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ON AN INTEGRAL EQUATION OF THE FIRST KIND ARISING IN THE THEORY OF COSSERAT

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In this paper we study an integral equation of the first kind concerning an indirect boundary integral method for the Dirichlet problem in the theory of Cosserat continuum. Our method hinges on the theory of reducible operators and on the theory of differential forms.

Keywords: Dirichlet boundary value problem; boundary integral representations; layer potentials; Cosserat theory.

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1. Introduction

In the theory of classical elasticity, a material point has only three degrees of freedom corresponding to its position in Euclidean space. In the couple-stress elasticity theory there are three additional, independent degrees of freedom, related to the rotation of each particle. The couple-stress or Cosserat theory of elasticity has emerged from the work of the brothers François and Eugène Cosserat at the turn of the last century [8].

Potential methods for couple-stress elasticity have been developed (see [9, 13]), extending the classical methods used in linear elasticity. In particular the representability of the solution of the first and the second boundary value problem have been obtained by means of a double layer potential and a simple layer potential respectively.

We mention also that corresponding problems in plane, anti-plane deformations and in the bending of plates have been studied by means of boundary integral methods in different papers (see [15–19] and the reference therein).

In the present paper we consider the three-dimensional Dirichlet problem in Cosserat theory. When we try to solve it by means of a simple layer potential, we get an integral system of the first kind on the boundary. Our aim is to show that such system can be reduced to a Fredholm one and that the latter is equivalent to the Dirichlet problem in a precise sense.

Our method, extending the one given in [1] for Laplace operator, hinges on the theory of reducible operators (see, e.g. [10, 14]) and on the theory of differential forms (see, e.g. [11,12]). The basic idea is to treat the arising integral system of the first kind (see (4.1) below) taking the differential of both sides. In this way we obtain the singular integral system (4.2) in which the unknown is a usual vector function, while the data is a vector whose components are differential forms of degree one. We show that this system can be reduced to a Fredholm one, which is equivalent to the singular integral system in a precise sense. We remark that our method uses neither the theory of pseudodifferential operators nor the concept of hypersingular integrals. For a sketch of the method in the simple but significant case of Laplacian we refer to [7, Sec. 2, p. 2].

Even if in this paper we have considered only connected boundaries, it is very likely that such a method could be used also in not simply connected domains, in analogy to other problems (see [5–7]).

The paper is organized as follows. In Sec. 2 we give some notations and definitions. In particular, we generalize the stress operator by analogy with the theory of elasticity. In Sec. 3 we prove some preliminary results regarding the fundamental solution and the first derivatives of a double layer potential. In Sec. 4 we will construct a reducing operator which will be useful in the study of the integral system of the first kind arising in the Dirichlet problem. In Sec. 5 we find the solution of the Dirichlet problem in terms of a simple layer potential. We show how to reduce the problem to an equivalent Fredholm system.

2. Definitions

In this paper Ω denotes a bounded domain of $\mathbb{R}^3$ such that its boundary $\partial\Omega$ is a Lyapunov surface Σ (i.e. Σ has a uniformly Hölder continuous normal field of some exponent $l \in (0,1]$) and such that $\mathbb{R}^3 - \overline{\Omega}$ is connected; $\nu(y) = (\nu_1(y), \nu_2(y), \nu_3(y))$ denotes the outwards unit normal vector at the point $y = (y_1, y_2, y_3) \in \Sigma$.

Given the set of constants $\lambda, \mu, \alpha, \varepsilon, v, \beta$ satisfying the conditions

$$\alpha, \beta, \mu, v > 0; \quad 3\lambda + 2\mu > 0; \quad 3\varepsilon + 2v > 0,$$

the homogeneous equation of statics of a Cosserat continuum has the form [13, p. 50]

$$\begin{cases} (\mu+\alpha)\Delta u + (\lambda+\mu-\alpha)\operatorname{grad}\operatorname{div} u + 2\alpha \operatorname{rot}\omega = 0 & \text{in } \Omega, \\ (v+\beta)\Delta\omega + (\varepsilon+v-\beta)\operatorname{grad}\operatorname{div}\omega + 2\alpha\operatorname{rot} u - 4\alpha\omega = 0 & \text{in } \Omega, \end{cases} \tag{2.1}$$

where $u = (u_1, u_2, u_3)$ is the displacement vector and $\omega = (\omega_1, \omega_2, \omega_3)$ is the rotation vector. It is convenient to write the basic equations (2.1) in a matrix form. To this end let us consider the block-matrix

$$M = \begin{pmatrix} M^1 & M^2 \\ M^3 & M^4 \end{pmatrix},$$

whose entries are (3×3)-matrices of differential operators given by

$$M^1_{ij} = (\mu + \alpha)\delta_{ij}\Delta + (\lambda + \mu - \alpha)\frac{\partial^2}{\partial x_i \partial x_j},$$

$$M^2_{ij} = M^3_{ij} = -2\alpha \sum_{k=1}^{3} \delta_{ijk} \frac{\partial}{\partial x_k},$$

$$M^4_{ij} = \delta_{ij}[(v + \beta)\Delta - 4\alpha] + (\varepsilon + v - \beta)\frac{\partial^2}{\partial x_i \partial x_j}$$

for $i, j = 1, 2, 3$, where δ_{kj} and δ_{jkp} denote the Kronecker delta and the Levi-Civita symbol, respectively. Equations (2.1) become

$$M\mathcal{U} = 0, \tag{2.2}$$

where $\mathcal{U} = (u, \omega)'$ is a six-components column vector.

We denote by T the stress operator (see [13, p. 59])

$$T = \begin{pmatrix} T^1 & T^2 \\ 0 & T^4 \end{pmatrix}, \quad T^i = (T^i_{kj}), \ \ k, j = 1, 2, 3, \ \ i = 1, 2, 4,$$

where

$$T^1 u = \lambda(\operatorname{div} u)\nu + (2\mu)\frac{\partial u}{\partial \nu} + (\mu - \alpha)(\nu \wedge \operatorname{rot} u),$$

$$T^2 u = 2\alpha(\nu \wedge u),$$

$$T^4 u = \varepsilon(\operatorname{div} u)\nu + (2v)\frac{\partial u}{\partial \nu} + (v - \beta)(\nu \wedge \operatorname{rot} u).$$

By analogy with the theory of elasticity, we introduce the generalized stress operator defined as the block-matrix

$$S = \begin{pmatrix} S^1 & S^2 \\ 0 & S^4 \end{pmatrix}, \quad S^i = (S^i_{kj}), \ \ k, j = 1, 2, 3, \ \ i = 1, 2, 4, \tag{2.3}$$

where each entry is an (3×3)-matrix given by

$$\begin{aligned} S^1 u &= (\lambda + \mu - \xi)(\operatorname{div} u)\nu + (\mu + \xi)\frac{\partial u}{\partial \nu} + (\xi - \alpha)(\nu \wedge \operatorname{rot} u), \\ S^2 u &= 2\alpha(\nu \wedge u), \\ S^4 u &= (\varepsilon + v - \chi)(\operatorname{div} u)\nu + (v + \chi)\frac{\partial u}{\partial \nu} + (\chi - \beta)(\nu \wedge \operatorname{rot} u), \end{aligned} \tag{2.4}$$

ξ, χ being real parameters. If $\xi = \mu$ and $\chi = v$, then $S = T$ is the stress operator. Further when

$$\xi = \left[\frac{2(\mu+\alpha)(\lambda+2\mu)}{\lambda+3\mu+\alpha} - \mu\right] \quad \text{and} \quad \chi = \left[\frac{2(v+\beta)(\varepsilon+2v)}{\varepsilon+3v+\beta} - v\right], \tag{2.5}$$

we call S pseudostress operator and we denote it by

$$T^0 = \begin{pmatrix} (T^0)^1 & (T^0)^2 \\ 0 & (T^0)^4 \end{pmatrix}.$$

By $W^{1,p}(\Sigma)$ we denote the usual Sobolev space. By $L^p_1(\Sigma)$ we mean the space of the differential forms of degree 1 whose components are L^p real-valued functions in a coordinate system of class C^1 (and then in every coordinate system of class C^1). We recall that if v is a k-form in Ω, the symbol dv denotes the differential of v and $*v$ denotes the star Hodge operator. In the sequel, we shall use the symbol $\underset{\Sigma}{*}$; it means that, if w is an 2-form on Σ and $w = w_0 d\sigma$, then $\underset{\Sigma}{*}w = w_0$.

Finally, we recall that if B and $\widetilde{B}$ are two Banach spaces and $R : B \to \widetilde{B}$ is a continuous linear operator, we say that R can be reduced on the left if there exists a continuous linear operator $\widetilde{R} : \widetilde{B} \to B$ such that $\widetilde{R}R = I + T$, where I stands for the identity operator on B and $T : B \to B$ is compact. One of the main properties of such operators is that the equation $R\alpha = \beta$ has a solution if and only if $\langle\gamma, \beta\rangle = 0$ for any γ such that $R^*\gamma = 0$, R^* being the adjoint of R. A left reduction is said to be equivalent if $\mathrm{N}(\widetilde{R}) = \{0\}$, where $\mathrm{N}(\widetilde{R})$ denotes the kernel of $\widetilde{R}$ (see, e.g. [14, pp. 19–20]). Obviously this means that $R\alpha = \beta$ if and only if $\widetilde{R}R\alpha = \widetilde{R}\beta$.

In the problem considered in the present paper the condition $\mathrm{N}(\widetilde{R}) = \{0\}$ is not satisfied. Nevertheless, we still have a kind of equivalence (see Theorem 5.4 below).

3. Preliminaries

3.1. *The fundamental solution* Ψ

The block-matrix of the fundamental solution of the homogeneous system (2.2) is given by:

$$\Psi(x) = \begin{pmatrix} \Psi^1(x) & \Psi^2(x) \\ \Psi^3(x) & \Psi^4(x) \end{pmatrix}, \quad x \in \mathbb{R}^3 \backslash \{(0,0,0)\}, \tag{3.1}$$

where $\Psi^i(x) = (\Psi^i_{kj}(x))$ $k, j = 1, 2, 3$, $i = 1, \ldots, 4$ are the following (3×3)-matrices (see [13, p. 93]):

$$\Psi^1_{kj}(x) = \frac{\delta_{kj}}{2\pi}\left[\frac{1}{\mu|x|} - \frac{\alpha}{\mu(\alpha+\mu)}\frac{e^{-\sigma|x|}}{|x|}\right] + \frac{1}{2\pi\mu}\frac{\partial^2}{\partial x_k \partial x_j}\left[-\frac{(\lambda+\mu)}{2(\lambda+2\mu)}|x| + \frac{\beta+v}{4\mu}\frac{e^{-\sigma|x|}-1}{|x|}\right],$$

$$\Psi^2_{kj}(x) = \Psi^3_{kj}(x) = \frac{1}{4\pi\mu}\sum_{p=1}^{3}\delta_{jkp}\frac{\partial}{\partial x_p}\frac{1-e^{-\sigma|x|}}{|x|},$$

$$\Psi^4_{kj}(x) = \frac{\delta_{kj}}{2\pi(\beta+v)}\frac{e^{-\sigma|x|}}{|x|} + \frac{1}{8\pi}\frac{\partial^2}{\partial x_k\partial x_j}\left[\frac{e^{-\rho|x|}-e^{-\sigma|x|}}{\alpha|x|} - \frac{e^{-\sigma|x|}-1}{\mu|x|}\right],$$

$$\sigma = \sqrt{\frac{4\alpha\mu}{(\mu+\alpha)(v+\beta)}} \quad \text{and} \quad \rho = \sqrt{\frac{4\alpha}{\varepsilon+2v}}. \tag{3.2}$$

Lemma 3.1 ([13, p. 94]). *Each column of* $\Psi(x)$ (3.1) *satisfies* (2.2) *for any* $x \in \mathbb{R}^3\backslash\{(0,0,0)\}$.

Lemma 3.2. *The matrix* $\Psi(x)$ *defined by* (3.2) *can be written as*

$$\Psi^1_{kj}(x) = \frac{1}{4\pi}\left[\frac{\lambda+3\mu+\alpha}{(\mu+\alpha)(\lambda+2\mu)}\frac{\delta_{kj}}{|x|} + \frac{\lambda+\mu-\alpha}{(\mu+\alpha)(\lambda+2\mu)}\frac{x_kx_j}{|x|^3}\right] + C_{kj}(x),$$

$$\Psi^2_{kj}(x) = \Psi^3_{kj}(x) = \mathcal{O}(1),$$

$$\Psi^4_{kj}(x) = \frac{1}{4\pi}\left[\frac{\varepsilon+3v+\beta}{(v+\beta)(\varepsilon+2v)}\frac{\delta_{kj}}{|x|} + \frac{\varepsilon+v-\beta}{(v+\beta)(\varepsilon+2v)}\frac{x_kx_j}{|x|^3}\right] + D_{kj}(x),$$

where

$$C_{kj}(x) = \frac{e^{-\sigma|x|}-1}{|x|}\left[-\frac{\delta_{kj}}{2\pi\mu}\frac{\alpha}{\mu+\alpha} + \frac{\alpha}{2\pi\mu(\mu+\alpha)}\frac{x_kx_j}{|x|^2}\right] + \frac{1}{2\pi\mu}\frac{\beta+v}{4\mu}\left(\frac{3x_kx_j}{|x|^2} - \delta_{kj}\right)\left[\frac{(1+\sigma|x|)e^{-\sigma|x|}-1+\frac{1}{2}\sigma^2|x|^2}{|x|^3}\right], \tag{3.3}$$

$$\begin{aligned} D_{kj}(x) = {} & \left[\frac{\delta_{kj}}{2\pi(\beta+v)} - \frac{1}{8\pi}\left(\frac{1}{\alpha}+\frac{1}{\mu}\right)\frac{x_kx_j}{|x|^2}\sigma^2\right]\left[\frac{e^{-\sigma|x|}-1}{|x|}\right] \\ & + \frac{1}{8\pi\alpha}\frac{x_kx_j}{|x|^2}\rho^2\left[\frac{e^{-\rho|x|}-1}{|x|}\right] + \left(\frac{3}{8\pi\alpha}\frac{x_kx_j}{|x|^2} - \frac{1}{8\pi\alpha}\delta_{kj}\right) \\ & \times \left[\frac{(1+\rho|x|)e^{-\rho|x|}-1+\frac{1}{2}\rho^2|x|^2}{|x|^3}\right] + \left[\frac{1}{8\pi}\left(\frac{1}{\alpha}+\frac{1}{\mu}\right)\delta_{kj}\right. \\ & \left. - \frac{3}{8\pi}\left(\frac{1}{\alpha}+\frac{1}{\mu}\right)\frac{x_kx_j}{|x|^2}\right]\left[\frac{(1+\sigma|x|)e^{-\sigma|x|}-1+\frac{1}{2}\sigma^2|x|^2}{|x|^3}\right]. \end{aligned} \tag{3.4}$$

The functions $C_{kj}(x)$ *and* $D_{kj}(x)$ *are bounded.*

The previous lemma can be verified by a straightforward calculation. In particular we use the following limits

$$\lim_{x\to 0}\frac{(1+\sigma|x|)e^{-\sigma|x|}-1}{|x|^2}=-\frac{\sigma^2}{2}, \tag{3.5}$$

$$\begin{gathered}\lim_{x\to 0}\frac{(1+\sigma|x|)e^{-\sigma|x|}-1+\frac{1}{2}\sigma^2|x|^2}{|x|^3}=\frac{\sigma^3}{3},\\ \lim_{x\to 0}\frac{e^{-\sigma|x|}-1}{|x|}=-\sigma.\end{gathered} \tag{3.6}$$

Denoting by

$$M_x^{ih}=\left(\nu_i\frac{\partial}{\partial x_h}-\nu_h\frac{\partial}{\partial x_i}\right) \tag{3.7}$$

we have the following lemma.

Lemma 3.3. *Let S be the generalized stress* (2.3) *and let Ψ be the fundamental solution* (3.1). *The matrix $S\Psi$ can be written as*

$$S\Psi=\begin{pmatrix}(S\Psi)^1 & (S\Psi)^2\\ (S\Psi)^3 & (S\Psi)^4\end{pmatrix},$$

where

$$(S\Psi)^1_{kj}=\frac{1}{4\pi}\left[\frac{(\mu+\xi)(\lambda+3\mu+\alpha)}{(\mu+\alpha)(\lambda+2\mu)}-2\right]M_x^{jk}\left(\frac{1}{|x|}\right)+\mathcal{O}\left(\frac{1}{|x|^{2-l}}\right), \tag{3.8}$$

$$(S\Psi)^2_{kj}=\mathcal{O}\left(\frac{1}{|x|}\right), \tag{3.9}$$

$$(S\Psi)^3_{kj}=\mathcal{O}\left(\frac{1}{|x|}\right), \tag{3.10}$$

$$(S\Psi)^4_{kj}(x)=\frac{1}{4\pi}\left[\frac{(\chi+v)(\varepsilon+3v+\beta)}{(v+\beta)(\varepsilon+2v)}-2\right]M_x^{jk}\left(\frac{1}{|x|}\right)+\mathcal{O}\left(\frac{1}{|x|^{2-l}}\right), \tag{3.11}$$

$l\in(0,1]$ *being the Lyapunov exponent of the surface* Σ.

Proof. Keeping in mind (2.4), (3.2) and (3.7) we find

$$(S\Psi)^1_{kj}(x)=-(\mu+\xi)M_x^{kp}\Psi^1_{pj}(x)+\frac{1}{2\pi}M_x^{kj}\left(\frac{1}{|x|}\right)+\frac{1}{2\pi}\delta_{kj}\frac{\partial}{\partial\nu}\frac{1}{|x|},$$

$$\begin{aligned}(S\Psi)^2_{kj}(x)={}&-(\mu+\xi)M_x^{kp}\Psi^2_{pj}(x)-\frac{(\mu+\alpha)}{4\pi\mu}\delta_{jpq}\nu_p\frac{\partial}{\partial x_k}\frac{\partial}{\partial x_q}\frac{1-e^{-\sigma|x|}}{|x|}\\ &-\frac{\alpha}{4\pi}\delta_{kpq}\nu_q\frac{\partial^2}{\partial x_p\partial x_j}\left(\frac{e^{-\rho|x|}-e^{-\sigma|x|}}{\alpha|x|}-\frac{e^{-\sigma|x|}-1}{\mu|x|}\right),\end{aligned}$$

$$(S\Psi)^3_{kj}(x) = -(\chi+v)M_x^{kp}\Psi^3_{pj}(x) - \frac{(v+\beta)}{4\pi\mu}\nu_p\delta_{jpq}\frac{\partial^2}{\partial x_k\partial x_q}\frac{1-e^{-\sigma|x|}}{|x|}$$
$$-\frac{(v+\beta)}{4\pi\mu}\delta_{jkq}\nu_p\frac{\partial^2}{\partial x_p\partial x_q}\frac{1-e^{-\sigma|x|}}{|x|},$$
$$(S\Psi)^4_{kj}(x) = -(\chi+v)M_x^{kp}\Psi^4_{pj}(x) + \frac{1}{2\pi}\nu_k\frac{\partial}{\partial x_j}\frac{e^{-\rho|x|}}{|x|}$$
$$+\frac{1}{2\pi}\delta_{kj}\frac{\partial}{\partial\nu}\frac{e^{-\sigma|x|}}{|x|} - \frac{1}{2\pi}\nu_j\frac{\partial}{\partial x_k}\frac{e^{-\sigma|x|}}{|x|}.$$

By means of the expressions of Ψ^i_{kj} found in Lemma 3.2 we obtain

$$(S\Psi)^1_{kj}(x) = \frac{\delta_{kj}}{2\pi}\frac{\partial}{\partial\nu}\frac{1}{|x|} + \frac{1}{4\pi}\left[\frac{(\mu+\xi)(\lambda+3\mu+\alpha)}{(\mu+\alpha)(\lambda+2\mu)} - 2\right]M_x^{jk}\left(\frac{1}{|x|}\right)$$
$$+\frac{1}{4\pi}\frac{(\mu+\xi)(\lambda+\mu-\alpha)}{(\mu+\alpha)(\lambda+2\mu)}M_x^{pk}\left(\frac{x_px_j}{|x|^3}\right) + (\mu+\xi)M_x^{pk}[C_{pj}(x)],$$
$$(S\Psi)^2_{kj}(x) = -(\mu+\xi)M_x^{kp}\left[\frac{1}{4\pi\mu}\delta_{jpk}\frac{\partial}{\partial x_k}\frac{1-e^{-\sigma|x|}}{|x|}\right]$$
$$-\frac{(\mu+\alpha)}{4\pi\mu}\delta_{jpq}\nu_p\frac{\partial}{\partial x_k}\frac{\partial}{\partial x_q}\frac{1-e^{-\sigma|x|}}{|x|}$$
$$-\frac{\alpha}{4\pi}\delta_{kpq}\nu_q\frac{\partial^2}{\partial x_p\partial x_j}\left(\frac{e^{-\rho|x|}-e^{-\sigma|x|}}{\alpha|x|} - \frac{e^{-\sigma|x|}-1}{\mu|x|}\right), \tag{3.12}$$
$$(S\Psi)^3_{kj}(x) = -(\chi+v)M_x^{kp}\left[\frac{1}{4\pi\mu}\delta_{jpk}\frac{\partial}{\partial x_k}\frac{1-e^{-\sigma|x|}}{|x|}\right] - \frac{(v+\beta)}{4\pi\mu}\nu_p\delta_{jpq}$$
$$\times\frac{\partial^2}{\partial x_k\partial x_q}\frac{1-e^{-\sigma|x|}}{|x|} - \frac{(v+\beta)}{4\pi\mu}\delta_{jkq}\nu_p\frac{\partial^2}{\partial x_p\partial x_q}\frac{1-e^{-\sigma|x|}}{|x|}, \tag{3.13}$$
$$(S\Psi)^4_{kj}(x) = \frac{1}{2\pi}\delta_{kj}\frac{\partial}{\partial\nu}\frac{1}{|x|} + \frac{1}{4\pi}\left[\frac{(\chi+v)(\varepsilon+3v+\beta)}{(v+\beta)(\varepsilon+2v)} - 2\right]M_x^{jk}\left(\frac{1}{|x|}\right)$$
$$-\frac{1}{4\pi}\frac{(\chi+v)(\varepsilon+v-\beta)}{(v+\beta)(\varepsilon+2v)}M_x^{kp}\left(\frac{x_px_j}{|x|^3}\right) + \widetilde{D}_{kj}(x) - (\chi+v)M_x^{kp}[D_{pj}(x)], \tag{3.14}$$

where

$$\widetilde{D}_{kj}(x) = \frac{1}{2\pi}\nu_k(x)\frac{\partial}{\partial x_j}\frac{e^{-\rho|x|}-1}{|x|} - \frac{1}{2\pi}\nu_j(x)\frac{\partial}{\partial x_k}\frac{e^{-\sigma|x|}-1}{|x|}$$
$$+\frac{1}{2\pi}\delta_{kj}\frac{\partial}{\partial\nu}\frac{e^{-\sigma|x|}-1}{|x|}. \tag{3.15}$$

Regarding $(S\Psi)^1_{kj}(x)$ we have

$$\frac{\partial}{\partial\nu}\frac{1}{|x|} = -\frac{x_h}{|x|^3}\nu_h = \mathcal{O}\left(\frac{1}{|x|^{2-l}}\right),$$

$$\begin{aligned} M^{pk}\left(\frac{x_p x_j}{|x|^3}\right) &= \left(\nu_p\frac{\partial}{\partial x_k} - \nu_k\frac{\partial}{\partial x_p}\right)\left(\frac{x_p x_j}{|x|^3}\right) \\ &= \left(\delta_{kj} - 3\frac{x_k x_j}{|x|^2}\right)\nu_p\frac{x_p}{|x|^3} = \mathcal{O}\left(\frac{1}{|x|^{2-l}}\right), \end{aligned}$$

$$M^{pk}_x[C_{pj}(x)] = \mathcal{O}\left(\frac{1}{|x|}\right).$$

The last equality holds because we have, in view of (3.5) and (3.6),

$$\begin{aligned} \frac{\partial}{\partial x_j}\frac{e^{-\sigma|x|}-1}{|x|} &= \frac{x_j}{|x|^3} - \frac{x_j\sigma e^{-\sigma|x|}}{|x|^2} - \frac{x_j e^{-\sigma|x|}}{|x|^3} \\ &= -\frac{x_j}{|x|}\left[\frac{(1+\sigma|x|)e^{-\sigma|x|}-1}{|x|^2}\right] \\ &= \mathcal{O}(1), \end{aligned}$$

$$\begin{aligned} \frac{\partial}{\partial x_j}\frac{x_k x_j}{|x|^2} &= \frac{x_k}{|x|^2} + \delta_{kj}\frac{x_j}{|x|^2} - \frac{2x_j x_j x_k}{|x|^4} \\ &= \mathcal{O}\left(\frac{1}{|x|}\right), \end{aligned}$$

$$\begin{aligned} &\frac{\partial}{\partial x_j}\frac{(1+\sigma|x|)e^{-\sigma|x|}-1+\frac{1}{2}\sigma^2|x|^2}{|x|^3} \\ &\quad = -3\frac{x_j}{|x|^2}\left[\frac{(1+\sigma|x|)e^{-\sigma|x|}-1+\frac{1}{2}\sigma^2|x|^2}{|x|^3}\right] \\ &\qquad -\sigma^2\frac{x_j}{|x|^2}\left[\frac{e^{-\sigma|x|}-1}{|x|}\right] = \mathcal{O}\left(\frac{1}{|x|}\right). \end{aligned}$$

As far as $(S\Psi)^2_{kj}(x)$ is concerned, since

$$\frac{\partial^2}{\partial x_p\partial x_j}\left(\frac{e^{-\rho|x|}-e^{-\sigma|x|}}{|x|}\right) = \frac{\partial^2}{\partial x_p\partial x_j}\left[\frac{(e^{-\rho|x|}-1)-(e^{-\sigma|x|}-1)}{|x|}\right],$$

the right-hand side of (3.12) is a finite linear combination of the following derivatives:

$$\frac{\partial^2}{\partial x_p\partial x_j}\frac{e^{-\sigma|x|}-1}{|x|}.$$

Thanks to (3.5) and (3.6), we can write

$$\begin{aligned}\frac{\partial^2}{\partial x_p \partial x_j}\frac{e^{-\sigma|x|}}{|x|} &= -\frac{\delta_{pj}}{|x|^3} + \frac{3x_p x_j}{|x|^5} + \sigma^2\frac{x_p x_j}{|x|^3} - \frac{\delta_{pj}}{|x|}\left[\frac{(\sigma|x|+1)e^{-\sigma|x|}-1}{|x|^2}\right] \\ &\quad + 3\frac{x_p x_j}{|x|^3}\left[\frac{(\sigma|x|+1)e^{-\sigma|x|}-1}{|x|^2}\right] + \sigma^2\frac{x_p x_j}{|x|^2}\left[\frac{e^{-\sigma|x|}-1}{|x|}\right] \\ &= -\frac{\delta_{pj}}{|x|^3} + \frac{3x_p x_j}{|x|^5} + \mathcal{O}\left(\frac{1}{|x|}\right) = \frac{\partial^2}{\partial x_p \partial x_j}\frac{1}{|x|} + \mathcal{O}\left(\frac{1}{|x|}\right).\end{aligned}$$

This implies (3.9).

Similar arguments show that (3.13) and (3.14) lead to (3.10) and (3.11), respectively. □

The vector function $\mathcal{U}(x) = (u,\omega)' \in \mathbb{R}^6$ defined as

$$\mathcal{U}(x) = \int_\Sigma \Psi(y-x)\Phi(y)d\sigma_y, \quad x \in \mathbb{R}^3 \tag{3.16}$$

is the simple layer potential with density $\Phi = (\varphi,\vartheta)'$ whose components are written as

$$\begin{cases} u_j(x) = \displaystyle\int_\Sigma [\Psi^1_{jh}(y-x)\varphi_h(y) + \Psi^2_{jh}(y-x)\vartheta_h(y)]d\sigma_y, & x \in \mathbb{R}^3, \\ \omega_j(x) = \displaystyle\int_\Sigma [\Psi^3_{jh}(y-x)\varphi_h(y) + \Psi^4_{jh}(y-x)\vartheta_h(y)]d\sigma_y, & x \in \mathbb{R}^3 \end{cases}$$

for $h, j = 1,2,3$.

The vector function $\mathcal{W}(x) \in \mathbb{R}^6$

$$\mathcal{W}(x) = \int_\Sigma [(S\Psi)(y-x)]'\Phi(y)d\sigma_y, \quad x \in \mathbb{R}^3 \tag{3.17}$$

is the double layer potential with density $\Phi = (\varphi,\vartheta)'$ whose components are:

$$\begin{cases} W_i(x) = \displaystyle\int_\Sigma [(S\Psi)^1_{ji}(y-x)\varphi_j(y) + (S\Psi)^3_{ji}\vartheta_j(y)]d\sigma_y, & i,j = 1,2,3, \\ W_{i+3}(x) = \displaystyle\int_\Sigma [(S\Psi)^2_{ji}(y-x)\varphi_j(y) + (S\Psi)^4_{ji}\vartheta_j(y)]d\sigma_y, & i,j = 1,2,3, \end{cases} \tag{3.18}$$

S being given by (2.3).

3.2. *On the first derivatives of a double layer potential*

We define the operator Θ_s as

$$\Theta_s(\varphi)(x) = *\int_\Sigma d_x[s_1(y-x)] \wedge \varphi(y) \wedge dx^s, \quad \varphi \in L^p_1(\Sigma), \tag{3.19}$$

where

$$s_1(y-x) = -\frac{1}{4\pi|y-x|}\sum_j dx^j\, dy^j.$$

In the sequel we use the following identity proved in [1, p. 187]:

$$\frac{1}{4\pi}\frac{\partial}{\partial x_s}\int_\Sigma u(y)\frac{\partial}{\partial \nu_y}\frac{1}{|x-y|}d\sigma_y = \Theta_s(du)(x), \quad x\in\Omega,\ \ u\in W^{1,p}(\Sigma). \tag{3.20}$$

Theorem 3.1. *Let $\mathcal{W}$ be the double layer potential* (3.17) *with density $\mathcal{U} = (u,\omega)' \in [W^{1,p}(\Sigma)]^6$. We have for any $x\in\Omega$*

$$\frac{\partial}{\partial x_s}W_j(x) = \mathcal{K}_{js}(du)(x) + \frac{\partial}{\partial x_s}\int_\Sigma (S\Psi)^3_{kj}(y-x)\omega_k(y)d\sigma_y, \tag{3.21}$$

$$\begin{aligned}\frac{\partial}{\partial x_s}W_{j+3}(x) &= \mathcal{F}_{js}(d\omega)(x)\\ &\quad + \frac{\partial}{\partial x_s}\int_\Sigma [\widetilde{D}_{kj}(y-x)\omega_k(y) + (S\Psi)^2_{kj}(y-x)u_k(y)]d\sigma_y,\end{aligned} \tag{3.22}$$

where $du = (du_1, du_2, du_3)$, $d\omega = (d\omega_1, d\omega_2, d\omega_3)$,

$$\begin{aligned}\mathcal{K}_{js}(\psi)(x) &= 2\Theta_s(\psi_j)(x) - \delta^{123}_{pkq}\int_\Sigma \frac{\partial}{\partial x_s}[H^1_{jp}(y-x)]\wedge\psi_k(y)\wedge dy^q,\\ &\qquad\qquad\qquad\qquad \psi\in[L^p_1(\Sigma)]^3,\end{aligned} \tag{3.23}$$

$$\begin{aligned}\mathcal{F}_{js}(\varphi)(x) &= 2\Theta_s(\varphi_j)(x) - \delta^{123}_{pkq}\int_\Sigma \frac{\partial}{\partial x_s}[H^2_{jp}(y-x)]\wedge\varphi_k(y)\wedge dy^q,\\ &\qquad\qquad\qquad\qquad \varphi\in[L^p_1(\Sigma)]^3,\end{aligned} \tag{3.24}$$

$$\begin{aligned}H^1_{jp}(y-x) &= \frac{1}{4\pi}\left[\frac{(\mu+\xi)(\lambda+3\mu+\alpha)}{(\mu+\alpha)(\lambda+2\mu)} - 2\right]\frac{\delta_{jp}}{|y-x|}\\ &\quad + \frac{1}{4\pi}\frac{(\mu+\xi)(\lambda+\mu-\alpha)}{(\mu+\alpha)(\lambda+2\mu)}\frac{1}{|y-x|}\frac{\partial}{\partial y_j}|y-x|\frac{\partial}{\partial y_p}|y-x|\\ &\quad + (\mu+\xi)C_{jp}(y-x),\end{aligned} \tag{3.25}$$

$$\begin{aligned}H^2_{jp}(y-x) &= \frac{1}{4\pi}\left[\frac{(\chi+v)(\varepsilon+3v+\beta)}{(v+\beta)(\varepsilon+2v)} - 2\right]\frac{\delta_{jp}}{|y-x|}\\ &\quad + \frac{1}{4\pi}\frac{(\chi+v)(\varepsilon+v-\beta)}{(v+\beta)(\varepsilon+2v)}\frac{1}{|y-x|}\frac{\partial}{\partial y_j}|y-x|\frac{\partial}{\partial y_p}|y-x|\\ &\quad + (\chi+v)D_{jp}(y-x).\end{aligned} \tag{3.26}$$

Here S, $\widetilde{D}_{jp}$, C_{jp} and D_{jp} are given by (2.3), (3.15), (3.3) *and* (3.4), *respectively.*

Proof. It follows from (3.18) and (3.8) that

$$\begin{aligned} W_j(x) &= \frac{1}{2\pi}\int_\Sigma u_j(y)\frac{\partial}{\partial\nu_y}\frac{1}{|y-x|}d\sigma_y \\ &\quad + \frac{1}{4\pi}\left[\frac{(\mu+\xi)(\lambda+3\mu+\alpha)}{(\mu+\alpha)(\lambda+2\mu)} - 2\right]\int_\Sigma u_k(y)M_y^{jk}\left(\frac{1}{|y-x|}\right)d\sigma_y \\ &\quad + \frac{1}{4\pi}\frac{(\mu+\xi)(\lambda+\mu-\alpha)}{(\mu+\alpha)(\lambda+2\mu)}\int_\Sigma u_k(y)M_y^{pk}\left[\frac{(y_j-x_j)(y_p-x_p)}{|y-x|^3}\right]d\sigma_y \\ &\quad + (\mu+\xi)\int_\Sigma u_k(y)M_y^{pk}[C_{jp}(y-x)]d\sigma_y + \int_\Sigma (S\Psi)^3_{kj}(y-x)\omega_k(y)d\sigma_y. \end{aligned}$$

The first and last integrals on the right-hand side are compact operators because of the weak singularities. Integrating by parts and keeping in mind that M_y^{ih} are tangential operators, we find

$$\begin{aligned} W_j(x) &= \frac{1}{2\pi}\int_\Sigma u_j(y)\frac{\partial}{\partial\nu_y}\frac{1}{|y-x|}d\sigma_y - \int_\Sigma H^1_{jp}(y-x)M_y^{pk}[u_k(y)]d\sigma_y \\ &\quad + \int_\Sigma (S\Psi)^3_{kj}(y-x)\omega_k(y)d\sigma_y. \end{aligned}$$

Finally, because of

$$M_y^{ih}u(y)d\sigma = \delta^{123}_{ihp}du(y)\wedge dy^p$$

and in view of (3.20), we obtain (3.21). With similar calculations we achieve (3.22). □

4. The Integral Equation of the First Kind and Its Reduction

If we look for the solution of the Dirichlet problem $M\mathcal{U} = 0$ in Ω, $\mathcal{U} = f$ on Σ, $f \in [W^{1,p}(\Sigma)]^6$, in the form of a simple layer potential (3.16), we get the following integral system of the first kind

$$\int_\Sigma \Psi_{ij}(y-x)\gamma_j(y)\,d\sigma_y = f_i(x) \tag{4.1}$$

on Σ. Our method consists at first in taking the differential of both sides, obtaining the following singular integral system

$$\int_\Sigma d_x[\Psi_{ij}(y-x)]\gamma_j(y)\,d\sigma_y = df_i(x). \tag{4.2}$$

Note that in this system the unknown is a usual vector $(\gamma_1,\dots,\gamma_6)$ whose components are scalar functions, while the data is the vector $(df_1,\dots,df_6)$ whose components are differential forms of degree 1.

Then we show that system (4.2) can be reduced to an equivalent Fredholm system.

We recall that if $\eta \in \Sigma$ is a Lebesgue point for $f \in L^1(\Sigma)$, we get

$$\begin{aligned}\lim_{x\to\eta} \int_\Sigma f(y)\partial_{x_s} \frac{(y_l - x_l)(y_j - x_j)}{|x-y|^3} d\sigma_y \\ = 2\pi(\delta_{lj} - 2\nu_j(\eta)\nu_l(\eta))\nu_s(\eta) f(\eta) \\ + \int_\Sigma f(y)\partial_{\eta_s} \frac{(y_l - \eta_l)(y_j - \eta_j)}{|\eta - y|^3}\, d\sigma_y,\end{aligned} \tag{4.3}$$

where the limit has to be understood as an internal angular boundary value[a] and the integral in the right-hand side is a singular integral. Further, let $\psi \in L^p_1(\Sigma)$ and write ψ as $\psi = \psi_h dx^h$ with[b]

$$\nu_h \psi_h = 0, \tag{4.4}$$

then, for almost every $\eta \in \Sigma$,

$$\lim_{x\to\eta} \Theta_h(\psi)(x) = -\frac{1}{2}\psi_h(\eta) + \Theta_h(\psi)(\eta), \tag{4.5}$$

where Θ_h is given by (3.19) and the limit has to be understood again as an internal angular boundary value.

Jump formulas (4.3) and (4.5) are proved in [5, Lemmas 3.2 and 3.3].

Lemma 4.1. *Let $\psi \in L^p_1(\Sigma)$. Let us write ψ as $\psi = \psi_h dx^h$ and suppose that* (4.4) *holds. Then, for almost every $\eta \in \Sigma$,*

$$\begin{aligned}&\lim_{x\to\eta} \delta^{123}_{lik} \int_\Sigma \partial_{x_s} H^1_{lj}(y-x) \wedge \psi(y) \wedge dy^k \\ &\quad = -\left[\frac{\mu+\lambda-\xi}{2\mu+\lambda}\nu_j(\eta)\psi_i(\eta) + \frac{\xi-\alpha}{\mu+\alpha}\nu_i(\eta)\psi_j(\eta)\right]\nu_s(\eta) \\ &\qquad + \delta^{123}_{lik} \int_\Sigma \partial_{\eta_s} H^1_{lj}(y-\eta) \wedge \psi(y) \wedge dy^k, \\ &\lim_{x\to\eta} \delta^{123}_{lik} \int_\Sigma \partial_{x_s} H^2_{lj}(y-x) \wedge \psi(y) \wedge dy^k \\ &\quad = -\left[\frac{\upsilon+\epsilon-\chi}{2\upsilon+\epsilon}\nu_j(\eta)\psi_i(\eta) + \frac{\chi-\beta}{\upsilon+\beta}\nu_i(\eta)\psi_j(\eta)\right]\nu_s(\eta) \\ &\qquad + \delta^{123}_{lik} \int_\Sigma \partial_{\eta_s} H^2_{lj}(y-\eta) \wedge \psi(y) \wedge dy^k,\end{aligned} \tag{4.6}$$

where H^1_{lj} and H^2_{lj} are defined by (3.25) *and* (3.26) *respectively and the limits have to be understood as internal angular boundary values.*

[a] For the definition of internal (external) angular boundary values see, e.g. [4, p. 53] or [13, p. 293].
[b] Assumption (4.4) is not restrictive, because, given the 1-form ψ on Σ, there exist scalar functions ψ_h defined on Σ such that $\psi = \psi_h dx^h$ and (4.4) holds (see [3, p. 41]).

Proof. Let us consider

$$\delta_{lik}^{123}\int_\Sigma \partial_{x_s}H_{lj}^1(y-x)\wedge\psi(y)\wedge dy^k = \delta_{lik}^{123}\delta_{rhk}^{123}\int_\Sigma \partial_{x_s}H_{lj}^1(y-x)\psi_h(y)\nu_r(y)d\sigma_y$$

$$= \delta_{rh}^{li}\int_\Sigma \partial_{x_s}H_{lj}^1(y-x)\psi_h(y)\nu_r(y)d\sigma_y.$$

In view of (3.25) and (4.3) we have that

$$\begin{aligned}
&\lim_{x\to\eta}\delta_{lik}^{123}\int_\Sigma \partial_{x_s}H_{lj}^1(y-x)\wedge\psi(y)\wedge dy^k\\
&\quad=\frac{\delta_{rh}^{li}}{2}\left[\frac{(\mu+\xi)(\lambda+\mu-\alpha)}{(\mu+\alpha)(\lambda+2\mu)}(\delta_{lj}-2\nu_j(\eta)\nu_l(\eta))\right.\\
&\qquad+\left.\left[\frac{(\mu+\xi)(\lambda+3\mu+\alpha)}{(\mu+\alpha)(\lambda+2\mu)}-2\right]\delta_{lj}\right]\nu_s(\eta)\nu_r(\eta)\psi_h(\eta)\\
&\qquad+\delta_{lik}^{123}\int_\Sigma \partial_{\eta_s}H_{lj}^1(y-\eta)\wedge\psi(y)\wedge dy^k\\
&\quad=\left[\frac{(\xi-\alpha)}{(\mu+\alpha)}\delta_{lj}-\frac{(\mu+\xi)(\lambda+\mu-\alpha)}{(\mu+\alpha)(\lambda+2\mu)}\nu_j(\eta)\nu_l(\eta)\right]\nu_s(\eta)(\nu_l(\eta)\psi_i(\eta)\\
&\qquad-\nu_i(\eta)\psi_l(\eta))+\delta_{lik}^{123}\int_\Sigma \partial_{\eta_s}H_{lj}^1(y-\eta)\wedge\psi(y)\wedge dy^k.
\end{aligned}$$

Because of (4.4) we can write this expression as

$$\begin{aligned}
&\lim_{x\to\eta}\delta_{lik}^{123}\int_\Sigma \partial_{x_s}H_{lj}^1(y-x)\wedge\psi(y)\wedge dy^k\\
&\quad=\frac{(\xi-\alpha)}{(\mu+\alpha)}\nu_s(\eta)\nu_j(\eta)\psi_i(\eta)-\frac{(\xi-\alpha)}{(\mu+\alpha)}\nu_s(\eta)\nu_i(\eta)\psi_j(\eta)\\
&\qquad-\frac{(\mu+\xi)(\lambda+\mu-\alpha)}{(\mu+\alpha)(\lambda+2\mu)}\nu_s(\eta)\nu_j(\eta)\psi_i(\eta)+\delta_{lik}^{123}\int_\Sigma \partial_{\eta_s}H_{lj}^1(y-\eta)\wedge\psi(y)\wedge dy^k\\
&\quad=-\frac{(\lambda+\mu-\xi)}{\lambda+2\mu}\nu_s(\eta)\nu_j(\eta)\psi_i(\eta)-\frac{\xi-\alpha}{\mu+\alpha}\nu_s(\eta)\nu_i(\eta)\psi_j(\eta)\\
&\qquad+\delta_{lik}^{123}\int_\Sigma \partial_{\eta_s}H_{lj}^1(y-\eta)\wedge\psi(y)\wedge dy^k.
\end{aligned}$$

The jump formula (4.6) is obtained with similar calculations. □

Lemma 4.2. *Let $(\psi,\varphi)\in[L_1^p(\Sigma)]^3\times[L_1^p(\Sigma)]^3$. Then, for almost every $\eta\in\Sigma$,*

$$\begin{aligned}
&\lim_{x\to\eta}(\lambda+\mu-\xi)\mathcal{K}_{jj}(\psi)(x)\nu_i(\eta)+(\mu+\alpha)\mathcal{K}_{ij}(\psi)(x)\nu_j(\eta)+(\xi-\alpha)\mathcal{K}_{ji}(\psi)(x)\nu_j(\eta)\\
&\quad=(\lambda+\mu-\xi)\mathcal{K}_{jj}(\psi)(\eta)\nu_i(\eta)+(\mu+\alpha)\mathcal{K}_{ij}(\psi)(\eta)\nu_j(\eta)\\
&\qquad+(\xi-\alpha)\mathcal{K}_{ji}(\psi)(\eta)\nu_j(\eta),
\end{aligned}\tag{4.7}$$

$$\lim_{x\to\eta}(\epsilon+\upsilon-\chi)\mathcal{F}_{jj}(\psi)(x)\nu_i(\eta)+(\upsilon+\beta)\mathcal{F}_{ij}(\psi)(x)\nu_j(\eta)+(\chi-\beta)\mathcal{F}_{ji}(\psi)(x)\nu_j(\eta)$$
$$=(\epsilon+\upsilon-\chi)\mathcal{F}_{jj}(\psi)(\eta)\nu_i(\eta)+(\upsilon+\beta)\mathcal{F}_{ij}(\psi)(\eta)\nu_j(\eta)$$
$$+(\chi-\beta)\mathcal{F}_{ji}(\psi)(\eta)\nu_j(\eta), \tag{4.8}$$

$\mathcal{K}$ and $\mathcal{F}$ being as in (3.23) *and* (3.24), *respectively, and the limits have to be understood as internal angular boundary values.*

Proof. Let us write ψ_i as $\psi_i=\psi_{ih}dx^h$ with

$$\nu_h\psi_{ih}=0, \quad i=1,2,3. \tag{4.9}$$

Keeping in mind (4.5) and Lemma 4.1 we have that

$$\lim_{x\to\eta}\mathcal{K}_{js}(\psi)(x)=-\psi_{js}(\eta)+\left[\frac{(\lambda+\mu-\xi)}{(\lambda+2\mu)}\nu_j(\eta)\psi_{hh}(\eta)+\frac{\xi-\alpha}{\mu+\alpha}\nu_h(\eta)\psi_{hj}(\eta)\right]\nu_s(\eta)$$
$$+\mathcal{K}_{js}(\psi)(\eta).$$

Therefore

$$\lim_{x\to\eta}(\lambda+\mu-\xi)\mathcal{K}_{jj}(\psi)(x)\nu_i(\eta)+(\mu+\alpha)\mathcal{K}_{ij}(\psi)(x)\nu_j(\eta)+(\xi-\alpha)\mathcal{K}_{ji}(\psi)(x)\nu_j(\eta)$$
$$=\Phi_i(\psi)(\eta)+(\lambda+\mu-\xi)\mathcal{K}_{jj}(\psi)(\eta)\nu_i(\eta)$$
$$+(\mu+\alpha)\mathcal{K}_{ij}(\psi)(\eta)\nu_j(\eta)+(\xi-\alpha)\mathcal{K}_{ji}(\psi)(\eta)\nu_j(\eta).$$

From (4.9) we get

$$\begin{aligned}\Phi_i(\psi)&=(\lambda+\mu-\xi)\left[-\psi_{jj}+\left(\frac{\mu-\xi+\lambda}{2\mu+\lambda}\nu_j\psi_{hh}+\frac{\alpha-\xi}{\mu+\alpha}\nu_h\psi_{hj}\right)\nu_j\right]\nu_i\\
&\quad+(\mu+\alpha)\left[-\psi_{ij}+\left(\frac{\mu+\lambda-\xi}{2\mu+\lambda}\nu_i\psi_{hh}+\frac{\alpha-\xi}{\mu+\alpha}\nu_h\psi_{hi}\right)\nu_j\right]\nu_j\\
&\quad+(\xi-\alpha)\left[-\psi_{ji}+\left(\frac{\mu+\lambda-\xi}{2\mu+\lambda}\nu_j\psi_{hh}+\frac{\alpha-\xi}{\mu+\alpha}\nu_h\psi_{hj}\right)\nu_i\right]\nu_j\\
&=(\lambda+\mu-\xi)\left[-\psi_{jj}\nu_i+\frac{\mu+\lambda-\xi}{2\mu+\lambda}\nu_i\psi_{hh}\right]\\
&\quad+(\mu+\alpha)\left[\frac{\mu+\lambda-\xi}{2\mu+\lambda}\nu_i\psi_{hh}\right]+(\xi-\alpha)\nu_h\psi_{hi}\\
&\quad+(\xi-\alpha)\left[-\psi_{ji}\nu_j+\frac{\mu+\lambda-\xi}{2\mu+\lambda}\psi_{hh}\nu_i+\frac{\alpha-\xi}{\mu+\alpha}\nu_h\psi_{hj}\nu_i\nu_j\right]\\
&=\left[(\lambda+\mu-\xi)\left(-1+\frac{(\lambda+\mu-\xi)}{\lambda+2\mu}\right)+\frac{\lambda+\mu-\xi}{2\mu+\lambda}(\mu+\alpha+\xi-\alpha)\right]\psi_{hh}\nu_i\\
&\equiv 0\end{aligned}$$

and (4.7) is proved. Analogously we obtain (4.8). □

Lemma 4.3. *Let $\mathcal{W} = (w, \zeta)'$ be the double layer potential* (3.17) *with density $(u, \omega)' \in [W^{1,p}(\Sigma)]^3 \times [W^{1,p}(\Sigma)]^3$. Let*

$$\mathcal{L}_j(x) = \begin{pmatrix} \displaystyle\int_\Sigma (S\Psi)^3_{kj}(y-x)\omega_k(y)d\sigma_y \\ \displaystyle\int_\Sigma [\widetilde{D}_{kj}(y-x)\omega_k(y) + (S\Psi)^2_{kj}(y-x)u_k(y)]d\sigma_y \end{pmatrix} \tag{4.10}$$

and let $\mathcal{W}^0 = \mathcal{W} - \mathcal{L}$ be the vector whose components are $(w^0, \zeta^0)'$. Then

$$\begin{aligned} S^1_{+,i}(w^0) = S^1_{-,i}(w^0) &= (\lambda + \mu - \xi)\mathcal{K}_{jj}(du)\nu_i \\ &\quad + (\mu + \alpha)\mathcal{K}_{ij}(du)\nu_j + (\xi - \alpha)\mathcal{K}_{ji}(du)\nu_j, \\ S^4_{+,i}(\zeta^0) = S^4_{-,i}(\zeta^0) &= (\epsilon + \upsilon - \chi)\mathcal{F}_{jj}(d\omega)\nu_i \\ &\quad + (\upsilon + \beta)\mathcal{F}_{ij}(d\omega)\nu_j + (\chi - \beta)\mathcal{F}_{ji}(d\omega)\nu_j, \end{aligned}$$

a.e. on Σ, where S^h_+, S^h_- denote the internal and external angular boundary limit of S^h, $h = 1, 4$ respectively and $\mathcal{K}_{ij}$ and $\mathcal{F}_{ij}$ are given by (3.23) *and* (3.24)*, respectively.*

Proof. The results are immediate consequence of Theorem 3.1 and Lemma 4.2. □

Let us introduce now the following block-matrix of singular operators:

$$\widetilde{R} = \begin{pmatrix} \widetilde{R}^1 & 0 \\ 0 & \widetilde{R}^4 \end{pmatrix}, \tag{4.11}$$

where $\widetilde{R}^1, \widetilde{R}^4 : [L^p_1(\Sigma)]^3 \to [L^p(\Sigma)]^3$ are defined as

$$(\widetilde{R}^1\psi)_i = (\lambda + \mu - \xi)\mathcal{K}_{jj}(\psi)\nu_i + (\mu + \alpha)\mathcal{K}_{ij}(\psi)\nu_j + (\xi - \alpha)\mathcal{K}_{ji}(\psi)\nu_j,$$

$$(\widetilde{R}^4\psi)_i = (\varepsilon + \upsilon - \chi)\mathcal{F}_{jj}(\psi)\nu_i + (\upsilon + \beta)\mathcal{F}_{ij}(\psi)\nu_j + (\chi - \beta)\mathcal{F}_{ji}(\psi)\nu_j,$$

$i, j = 1, 2, 3.$

Proposition 4.1. *Let R be the matrix*

$$R = \begin{pmatrix} R^1 \\ R^4 \end{pmatrix}, \tag{4.12}$$

where $R^1, R^4 : [L^p(\Sigma)]^6 \to [L^p_1(\Sigma)]^3$ are given by

$$(R^1\Phi)_j(x) = \int_\Sigma [d_x[\Psi^1_{jh}(y-x)]\varphi_h(y) + d_x[\Psi^2_{jh}(y-x)]\vartheta_h(y)]d\sigma_y,$$

$$(R^4\Phi)_j(x) = \int_\Sigma [d_x[\Psi^3_{jh}(y-x)]\varphi_h(y) + d_x[\Psi^4_{jh}(y-x)]\vartheta_h(y)]d\sigma_y.$$

Let $\widetilde{R}$ be the matrix (4.11)*. Then*

$$\widetilde{R}R\Phi = -\Phi + H^2\Phi + J\Phi, \tag{4.13}$$

where H is the integral operator

$$H\Phi(x) = \int_\Sigma S_x[\Psi(y-x)]\Phi(y)d\sigma_y,$$

and J is a compact operator from $[L^p(\Sigma)]^6$ into itself.

Proof. Let $\mathcal{U} = (u,\omega)'$ be the simple layer potential (3.16) with density $\Phi = (\varphi,\vartheta)' \in [L^p(\Sigma)]^3 \times [L^p(\Sigma)]^3$. From (4.12) we have that

$$(\widetilde{R}^1 R^1\varphi)_i = (\lambda+\mu-\xi)\mathcal{K}_{jj}(du)\nu_i + (\mu+\alpha)\mathcal{K}_{ij}(du)\nu_j + (\xi-\alpha)\mathcal{K}_{ji}(du)\nu_j,$$

$$(\widetilde{R}^4 R^4\vartheta)_i = (\epsilon+v-\chi)\mathcal{F}_{jj}(d\omega)\nu_i + (v+\beta)\mathcal{F}_{ij}(d\omega)\nu_j + (\chi-\beta)\mathcal{F}_{ji}(d\omega)\nu_j.$$

From Lemma 4.3 we have

$$(\widetilde{R}^1 R^1\varphi)_i = S_i^1(w^0), \quad i=1,2,3, \text{ a.e. on } \Sigma,$$

$$(\widetilde{R}^4 R^4\vartheta)_i = S_i^4(\zeta^0), \quad i=1,2,3, \text{ a.e. on } \Sigma.$$

Since

$$S\mathcal{W}^0 = \begin{pmatrix} S^1 w^0 + S^2\zeta^0 \\ S^4\zeta^0 \end{pmatrix},$$

$$\widetilde{R}R\Phi = \begin{pmatrix} S^1 w^0 \\ S^4\zeta^0 \end{pmatrix} = S\mathcal{W}^0 - \begin{pmatrix} S^2\zeta^0 \\ 0 \end{pmatrix}.$$

In view of (4.10), we have $\mathcal{W} = \mathcal{W}^0 + \mathcal{L}$, and then

$$\widetilde{R}R\Phi = S\mathcal{W} + J\Phi,$$

where $J : [L^p(\Sigma)]^6 \to [L^p(\Sigma)]^6$, defined as

$$J\Phi = -\begin{pmatrix} S^2\zeta^0 \\ 0 \end{pmatrix} - S\mathcal{L},$$

is a compact operator. On the other hand, from the Green representation formula [13]

$$\mathcal{W}(x) = -2\mathcal{U}(x) + \int_\Sigma \Psi(y-x)S\mathcal{U}(y)d\sigma_y, \quad x \in \Omega$$

and from the following jump formula [13, p. 493]

$$\left[S\left(\int_\Sigma \Psi(y-x)\varphi(y)d\sigma_y\right)\right]^+ = \varphi(x) + \int_\Sigma S_x(\Psi(y-x))\varphi(y)d\sigma_y, \quad x \in \Sigma,$$

we have

$$\begin{aligned} S\mathcal{W}(x) &= S\left[-2\mathcal{U}(x) + \int_\Sigma \Psi(y-x)S\mathcal{U}(y)d\sigma_y\right] \\ &= -S[\mathcal{U}(x)] + \int_\Sigma S_x[\Psi(y-x)]S\mathcal{U}(y)d\sigma_y \end{aligned}$$

$$= -S\left[\int_\Sigma \Psi(y-x)\Phi(y)d\sigma_y\right]$$
$$+ \int_\Sigma S_x[\Psi(y-x)]S_y\left[\int_\Sigma \Psi(z-y)\Phi(z)d\sigma_z\right]d\sigma_y.$$

Then

$$\begin{aligned} S\mathcal{W}(x) &= -\Phi(x) - \int_\Sigma S_x[\Psi(y-x)]\Phi(y)d\sigma_y + \int_\Sigma S_x[\Psi(y-x)]\Phi(y)d\sigma_y \\ &\quad + \int_\Sigma S_x[\Psi(y-x)]\int_\Sigma S_y[\Psi(z-y)]\Phi(z)d\sigma_z d\sigma_y \\ &= -\Phi(x) + \int_\Sigma S_x[\Psi(y-x)]\int_\Sigma S_y[\Psi(z-y)]\Phi(z)d\sigma_z d\sigma_y \\ &= -\Phi(x) + H^2\Phi(x). \end{aligned}$$ □

Theorem 4.1. *The operator R defined by* (4.12) *can be reduced on the left. A reducing operator is given by $\widetilde{R}$ with ξ and χ as in* (2.5).

Proof. Replacing in (4.13) χ and ξ given by (2.5), we obtain that the operator H is compact, because the coefficients of the singular parts in (3.8) $[\frac{(\mu+\xi)(\lambda+3\mu+\alpha)}{(\mu+\alpha)(\lambda+2\mu)} - 2]$ and $[\frac{(\chi+v)(\varepsilon+3v+\beta)}{(v+\beta)(\varepsilon+2v)} - 2]$ vanish. Then, the kernel of H has only a weak singularity:

$$T_x^0[\Psi(y-x)] = \mathcal{O}(|y-x|^{l-2}),$$

$l \in (0,1]$ being the Lyapunov exponent of surface Σ. □

5. Representation Theorem

As a by-product of our method, we obtain the representability of the solution of the Dirichlet problem with datum f given in $[W^{1,p}(\Sigma)]^6$ by means of a simple layer potential. The density of such a potential is obtained as a solution of a Fredholm equation.

Theorem 5.1. *Given $w \in [L_1^p(\Sigma)]^6$, there exists a solution $\Phi \in [L^p(\Sigma)]^6$ of the following singular integral system*

$$R\Phi = w \quad a.e.\ x \in \Sigma, \tag{5.1}$$

where R is defined as in (4.12) *if and only if*

$$\int_\Sigma \gamma_i \wedge w_i = 0 \tag{5.2}$$

for every $\gamma = (\gamma_1, \ldots, \gamma_6) \in [L_1^q(\Sigma)]^6$, $(q = p/(p-1))$ such that γ_i, $i = 1, \ldots, 6$, is a weakly closed 1-form.

Proof. Consider the adjoint of R (see (4.12)), $R^* : [L^q_1(\Sigma)]^6 \to [L^q(\Sigma)]^6$, i.e. the operator whose components are given by

$$R^*_j\psi(x) = \int_\Sigma \psi_i(y) \wedge d_y[\Psi_{ij}(y-x)].$$

Theorem 4.1 implies that the integral system (5.1) has a solution $\Phi \in [L^p(\Sigma)]^6$ if and only if the compatibility conditions

$$\int_\Sigma \psi_i \wedge w_i = 0$$

hold for any $\psi = (\psi_1, \dots, \psi_6) \in [L^q_1(\Sigma)]^6$ such that $R^*\psi = 0$. On the other hand $R^*\psi = 0$ if and only if ψ_i is a weakly closed 1-form, i.e.

$$\int_\Sigma \psi_i \wedge dg = 0 \quad \forall\, g \in \mathring{C}^\infty(\mathbb{R}^3).$$

In fact, if

$$\int_\Sigma \psi_j(y) \wedge d_y[\Psi_{ij}(y-x)] = 0 \quad \text{a.e. } x \in \Sigma, \tag{5.3}$$

we have

$$\int_\Sigma p_i(x) d\sigma_x \int_\Sigma \psi_j(y) \wedge d_y[\Psi_{ij}(y-x)] = 0 \quad \forall\, p_i \in C^\lambda(\Sigma)$$

and then

$$0 = \int_\Sigma \psi_j(y) \wedge d_y \int_\Sigma p_i(x)\Psi_{ij}(y-x) d\sigma_x = \int_\Sigma \psi_j \wedge d\mathcal{U}_j$$

for any smooth solution $\mathcal{U}$ of (2.2). Therefore we have

$$\int_\Sigma \psi_j(y) \wedge d_y[\Psi_{ij}(y-x)] = 0 \quad \forall\, x \in \mathcal{C}\overline{\Omega}.$$

Let us consider

$$z_i(x) = \int_\Sigma \psi_j(y) \wedge d_y[\Psi_{ij}(y-x)].$$

If $v \in [C^\infty(\mathbb{R}^3)]^6$ and $\eta \in [C^1(\overline{\Omega}) \cap C^2(\Omega)]^6$ are such that $M\eta = Mv$ in Ω and $\eta = 0$ on Σ, we have

$$\begin{aligned}\int_\Omega z_i(Mv)_i dx = \int_\Omega z_i(M\eta)_i dx &= \int_\Omega (M\eta)_i(x) dx \int_\Sigma \psi_j(y) \wedge d_y[\Psi_{ij}(y-x)] \\ &= \int_\Sigma \psi_j(y) \wedge d_y \int_\Omega (M\eta)_i(x)\Psi_{ij}(y-x) dx.\end{aligned}$$

From the Gauss–Green formula (see, e.g. [13, p. 146]) we have

$$\int_\Omega (\mathcal{U}M\mathcal{V} - \mathcal{V}M\mathcal{U}) dx = \int_\Sigma (\mathcal{U}T\mathcal{V} - \mathcal{V}T\mathcal{U}) d\sigma_x,$$

where $\mathcal{U}$ and $\mathcal{V}$ are smooth vector functions. Keeping in mind that $\eta = 0$ on Σ and that the column vectors $(\Psi_{1j}, \Psi_{2j}, \dots, \Psi_{6j})'$, $j = 1, \dots, 6$, are solutions of (2.2) (see

Lemma 3.1), it follows that

$$\int_{\Omega} \Psi_{ij}(M\eta)_i dx = \int_{\Sigma} \Psi_{ij}(T\eta)_i d\sigma.$$

In view of (5.3), we find

$$\begin{aligned}\int_{\Omega} z_i(Mv)_i dx &= \int_{\Sigma} \psi_j(y) \wedge d_y \int_{\Sigma} (T\eta)_i(x)\Psi_{ij}(y-x)d\sigma_y \\ &= \int_{\Sigma} (T\eta)_i(x)d\sigma_y \int_{\Sigma} \psi_j(y) \wedge d_y[\Psi_{ij}(y-x)] = 0.\end{aligned}$$

Therefore

$$\begin{aligned}0 = \int_{\mathbb{R}^3} z_i(M\varphi)_i dx &= \int_{\mathbb{R}^3} (M\varphi)_i dx \int_{\Sigma} \psi_j(y) \wedge d_y[\Psi_{ij}(y-x)] \\ &= \int_{\Sigma} \psi_j(y) \wedge d_y \int_{\mathbb{R}^3} (M\varphi)_i(x)\Psi_{ij}(y-x)dx = \int_{\Sigma} \psi_j(y) \wedge d\varphi_j,\end{aligned}$$

for any $\varphi \in [\mathring{C}^{\infty}(\mathbb{R}^3)]^6$. This shows that ψ_j is weakly closed form and the theorem is proved. □

By $\mathcal{S}^p$ we denote the class of simple layer potentials (3.16) with density belonging to $[L^p(\Sigma)]^6$.

Theorem 5.2. *Given $f \in [W^{1,p}(\Sigma)]^6$, the BVP*

$$\begin{cases}\mathcal{U} \in \mathcal{S}^p, \\ M\mathcal{U} = 0 \quad \text{in } \Omega, \\ d\mathcal{U} = df \quad \text{on } \Sigma\end{cases} \tag{5.4}$$

admits solution. It is given by (3.16) where its density γ solves the singular integral system $R\gamma = df$, where R is given by (4.12).

Proof. There exists a solution of (5.4) if and only if there exists a solution $\gamma \in [L^p(\Sigma)]^6$ of the singular integral system (4.2). In view of Theorem 5.1, there exists a solution $\gamma \in [L^p(\Sigma)]^6$ of such a system because conditions (5.2) are satisfied. □

Lemma 5.1. *Let $C \in \mathbb{R}^6$. The following BVP*

$$\begin{cases}\mathcal{V} \in \mathcal{S}^p, \\ M\mathcal{V} = 0 \quad \text{in } \Omega, \\ \mathcal{V} = C \quad \text{on } \Sigma\end{cases} \tag{5.5}$$

has one and only one solution $\mathcal{V}$ given by a simple layer potential with density belonging to $[C^{\lambda}(\Sigma)]^6$, $0 < \lambda < l \leq 1$.

Proof. It is known that there exists a unique solution of $M\mathcal{V} = 0$ in Ω, $\mathcal{V} = C$ on Σ belonging to $[C^{1,\lambda}(\overline{\Omega})]^6$, $0 < \lambda < l \leq 1$. This result can be proved as in [13, Theorem 5.3, p. 367]. It is also known that the following BVP

$$\begin{cases} \mathcal{U} \in S^p, \\ M\mathcal{U} = 0 & \text{in } \Omega, \\ T\mathcal{U} = T\mathcal{V} & \text{on } \Sigma \end{cases}$$

has solution. This means that the solution $\mathcal{U}$ of the problem $M\mathcal{U} = 0$ in Ω, $T\mathcal{U} = T\mathcal{V}$ on Σ can be represented by a simple layer potential (see [13, p. 501]). It follows from the uniqueness theorem (see [13, Theorem 4.2, p. 148]) that there exist $a, b \in \mathbb{R}^3$ such that

$$\mathcal{U} = \mathcal{V} + \begin{pmatrix} (a \wedge x) + b \\ a \end{pmatrix}.$$

Since the column vector $\mathcal{Z} = (a \wedge x) + b, a)'$ is a solution of the problem: $M\mathcal{Z} = 0$ in Ω, $T\mathcal{Z} = 0$ on Σ, it can be represented by means of a simple layer potential with an Hölder continuous density. Therefore, $\mathcal{V} = \mathcal{U} - \mathcal{Z}$ satisfies the assertion of the lemma. □

Theorem 5.3. *The following Dirichlet BVP*

$$\begin{cases} \mathcal{U} \in \mathcal{S}^p, \\ M\mathcal{U} = 0 & \text{in } \Omega, \\ \mathcal{U} = f & \text{on } \Sigma, \quad f \in [W^{1,p}(\Sigma)]^6 \end{cases} \tag{5.6}$$

admits a unique solution $\mathcal{U}$. In particular, the density Φ of $\mathcal{U}$ can be written as $\Phi = \Phi_0 + \Gamma_0$, where Φ_0 solves the singular integral system

$$\int_\Sigma d_x[\Psi_{ij}(y - x)]\Phi_{0j}(y)d\sigma_y = df_i(x), \quad i = 1, \ldots, 6, \quad a.e.\ x \in \Sigma$$

and Γ_0 is the density of a simple layer potential which is constant on Σ.

Proof. Let $\widetilde{\mathcal{U}}$ be a solution of (5.4). Since $d\widetilde{\mathcal{U}} = df$ on Σ and Σ is connected, $\widetilde{\mathcal{U}} = f - C$ on Σ, $C \in \mathbb{R}^6$. Then $\mathcal{U} = \widetilde{\mathcal{U}} + \mathcal{V}$, $\mathcal{V}$ being solution of (5.5), solves (5.6). The uniqueness follows from [13, Theorem 4.1, p. 148]. □

Theorem 5.4. *If $f \in [W^{1,p}(\Sigma)]^6$, the singular integral system $R\Phi = df$ is equivalent to the Fredholm system $\widetilde{R}R\Phi = \widetilde{R}(df)$.*

Proof. As in [2, pp. 253–254], one can show that $N(\widetilde{R}R) = N(R)$. This implies that, if g is such that there exists a solution Φ of the equation $R\Phi = g$, then this equation is satisfied if and only if $\widetilde{R}R\Phi = \widetilde{R}g$. Since we know that the equation $R\Phi = df$ is solvable, we have that $R\Phi = df$ if and only if $\widetilde{R}R\Phi = \widetilde{R}(df)$. □

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