PARSEVAL FRAMES BUILT UP FROM GENERALIZED SHIFT INVARIANT SYSTEMS

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ABSTRACT. Wavelet systems, and many of its generalizations such as wavelet packets, shearlets, and composite dilation wavelets are generalized shift invariant systems (GSI) in the sense of the work by Ron and Shen.

It is well known that a wavelet system is never **Z**-shift invariant (SI). Nevertheless, one can modify it and construct a **Z**-SI system, called a *quasi-affine system*, which shares most of the frame properties of the wavelet system. The analogue of a quasi-affine system for a GSI system is called an oblique oversampling: it is shift invariant with respect to a fixed lattice.

Assumptions on a GSI system X were given by Ron and Shen to ensure that any oblique oversampling is a Parseval frame for $L^2(\mathbb{R}^n)$ whenever X is.

We show that these assumptions are not satisfied for some of the wavelet generalizations mentioned above and that elements implicit in their work provide other sufficient conditions on the system under which any oblique oversampling is a Parseval frame for $L^2(\mathbb{R}^n)$ (shift invariant with respect to a fixed lattice).

Moreover, in the orthonormal setting it is shown that completeness yields a shift invariant Parseval frame for suitable proper subspaces of $L^2(\mathbb{R}^n)$, too.

1. INTRODUCTION

Let J be a countable index set. For any $j \in J$, let $\Gamma_j = A_j\mathbb{Z}^n$ be a lattice of full rank, where A_j is a $n \times n$ nonsingular real matrix. Let us denote the determinant of the lattice by $|\Gamma_j| = |\det A_j|$. For any $x \in \mathbb{R}^n$, and $f \in L^2(\mathbb{R}^n)$ let $T_x f(y) = f(y - x)$ be the translation of f by x. The following object is defined in a paper by Ron and Shen, [8].

Definition 1.1. Let $\varphi_j \in L^2(\mathbb{R}^n)$ be a fixed function for any $j \in J$.
The set

$$X = \{T_\gamma \varphi_j,\, j \in J,\, \gamma \in \Gamma_j\},$$

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is called a generalized shift invariant (GSI) system.

The name is justified by the fact that if we set $X_j = \{T_\gamma \varphi_j,\ \gamma \in \Gamma_j\}$, then $X = \bigcup_{j\in J} X_j$ and X_j is Γ_j-shift invariant. Any φ_j is called a generating function.

Wavelet systems, and many of its generalizations such as wavelet packets [2], shearlets [4], and composite dilation wavelets [5], are generalized shift invariant systems.

For example, in the case of composite dilation wavelets [5], starting from two (countable) subgroups A, B of $GL_n(\mathbb{R})$ and a lattice $\Gamma = c\mathbb{Z}^n$, $c \in GL_n(\mathbb{R})$, one considers

$$X = \{D_a\, D_b\, T_\gamma \psi, \quad a \in A, \quad b \in B, \quad \gamma \in \Gamma\}, \quad \psi \in L^2(\mathbb{R}^n),$$

where $D_a f(x) = |\det a|^{1/2} f(ax)$ is the dilation of f by the matrix $a \in A$.

The rule which allows to exchange dilations with translations can be applied twice

$$D_a\, D_b\, T_\gamma \psi = D_a\, T_{b^{-1}\gamma} D_b \psi = T_{a^{-1}\,b^{-1}\,\gamma} D_a\, D_b \psi,$$

and so, by taking $J = A \times B$ as index set, $\Gamma_{(a,b)} = a^{-1}\, b^{-1}\, c\,\mathbb{Z}^n$ as lattice, and $\varphi_{(a,b)} = D_a\, D_b \psi$ as generating function, X can be viewed as a GSI system.

It is well known that a wavelet (affine) system

$$\{2^{j/2}\, \psi(2^j \cdot -k), \quad j, k \in \mathbb{Z}\}, \quad \psi \in L^2(\mathbb{R})\ ,$$

is never **Z**-shift invariant (SI). Nevertheless, one can modify it and construct a **Z**-SI system, called a *quasi-affine system*

$$\{2^{j/2}\, \psi(2^j \cdot -k),\ j \geq 0,\ k \in \mathbb{Z}\} \cup \{2^j\, \psi(2^j(\cdot - k)),\ j < 0,\ k \in \mathbb{Z}\}, \tag{1}$$

which shares most of the frame properties of the wavelet system [7], [3].

Definition 1.2. A frame for a Hilbert space $\mathcal{H}$ is a collection of vectors $x_i \in \mathcal{H}$, $i \in I$, such that there exist $A, B > 0$

$$A\|x\|^2 \leq \sum_{i\in I} |< x, x_i >|^2 \leq B\|x\|^2 \tag{2}$$

for all $x \in \mathcal{H}$.

If $A = B = 1$ the frame is called a Parseval frame.

If the system verifies only the right hand side in equation (2) it is called a Bessel system with Bessel bound B.

Theorem 1.3 (Ron, Shen [7], Chui, Shi, Stöckler [3]). *A wavelet system is a frame $\Longleftrightarrow$ its quasi-affine counterpart (1) is a frame (with the same frame bounds).*

There is an analogue of a quasi-affine system for a GSI system, it is called an *oblique oversampling*, (see Definition 2.3): it is shift invariant with respect to a fixed lattice.

Assumptions on a GSI system X were given by Ron and Shen to ensure that any oblique oversampling is a Parseval frame for $L^2(\mathbb{R}^n)$ whenever X is;

Unfortunately these assumptions (small tail condition and temperateness, we remand to [8] for their definition) are not satisfied for some of the wavelet generalizations mentioned above and we look for other conditions motivated by the following.

It has been shown in [9] that there is a wavelet packet, constructed from the Lemarié-Meyer wavelet, and corresponding to an a.e. partition of $[0, +\infty)$ which is not an orthonormal basis of $L^2(\mathbb{R})$. The proof relies on the fact that it possible to obtain, from that particular wavelet packet, a $\mathbb{Z}$-shift invariant Parseval frame of $L^2(\mathbb{R})$; moreover the assumption to be a basis implies that even suitable subspaces of $L^2(\mathbb{R})$ have a $\mathbb{Z}$-shift invariant Parseval frame, this fact yields a contradiction.

The purpose of this paper is to show that the existence of such a Parseval frame does not depend on the particular choice of the wavelet packet. Indeed, after the needed definitions, we prove the main result in Section 2

Theorem 1.4. *Let X be a GSI system with the following properties*

1. *X is nested;*
2. *X is Bessel with Bessel bound less then or equal to 1;*
3. *The diagonal function of X satisfies $\widetilde{g}(\xi) \geq 1$, a.e.,*
   (and so $\widetilde{g}(\xi) = 1$).

Then any oblique oversampling of X is a Parseval frame for $L^2(\mathbb{R}^n)$.

The following corollary follows

Corollary 1.5. *Let X be a nested GSI system with the following properties*

1. *X is a Parseval frame for $L^2(\mathbb{R}^n)$;*
2. *The diagonal function of X satisfies $\widetilde{g}(\xi) \geq 1$, a.e.*

Then any oblique oversampling of X is a Parseval frame for $L^2(\mathbb{R}^n)$.

The paper is organized as follows. In Section 2 we recall the main definitions and we prove Theorem 1.4. In Section 3 we show that a wavelet packets system based on the Shannon multiresolution analysis provides an example of a GSI system verifying the hypotheses of Theorem 1.4 which does not satisfies either the small tail condition nor the temperateness condition. The case of proper subspaces of $L^2(\mathbb{R}^n)$ is treated in Section 4. When *orthonormal basis* replaces *Bessel system* it is shown that a shift invariant Parseval frame can be obtained for suitable proper subspaces of $L^2(\mathbb{R}^n)$, too. The passage is not straightforward and it will be clear the role played by the completeness of the system, thus enlightening the analogous result in [9].

2. Generalized shift-invariant systems

In this section we give a quick review of needed definitions and known results by Ron and Shen in [8], and prove the main result. We stress the fact that the proof is implicit in their work. We remand the interested reader to their paper for the meaning of the objects involved and the missing details.

Definition 2.1. Let X be a GSI system, and let us assume $\leq$ is a total ordering of the index set J. We say that X is nested if, for every $j, j' \in J$,

$$j \leq j' \Longrightarrow \Gamma_j \subset \Gamma_{j'}. \tag{3}$$

It should be said that in the original definition of nested systems an equivalence arrow replaces the right arrow, but (3) is what actually needs in the proofs.

Let us recall that Γ^* denotes the dual lattice of Γ, i.e.

$$\Gamma^* = \{x \in \mathbb{R}^n,\, x \cdot \gamma \in \mathbb{Z}, \forall \gamma \in \Gamma\} = (A^{*-1})\mathbb{Z}^n.$$

Definition 2.2. Let X be a GSI system. We say that X is tailless if, for every compact $\Omega \subset \mathbb{R}^n$ that does not contain the origin, the number of different lattices Γ_j that satisfy $2\pi\Gamma_j^* \cap \Omega \neq \emptyset$ is finite.

Definition 2.3. Let X and X^0 be two GSI systems with the same index set J. Let φ_j and φ_j^0 be the relative generating functions. We say that X^0 is an oversampling of X if, for every $j \in J$ the following holds:

1) The lattice $\Gamma_j^0 = A_j^0\mathbb{Z}^n$ is a superlattice of $\Gamma_j = A_j\mathbb{Z}^n$;
2) The following relation holds

$$\varphi_j^0 = \left(\frac{|\Gamma_j^0|}{|\Gamma_j|}\right)^{\frac{1}{2}} \varphi_j.$$

The notation X^0 will always denote an oversampling of X.

If X is nested we say that the oversampling is oblique if there exists $j_0 \in J$ such that, for every $j \in J$,

$$\Gamma_j^0 = \begin{cases} \Gamma_j, & j > j_0, \\ \\ \Gamma_{j_0}, & j \leq j_0. \end{cases}$$

Let us note that the oblique oversampling X^0 admits an underling lattice, namely Γ_{j_0} which is contained in every Γ_j^0, for every $j \in J$, hence X^0 is Γ_{j_0}-shift invariant (Γ_{j_0}-SI in short). As a consequence, X^0 is tailless.

The main tool in the study of GSI systems in [8] is the dual Gramian of X, $\widetilde{G} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$,

$$\widetilde{G}(\xi, \eta) = \sum_{j \in \kappa(\xi - \eta)} \frac{\hat{\varphi}_j(\xi)\overline{\hat{\varphi}_j(\eta)}}{|\det A_j|},$$

where the valuation function κ is defined as

$$\kappa(\xi) := \{j \in J, \xi \in 2\pi\Gamma_j^*\} = \{j \in J, \xi \in 2\pi(A_j^{*-1})\mathbb{Z}^n\}.$$

In other words, for a fixed $\xi \in \mathbb{R}^n$, $\widetilde{G}(\xi, \eta) = 0$ unless η lies in the countable set $\xi + \bigcup_{j \in J}(2\pi(A_j^{*-1})\mathbb{Z}^n)$.

Another important tool is the diagonal function,

$$\widetilde{g}(\xi) = \widetilde{G}(\xi, \xi) = \sum_{j \in J} \frac{|\hat{\varphi}_j(\xi)|^2}{|\det A_j|},$$

since $\kappa(0) = \{j \in J, 0 \in 2\pi(A_j^{*-1})\mathbb{Z}^n\} = J$.

The following result appears in [6, Proposition 4.1].

Proposition 2.4. *Let the GSI system* $\{T_\gamma \varphi_j, j \in J, \gamma \in \Gamma_j\}$ *be a Bessel system with Bessel bound less then or equal to* 1.

Then for almost all $\xi \in \mathbb{R}^n$,

$$\widetilde{g}(\xi) = \sum_{j \in J} \frac{|\hat{\varphi}_j(\xi)|^2}{|\det A_j|} \leq 1.$$

We are ready to prove the main result.

Proof of Theorem 1.4

The proof makes use of some results in [8] applied to X and X^0. By [8, Corollary 3.18] X dominates X^0, (in the original version of the result there is an additional requirement on the GSI system: to have a small tail; it is needed only to obtain another result).

Since X^0 is tailless and X is Bessel with Bessel bound less then or equal to 1, the first part of the proof of [8, Proposition 3.15], together with [8, Theorem 2.14, and 3.4] applied to X^0, yield that X^0 is Bessel too, with bound less then or equal to 1. So [8, Lemma 3.6] applies to X^0, and together with $\widetilde{g}_{X^0}(\xi) = \widetilde{g}_X(\xi) = 1$, [8, Proposition 3.9] and Proposition 2.4, shows that X^0 is scalar. Finally taillessness and [8, Corollary 3.7] ensure that X^0 is a Parseval frame. □

3. Wavelet packets, temperateness, and the small tail condition

We show first that wavelet packets yield nested GSI systems. Then we consider the simple case of Shannon wavelet packets, and we give an example of a Bessel system (with Bessel bound less then or equal to 1) that does not satisfy both the small tail and temperateness condition; moreover its diagonal function is equal to 1 (and so Theorem 1.4 applies).

Let us first recall the definition of wavelet packets, [2]. Consider a pair of quadratic mirror filters (QMF) with transfer functions $m_0(\theta)$ and $m_1(\theta) = e^{-i\theta}\overline{m_0(\theta+\pi)}$ associated to a multiresolution analysis (MRA) with wavelet $\psi \in L^2(\mathbb{R})$ and scaling function $\varphi \in L^2(\mathbb{R})$. The basic wavelet packets are elements of $L^2(\mathbb{R})$ and are defined recursively by the formulas (for the Fourier transform):

$$\hat{w}_0(\theta) = \hat{\varphi}(\theta), \quad \hat{w}_1(\theta) = \hat{\psi}(\theta),$$
$$\hat{w}_{2n}(\theta) = m_0(\frac{\theta}{2})\hat{w}_n(\frac{\theta}{2}),$$
$$\hat{w}_{2n+1}(\theta) = m_1(\frac{\theta}{2})\hat{w}_n(\frac{\theta}{2}).$$

The general wavelet packets are defined by taking some of the dilation and translation of the basic ones, i.e.,

$$2^{q/2}w_n(2^q x - k), \quad k \in \mathbb{Z}, \quad (n,q) \in F \subset \mathbb{N}\times\mathbb{Z}. \tag{4}$$

By sake of brevity we shall call (4) wavelet packets again.

Any system made of wavelet packets

$$X = \{2^{q/2}w_n(2^q x - k), k \in \mathbb{Z},\ (n,q) \in F\}, \tag{5}$$

$F \subset \mathbb{N}\times\mathbb{Z}$, can be viewed as a nested GSI system. Indeed it is sufficient to take $J = F$ as index set, and for any $(n,q) \in F$, $\Gamma_{(n,q)} = A_{(n,q)}\mathbb{Z} = 2^{-q}\mathbb{Z}$ as lattice, and $\varphi_{(n,q)} = D_{2^q}w_n$, as the generating function, so

$$\{\varphi_{(n,q)}(x - 2^{-q}k),\ 2^{-q}k \in 2^{-q}\mathbb{Z},\ (n,q) \in F\} \quad = \quad X.$$

If we define a total ordering on $F \subset \mathbb{N}\times\mathbb{Z}$ as the lexicographical order

$$(n,q) \le (n',q') \Leftrightarrow \text{either } q < q',\ \text{or, in the case } q = q',\ n \le n',$$

it is easy to see that X is nested since

$$(n,q) \leq (n',q') \Longrightarrow 2^{-q'} \leq 2^{-q} \Longrightarrow 2^{-q}\mathbb{Z} \subset 2^{-q'}\mathbb{Z}.$$

It should be noted that if the set of indexes is such that all the intervals

$$I_{n,q} = [\frac{n}{2^q}, \frac{n+1}{2^q}),$$

are mutually disjoint, then the associated closed subspaces of $L^2(\mathbb{R})$

$$W_{n,q} = \overline{\operatorname{span}}\{2^{q/2} w_n(2^q x - k), k \in \mathbb{Z}\},$$

are mutually orthogonal and the GSI system X is Bessel, with bound less then or equal to 1.

We pass to construct an oblique oversampling for any system X given by (5).

Let $(n_0, q_0) \in F$ be fixed. Actually we focus on q_0 and for any $(n,q) \in F$ we make a choice as follows.

a) If $q > q_0$ we leave things unchanged: we take generating function $\varphi^0_{(n,q)} = D_{2^q} w_n$, and lattice $\Gamma^0_{(n,q)} = 2^{-q}\mathbb{Z}$.

b) If $q \leq q_0$ we modify the norm by taking generating function $\varphi^0_{(n,q)} = 2^{(q-q_0)/2} D_{2^q} w_n$, and we take the fixed lattice $\Gamma^0_{(n,q)} = 2^{-q_0}\mathbb{Z}$, for all $q \leq q_0$.

It is easy to see that, in both cases, $\Gamma^0_{(n,q)}$ is a superlattice of $\Gamma_{(n,q)} = 2^{-q}\mathbb{Z}$. The equality 2) of Definition 2.3 is also satisfied, since, if $q < q_0$,

$$\varphi^0_{(n,q)}(x) = \left(\frac{2^{-q_0}}{2^{-q}}\right)^{\frac{1}{2}} 2^{q/2} w_n(2^q x) = \left(\frac{|\,\Gamma^0_{(n,q)}\,|}{|\,\Gamma_{(n,q)}\,|}\right)^{\frac{1}{2}} \varphi_{(n,q)},$$

the other case being trivial.

The oversampling is oblique, with respect to the total ordering of F, with (n_0, q_0) acting as the right index, since for every $(n,q) \in F$,

$$\Gamma^0_{(n,q)} = \begin{cases} 2^{-q}\mathbb{Z} = \Gamma_{(n,q)}, & (n,q) > (n_0,q_0), \\ \\ 2^{-q_0}\mathbb{Z} = \Gamma_{(n_0,q_0)}, & (n,q) \leq (n_0,q_0). \end{cases}$$

Hence the oversampling is tailless. Let us recall that taillessness in one dimension means that the set of all different numbers in the set $\{2^{-q}, q \geq q_0\}$ is bounded and has no accumulation points other then 0. Let us note also that the original system we started with might not be tailless.

The Gramian of any wavelet packet system is $\widetilde{G} : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$,

$$\widetilde{G}(\xi, \eta) = \sum_{(n,q) \in \kappa(\xi - \eta)} 2^q D_{2^{-q}} \hat{w}_n(\xi) \overline{D_{2^{-q}} \hat{w}_n(\eta)},$$

where the valuation function κ is given by

$$\kappa(\xi) = \{(n, q) \in F, \xi \in 2\pi 2^q \mathbb{Z}\}.$$

Furthermore the diagonal function is

$$\widetilde{g}(\xi) = \widetilde{G}(\xi, \xi) = \sum_{(n,q)\in F} |\hat{w}_n(2^{-q}\xi)|^2.$$

We consider now Shannon wavelet packets, i.e. those generated from the Shannon wavelet, and we provide an example of a Bessel system (with Bessel bound less then or equal to 1) for which both the temperateness and the small tail condition do no hold, while the diagonal function satisfies the condition $\widetilde{g}(\xi) = 1$, a.e. $\xi \in \mathbb{R}$.

It will be useful to recall some known properties of Shannon wavelet packets.

Every wavelet packet w_n arising from the Shannon wavelet satisfies the following equality for its Fourier transform, [10]:

$$|\hat{w}_n(\xi)| = \chi_{A_{0,n}}(\xi) + \chi_{A_{0,n}}(-\xi), \tag{6}$$

where the integers $k(n)$, defining intervals $A_{0,n} = \left[\frac{k(n)}{2}, \frac{k(n)+1}{2}\right)$, are given in the next theorem. Before stating it we recall that for any diadic decomposition $n = \varepsilon_1 + 2\varepsilon_2 + \cdots + 2^{j-1}\varepsilon_j$, $\varepsilon_i = 0, 1$, we say that n has an even sequence if $\sum_{i=0}^{j} \varepsilon_i$ is even, otherwise we say that it has an odd sequence.

Theorem 3.1. (1) *$k(0) = 0$;*

(2) *If $n \in \mathbb{N}$ has an even sequence, then $k(n) = 2k\left(\left[\frac{n}{2}\right]\right)$;*

(3) *If $n \in \mathbb{N}$ has an odd sequence, then $k(n) = 2k\left(\left[\frac{n}{2}\right]\right) + 1$.*

(As usual $\left[\frac{n}{2}\right]$ denotes the floor function of $\frac{n}{2}$).

Example. The example we consider is given by the following GSI system

$$Y = \{2^{q/2} w_n(2^q x - k),\ k \in \mathbb{Z},\ (n, q) \in E\},$$

where we take $(n, q) \in E \subset \mathbb{N} \times \mathbb{Z}$ if and only if (see Figure 1)

- $q = 0$, and $n \geq 1$;
- $q = -2p$, $p \in \mathbb{N}^*$, and $n = \sum_{i=1}^{2p} \varepsilon_i 2^{i-1}$, where $\varepsilon_1 = 0$, $\varepsilon_{2i+1} = 1$, $i = 1, \ldots, p-1$, and $\varepsilon_i = 0, 1$.

The small tail condition is equivalent to the following, see [8, Definition 2.24, Remark 2.26]: Y has a small tail if and only if for every compact set Ω that excludes the origin, there exists a decomposition

$$E = E_1 \cup E_2$$

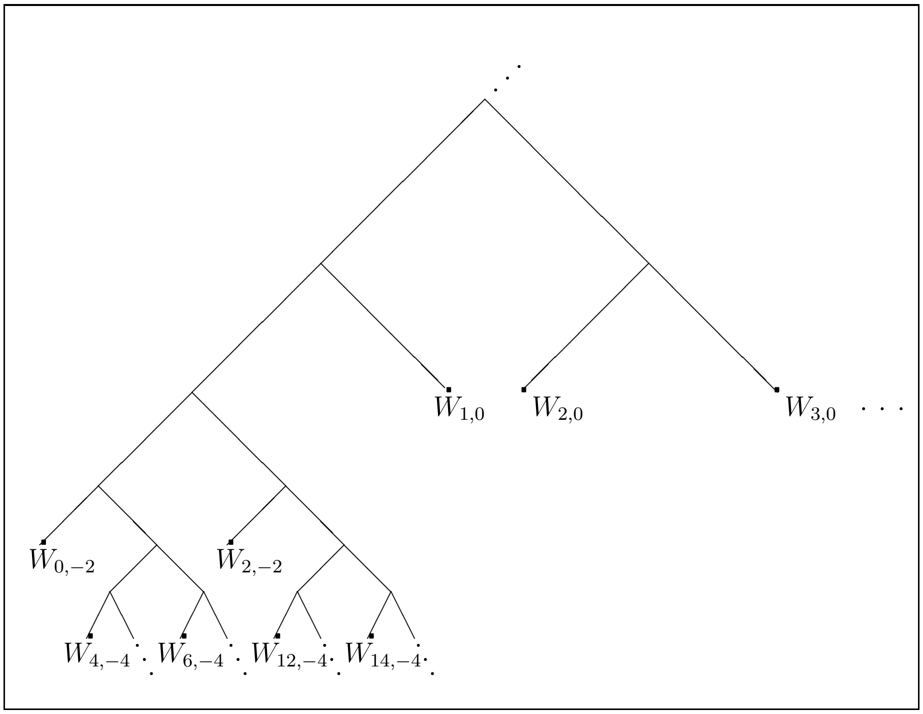

FIGURE 1. The closed subspaces of $L^2(\mathbb{R})$ corresponding to our choice of E.

such that

(i) $Y_1 := \{2^{q/2} w_n(2^q x - k),\ k \in \mathbb{Z}, (n,q) \in E_1\}$ is tailless;

(ii) $\sum_{(n,q)\in E_2} 2^q \, \|\sum_{k\in\mathbb{Z}} \chi_\Omega(\cdot + 2^q k) \mid D_{2^{-q}} \hat{w}_n(\cdot + 2^q k) \mid^2\|_\infty < +\infty.$

It is easy to see that (ii) is equivalent to the following

$$\sum_{(n,q)\in E_2} \|\sum_{k\in\mathbb{Z}} \chi_{2^{-q}\Omega - k}(\chi_{A_{0,n}-k} + \chi_{-A_{0,n}-k})^2\|_\infty < +\infty,$$

which, in turn, being $A_{0,n} - k$ and $-A_{0,n} - k$ disjoint, is equivalent to

$$\sum_{(n,q)\in E_2} \|\sum_{k\in\mathbb{Z}} (\chi_{(A_{0,n}\cap 2^{-q}\Omega)-k} + \chi_{(-A_{0,n}\cap 2^{-q}\Omega)-k})\|_\infty < +\infty.$$

Also let us note that (i) is verified if and only if the set $(2^{-q})_{(n,q)\in E_1}$ is bounded (see [8, Discussion 2.18]) that means that the set of indexes $(n,q) \in E_1$, $q < 0$ is finite. So, in order to prove that Y has not a small tail, it is sufficient to show that there exists a compact set Ω, that excludes the origin, such that, for every $q_0 < 0$

$$\sum_{(n,q)\in E, q<q_0} \|\sum_{k\in\mathbb{Z}} (\chi_{(A_{0,n}\cap 2^{-q}\Omega)-k} + \chi_{(-A_{0,n}\cap 2^{-q}\Omega)-k})\|_\infty = +\infty.$$

Proposition 3.2. *Let*

$$\Omega = \left[\frac{1}{4}, \frac{1}{2}\right],$$

then for any $q \in \mathbb{Z}$, $q \leq -2$, there exists $n \in \mathbb{N}$, $(n,q) \in E$ such that $A_{0,n} \subset 2^{-q}\Omega$.

Proof. It will be showed, by induction on $q = -2p$, $p \geq 1$, that there exists $n \in \mathbb{N}$

$$n = 2\varepsilon_2 + 4\varepsilon_3 + \cdots + 2^{-q-1}\varepsilon_{-q}, \quad \varepsilon_{2i+1} = 1,\ i = 1, \ldots, p-1,$$

such that

$$2^{-q-1} \leq k(n) < 2^{-q}.$$

The statement is true for $q = -2$ by taking $n = 2$.
Assuming it is true for q, let us consider $q-2$. By induction hypothesis, let

$$n = 2\varepsilon_2 + 2^2 + 2^3\varepsilon_4 + \cdots + 2^{-q-2} + 2^{-q-1}\varepsilon_{-q},$$

such that

$$2^{-q-1} \leq k(n) < 2^{-q},$$

i.e.

$$2^{-(q-2)-1} \leq 4k(n) < 4k(n) + 3 \leq 2^{-(q-2)} - 1 < 2^{-(q-2)}.$$

Now n, $2n$ and $4n$ have a sequence of the same type. In the even case

$$k(4n) = 2k\left(\left[\frac{4n}{2}\right]\right) = 2k(2n) = 4k(n),$$

while in the odd case

$$k(4n) = 2k\left(\left[\frac{4n}{2}\right]\right) + 1 = 2k(2n) + 1 = 2(2k(n)+1) + 1 = 4k(n) + 3.$$

Since $(q-2, 4n) \in E$ the proof is completed. $\square$

Corollary 3.3. *The GSI system Y does not have a small tail.*

Proof. Let us take $\Omega = \left[\frac{1}{4}, \frac{1}{2}\right]$ and, for any $q \leq -2$, let us select an integer n_q provided by Proposition 3.2. Then

$$\sum_{(n,q)\in E, q<q_0} \|\sum_{k\in\mathbb{Z}} (\chi_{(A_{0,n}\cap 2^{-q}\Omega)-k} + \chi_{(-A_{0,n}\cap 2^{-q}\Omega)-k})\|_\infty$$

$$\geq \sum_{q<q_0, q \text{ even}} \|\sum_{k\in\mathbb{Z}} \chi_{(A_{0,n_q}\cap 2^{-q}\Omega)-k}\|_\infty = \sum_{q<q_0, q \text{ even}} 1 = +\infty.$$

□

Remark 3.4. Another example of a wavelet packet system which does not have a small tail is the one generated by the Lemariè-Meyer wavelet, with the same choice of index set E. We can apply the same reasoning as before since in $A_{0,n} \cup -A_{0,n}$

$$|\hat{w}_n(\xi)| > \frac{\sqrt{2}}{2}.$$

Indeed

$$\begin{aligned} & \sum_{(n,q)\in E_2} 2^q \, \|\sum_{k\in\mathbb{Z}} \chi_\Omega(\cdot + 2^q k) \mid D_{2^{-q}}\hat{w}_n(\cdot + 2^q k) \mid^2\|_\infty \\ = & \sum_{(n,q)\in E_2} \|\sum_{k\in\mathbb{Z}} \chi_{2^{-q}\Omega}(\cdot + k) \mid \hat{w}_n(\cdot + k) \mid^2\|_\infty \\ \geq & \frac{1}{2}\sum_{(n,q)\in E_2} \|\sum_{k\in\mathbb{Z}} (\chi_{(A_{0,n}\cap 2^{-q}\Omega)-k} + \chi_{(-A_{0,n}\cap 2^{-q}\Omega)-k})\|_\infty = +\infty. \end{aligned}$$

We pass now to explore the temperateness of the system Y. Let us recall the definition.

Y is tempered if for every compact set Ω that excludes the origin, there exists a decomposition

$$E = E_1 \cup E_2,$$

such that

(iii) $Y_1 := \{2^{q/2} w_n(2^q x - k),\ k \in \mathbb{Z}, (n,q) \in E_1\}$ is tailless;
(iv)

$$\sum_{(n,q)\in E_2} \|D_{2^{-q}}\hat{w}_n\|^2_{L^2(\Omega)} < +\infty.$$

Corollary 3.5. *The GSI system Y is not tempered.*

Proof. Again let us take $\Omega = \left[\frac{1}{4}, \frac{1}{2}\right]$ and, for any $q \leq -2$, let us select an integer n_q provided by Proposition 3.2. Then

$$\begin{aligned} \sum_{(n,q)\in E_2} \|D_{2^{-q}}\hat{w}_n\|^2_{L^2(\Omega)} &\geq \sum_{q<q_0,\, q \text{ even}} \int_{2^{-q}\Omega} \mid \hat{w}_{n_q}(\xi) \mid^2 \, d\xi \\ \geq \sum_{q<q_0,\, q \text{ even}} \int_{2^{-q}\Omega} \chi_{A_{0,n_q}}(\xi)\, d\xi &= \sum_{q<q_0,\, q \text{ even}} 2^{-1} = +\infty. \end{aligned}$$

□

Concerning again the Shannon wavelet, due to the simple form of its Fourier transform, given in (6), it is not hard to see that, whenever the index set F verifies the identity

$$\bigcup_{(n,q)\in F} [\frac{n}{2^q}, \frac{n+1}{2^q}) = [0, +\infty), \quad (n,q) \in F, \tag{7}$$

(here the dot means disjoint union), except for a set of zero Lebesgue measure, then the diagonal function of the corresponding system of wavelet packets satisfies the condition $\widetilde{g}(\xi) = 1$. It turns out that any such system is an orthonormal basis of $L^2(\mathbb{R})$, too. So is our example.

On the other hand, in [9] we have proved that there exists a (Lemariè-Meyer) orthonormal system of wavelet packets verifying (7), except for a set of zero Lebesgue measure, and $\widetilde{g}(\xi) = 1$, a.e., which is not a basis, thus showing that the latter property on the diagonal function is independent from completeness.

It is our belief, however, that (7) implies $\widetilde{g}(\xi) \geq 1$ whatever the wavelet packet considered.

4. Shift invariant NTF for subspaces

In Theorem 1.4 we have given a sufficient condition for a nested GSI system to generate a Parseval frame for $L^2(\mathbb{R}^n)$ which is shift invariant with respect to a certain lattice. In this section we add orthonormality and we explore whether we can obtain a shift invariant Parseval frame for proper subspaces of $L^2(\mathbb{R}^n)$ of the form $\overline{\text{span}}\{T_\gamma\varphi_j, j \in F, \gamma \in \Gamma_j\}$, where F is a subspace of indexes. These kind of spaces are of great interest, for example the space generated by negative dilates of an affine system studied by Bownik and Speegle in [1] is of this type.

It turns out that we need to impose an additional condition on the original system: completeness.

Proposition 4.1. *Let*

$$Y = \{T_\gamma\varphi_j, j \in J, \gamma \in \Gamma_j\},$$

be a nested GSI system with the following properties

(1) *Y is an orthonormal basis for $L^2(\mathbb{R}^n)$;*
(2) $\widetilde{g}(\xi) \geq 1$.

Let Y^0 be any oblique oversampling of Y, with generating functions φ_j^0; let j_0 denote the index provided by Definition 2.3.

Let $F \subset \{j \in J, j < j_0\}$ be any subset of indexes.

Then the Γ_{j_0}-SI system $\{T_\gamma\varphi_j^0, j \in F, \gamma \in \Gamma_{j_0}\}$ is a Parseval frame for $\overline{span}\{T_\gamma\varphi_j, j \in F, \gamma \in \Gamma_j\}$.

Proof. Let Y^0 be any oblique oversampling of Y with generating functions φ_j^0. Recall that the lattices of Y^0 are, for a fixed index j_0,

$$\Gamma_j^0 = \begin{cases} \Gamma_j, & j > j_0, \\ \\ \Gamma_{j_0}, & j \leq j_0. \end{cases}$$

Since Y verifies hypotheses of Proposition 1.4, Y^0 is a Parseval frame for $L^2(\mathbb{R}^n)$.

We first prove that $T_\gamma \varphi_j^0$, for $\gamma \in \Gamma_{j_0}$, and $j \in F$, belongs to

$$V = \overline{\text{span}}\{T_\gamma \varphi_j, j \in F, \gamma \in \Gamma_j\}.$$

Indeed, since $\{T_\delta \varphi_i, i \in J, \delta \in \Gamma_i\}$ is an orthonormal basis, we can write

$$T_\gamma \varphi_j^0 = \sum_{i \in J, \delta \in \Gamma_i} < T_\gamma \varphi_j^0, T_\delta \varphi_i > T_\delta \varphi_i. \tag{8}$$

Now let us compute the coefficients for $i \notin F$ and $\delta \in \Gamma_i$. We have:

$$\begin{aligned} < T_\gamma \varphi_j^0, T_\delta \varphi_i > &= \int_{\mathbb{R}^n} \varphi_j^0(x-\gamma)\overline{\varphi_i(x-\delta)}\, dx \\ &= \left(\frac{|\Gamma_j^0|}{|\Gamma_j|}\right)^{\frac{1}{2}} \int_{\mathbb{R}^n} \varphi_j(x-\gamma)\overline{\varphi_i(x-\delta)}\, dx \\ &= \left(\frac{|\Gamma_j^0|}{|\Gamma_j|}\right)^{\frac{1}{2}} \int_{\mathbb{R}^n} \varphi_j(x)\overline{\varphi_i(x-(\delta-\gamma))}\, dx \\ &= 0, \end{aligned}$$

since $\Gamma_{j_0} \subset \Gamma_i$ and so $\delta - \gamma \in \Gamma_i$, also $i \notin F$ implies $i \geq j_0 > j$ so that orthonormality condition implies $\varphi_j \perp \{T_\eta \varphi_i, \eta \in \Gamma_i\}$.

So that, we can rewrite expansion (8) as

$$T_\gamma \varphi_j^0 = \sum_{i \in F, \delta \in \Gamma_i} < T_\gamma \varphi_j^0, T_\delta \varphi_i > T_\delta \varphi_i \in V.$$

Finally by the Parseval frame condition and similar calculations as above, for any $f \in V$ we have

$$\begin{aligned} \|f\|_2^2 &= \sum_{i \in J, \delta \in \Gamma_i^0} |< f, T_\delta \varphi_i^0 >|^2 = \sum_{i \in F, \delta \in \Gamma_i^0} |< f, T_\delta \varphi_i^0 >|^2 \\ &= \sum_{i \in F, \delta \in \Gamma_{j_0}} |< f, T_\delta \varphi_i^0 >|^2. \end{aligned}$$

□

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