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FREE VIBRATIONS OF TAPERED BEAMS WITH FLEXIBLE ENDS

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Abstract—The dynamic behaviour of beams with linearly varying cross-section is examined, in the presence of rotationally and axially flexible ends. The equation of motion is solved in terms of Bessel functions, and the boundary conditions lead to the frequency equation which is a function of four flexibility coefficients. For some particular cases of perfect constraints some known results of the literature can be recovered. Numerical results end the paper. Copyright © 1996 Elsevier Science Ltd

1. INTRODUCTION

The dynamic analysis of tapered beams has been the subject of countless scientific investigations. Starting from the early 1960s, a number of papers by Mabie and Rogers [1–4] presented the exact frequency equations for various tapered beams with classical boundary conditions, by using the Bessel function theory. The presence of a concentrated mass should be noted. Essentially, they based their analysis on some results from Ref. [5], and on the monumental treatise by Watson [6],

Later on, Goel generalized the Mabie–Rogers analysis [7] in order to take into account more complex structures with rotational springs at the ends, and Craver and Jampala [8] were able to deduce the frequency equation for a cantilever beam with intermediate translational spring.

The same structural system as in Ref. [7] has been treated by means of the approximate Rayleigh–Ritz and Rayleigh–Schmidt techniques by Grossi and co-workers [9, 10], together with some other classical boundary conditions. The performances of these approximate methods were surprisingly good.

The optimized Rayleigh–Ritz method has been used by Alvarez *et al.* [11], in order to study the vibrations of an elastically restrained, non-uniform beam with translational and rotational springs, and with a tip mass.

Finally, a different approach, by means of Fourier expansions of the mass and moment of inertia should be noted [12], because it is suitable for all variations of cross-sectional properties, including both continuously varying systems and stepped beams.

In this paper a tapered beam with rotational and translational springs at the ends is examined, in which both the height and the depth of the cross section are supposed to vary according to a linear law. The differential equation of motion is solved in terms of Bessel functions, and the boundary conditions are imposed, in order to deduce the frequency equation. A number of particular cases with classical constraints can be immediately deduced for limiting values of the flexibilities, and some result from the literature has also been recovered.

Some numerical examples have been presented, in which the frequency equation has been solved by means of the false position method, in order to deduce the values of the first nondimensional frequency coefficients.

2. THE EQUATION OF MOTION

The differential equation of motion of a slender Euler–Bernoulli beam is given by

$$\frac{\partial^2}{\partial z^2} EI(z)\left(\frac{\partial^2 v(z,t)}{\partial z^2}\right) + \rho A(z)\frac{\partial^2 v(z,t)}{\partial t^2} = 0, \quad (1)$$

where $v(z,t)$ is the transverse displacement, z is the abscissa, t is the time, E is the Young modulus, $I(z)$ is the cross-sectional inertia, $A(z)$ is cross-sectional area, and L is the span of the beam.

The solution of the previous equation can be expressed as

$$v(z, t) = V(z)\,e^{i\omega t}, \quad (2)$$

where ω is the circular frequency, and $i = \sqrt{-1}$, so that eqn (1) becomes

$$\frac{d^2}{dz^2}\left(EI(z)\frac{d^2V(z)}{dz^2}\right) - \rho A(z)\omega^2 V(z) = 0. \quad (3)$$

If the cross-section of the beam is supposed to vary according to a linear law, as illustrated in Fig. 1, then

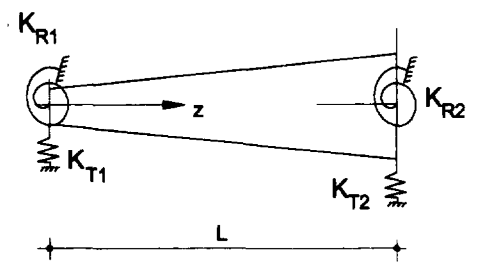

Fig. 1. The structural system.

area and inertia of the section will vary according to the following laws:

$$A(z)=A_0\left((\alpha-1)\frac{z}{L}+1\right)^2$$
$$I(z)=I_0\left((\alpha-1)\frac{z}{L}+1\right)^4, \qquad (4)$$

where A_0 and I_0 are the cross-sectional area and inertia at $z=0$, and $\alpha=h_1/h_0=b_1/b_0$ is the taper ratio of the beam. Finally, h_0 and h_1 are the cross-section heights at $z=0$ and $z=L$, respectively, and b_0, b_1 are the cross-section widths at the same abscissae.

If eqns (4) and (5) are inserted into the differential equation of motion (3) it is possible to write

$$\left[(\alpha-1)\frac{z}{L}+1\right]^4\frac{d^4V}{dz^4}+8\left[(\alpha-1)\frac{z}{L}+1\right]^3\left[\frac{\alpha-1}{L}\right]\frac{d^3V}{dz^3}+12\left[(\alpha-1)\frac{z}{L}+1\right]^2\left[\frac{\alpha-1}{L}\right]^2\frac{d^2V}{dz^2}-\frac{\rho A_0}{EI_0}\left[(\alpha-1)\frac{z}{L}+1\right]^2\omega^2V=0. \qquad (5)$$

It is convenient to define the following non-dimensional coefficient:

$$X=(\alpha-1)\frac{z}{L}+1 \qquad (6)$$

which is equal to 1 at $z=0$ and to α at $z=L$. Equation (6) simplifies as follows:

$$X^4\frac{d^4V}{dX^4}+8X^3\frac{d^3V}{dX^3}+12X^2\frac{d^2V}{dX^2}-\left[\frac{Lk}{(\alpha-1)}\right]^4X^2V=0 \qquad (7)$$

with

$$k^4=\frac{\rho A_0\omega^2}{EI_0}. \qquad (8)$$

The general solution of this equation is

$$V(X)=\frac{1}{X}[AJ_2[q\sqrt{X}]+BY_2[q\sqrt{X}]+CI_2[q\sqrt{X}]+DK_2[q\sqrt{X}]], \qquad (9)$$

where A, B, C, D are integration constants, $q=2Lk/(\alpha-1)$, J_2 is the Bessel function of I degree and order 2, Y_2 is the Bessel function of II degree and order 2, I_2 is the modified Bessel function of I degree and order 2, and finally K_2 is the modified Bessel function of II degree and order 2.

3. THE BOUNDARY CONDITIONS

The boundary conditions, in the presence of constraints with rotational stiffnesses k_{R1}, k_{R2} and transverse stiffnesses k_{T1}, and k_{T2} are given by

$$\frac{d^2V}{dX^2}-\frac{k_{R1}L}{EI_0(\alpha-1)}\frac{dV}{dX}=0 \qquad (10)$$

$$4\frac{d^2V}{dX^2}+\frac{d^3V}{dX^3}+\frac{k_{T1}L^3}{EI_0(\alpha-1)^3}V=0 \qquad (11)$$

at $z=0$, and:

$$\frac{d^2V}{dX^2}+\frac{k_{R2}L}{\alpha^4EI_0(\alpha-1)}\frac{dV}{dX}=0 \qquad (12)$$

$$\frac{4}{\alpha}\frac{d^2V}{dX^2}+\frac{d^3V}{dX^3}-\frac{k_{T2}L^3}{\alpha^4EI_0(\alpha-1)^3}V=0 \qquad (13)$$

at $z=L$.

It is convenient to define the following non-dimensional stiffness coefficients:

$$R_1=\frac{k_{R1}L}{EI_0},\quad R_2=\frac{k_{R2}L}{EI_1},$$
$$T_1=\frac{k_{T1}L^3}{EI_0},\quad T_2=\frac{k_{T2}L^3}{EI_1}, \qquad (14)$$

so that the boundary conditions become

$$\frac{d^2V}{dX^2}-\frac{R_1}{(\alpha-1)}\frac{dV}{dX}=0 \qquad (15)$$

$$4\frac{d^2V}{dX^2}+\frac{d^3V}{dX^3}+\frac{T_1}{(\alpha-1)^3}V=0 \qquad (16)$$

$$\frac{d^2V}{dX^2}+\frac{R_2}{(\alpha-1)}\frac{dV}{dX}=0 \qquad (17)$$

$$\frac{4}{\alpha}\frac{d^2V}{dX^2}+\frac{d^3V}{dX^3}-\frac{T_2L^3}{(\alpha-1)^3}V=0. \qquad (18)$$

If the general solution (9) is inserted into these boundary conditions, then a homogeneous system of four equations is obtained, for the four integration constants. In order to have a non-trivial solution, the determinant of this system must be zero. After some algebra, the terms of the determinant can be written down as follows:

$$a_{11} = qJ_4[q] + 2\frac{R_1}{(\alpha - 1)}J_3[q] \quad (19)$$

$$a_{12} = qY_4[q] + 2\frac{R_1}{(\alpha - 1)}Y_3[q] \quad (20)$$

$$a_{13} = qI_4[q] + 2\frac{R_1}{(\alpha - 1)}I_3[q] \quad (21)$$

$$a_{14} = qK_4[q] + 2\frac{R_1}{(\alpha - 1)}K_3[q] \quad (22)$$

$$a_{21} = 8q^2J_4[q] - q^3J_5[q] + 8\frac{T_1}{(\alpha - 1)^3}J_2[q] \quad (23)$$

$$a_{22} = 8q^2Y_4[q] - q^3Y_5[q] + 8\frac{T_1}{(\alpha - 1)^3}Y_2[q] \quad (24)$$

$$a_{23} = 8q^2I_4[q] + q^3I_5[q] + 8\frac{T_1}{(\alpha - 1)^3}I_2[q] \quad (25)$$

$$a_{24} = 8q^2K_4[q] - q^3K_5[q] + 8\frac{T_1}{(\alpha - 1)^3}K_2[q] \quad (26)$$

$$a_{31} = qJ_4[q\sqrt{\alpha}] - 2\frac{R_2\sqrt{\alpha}}{(\alpha - 1)}J_3[q\sqrt{\alpha}] \quad (27)$$

$$a_{32} = qY_4[q\sqrt{\alpha}] - 2\frac{R_2\sqrt{\alpha}}{(\alpha - 1)}Y_3[q\sqrt{\alpha}] \quad (28)$$

$$a_{33} = qI_4[q\sqrt{\alpha}] + 2\frac{R_2\sqrt{\alpha}}{(\alpha - 1)}I_3[q\sqrt{\alpha}] \quad (29)$$

$$a_{34} = qK_4[q\sqrt{\alpha}] - 2\frac{R_2\sqrt{\alpha}}{(\alpha - 1)}K_3[q\sqrt{\alpha}] \quad (30)$$

$$a_{41} = 8q^2J_4[q\sqrt{\alpha}] - q^3\sqrt{\alpha}J_5[q\sqrt{\alpha}] - 8\frac{T_2\alpha^2}{(\alpha - 1)^3}J_2[q] \quad (31)$$

$$a_{42} = 8q^2Y_4[q\sqrt{\alpha}] - q^3Y_5[q\sqrt{\alpha}] - 8\frac{T_2\alpha^2}{(\alpha - 1)^3}Y_2[q\sqrt{\alpha}] \quad (32)$$

$$a_{43} = 8q^2I_4[q\sqrt{\alpha}] + q^3I_5[q\sqrt{\alpha}] - 8\frac{T_2\alpha^2}{(\alpha - 1)^3}I_2[q\sqrt{\alpha}] \quad (33)$$

$$a_{44} = 8q^2K_4[q\sqrt{\alpha}] - q^3K_5[q\sqrt{\alpha}] - 8\frac{T_2\alpha^2}{(\alpha - 1)^3}K_2[q\sqrt{\alpha}]. \quad (34)$$

4. PARTICULAR CASES

If the nondimensional stiffness coefficients are allowed to become zero or infinity, then the limiting cases of perfect constraints can be easily recovered. For example, if $R_1 = R_2 = T_2 = 0$ and $T_1 \to \infty$, then the frequency equation of the simply supported–clamped beam is obtained, if $R_1 = T_1 = 0$ and $R_2 \to \infty$, $T_2 \to \infty$, then the frequency equation of the cantilever beam can be studied, both the determinants are equal to the determinants given by Mabie–Rogers in Ref. [1]. Other interesting limiting cases can be listed as shown on the next page.

5. NUMERICAL EXAMPLES

Let k_iL be the nondimensional free frequency coefficient. The frequency equation will be numerically solved by the modified bisection method (false position method), and the whole procedure has been greatly simplified by using the powerful *Mathematica* software [13].

In Table 1 the first five nondimensional free frequency coefficients are given for $T_1 = T_2 \to \infty$ and $\alpha = 2$, for various values of the nondimensional rotational stiffness coefficients R_1 and R_2. For the sake of comparison, in the same table the approximate values given in Ref. [7] are also reported.

In Table 2 the first four nondimensional coefficients are given for $R_2 = T_2 \to \infty$ and $\alpha = 1.4$, for various values of R_1 and T_1. Two limiting cases, in which the cantilever beam and the simply supported clamped beam are recovered, can be compared with the results given in Ref. [2].

Table 1. First five nondimensional frequency coefficients for $T_1 = T_2 \to \infty$ and $\alpha = 2$, for various values of the nondimensional rotational stiffness coefficients R_1 and R_2

R_1	R_2	k_1L	k_2L	k_3L	k_4L	k_5L
0	0	3.7300	7.6302	11.4217	15.2083	18.9954
0	0.01	3.7345	7.6317	11.4226	15.2089	18.9959
0	0.1	3.7737	7.6447	11.4306	15.2147	19.0004
0	1	4.0635	7.7619	11.5054	15.2695	19.0436
0	10	4.7549	8.2846	11.9277	15.6221	19.3456
1	0	3.7984	7.6803	11.4604	15.2397	19.0218
1	0.1	3.8409	7.6946	11.4693	15.2461	19.0267
1	1	3.1249	7.8105	11.5436	15.3007	19.0698

Table 2. First four nondimensional coefficients for $R_2 = T_2 \to \infty$ and $\alpha = 1.4$, for various values of R_1 and T_1

R_1	T_1	k_1L	k_2L	k_3L	k_4L
	∞	4.4329	7.8008	11.2061	14.6219
0	1000	4.3755	7.4772	10.2215	12.8572
	10	2.8554	5.4414	8.7426	12.1196
	1	2.4420	5.3805	8.7280	12.1141
	0.1	2.3834	5.3745	8.7265	12.1136
	0	2.3766	5.3739	8.7264	12.1135
∞	100	5.1171	8.4939	11.8929	15.2933
	10	4.8490	8.1441	11.4959	14.8703
	1	4.5172	7.8562	11.2476	14.6550

(1) Simply supported beam ($T_1 = T_2 \to \infty$, $R_1 = R_2 = 0$)

$$\begin{vmatrix} J_4[q] & Y_4[q] & I_4[q] & K_4[q] \\ J_2[q] & Y_2[q] & I_2[q] & K_2[q] \\ J_4[q\sqrt{\alpha}] & Y_4[q\sqrt{\alpha}] & I_4[q\sqrt{\alpha}] & K_4[q\sqrt{\alpha}] \\ -J_2[q\sqrt{\alpha}] & -Y_2[q\sqrt{\alpha}] & -I_2[q\sqrt{\alpha}] & -K_2[q\sqrt{\alpha}] \end{vmatrix}. \tag{35}$$

(2) Clamped–sliding beam ($R_1 = T_1 = R_2 \to \infty$, $T_2 = 0$)

$$\begin{vmatrix} J_3[q] & Y_3[q] & -I_3[q] & K_3[q] \\ J_2[q] & Y_2[q] & I_2[q] & K_2[q] \\ -J_3[q\sqrt{\alpha}] & -Y_3[q\sqrt{\alpha}] & I_3[q\sqrt{\alpha}] & -K_3[q\sqrt{\alpha}] \\ 8J_4[q\sqrt{\alpha}] - q\sqrt{\alpha}J_5[q\sqrt{\alpha}] & 8Y_4[q\sqrt{\alpha}] - q\sqrt{\alpha}Y_5[q\sqrt{\alpha}] & 8I_4[q\sqrt{\alpha}] + q\sqrt{\alpha}I_5[q\sqrt{\alpha}] & 8K_4[q\sqrt{\alpha}] - q\sqrt{\alpha}K_5[q\sqrt{\alpha}] \end{vmatrix}. \tag{36}$$

(3) Clamped beam ($R_1 = T_1 = R_2 = T_2 \to \infty$)

$$\begin{vmatrix} J_3[q] & Y_3[q] & -I_3[q] & K_3[q] \\ J_2[q] & Y_2[q] & I_2[q] & K_2[q] \\ -J_3[q\sqrt{\alpha}] & -Y_3[q\sqrt{\alpha}] & I_3[q\sqrt{\alpha}] & -K_3[q\sqrt{\alpha}] \\ -J_2[q\sqrt{\alpha}] & -Y_2[q\sqrt{\alpha}] & -I_2[q\sqrt{\alpha}] & -K_2[q\sqrt{\alpha}] \end{vmatrix}. \tag{37}$$

(4) Simply supported–sliding beam ($R_1 = T_2 = 0$, $R_2 = T_1 \to \infty$)

$$\begin{vmatrix} J_4[q] & Y_4[q] & I_4[q] & K_4[q] \\ J_2[q] & Y_2[q] & I_2[q] & K_2[q] \\ -J_3[q\sqrt{\alpha}] & -Y_3[q\sqrt{\alpha}] & I_3[q\sqrt{\alpha}] & -K_3[q\sqrt{\alpha}] \\ 8J_4[q\sqrt{\alpha}] - q\sqrt{\alpha}J_5[q\sqrt{\alpha}] & 8Y_4[q\sqrt{\alpha}] - q\sqrt{\alpha}Y_5[q\sqrt{\alpha}] & 8I_4[q\sqrt{\alpha}] + q\sqrt{\alpha}I_5[q\sqrt{\alpha}] & 8K_4[q\sqrt{\alpha}] - q\sqrt{\alpha}K_5[q\sqrt{\alpha}] \end{vmatrix}. \tag{38}$$

(5) Free–sliding beam ($R_1 = T_1 = T_2 = 0,\ R_2 \to \infty$)

$$\begin{vmatrix} J_4[q] & Y_4[q] & I_4[q] & K_4[q] \\ 8J_4[q] - qJ_5[q] & 8Y_4[q] - qY_5[q] & 8I_4[q] + qI_5[q] & 8K_4[q] - qK_5[q] \\ -J_3[q\sqrt{\alpha}] & -Y_3[q\sqrt{\alpha}] & I_3[q\sqrt{\alpha}] & -K_3[q\sqrt{\alpha}] \\ 8J_4[q\sqrt{\alpha}] - q\sqrt{\alpha}J_5[q\sqrt{\alpha}] & 8Y_4[q\sqrt{\alpha}] - q\sqrt{\alpha}Y_5[q\sqrt{\alpha}] & 8I_4[q\sqrt{\alpha}] + q\sqrt{\alpha}I_5[q\sqrt{\alpha}] & 8K_4[q\sqrt{\alpha}] - q\sqrt{\alpha}K_5[q\sqrt{\alpha}] \end{vmatrix}. \tag{39}$$

(6) Sliding–sliding beam ($R_1 = R_2 \to \infty,\ T_1 = T_2 = 0$)

$$\begin{vmatrix} J_3[q] & Y_3[q] & -I_3[q] & K_3[q] \\ 8J_4[q] - qJ_5[q] & 8Y_4[q] - qY_5[q] & 8I_4[q] + qI_5[q] & 8K_4[q] - qK_5[q] \\ -J_3[q\sqrt{\alpha}] & -Y_3[q\sqrt{\alpha}] & I_3[q\sqrt{\alpha}] & -K_3[q\sqrt{\alpha}] \\ 8J_4[q\sqrt{\alpha}] - q\sqrt{\alpha}J_5[q\sqrt{\alpha}] & 8Y_4[q\sqrt{\alpha}] - q\sqrt{\alpha}Y_5[q\sqrt{\alpha}] & 8I_4[q\sqrt{\alpha}] + q\sqrt{\alpha}I_5[q\sqrt{\alpha}] & 8K_4[q\sqrt{\alpha}] - q\sqrt{\alpha}K_5[q\sqrt{\alpha}] \end{vmatrix}. \tag{40}$$

(7) Simply supported–free beam ($R_1 = R_2 = T_2 = 0,\ T_1 \to \infty$)

$$\begin{vmatrix} J_4[q] & Y_4[q] & I_4[q] & K_4[q] \\ J_2[q] & Y_2[q] & I_2[q] & K_2[q] \\ J_4[q\sqrt{\alpha}] & Y_4[q\sqrt{\alpha}] & I_4[q\sqrt{\alpha}] & K_4[q\sqrt{\alpha}] \\ 8J_4[q\sqrt{\alpha}] - q\sqrt{\alpha}J_5[q\sqrt{\alpha}] & 8Y_4[q\sqrt{\alpha}] - q\sqrt{\alpha}Y_5[q\sqrt{\alpha}] & 8I_4[q\sqrt{\alpha}] + q\sqrt{\alpha}I_5[q\sqrt{\alpha}] & 8K_4[q\sqrt{\alpha}] - q\sqrt{\alpha}K_5[q\sqrt{\alpha}] \end{vmatrix}. \tag{41}$$

(8) Free–free beam ($R_1 = T_1 = R_2 = T_2 = 0$)

$$\begin{vmatrix} J_4[q] & Y_4[q] & I_4[q] & K_4[q] \\ 8J_4[q] - qJ_5[q] & 8Y_4[q] - qY_5[q] & 8I_4[q] + qI_5 & 8K_4[q] - qK_5[q] \\ J_4[q\sqrt{\alpha}] & Y_4[q\sqrt{\alpha}] & I_4[q\sqrt{\alpha}] & K_4[q\sqrt{\alpha}] \\ 8J_4[q\sqrt{\alpha}] - q\sqrt{\alpha}J_5[q\sqrt{\alpha}] & 8Y_4[q\sqrt{\alpha}] - q\sqrt{\alpha}Y_5[q\sqrt{\alpha}] & 8I_4[q\sqrt{\alpha}] + q\sqrt{\alpha}I_5[q\sqrt{\alpha}] & 8K_4[q\sqrt{\alpha}] - q\sqrt{\alpha}K_5[q\sqrt{\alpha}] \end{vmatrix}. \tag{42}$$

Finally, the determinant given by Goel [7] for a beam with rotationally flexible ends should be corrected as follows ($R_1 \neq 0, R_2 \neq 0, T_1 \to \infty, T_2 \to \infty$):

$$\begin{vmatrix} qJ_4[q] + 2\frac{R_1}{(\alpha-1)}J_3[q] & qY_4[q] + 2\frac{R_1}{(\alpha-1)}Y_3[q] & qI_4[q] - 2\frac{R_1}{(\alpha-1)}I_3[q] & qK_4[q] + 2\frac{R_1}{(\alpha-1)}K_3[q] \\ J_2[q] & Y_2[q] & I_2[q] & K_2[q] \\ qJ_4[q\sqrt{\alpha}] - 2\frac{R_2\sqrt{\alpha}}{(\alpha-1)}J_3[q\sqrt{\alpha}] & qY_4[q\sqrt{\alpha}] - 2\frac{R_2\sqrt{\alpha}}{(\alpha-1)}Y_3[q\sqrt{\alpha}] & qI_4[q\sqrt{\alpha}] + 2\frac{R_2\sqrt{\alpha}}{(\alpha-1)}I_3[q\sqrt{\alpha}] & qK_4[q\sqrt{\alpha}] - 2\frac{R_2\sqrt{\alpha}}{(\alpha-1)}K_3[q\sqrt{\alpha}] \\ -J_2[q\sqrt{\alpha}] & -Y_2[q\sqrt{\alpha}] & -I_2[q\sqrt{\alpha}] & -K_2[q\sqrt{\alpha}] \end{vmatrix}. \quad (43)$$

Table 3. First four nondimensional frequencies as functions of T_1 and T_2 with $\alpha = 1.4$, $R_1 = R_2 = 0$

R_1	T_1	k_1L	k_2L	k_3L	k_4L
∞	∞	3.4159	6.8687	10.2978	13.7260
1000	1000	3.3755	6.5696	9.2888	11.5626
100	100	3.0724	5.0667	6.7115	9.0709
10	10	2.1010	3.1302	5.3938	8.6415
1	1	1.2140	1.7851	5.2122	8.6003

Finally, in Table 3 the rotational stiffness coefficients are set equal to zero, α is assumed to be equal to 1.4, and the first four nondimensional frequencies are given as functions of T_1 and T_2.

6. CONCLUSIONS

The frequency equation for a slender tapered beam in the presence of elastic ends has been deduced, and various classical boundary conditions have been treated as limiting cases of the general system. Numerical examples and comparisons from the literature end the paper.

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