

Lagrange interpolation on the semiaxis. A survey

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Abstract

In this brief survey are collected some recent results about optimal interpolation processes of Lagrange type based on the zeros of *generalized Laguerre polynomials*, i.e. the sequence of orthogonal polynomials $\{p_m(w_\alpha)\}_m$ where $w_\alpha(x) = e^{-x^\beta} x^\alpha$. A new extended Lagrange process having optimal Lebesgue constants is also introduced.

Keywords : Lagrange Interpolation; Orthogonal Polynomials; Approximation by polynomials.

1 Introduction

Lagrange interpolation is an useful tool in many applications of numerical analysis, since it is flexible, easily computable and, differently from the most popular piecewise polynomials approximation, its accuracy increases for smoother functions. It appears in numerical integration and differentiation processes and also in the numerical treatment of functional equations by collocation methods.

In the case of the finite interval Lagrange interpolation on the zeros of orthogonal polynomials was extensively studied and the knowledge of "good" interpolation processes appears to be sufficiently complete (see [13],[16] and the references therein). In this short survey we limit ourselves to some recent developments on Lagrange interpolation of functions on the positive semi-axis.

To be more precise, given a generalized Laguerre weight $w_\alpha(x) = e^{-x^\beta} x^\alpha$, $\alpha > -1, \beta > \frac{1}{2}$, let $\mathcal{L}_m(w_\alpha)(f)$ be the Lagrange polynomial interpolating a given function f at the zeros of the m -th Laguerre polynomial $p_m(w_\alpha)$.

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We will assume $f \in C_u$ which is the space of the functions f satisfying the limit conditions $\lim_{x \rightarrow 0^+} f(x)u_\gamma(x) = 0$, $\lim_{x \rightarrow +\infty} f(x)u_\gamma(x) = 0$, with $u_\gamma(x) = e^{-x^\beta/2}x^\gamma$, $\gamma \geq 0$ and equipped with the uniform norm $\|f u_\gamma\|_\infty$. Roughly speaking, functions in C_u can increase exponentially for $x \rightarrow +\infty$ and to have an algebraic growth for $x \rightarrow 0^+$.

We start from a result about the Lebesgue constant proved by P. Vértesi in [24]: for any matrix \mathcal{X} of knots defined in $(0, +\infty)$ one has $\|\mathcal{L}_m(\mathcal{X})\|_{C_u} \geq C \log m$. However, $\mathcal{L}_m(w_\alpha)$ as map of C_u into \mathbb{P}_m is a "bad" process since there exist choices of α, γ such that $\|\mathcal{L}_m(w_\alpha)u_\gamma\|_\infty \sim m^\tau$ [15],[19].

To overcome this bad behavior some different processes essentially based on the zeros of generalized Laguerre polynomials were proposed in [14],[15],[12],[8],[19],[20]. Under suitable conditions, some new modified Lagrange sequences were recently introduced to approximate functions $f \in C_u$ like the best approximation of this space, except the factor $\log m$.

To complete in some sense the study of optimal Lagrange interpolation processes, we collect recent results about the so-called *extended interpolation*. Setting $S_{m+n} = p_n(\sigma)p_m(\rho)$, being σ, ρ two generalized Laguerre weights, the Lagrange polynomial $\mathcal{L}_{m+n}(\rho, \sigma, f)$ based on the zeros of S_{m+n} is known as "extended Lagrange polynomial". As we shall see, the parameters n, m, ρ, σ have to be chosen in such a way that any polynomial of the sequence $\{S_k\}_k$ have not only simple zeros but also "far enough" (see [21]). Indeed, the existence of too much "close" consecutive interpolation nodes has a negative consequence on the behavior of the corresponding interpolation process. Since the research of "good" interpolation matrices of knots is still an open problem, here we propose a new interpolating process essentially based on the zeros of $p_m(w_\alpha)p_{m+1}(w_\alpha)$, proving conditions under which the sequence of the corresponding Lebesgue constants behaves like $\log m$ in some weighted uniform spaces.

The plan of the paper is the following: Section 2 contains an excursion on recent results about Lagrange interpolation processes and their convergence. Section 3 contains some results about the extended Lagrange interpolation with a new interpolation process. Finally Section 4 contains the proofs of the new results.

2 Lagrange Interpolation

In the sequel \mathcal{C} will denote any positive constant which can be different in different formulas. Moreover $\mathcal{C} \neq \mathcal{C}(a, b, \dots)$ will be used to mean the constant \mathcal{C} is independent of a, b, \dots . The notation $A \sim B$, where A and B are positive quantities depending on some parameters, will be used if and only if $(A/B)^{\pm 1} \leq \mathcal{C}$, with \mathcal{C} positive constant independent of the above parameters.

Throughout the paper θ will denote a fixed real number, with $0 < \theta < 1$, which can be different in different formulas and \mathbb{P}_m will be the space of all algebraic polynomials of degree at most m .

Consider the weight

$$w_\alpha(x) = e^{-x^\beta} x^\alpha, \quad \alpha > -1, \quad \beta > \frac{1}{2}. \quad (2.1)$$

Let $\{p_m(w_\alpha)\}_m$ be the corresponding sequence of orthonormal polynomials having positive leading coefficients, i.e.

$$p_m(w_\alpha, x) = \gamma_m(w_\alpha)x^m + \text{terms of lower degree}, \quad \gamma_m(w_\alpha) > 0$$

and let $\{x_{m,k}\}_{k=1}^m$ be the zeros of $p_m(w_\alpha)$ in increasing order.

As proved in [8],[7] the zeros of $p_m(w_\alpha)$ lie in the range $(0, a_m(\sqrt{w_\alpha}))$, being $a_m := a_m(\sqrt{w_\alpha})$ the Mhaskar-Rakhmanov-Saff number (shortly M-R-S number) w.r.t. w_α . We recall that a_m is defined as the smallest positive number satisfying

$$\max_{x \in \mathbb{R}} |P_m(x) \sqrt{w_\alpha(x)}| = \max_{x \in [0, a_m(\sqrt{w_\alpha})]} |P_m(x) \sqrt{w_\alpha(x)}|, \quad \forall P_m \in \mathbb{P}_m,$$

being

$$a_m = a_m(\sqrt{w_\alpha}) = \left[\frac{2^{2\beta} (\Gamma(\beta))^2}{\Gamma(2\beta)} \right]^{\frac{1}{\beta}} \left(1 + \frac{2\alpha + 1}{8m} \right)^{\frac{1}{\beta}} m^{\frac{1}{\beta}} \quad (2.2)$$

where Γ denotes the Gamma function. Therefore $a_m(\sqrt{w_\alpha}) \sim m^{\frac{1}{\beta}}$.

We observe that in view of (2.2) generalized Laguerre weights having the same exponential part e^{-bx^β} , eventually with a different constant b , have M-R-S numbers differing for a constant. Therefore, in the next we use only a_m for any generalized Laguerre weight. Setting

$$u_\gamma(x) = e^{-x^\beta/2} x^\gamma, \quad \gamma \geq 0, \quad \beta > \frac{1}{2},$$

C_u is the functional space defined as follows

$$C_u = \left\{ f \in C((0, +\infty)) : \lim_{x \rightarrow 0^+} f(x)u_\gamma(x) = 0 = \lim_{x \rightarrow +\infty} f(x)u_\gamma(x) \right\},$$

and equipped with the norm

$$\|f\|_{C_u} = \max_{x \geq 0} u_\gamma(x) |f(x)| = \|f u_\gamma\|_\infty.$$

We remark that the assumption $\beta > \frac{1}{2}$ is necessary to assure the density of the polynomials in the space C_u [9].

For any continuous function f , let $\mathcal{L}_m(w_\alpha, f)$ be the Lagrange polynomial interpolating f at $\{x_{m,k}\}_{k=1}^m$, i.e.

$$\mathcal{L}_m(w_\alpha, f; x) = \sum_{k=1}^m \ell_{m,k}(x) f(x_{m,k}), \quad \ell_{m,k}(x) = \frac{p_m(w_\alpha, x)}{p'_m(w_\alpha, x_{m,k})(x - x_{m,k})}. \quad (2.3)$$

The m -th Lebesgue constant is defined in the usual way,

$$\|\mathcal{L}_m(w_\alpha)\|_{C_u} = \sup_{\|f\|_{C_u}=1} \|\mathcal{L}_m(w_\alpha, f)\|_{C_u}.$$

As it is well known, the behavior of the sequence $\{\|\mathcal{L}_m(w_\alpha)\|_{C_u}\}_m$ determines the order of the interpolation error, since

$$\|f - \mathcal{L}_m(w_\alpha, f)\|_{C_u} \leq E_m(f)_{u_\gamma} (1 + \|\mathcal{L}_m(w_\alpha)\|_{C_u}), \quad (2.4)$$

being $E_m(f)_{u_\gamma} = \inf_{P_m \in \mathbb{P}_m} \|(f - P_m)u_\gamma\|$ the error of best approximation of $f \in C_u$.

Similarly to the classical result of Faber and Bernstein in the case of the finite interval, P. Vértesi [24], [25], in a more general context, proved that for any set of knots in the interval $[0, a_m]$,

$$\|\mathcal{L}_m(w_\alpha)u_\gamma\|_\infty \geq \mathcal{C} \log m, \quad \mathcal{C} \neq \mathcal{C}(m). \quad (2.5)$$

Successively, the following "negative" result was proved (see [15], [19]):

Proposition 2.1. *For any choice of $\alpha, \gamma \geq 0$ and $\beta > \frac{1}{2}$, there exists a positive τ s.t.*

$$\|\mathcal{L}_m(w)\|_{C_u} \sim m^\tau,$$

with $0 < \mathcal{C} \neq \mathcal{C}(m)$.

The "bad" behavior stressed in the Proposition 2.1 can be overcome by slightly modifying the usual interpolation process, according to some different approaches that we go to describe.

2.1 Interpolation with additional knots

Consider the Lagrange polynomial $L_{m+1}(w_\alpha, f)$ interpolating f at the zeros of $p_m(w_\alpha, x)$ and on the special knot a_m ,

$$\begin{aligned} L_{m+1}(w_\alpha, f, x_{m,k}) &= f(x_{m,k}), \quad k = 1, 2, \dots, m, \\ L_{m+1}(w_\alpha, f, a_m) &= f(a_m). \end{aligned} \quad (2.6)$$

Setting $x_{m,m+1} = a_m$, an expression is the following

$$L_{m+1}(w_\alpha, f, x) = \sum_{k=1}^{m+1} l_{m,k}^*(x) f(x_{m,k}),$$

$$l_{m,k}^*(x) = \frac{a_m - x}{a_m - x_{m,k}} \ell_{m,k}(x), \quad k = 1, 2, \dots, m, \quad l_{m,m+1}^*(x) = \frac{p_m(w_\alpha, x)}{p_m(w_\alpha, a_m)},$$

being $\ell_{m,k}$, $k = 1, 2, \dots, m$ defined in (2.3). Denoting by $\|L_{m+1}(w_\alpha)\|_{C_u}$ the corresponding Lebesgue constant, the following result holds [15],[12]:

Theorem 2.1. *There exists a positive constant $\mathcal{C} \neq \mathcal{C}(m)$ such that*

$$\|L_{m+1}(w_\alpha)\|_{C_u} \sim \log m,$$

if and only if the parameters $\alpha > -1$ and $\gamma \geq 0$ satisfy the conditions:

$$\max\left(0, \frac{\alpha}{2} + \frac{1}{4}\right) \leq \gamma \leq \frac{\alpha}{2} + \frac{5}{4} \quad (2.7)$$

This result is sharp, taking into account (2.5).

In view of the Theorem 2.1, $L_{m+1}(w_\alpha, f)$ approximates $f \in C_u$ like the best approximation of this space, except the factor $\log m$:

$$\|[f - L_{m+1}(w_\alpha, f)]u_\gamma\|_\infty \leq \mathcal{C}E_m(f)_{u_\gamma} \log m. \quad (2.8)$$

In particular for smoother functions, say $f \in W_{\gamma,r}^\infty$, being

$$W_{\gamma,r}^\infty = \left\{ f \in C_u : \|f^{(r)}\varphi^r u_\gamma\|_\infty < \infty \right\}, \quad r \geq 1, \quad \varphi(x) = \sqrt{x}$$

a Sobolev-type space, equipped with the norm

$$\|f\|_{W_{\gamma,r}^\infty} = \|f u_\gamma\|_\infty + \|f^{(r)}\varphi^r u_\gamma\|_\infty,$$

the following estimate holds [19]

$$E_m(f)_{u_\gamma} \leq \mathcal{C} \left(\frac{\sqrt{a_m}}{m} \right)^r \|f^{(r)}\varphi^r u_\gamma\|_\infty, \quad \mathcal{C} \neq \mathcal{C}(m, f). \quad (2.9)$$

Therefore by (2.8) it follows that for any $f \in W_{\gamma,r}^\infty$

$$\|[f - L_{m+1}(w_\alpha, f)]u_\gamma\|_\infty \leq \mathcal{C} \left(\frac{\sqrt{a_m}}{m} \right)^r \|f^{(r)}\varphi^r u_\gamma\|_\infty \log m \quad (2.10)$$

with $\mathcal{C} \neq \mathcal{C}(m, f)$.

Assuming now that the parameters α, γ are both fixed, the assumption in (2.7) could be never satisfied. In this case it can be useful to add some interpolation knots t_1, \dots, t_s "close" to the endpoint 0. Indeed the Lagrange polynomial based on the previous knots $\{x_{m,k}\}_{k=1}^{m+1} \cup \{t_i\}_{i=1}^s$, under suitable assumptions, is an optimal interpolation process again.

To be more precise, let $\{t_i\}_{i=1}^s$ be s simple knots added in the range $(0, x_{m,1})$, for instance $t_i = \frac{i}{s+1}x_{m,1}$, $i = 1, 2, \dots, s$ and let $B_s(x) = \prod_{i=1}^s (x-t_i)$. Denote by $L_{m+1,s}(w_\alpha, f)$ the Lagrange polynomial interpolating f at the zeros of $p_m(w_\alpha, x)B_s(x)(a_m - x)$. By the same arguments used in the proof of Theorem 3.5 in [15], it is no hard to prove the following result:

Theorem 2.2. *For any function $f \in C_u$, if there exists an integer s such that*

$$\frac{1}{4} \leq \gamma - \frac{\alpha}{2} + s \leq \frac{5}{4}, \quad (2.11)$$

then we have

$$\|L_{m+1,s}(w_\alpha, f)u_\gamma\|_\infty \leq \mathcal{C}\|f u_\gamma\|_\infty \log m, \quad (2.12)$$

where $0 < \mathcal{C} \neq \mathcal{C}(m, f)$.

Remark 2.1. *In the case $\gamma = \alpha = 0$, the assumption of Theorem 2.1 is not verified, whereas adding one additional point, for instance $t_1 = \frac{x_{m,1}}{2}$, the sequence $\{\|L_{m+1,1}(w_\alpha)\|_{C_u}\}_m$ has a logarithmic behavior.*

We conclude this section with a short remark on the method of *additional points*. In the case of Lagrange interpolation on the real axis, J. Szabados [22] introduced the so called "method of additional points," adding two extra knots near to $\pm \bar{a}_m$, being \bar{a}_m the M-R-S number related to a Freud weight. Upon this idea, in [15] it was introduced the interpolation polynomial $L_{m+1}(w_\alpha, f)$ based on the zeros of classical Laguerre polynomials and on the additional point $4m$ which is, more or less, the M-R-S number w.r.t to a

Laguerre weight $e^{-x}x^\alpha$, $\alpha > -1$. The extension to the case of a generalized Laguerre weight was studied in [12]. Similarly to what happens in the Freud weight case (see [9],[10],[11]), the factors $\frac{a_m-x}{a_m-x_{m,j}}$ influence substantially the behavior of the Lebesgue constants. Indeed, recalling that the Laguerre polynomial in a neighborhood of a_m is estimated as follows

$$|p_m(w_\alpha, x)|\sqrt{w_\alpha(x)} \leq C \left(\left| 1 - \frac{x}{a_m} \right| + m^{-\frac{2}{3}} \right)^{-\frac{1}{4}}, \quad a_m(1-\varepsilon) \leq x \leq a_m(1+\delta), \delta, \varepsilon > 0,$$

the factor $(a_m - x)$ damp the growth of the polynomial in the range $a_m \leq x \leq a_m(1+\delta)$, $\delta > 0$.

The introduction of the additional points "close" to the endpoint 0 in the case of the classical Laguerre weight is referable to [15].

2.2 Truncated sequences of Lagrange polynomials

Let θ be fixed, with $\theta \in (0, 1)$ and denote by $\chi_{m,\theta}$ the characteristic function of the interval $[0, \theta a_m]$. With $\mathcal{L}_m(w_\alpha, f)$ defined in (2.3), let $\{\chi_{m,\theta}\mathcal{L}_m(w_\alpha, f\chi_{m,\theta})\}_m$ be a "truncated" sequence of Lagrange polynomials interpolating only a finite section of the function f on the interval $[0, \theta a_m]$. Indeed, since [20], [19]

$$\|f(1 - \chi_{m,\theta})u_\gamma\|_\infty \leq E_M(f)_{u_\gamma} + \mathcal{C}e^{-Am}\|fu_\gamma\|_\infty, \quad (2.13)$$

with $M = \left\lceil \frac{\theta}{1+\theta}m \right\rceil \sim m$, the neglected part of f behaves like the error of best approximation $E_M(f)_{u_\gamma}$ being M a fraction of m depending on θ . Defined

$$x_{m,j} = \min \{x_{m,k} : x_{m,k} \geq \theta a_m, \quad k = 1, 2, \dots, m\}. \quad (2.14)$$

consider the truncated function $f_j = f(1 - \Psi_j)$, being

$$\Psi(x) = \begin{cases} 0, & \text{if } x \leq 0 \\ 1, & \text{if } x \geq 1 \end{cases} \quad \text{and } \Psi_j(x) = \Psi\left(\frac{x-x_{m,j}}{x_{m,j+1}-x_{m,j}}\right)$$

By definition, f_j has the same smoothness of f , coincides with f in the interval $(0, x_{m,j}]$, it is identically null for $x \in [x_{m,j+1}, +\infty)$, and these two "pieces" are smoothly linked by the function Ψ_j in the interval $(x_{m,j}, x_{m,j+1})$.

Then, for this modified Lagrange process, the authors in [19] proved the following

Theorem 2.3. *For any $f \in C_u$, under the assumption*

$$\max\left(0, \frac{\alpha}{2} + \frac{1}{4}\right) \leq \gamma \leq \frac{\alpha}{2} + \frac{5}{4} \quad (2.15)$$

the following estimate holds

$$\|\chi_{m,\theta}\mathcal{L}_m(w_\alpha, f_j)u_\gamma\|_\infty \leq \mathcal{C}\|f_ju_\gamma\|_\infty \log m,$$

with $\mathcal{C} \neq \mathcal{C}(m, f)$.

[a] denotes the largest integer smaller than or equal to $a \in \mathbb{R}^+$

About the error estimate, by the previous theorem and (2.13) one has the following estimate [19] (see also [17]):

$$\| [f - \chi_{m,\theta} \mathcal{L}_m(w_\alpha, f_j)] u_\gamma \|_\infty \leq \mathcal{C} \log m (E_M(f)_{u_\gamma} + e^{-Am} \|f u_\gamma\|_\infty) \quad (2.16)$$

with $\mathcal{C} \neq \mathcal{C}(m, f)$.

By the previous result we can conclude that under the assumption (2.15) the sequence of polynomials $\{\mathcal{L}_m(w_\alpha, f)\}_m$ whose bad behavior on the whole semi-axis was evidenced by Proposition 2.1, can be successfully used to approximate $f \in C_u$ in the restricted range $[0, \theta a_m]$, with almost the same rate of convergence offered by the sequence $\{L_{m+1}(w_\alpha, f)\}_m$, since the degree M is a fraction of m . Moreover, truncated sequences afford the advantage of a reduced computational cost and a delayed possible overflow phenomenon in the computation of f whenever it grows exponentially.

Truncated sequences are a "remedium" in some sense on the growth of the polynomial $p_m(w_\alpha, x)$ in a neighborhood of a_m , since in the restricted range $(\frac{a_m}{m^2}, \theta a_m)$, with $0 < \theta < 1$, the following estimate holds

$$|p_m(w_\alpha, x)| \sqrt{w_\alpha(x)} \leq \frac{\mathcal{C}}{\sqrt[4]{a_m x}}, \quad \mathcal{C} \frac{a_m}{m^2} \leq x \leq \theta a_m.$$

2.3 Polynomial sequences of a finite section of f

Now we remark that $\{\chi_{m,\theta} \mathcal{L}_m(w_\alpha, f \chi_{m,\theta})\}_m$ is not a polynomial sequence and $\chi_{m,\theta} \mathcal{L}_m(w_\alpha)$ does not project C_u onto \mathbb{P}_{m-1} as well as required in some applications like the numerical treatment of functional equations (see [13]).

In order to overcome this problem, in [8] (see also [12][18]) the authors introduced a polynomial sequence interpolating a finite section of the function. To be more precise, for any fixed $\theta \in (0, 1)$, let $j = j(m)$ be the index of the zero of $p_m(w_\alpha)$ defined in (2.14) and $f_{j,\theta} := f \chi_{m,\theta}$. Then, the interpolating polynomial $\bar{L}_{m+1}^{**}(w_\alpha, f)$ is defined as

$$\bar{L}_{m+1}^{**}(w_\alpha, f, x) := L_{m+1}(w_\alpha, f_{j,\theta}), \quad (2.17)$$

being $L_{m+1}(w_\alpha, g)$ the Lagrange polynomial defined in (2.6) interpolating a given function g at $\{x_{m,k}\}_{k=1}^m \cup \{a_m\}$.

The operator $\bar{L}_{m+1}^{**}(w_\alpha)$ is a projector of C_u into the subspace $\mathcal{P}_m^* \subset \mathbb{P}_m$ defined as

$$\mathcal{P}_m^* = \{q \in \mathbb{P}_m : q(a_m) = 0 = q(x_{m,k}), \quad k > j\}.$$

For this polynomial process the Lebesgue constant in C_u have order $\log m$, under suitable hypothesis on α, γ . Indeed in [12] it was proved that

Theorem 2.4. *If the parameters satisfy the assumptions*

$$\max\left(0, \frac{\alpha}{2} + \frac{1}{4}\right) \leq \gamma \leq \frac{\alpha}{2} + \frac{5}{4} \quad (2.18)$$

then

$$\|\bar{L}_{m+1}^{**}(w_\alpha, f) u_\gamma\|_\infty \leq \mathcal{C} \|f_{j,\theta} u_\gamma\|_\infty \log m,$$

and

$$\| [f - \bar{L}_{m+1}^{**}(w_\alpha, f)] u_\gamma \|_\infty \leq \mathcal{C} \log m (E_M(f)_{u_\gamma} + e^{-Am} \|f u_\gamma\|_\infty). \quad (2.19)$$

where $M = \left\lceil \frac{\theta}{1+\theta} m \right\rceil$ and with $\mathcal{C} \neq \mathcal{C}(m, f)$.

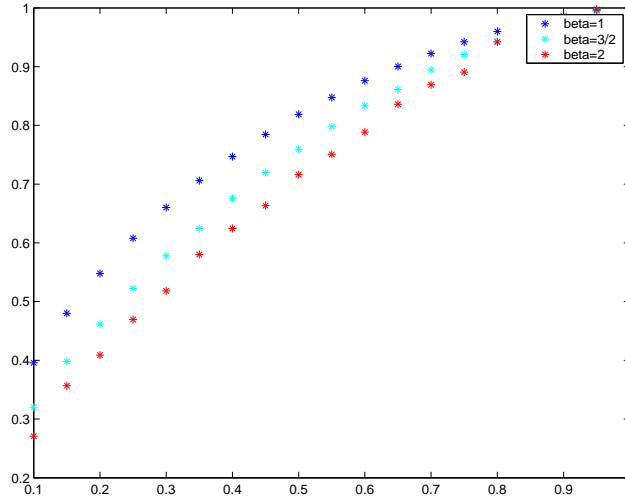


Figure 1: $\tilde{v}_m(\theta)$ for $\theta \in (0, 1)$

We remark that truncated Lagrange polynomial sequences were successfully applied in quadrature by introducing truncated Gaussian rules and truncated product integration formulae, which are more convenient and faster convergent (see [3],[14], [12], [17], [18]).

We conclude this section showing empirically how the number of the interpolation knots involved in truncated processes depends on θ .

Defined

$$N_m(a, b) = \text{Number of zeros of } p_m(w_\alpha) \text{ in } (a, b)$$

for any $\theta \in (0, 1)$ let

$$v_m(\theta) = \frac{N_m(0, \theta a_m)}{m}$$

and

$$\tilde{v}_m(\theta) = \frac{1}{N-1} \sum_{m=2}^N v_m(\theta), \quad N = 2048$$

In the case $\alpha = 0$, the values obtained for different choices of the parameter β are plotted in Figure 1. As we can see, for increasing values of $\beta > 1$ the number of zeros in $(0, \theta a_m)$ decreases.

3 Extended Lagrange interpolation

First we recall the main idea of extended interpolation based on the zeros of orthogonal polynomials.

Let ρ and σ be two weight functions, both of them supported in $-\infty \leq a < b \leq +\infty$ and let $\{p_m(\rho)\}_m, \{p_m(\sigma)\}_m$ be the corresponding sequences of orthonormal polynomials. Assume that the polynomial $Q_{m+n} = p_m(\rho)p_n(\sigma)$ has simple zeros $\zeta_i, i = 1, 2, \dots, m+n$. Then, for any continuous function f the Lagrange polynomial $\mathcal{L}_{m+n}(\rho, \sigma, f)$ interpolating f at the zeros of Q_{m+n} , i.e.

$$\mathcal{L}_{m+n}(\rho, \sigma, f; \zeta_i) = f(\zeta_i), \quad i = 1, 2, \dots, m+n.$$

is called *extended interpolation polynomial*. From

$$\mathcal{L}_{m+n}(\rho, \sigma, f) = p_m(\rho)\mathcal{L}_n\left(\sigma, \frac{f}{p_m(\rho)}\right) + p_n(\sigma)\mathcal{L}_m\left(\rho, \frac{f}{p_n(\sigma)}\right),$$

it follows that $\mathcal{L}_{m+n}(\rho, \sigma, f)$ can "extended" a previous interpolation polynomial $\mathcal{L}_n(\sigma, f)$, reusing the previous n function evaluations. We point out that when $m = n$ or $m = n + 1$, an "high" degree Lagrange polynomial can be constructed by two half degree Lagrange polynomials, which are separately computable. By this way difficulties in computing the zeros of "large" degree orthogonal polynomials are overcome in some sense. Another relevant application of extended interpolation processes is the aid in the construction of extended positive quadratures rules (see [6] and the references therein).

At first we introduce a new extended Lagrange process w.r.t the same weight w_α based on the zeros of shifted degree polynomials. Set $\tilde{z}_{2i-1} = x_{m+1,i}$, $i = 1, 2, \dots, m+1$, $\tilde{z}_{2i} = x_{m,i}$, $i = 1, 2, \dots, m$, being $\{x_{m,k}\}_{k=1}^m$ and $\{x_{m+1,k}\}_{k=1}^{m+1}$ the zeros of $p_{m+1}(w_\alpha)$ and $p_m(w_\alpha)$, respectively. Therefore, the extended Lagrange polynomial $\mathcal{L}_{2m+1}(w_\alpha, w_\alpha, f)$ interpolating f at the zeros of $\tilde{Q}_{2m+1} := p_{m+1}(w_\alpha)p_m(w_\alpha)$ there exists and it can take the following expression:

$$\mathcal{L}_{2m+1}(w_\alpha, w_\alpha, f; x) = \sum_{k=1}^{2m+1} \tilde{\ell}_{2m+1,k}(x) f(\tilde{z}_k), \quad \tilde{\ell}_{2m+1,k}(x) = \frac{\tilde{Q}_{2m+1}(x)}{\tilde{Q}'_{m+1}(\tilde{z}_k)(x - \tilde{z}_k)}. \quad (3.20)$$

Now we recall the following result on extended interpolation which bind in some sense the "good" order of Lebesgue constants sequence to the "good" distance between consecutive interpolation knots. Indeed, setting $\mathcal{X} = \{\xi_{m,i}, \quad i = 1, 2, \dots, m, \quad m \in \mathcal{N}\}$, let $\mathcal{L}_m(\mathcal{X}, g)$ be the Lagrange polynomial interpolating g at the elements of the m -th row of \mathcal{X} . With $\sigma_\delta(x) = e^{-x^\beta} x^\delta$, $\delta \geq 0$, $\beta > \frac{1}{2}$, and

$$C_\delta = \left\{ f : f\sigma_\delta \in C^0(\mathbf{R}^+), \lim_{x \rightarrow 0^+} |f(x)|\sigma_\delta(x) = 0 = \lim_{x \rightarrow \infty} |f(x)|\sigma_\delta(x), \right\},$$

equipped with the norm $\|f\|_{C_\delta} = \sup_{x \geq 0} |f(x)|\sigma_\delta(x)$, the m -th Lebesgue constant in C_δ is defined as

$$\|\mathcal{L}_m(\mathcal{X})\|_{C_\delta} = \sup_{\|g\|_{C_\delta}=1} \|\mathcal{L}_m(\mathcal{X}, g)\sigma_\delta\|, \quad m = 1, 2, \dots \quad (3.21)$$

Then the following Proposition holds [21]:

Proposition 3.1. *If for m sufficiently large (say $m > m_0$), there exists $k := k(m)$ s. t.*

$$\Delta\xi_{m,k} \leq \left(\frac{\sqrt{a_m}}{m}\right)^{\eta+1} \sqrt{\xi_{m,k}}, \quad \eta > 0 \quad (3.22)$$

then

$$\|\mathcal{L}_m(\mathcal{X})\|_{C_\delta} \geq \mathcal{C} \left(\frac{m}{\sqrt{a_m}}\right)^\eta, \quad (3.23)$$

where $\mathcal{C} \neq \mathcal{C}(m)$.

In $[-1, 1]$ an analogous result was proved in [23].

Remark 3.1. Setting \tilde{z}_j the knot defined as

$$\tilde{z}_j = \tilde{z}_{j(m)} = \min \{ \tilde{z}_k : \tilde{z}_k \geq \theta a_{m+1}, \quad k = 1, 2, \dots, 2m+1 \}, \quad (3.24)$$

in our case it is no hard to prove that

$$\Delta \tilde{z}_k = \tilde{z}_{k+1} - \tilde{z}_k \sim \frac{\sqrt{a_{m+1}}}{m} \sqrt{\tilde{z}_{k+1}}, \quad k = 1, 2, \dots, j, \quad (3.25)$$

uniformly in $m \in \mathbb{N}$. In the special case $\beta = 1$, (3.25) was proved in [2].

Note that the distance between two consecutive zeros of \tilde{Q}_{2m+1} is comparable with those of consecutive zeros of $p_m(w_\alpha)$ (see [8],[7], [21].)

However a "good" distance is not enough to assure optimal order of the Lebesgue constants. Indeed the following "bad" result holds

Proposition 3.2. For any choice of α , $\delta \geq 0$ and $\beta > \frac{1}{2}$, there exists a positive τ s.t.

$$\|\mathcal{L}_{2m+1}(w_\alpha, w_\alpha)\|_{C_\delta} = \sup_{\|f\sigma_\delta\|_\infty=1} \|\mathcal{L}_{2m+1}(w_\alpha, w_\alpha, f)\sigma_\delta\|_\infty \geq C m^\tau, \quad (3.26)$$

with $0 < C \neq C(m)$.

Nevertheless the system of knots made up of the zeros of $\{\tilde{Q}_n\}_n$ can be proposed in order to obtain optimal Lebesgue constants too. This goal will be achieved by considering the Lagrange polynomial based on the zeros of $\tilde{Q}_{2m+1}(x)(a_{m+1} - x)$ and interpolating a finite section of the function f , being a_{m+1} the $(m+1)$ -th M-R-S number w.r.t. w_α . Denoting by $\chi_{m,\theta}$ the characteristic function of the segment $(0, \tilde{z}_j)$, let us introduce the extended Lagrange polynomial defined as

$$L_{2m+2}^*(w_\alpha, w_\alpha, f; x) := \sum_{k=1}^j \bar{l}_{2m+2,k}(x) f(\tilde{z}_k), \quad (3.27)$$

where

$$\begin{aligned} \bar{l}_{2m+2,k}(x) &= \tilde{l}_{2m+1,k}(x) \frac{(a_{m+1} - x)}{(a_{m+1} - \tilde{z}_k)}, \quad k = 1, 2, \dots, 2m+1, \\ \bar{l}_{2m+2,2m+2}(x) &= \frac{\tilde{Q}_{2m+1}(x)}{\tilde{Q}_{2m+1}(a_{m+1})}. \end{aligned}$$

Obviously, $L_{2m+2}^*(w_\alpha, w_\alpha, f)$ is a polynomial of degree $2m+1$ such that $L_{2m+2}^*(w_\alpha, w_\alpha, f; a_m) = 0 = L_{2m+2}^*(w_\alpha, w_\alpha, f; \tilde{z}_k)$, for $k > j$.

We are able to prove that under suitable relations between the weights w_α and σ , the corresponding Lebesgue constants grow logarithmically:

Theorem 3.1. For any function $f \in C_\delta$, with $\delta > 0$,

$$\|L_{2m+2}^*(w_\alpha, w_\alpha, f)\sigma_\delta\|_\infty \leq C\|f\sigma_\delta\|_\infty \log m \quad (3.28)$$

with $0 < C \neq C(m, f)$, if and only if

$$\frac{1}{2} \leq \delta - \alpha \leq \frac{3}{2}. \quad (3.29)$$

Moreover

$$\|[f - L_{2m+2}^*(w_\alpha, w_\alpha, f)]\sigma_\delta\|_\infty \leq C\{E_{\bar{M}}(f)\sigma_\delta \log m + e^{-Am}\|f\sigma_\delta\|_\infty\}, \quad (3.30)$$

where $\bar{M} = \left[2m \left(\frac{\theta}{1+\theta}\right)^\beta\right] \sim m$, $0 < C \neq C(m, f)$, $0 < A \neq A(m, f)$.

We conclude recalling an extended interpolation process introduced in [21] and involving two different weight functions, suitable related among them. Consider the following weights $w_\alpha(x) = e^{-x^\beta}x^\alpha$, $\alpha > -1$, $\beta > \frac{1}{2}$ and $w_{\alpha+1}(x) = xw_\alpha(x)$. Denote by $\{x_k\}_{k=1}^{m+1}$ the zeros of $p_{m+1}(w_\alpha)$ in increasing order and by $\{y_k\}_{k=1}^m$ the zeros of the corresponding m -th orthonormal polynomial $p_m(w_{\alpha+1})$.

Set $z_{2i-1} = x_i$, $i = 1, 2, \dots, m+1$, $z_{2i} = y_i$, $i = 1, 2, \dots, m$, $z_{2m+2} = a_{m+1}$. Since in [21] it was proved that the zeros of $p_m(w_{\alpha+1})$ interlace those of $p_{m+1}(w_\alpha)$, (see also [4]), let $L_{2m+2}(w_\alpha, w_{\alpha+1}, f)$ be the Lagrange polynomial interpolating f at the zeros of $Q_{2m+1} := p_{m+1}(w_\alpha)p_m(w_{\alpha+1})$ and at the special knot a_{m+1} .

Denoted by z_j the knot

$$z_j = z_{j(m)} = \min\{z_k : z_k \geq \theta a_{m+1}, \quad k = 1, 2, \dots, 2m+1\}, \quad (3.31)$$

and by $\chi_{m,\theta}$ be the characteristic function of the segment $(0, z_j)$, let be

$$L_{2m+2}^*(w_\alpha, w_{\alpha+1}, f) := L_{2m+2}(w_\alpha, w_{\alpha+1}, f\chi_{m,\theta}) \quad (3.32)$$

the Lagrange polynomial interpolating $f\chi_{m,\theta}$ at the zeros $\{z_i\}_{i=1}^{2m+2}$. Also in this case the polynomial sequence $\{L_{2m+2}^*(w_\alpha, w_{\alpha+1}, f)\}_m$, can be used to approximate $f \in C_\delta$ successfully, under suitable conditions [21]

Theorem 3.2. For any function $f \in C_\delta$, with $\delta > 0$,

$$\|L_{2m+2}^*(w_\alpha, w_{\alpha+1}, f)\sigma_\delta\|_\infty \leq C\|f\sigma_\delta\|_\infty \log m \quad (3.33)$$

with $0 < C \neq C(m, f)$, if and only if

$$1 \leq \delta - \alpha \leq 2. \quad (3.34)$$

Moreover

$$\|[f - L_{2m+2}^*(w_\alpha, w_{\alpha+1}, f)]\sigma_\delta\|_\infty \leq C\{E_M(f)\sigma_\delta \log m + e^{-Am}\|f\sigma_\delta\|_\infty\} \quad (3.35)$$

where $M = \left[2m \left(\frac{\theta}{1+\theta}\right)^\beta\right] \sim m$, $0 < C \neq C(m, f)$, $0 < A \neq A(m, f)$.

If the parameters δ and α don't satisfy (3.34), we can adopt the additional nodes method [21], introducing the polynomial $L_{2m+2,s}^*(w_\alpha, w_{\alpha+1}, f)$ interpolating $f\chi_{m,\theta}$ at the zeros of $Q_{2m+1}(x)B_s(x)(a_{m+1} - x)$, where $t_i = \frac{i}{s+1}x_1$, $i = 1, 2, \dots, s$ and $B_s(x) = \prod_{i=1}^s (x - t_i)$. The following result holds

Theorem 3.3. *For any function $f \in C_\delta$, if there exists an integer s such that*

$$1 \leq \delta - \alpha + s \leq 2, \quad (3.36)$$

then we have

$$\|L_{2m+2,s}^*(w_\alpha, w_{\alpha+1}, f)\sigma_\delta\|_\infty \leq C\|f\sigma_\delta\|_\infty \log m, \quad (3.37)$$

where $0 < C \neq C(m, f)$. Moreover,

$$\|[f - L_{2m+2,s}^*(w_\alpha, w_{\alpha+1}, f)]\sigma_\delta\|_\infty \leq C \{E_M(f)\sigma_\delta \log m + e^{-Am}\|f\sigma_\delta\|_\infty\}, \quad (3.38)$$

where $M = \left[2m \left(\frac{\theta}{1+\theta}\right)^\beta\right] \sim m$, $0 < C \neq C(m, f)$, $0 < A \neq A(m, f)$.

We conclude giving some details useful for the practical computation of the interpolation knots. We recall the three-term recurrence relation for the orthogonal polynomials w.r.t. the weight w_α .

$$\begin{aligned} p_{-1}(w_\alpha, x) &= 0, & p_0(w_\alpha, x) &= \left(\int_0^\infty w_\alpha(x)dx\right)^{-\frac{1}{2}} \\ b_{n+1}p_{n+1}(w_\alpha; x) &= (x - e_n)p_n(w_\alpha, x) - b_n p_{n-1}(w_\alpha, x) \\ b_n &= \frac{\gamma_{n-1}(w_\alpha)}{\gamma_n(w_\alpha)} & e_n &= \int_0^\infty x p_n^2(w_\alpha, x) w_\alpha(x) dx. \end{aligned} \quad (3.39)$$

Although the coefficients $\{b_k\}_k, \{e_k\}_k$ are not always known, there exist efficient numerical procedures to calculate them [1] (see also [5]). The computation of the zeros of Generalized Laguerre polynomials with parameter $\beta \neq 1$, requires an higher computational effort. Indeed, when $\beta \neq 1$ the coefficients in the three term recurrence relation for the polynomials $\{p_m(w)\}_m$ are not always known. However there exists the Mathematica Package OrthogonalPolynomials [1] to compute these zeros by using "high" variable precision.

4 The proofs

Now we collect some polynomial inequalities deduced in [18](see also [7]).

Let $x \in [x_{m,1}, x_{m,m}]$ and $d = d(x) \in \{1, \dots, m\}$ be an index of a zero of $p_m(w_\alpha)$ closest to x . Then, for some positive constant $C \neq C(m, x, d)$, we have

$$\frac{1}{C} \left(\frac{x - x_{m,d}}{x_d - x_{d\pm 1}} \right)^2 \leq p_m^2(w_\alpha, x) e^{-x^\beta} \left(x + \frac{a_m}{m^2} \right)^{\alpha + \frac{1}{2}} \sqrt{|a_m - x| + a_m m^{-\frac{2}{3}}} \leq C \left(\frac{x - x_d}{x_{m,d} - x_{m,d\pm 1}} \right)^2. \quad (4.40)$$

and for a fixed real number $0 < \delta < 1$,

$$|p_m(w_\alpha, x)|\sqrt{w_\alpha(x)} \leq \frac{\mathcal{C}}{\sqrt[4]{x} \sqrt[4]{|a_m - x| + a_m m^{-\frac{2}{3}}}}, \quad \frac{a_m}{m^2} \leq x \leq a_m(1 + \delta). \quad (4.41)$$

In particular, for a fixed $0 < \theta < 1$

$$|p_m(w_\alpha, x)|\sqrt{w_\alpha(x)} \leq \mathcal{C} \frac{1}{\sqrt[4]{a_m x}}, \quad \frac{a_m}{m^2} \leq x \leq \theta a_m. \quad (4.42)$$

Moreover, for $k = 1, 2, \dots, m$

$$\frac{1}{|p'_m(w_\alpha, x_{m,k})|\sqrt{w_\alpha(x_{m,k})}} \sim \Delta x_{m,k} \sqrt[4]{a_m x_{m,k}} \sqrt[4]{\left|1 - \frac{x_{m,k}}{a_m}\right| + m^{-\frac{2}{3}}}, \quad \Delta x_{m,k} = x_{m,k+1} - x_{m,k}. \quad (4.43)$$

For any polynomial $P_m \in \mathbb{P}_m$, the Bernstein inequality [7] [19]

$$\max_{x \geq 0} |P'_m(x)|\sqrt{w_\alpha(x)}\sqrt{x} \leq \mathcal{C} \frac{m}{\sqrt{a_m}} \max_{x \geq 0} |P_m(x)\sqrt{w_\alpha(x)}|, \quad \mathcal{C} \neq \mathcal{C}(m, P_m) \quad (4.44)$$

and the Remez-type inequality [19]

$$\max_{x \geq 0} |P_m(x)u_\gamma(x)| \leq \mathcal{C} \max_{x \geq \frac{a_m}{m^2}} |P_m(x)u_\gamma(x)| \quad (4.45)$$

hold. Finally we recall that for any polynomial $P_m \in \mathbb{P}_m$, one has [19]

$$\max_{x \geq a_m(1+\delta)} |P_m(x)u_\gamma(x)| \leq \mathcal{C} e^{-Am} \max_{x \leq a_m} |P_m(x)u_\gamma(x)| \quad (4.46)$$

where $\mathcal{C} \neq \mathcal{C}(m)$, $A \neq A(m)$.

In the next will be useful the following

Lemma 4.1. *Setting $\tilde{z}_{2i-1} = x_{m+1,i}$, $i = 1, 2, \dots, m+1$, $\tilde{z}_{2i} = x_{m,i}$, $i = 1, 2, \dots, m$, and $\tilde{Q}_{2m+1} := p_{m+1}(w_\alpha)p_m(w_\alpha)$, the following estimate holds true*

$$\frac{1}{|\tilde{Q}'_{2m+1}(\tilde{z}_k)|\sigma_\delta(\tilde{z}_k)} \leq \mathcal{C} \frac{\sqrt{a_m - \tilde{z}_k}}{\tilde{z}_k^{\delta-\alpha-1}} \Delta \tilde{z}_k, \quad \tilde{z}_k < \tilde{z}_j, \quad \mathcal{C} \neq \mathcal{C}(m) \quad (4.47)$$

Proof: Using (4.40)

$$\frac{1}{|p_m(w_\alpha, x_{m+1,k})|\sqrt{w_\alpha(x_{m+1,k})}} \leq \mathcal{C} \sqrt[4]{x_{m+1,k}(a_m - x_{m+1,k})}, \quad x_{m+1,k} \leq \tilde{z}_j, \quad (4.48)$$

so, by (4.43), it follows

$$\frac{1}{|Q'_{2m+1}(x_{m+1,k})|\sigma_\delta(x_{m+1,k})} \leq \mathcal{C} \frac{\sqrt{a_m - x_{m+1,k}}}{x_{m+1,k}^{\delta-\alpha-\frac{1}{2}}} \Delta x_{m+1,k}, \quad x_{m+1,k} \leq \tilde{z}_j.$$

An analogous estimate holds replacing $x_{m+1,k}$ with $x_{m,k}$. Using then $\Delta x_{m+1,k} \sim \Delta x_{m,k} \sim \Delta \tilde{z}_k$, $\tilde{z}_k < \tilde{z}_j$, the Lemma follows. \square

Lemma 4.2. For $x \in (\tilde{z}_1, \tilde{z}_{2m+1})$ and denoted with \tilde{z}_d the zero of \tilde{Q}_{2m+1} closest to x , we have

$$\frac{|\tilde{Q}_{2m+1}(x)|}{|\tilde{Q}'_{2m+1}(\tilde{z}_d)(x - \tilde{z}_d)|} \frac{\sigma_\delta(x)}{\sigma_\delta(\tilde{z}_d)} \leq \mathcal{C} \quad \mathcal{C} \neq \mathcal{C}(m, x). \quad (4.49)$$

Proof: Denoted by $x_{m+1,d}$ a zero of $p_{m+1}(w_\alpha)$ closest to $x \in [x_{m+1,1}, x_{m+1,m+1}]$, in [12] it was proved,

$$\left| \frac{p_{m+1}(w_\alpha, x)}{p'_{m+1}(w_\alpha, x_{m+1,d})(x - x_{m+1,d})} \right| \frac{\sqrt{w_\alpha(x)}}{\sqrt{w_\alpha(x_{m+1,d})}} \sim 1.$$

Therefore, assuming \tilde{z}_d is a zero of $p_{m+1}(w_\alpha)$, we have

$$\frac{|\tilde{Q}_{2m+1}(x)|}{|\tilde{Q}'_{2m+1}(\tilde{z}_d)(x - \tilde{z}_d)|} \frac{\sigma_\delta(x)}{\sigma_\delta(\tilde{z}_d)} \leq \mathcal{C} \frac{|p_m(w_\alpha, x)|}{|p_m(w_\alpha, \tilde{z}_d)|} \frac{\sigma_\delta(x)}{\sigma_\delta(\tilde{z}_d)} \frac{\sqrt{w_\alpha(\tilde{z}_d)}}{\sqrt{w_\alpha(x)}}$$

and using (4.48) and (4.41),

$$\frac{|\tilde{Q}_{2m+1}(x)|}{|\tilde{Q}'_{2m+1}(\tilde{z}_d)(x - \tilde{z}_d)|} \frac{\sigma_\delta(x)}{\sigma_\delta(\tilde{z}_d)} \leq \mathcal{C} \left(\frac{x}{\tilde{z}_d} \right)^{\delta - \alpha - \frac{1}{4}} \leq \mathcal{C},$$

since $x \sim \tilde{z}_d$. \square

Proof of Theorem 3.1

We prove the sufficient condition. By (4.45) we have

$$\|L_{2m+2}^*(w_\alpha, w_\alpha, f)\sigma_\delta\|_\infty \leq \mathcal{C} \max_{\frac{a_m}{m^2} \leq x \leq a_m} |L_{2m+2}^*(w_\alpha, w_\alpha, f; x)\sigma_\delta(x)| \quad (4.50)$$

and by (3.27)

$$\begin{aligned} \|L_{2m+2}^*(w_\alpha, w_\alpha, f)\sigma_\delta\|_\infty &\leq \mathcal{C} \|f\sigma_\delta\|_\infty \times \\ &\max_{\frac{a_m}{m^2} \leq x \leq a_m} \sum_{k=1}^j \left| \frac{\tilde{Q}_{2m+1}(x)(a_{m+1} - x)}{\tilde{Q}'_{2m+1}(\tilde{z}_k)(a_{m+1} - \tilde{z}_k)(x - \tilde{z}_k)} \right| \frac{\sigma_\delta(x)}{\sigma_\delta(\tilde{z}_k)} \\ &=: \mathcal{C} \|f\sigma_\delta\|_\infty \max_{\frac{a_m}{m^2} \leq x \leq a_m} \Sigma(x) \end{aligned} \quad (4.51)$$

By (4.42)

$$|\tilde{Q}_{2m+1}(x)|\sigma_\delta(x) \leq \mathcal{C} \frac{x^{\delta - \alpha - \frac{1}{2}}}{\sqrt{a_m}}, \quad \frac{a_m}{m^2} \leq x \leq \theta a_m, \quad (4.52)$$

and recalling (4.47) we have

$$\Sigma(x) \leq C \sum_{k=1, k \neq d}^j \frac{\Delta \tilde{z}_k}{|x - \tilde{z}_k|} \frac{\sqrt{a_m - x}}{\sqrt{a_m - \tilde{z}_k}} \frac{x^{\delta - \alpha - \frac{1}{2}}}{\tilde{z}_k^{\delta - \alpha - \frac{1}{2}}} + \left| \frac{\tilde{Q}_{2m+1}(x)(a_m - x)}{\tilde{Q}'_{2m+1}(\tilde{z}_d)(x - \tilde{z}_d)(a_m - \tilde{z}_d)} \right| \frac{\sigma_\delta(x)}{\sigma_\delta(\tilde{z}_d)}.$$

By Lemma 5.1 [21, p.19], (see also [15]) under the assumption $0 \leq \delta - \alpha - \frac{1}{2} \leq 1$, using Lemma 4.2 and taking into account $(a_m - x) \sim (a_m - \tilde{z}_d)$, we get

$$\Sigma(x) \leq C \log m. \quad (4.53)$$

Combining last inequality with (4.51), (3.28) follows.

We omit the proof of the necessary part, since it follows with a slight change in the proof of Theorem 3.1 in [21]. \square

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