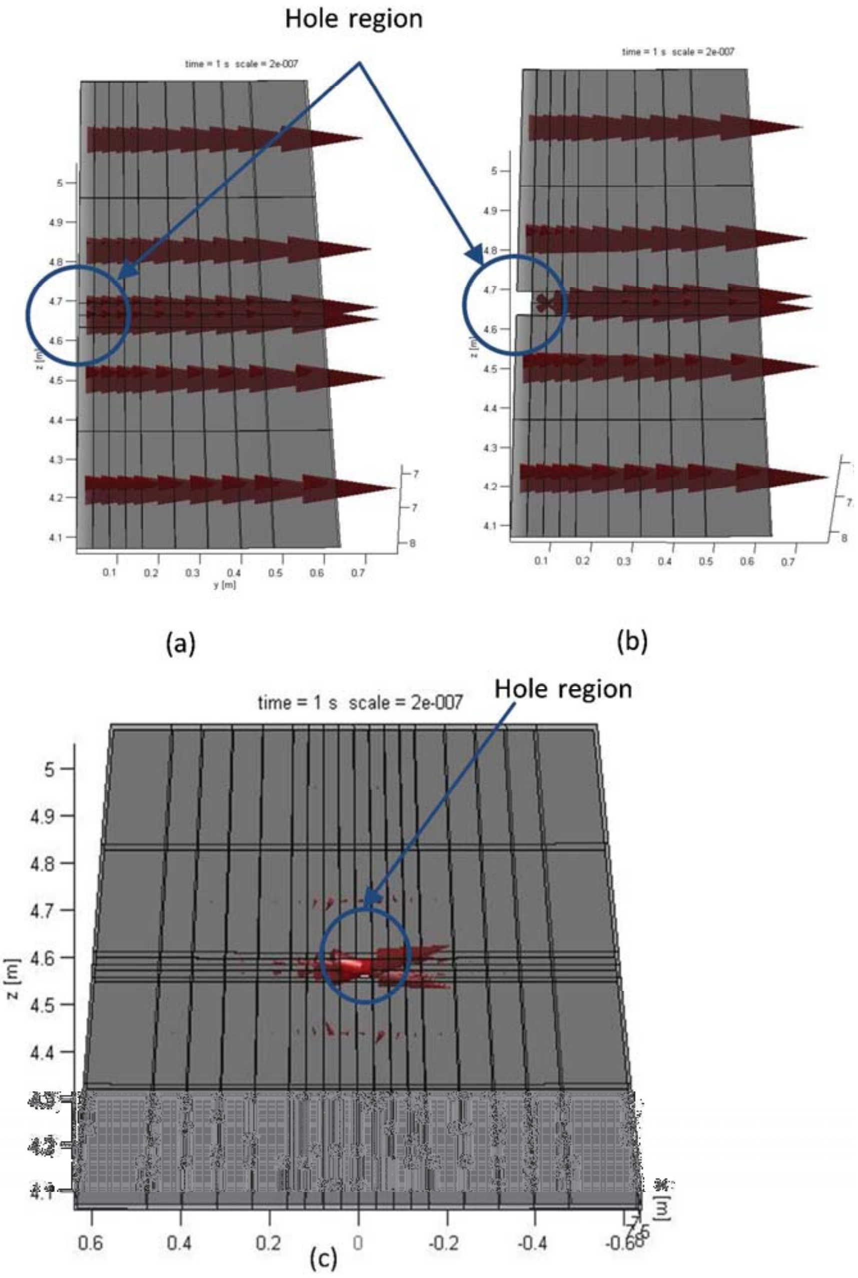

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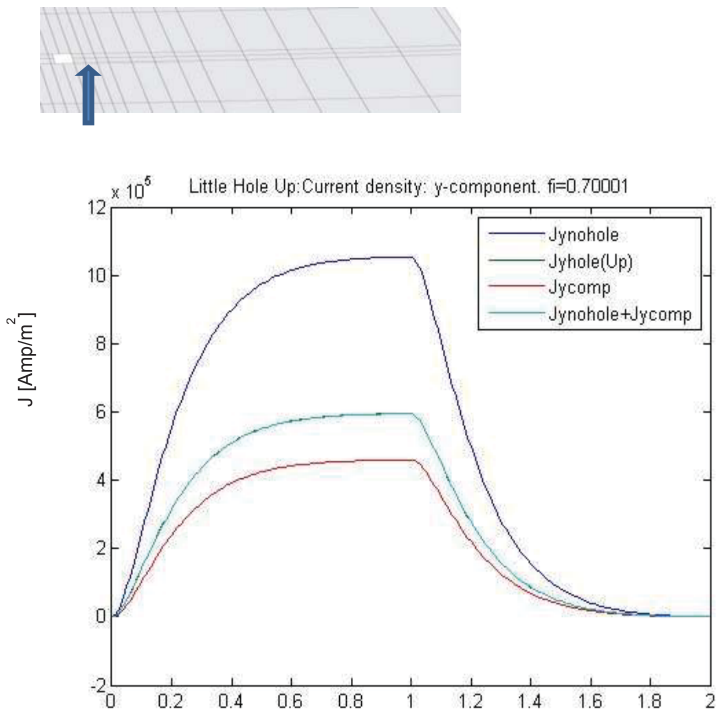

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