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The effect of rotatory inertia on the frequency and normal mode equations of non-uniform rotating beams

Nicola M. Auciello

Department of Structural Engineering, University of Basilicata, Via dell'Ateneo Lucano 10 – Potenza- Italy
E-mail: nicola.auciello@unibas.it

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SUMMARY. This paper presents a dynamic model for the vibration of rotating tapered beams including rotary inertia. The governing differential equations of motion of the beam in free vibration are derived using Lagrange's equations and include the effect of an arbitrary hub radius. Three linear partial differential equations are derived. Two of the linear differential equations are coupled through the stretch and chordwise deformation, the other equation is an uncoupled one for the flapwise deformation. An approximate method based on the Rayleigh-Ritz solution is proposed to solve the natural frequency of very slender rotating beam at high angular velocity. The parameters for the hub radius, rotational speed, tapered ratio, rotary inertia and slenderness ratio are incorporated into the equation of motion. The theory is valid for a wide range of applications in rotating machinery design.

1 INTRODUCTION

There are many engineering example which can be idealized as rotating beams, such as helicopter blades, turbine blades, satellite booms, aircraft rotary wings, etc. Rotating beam differs from a non-rotating beam in having additional centrifugal force and Coriolis effects on its dynamics. The stretching causes the increment of the bending stiffness of the structures, which naturally results in the variation of natural frequencies and mode shapes. Vibration in many cases greatly affects the nature of engineering designs. Consequently, considerable attention has been paid in free vibration analysis involving the study of natural frequencies and mode shapes of such structures. Identifying such structural properties is essential to the analysis of structural dynamics and the suppression of unwanted vibrations. Numerous methods such as experimental, analytical and numerical methods have been developed and used to analyse the structural dynamics of beam-like structures. In this respect, the modal analysis is a well-known practical technique for investigation of the dynamic response and vibrations of beams. Indeed transverse free vibrations of non-uniform beams have been studied by numerous researchers in both aeronautical and mechanical engineering fields either analytically or numerically. Added to this, several analytical solutions, most of which are applied for linearly tapered beams, have been represented in terms of orthogonal polynomials [1], power series by differential transformation method [2], and finite element analysis [3]. Recently, Gunda et al [4] used the linear combination of terms of the functions derived from the exact solution of the governing static differential equation of a stiff-string and that of a non-rotating beam

In the present study, the equations of motion of rotating Rayleigh beam are derived by the Lagrange's equation. In order to capture all inertia effect and coupling between extensional and flexural deformation, the consistent linearization of the fully geometrically non-linear beam theory. The problem with many discrete degrees of freedom is studied through the adoption of orthogonal polynomial functions satisfying the essential conditions only.

2 MATHEMATICAL FORMULATIONS

Considered a tapered Rayleigh beam length L rigidly mounted on the periphery of rigid hub with radius r rotating about its axis fixed in space at a constant angular velocity Ω. Figure 1 show the deformation of the neutral axis of the beam. The origin of the coordinate system is chosen to be the intersection of the centroid axes of the hub and the beam. A generic point P^0 the undeformed position is given of the vector:

$$\mathbf{r}_0 = \left[r_H + x_1,\ x_2,\ x_3\right]^T. \tag{1}$$

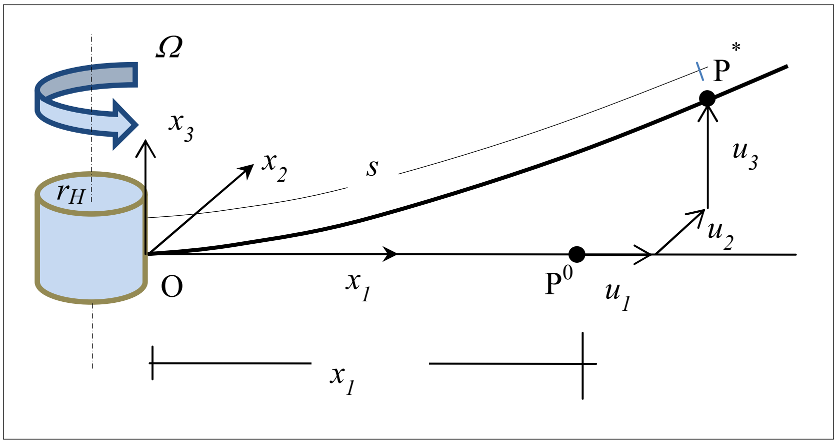

Figure 1: Deformed of the blade neutral axis

If the beam now in deforms as a result of flexure and also under tension due to the centrifugal force, the position vector of the deformed point would now be given of the $\mathbf{r}$:

$$\mathbf{r} = \left[r_H + x_1 + u_1 - x_3\, u_3^{'} - x_2\, u_2^{'},\ x_2 + u_2,\ x_3 + u_3\right]^T \tag{2}$$

The velocity of a material point in deformed state is given by: $\mathbf{v} = \dot{\mathbf{r}} + (\Omega\,\mathbf{e}_3 \times \mathbf{r})$

$$\mathbf{v} = \left[\dot{u}_1 - \Omega(x_2 + u_2) - x_3\,\dot{u}_3^{'} - x_2\,\dot{u}_2^{'},\ \ \dot{u}_2 + \Omega(r_H + x_1 + u_1) - \Omega(x_3\, u_3^{'} + x_2\, u_2^{'}),\ \ \dot{u}_3\right]^T \tag{3}$$

the time derivative are defined with a dot.

From a geometrical point of view the length, s, is a function of Cartesian coordinates and is given by the following relationship:

$$s = u_1 + \frac{1}{2}\int_0^L \left[\left(u_{2,1}\right)^2 + \left(u_{3,1}\right)^2\right] d\tau, \quad \Rightarrow \quad u_1 = s - \frac{1}{2}\int_0^L \left[\left(u_{2,1}\right)^2 + \left(u_{3,1}\right)^2\right] d\tau \tag{4}$$

where ($,_1$) represents the partial derivative with respect to the integral domain variable x_1 and τ is dummy variable. The governing differential equations of motion of the rotating tapered beam in free vibration are derived by applying Lagrange's equation which requires the expression for kinetic and strain energies. The kinetic energy of the system is given by

$$T = \frac{1}{2}\int_V m\left(\mathbf{v}\bullet\mathbf{v}\right) dV = \frac{1}{2}\int_0^L \frac{\rho}{A}\int_A \left[v_1^2 + v_2^2 + v_3^2\right] dA\, dx_1 \ . \tag{5}$$

The stain energy U of the rotating Rayleigh beam is defined [3]

$$U = \frac{1}{2}\int_0^L \Big[\ EA(s_{,1})^2 + EI_3(u_{2,11})^2 + EI_2(u_{3,11})^2 + EI_{23}(u_{2,11})(u_{3,11})\,dx_1 \Big], \qquad (6)$$

E is Young's modulus of the beam. For x_2, x_3 principal axes of inertia $I_{23} = 0$, therefore, I_2 and I_3 are the principal second area moments of the cross-section. In the present study, s, u_2 and u_3 are approximated by spatial functions and the corresponding coordinates. By employing the Rayleigh-Ritz method the variables are approximated as follows:

$$s(x_1,t) = \mathbf{\Phi}_1^T\,\mathbf{q}_1, \qquad u_2(x_1,t) = \mathbf{\Phi}_2^T\,\mathbf{q}_2, \qquad u_3(x_1,t) = \mathbf{\Phi}_3^T\,\mathbf{q}_3. \qquad (7)$$

$\mathbf{q}_1$, $\mathbf{q}_2$ and $\mathbf{q}_3$ are the generalized coordinates, $\mathbf{\Phi}_1$, $\mathbf{\Phi}_2$ and $\mathbf{\Phi}_3$ are the orthogonal polynomials for s, u_2 and u_3; [1].

2.1 *Equations of motion*

The Lagrange's equations for free vibration of a distributed parameter are given by

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial U}{\partial q_i} = 0, \qquad i = 1, 2....n \qquad (8)$$

where n is the total number of modal coordinates. The partial derivatives of T and U with respect to the generalized coordinates are needed. By substituting the partial derivatives into eq. (8), the linearized equations of motion can be obtained as follows:

$$\mathbf{M}^{11}\,\ddot{\mathbf{q}}_1 - 2\Omega\mathbf{M}^{12}\,\dot{\mathbf{q}}_2 + \left(\mathbf{K}^s - \Omega^2\,\mathbf{M}^{11}\right)\mathbf{q}_1 - \Omega^2\,(r\,\mathbf{P}_1 + \mathbf{Q}_1) = \mathbf{0} \qquad (9)$$

$$(\mathbf{M}^{R2} + \mathbf{M}^{22})\ddot{\mathbf{q}}_2 + 2\,\Omega\,\mathbf{M}^{21}\,\dot{\mathbf{q}}_1 + \left[\mathbf{K}^{B2} + \Omega^2\left(\mathbf{M}^{\rho 2} - \mathbf{M}^{22} - \mathbf{M}^{R2}\right)\right]\mathbf{q}_2 = \mathbf{0} \qquad (10)$$

$$\left(\mathbf{M}^{R3} + \mathbf{M}^{33}\right)\ddot{\mathbf{q}}_3 + \left(\mathbf{K}^{B3} + \Omega^2\mathbf{M}^{\rho 3} - \Omega^2\mathbf{M}^{R3}\right)\mathbf{q}_3 = \mathbf{0}. \qquad (11)$$

Introducing the following parameters

$$\xi = x_1/L, \quad \delta = r_H/L, \quad \gamma^2 = \rho\,A_0\,\Omega\,L^4/(E\,I_0), \quad \lambda^2 = \omega^2\,\rho A_0 L^4/E\,I_0, \quad r^2 = I_3/(A\,L^2).$$

2.2 *Flapwise bending vibration analysis*

The flapwise bending vibration of the rotating beam is governed by equation (11) which is not coupled with equations (9) end (10). From equation (11), an eigenvalue problem for the flapwise bending vibration of a rotating cantilever beam can be formulated by assuming that the u_3 ,s are harmonic function of t.

2.3 *Chordwise bending vibration analysis*

Equation (9) is couped with equation (10) through gyroscopic coupling terms. The coupling terms are often assumed negligible and ignored. This assumption is usually reasonable since the first stretching natural frequency of beams is far separated from the first bending natural frequency.

With this assumption, equation (10) can be simplified as

$$(\mathbf{M}^{R2} + \mathbf{M}^{22})\ddot{\mathbf{q}}_2 + \left[\mathbf{K}^{B2} + \Omega^2\left(\mathbf{M}^{\rho 2} - \mathbf{M}^{22} - \mathbf{M}^{R2}\right)\right]\mathbf{q}_2 = \mathbf{0}. \qquad (12)$$

3 NUMERICAL RESULTS

In order to obtain accurate numerical result s, several assumed modes are used to construct the matrices defined in Eq. (9-12). Any compact set of functions which satisfy the essential boundary condition of the Rayleigh beam can be used as the test functions ; [1]. The normalized modes of a non-rotating cantilever beam, the orthogonal polynomial can be used as test functions in the numerical calculation. The span-wise variation of the cross sectional are and the second moments of area of the beams are defined by:

$$A(\xi)=A_0\left(1-\alpha\,\xi\right)\left(1-\beta\,\xi\right)=A_0\,G(\xi),$$
$$I(\xi)=I_0\left(1-\alpha\,\xi\right)^3\left(1-\beta\,\xi\right)=I_0\,H(\xi). \qquad (13)$$

In Table 1, the first five natural frequencies of the rotating tapered Rayleigh beam is given for three rotational speeds, γ=0, γ=5 and γ=10.

β=0	Present			[2]		
α=0,5	γ=0	γ=5	γ=10	γ=0	γ=5	γ=10
λ_1	3,818	6,7391	11,4978	3,818	6,7356	11,4856
λ_2	18,1688	21,7362	29,9639	18,2245	21,7911	30,0232
λ_3	46,3265	49,9288	59,3794	46,5757	50,1876	59,6737
λ_4	87,1368	90,7623	100,8026	87,7974	91,4413	101,5422
λ_5	139,487	143,0885	153,3487	140,8192	144,4462	154,7865

Table 1: Natural frequency of vibration λ_I of rotating tapered cantilever Rayleigh beams as a function of the rotational speed parameter with δ=0 and r=1/30: flapwise bending.

4 CONCLUSIONS

In this work, three sets of linear equations of motion for rotating tapered Rayleigh cantilever beams. The Rayleigh-Ritz approach is used and the Boundary Characteristic Orthogonal Polynomials are chosen as trial functions; (BCOPs method). The natural frequencies were shown to increase as the angular speed and hub radius increase. In the dynamic analysis, this method is applied to determine the natural frequencies of tapered beams and the results compare very well with the published. The effects of the slenderness ratio, hub radius ration, and rotational speed on the natural frequencies are investigated; the frequencies increase with the increasing rotational speed due to the stiffening effect of the centrifugal force induced from the rotation.

The advantage of the procedure used is the generality of polynomial functions which only need to satisfy the essential conditions. The numerical examples have been completely carried through by means of the powerful symbolic software. Finally, the present method can be easily extended to non-uniform rotating beams with discontinuities, as well as other end conditions.

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