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GIUSEPPE DATTOLI
ENEA–Frascati, Frascati (RM), Italy

HARI M. SRIVASTAVA
University of Victoria, Victoria, British Columbia, Canada

and

CLEMENTE CESARANO
ENEA–Frascati, Frascati (RM), Italy/University of Ulm, Ulm, Germany

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www.aracneeditrice.it
info@aracneeditrice.it

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Introduction

This volume contains the contributions to the second Melfi Workshop on Advanced Special Function and Integration Methods.

As in the previous workshop we have tried to emphasize the role played by Special Functions (SF) in different research areas, covering fields from pure mathematics to applications.

The increasing importance of SF's, of conventional and generalized nature, in unrelated fields is due to the concurrence of many factors, which can however be traced back to a common thread.

One of the most attractive aspects of the modern point of view on the SF theory is perhaps a by product of the Lie treatment, namely the possibility of expressing a large number of complicated operational relations in terms of different types of SF. Very recently a breakthrough in this respect has been offered by elementary methods linked to the so called MONOMIALITY PRINCIPLE. Such a technique provides the tools to explore in a systematic and unified way a large class of operational identities and offers the keynote to construct SF of generalized nature. The interest in applications for this aspect of the problem relies upon the fact that

1) most of the operational identities occurring in this treatment appears quite naturally in quantum mechanical problems or in applications concerning wave propagation in classical optics

2) the SF of generalized nature are the suited tool to solve questions related to classical electromagnetism, like radiation emission by accelerated charge

3) a large body of the formalism, associated with the monomiality principle, is ideally suited for the study of differential equations of different nature, involving parabolic equations related to evolution problems.

Regarding this last point, the combined use of SF and operational techniques becomes a unique tool to solve Schroedinger and Liouville type equations by means of the evolution operator method.

Modern computer languages, devoted to the solution of practical problems as charged beam transport in accelerators or optical beam transport in laser cavities, utilize evolution operator and algebraic methods in which the SF's are an almost obliged step. Furthermore many transforms (Fourier, Hankel, Wigner, Gauss, Fresnell... and their fractional counterparts) are now being viewed as particular cases of the evolution operator and offers therefore a new field of application of conventional and generalized SF's.

In these proceedings we have tried to give an idea of how SF may combine with methods of algebraic nature and provide the answer to various problems in application. However we did not forget other aspects as the mutual influence between SF's and Combinatorics which is having during these last years a significant evolution due, among the other things, to applications in cryptography.

Even though we have stressed the applicative content of this book, we must also underline that the research in this field cannot proceed without the support from more speculative points of view.

The proceedings include therefore contributions on advanced topics dealing with new types of SF's belonging to the families of pseudo–SF including the so called d–orthogonal polynomials. During the workshop it has been pointed out that these last mentioned functions may be a crucial tool for problems in classical and quantum statistics.

As for the last year, we must underline that the success of the Workshop has been determined by the generous and enthusiastic support of the Melfi Town Council. In particular we owe our gratitude to the Major Prof. Ernesto A. Navazio, to On. dr. Nicola Pagliuca to Prof. Luigi Branchini, to prof. Sandro Calabrese members of the Town Council and to dr. Tania Lasala director of the culture department. It is also a pleasure to thank dr. G. Bartolomei, Mr. M. Pierotti for their heroic effort in solving all the logistic and bureaucratic problems.

It is finally a pleasure to recognize the financial support for the publication of the proceedings of the Dipartimento di Statistica, Probabilità e Statistiche Applicate of University of Rome La Sapienza.

Giuseppe Dattoli

THE EULER–KNOPP TRANSFORMATION AND ASSOCIATED FAMILIES OF GENERATING FUNCTIONS

H.M. Srivastava

Department of Mathematics and Statistics

University of Victoria – Victoria, British Columbia V8W 3P4 (Canada)

E-mail: harimsri@math.uvic.ca

Abstract The problem of summation of a slowly convergent series is usually handled by transforming the given series into another series which converges faster than the original series. One of many such series transformations, which are commonly employed for accelerating the convergence of slowly convergent series, is the familiar Euler–Knopp transformation which was recently generalized by making use of the classical Laguerre and related polynomials. The main object of this two-part series of lectures is to show how some of these generalized series transformations, which are presented here rather systematically, would lead naturally to the derivation of several general results on generating functions (involving the Stirling numbers of the second kind) for a fairly wide variety of special functions and polynomials in one, two, and more variables. Relevant connections of many of the families of generating functions, which are considered here, with various known results (given by earlier authors) are also pointed out.

1. Introduction, Definitions, and Preliminaries

In a large variety of widespread areas of applications, one encounters the need for summing a given series that usually does not converge as fast as may be desired. A well–exploited technique for summing a slowly converging series consists in attempting to transform the given series into another which converges more rapidly than the original series. One of several such transformations, which are commonly used for accelerating convergence of slowly converging series, is the familiar Euler–Knopp transformation (*cf.*, *e.g.*, Hardy [5, p. 178 *et seq.*]), which we recall here as

Theorem 1 (Euler–Knopp) *In terms of a given sequence* $\{a_n\}_{n=0}^{\infty}$, *let*

$$A_n^{(\nu)}(p) := \sum_{k=0}^{n} \binom{n+\nu}{n-k} (-p)^{n-k} a_k \tag{1.1}$$
$$\left(n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\};\ \mathbb{N} := \{1,2,3,\ldots\}\right),$$

where ν *is a constant and* p *is a suitable acceleration parameter.*

Then

$$\sum_{n=0}^{\infty} a_n z^n = \sum_{n=0}^{\infty} A_n^{(0)}(p)\, z^n (1-pz)^{-n-1} \tag{1.2}$$

for all values of z *and* p *for which both summands exist and each of the series converges.*

The *classical* Euler transformation is a special case of Theorem 1 when

$$z = -1 \quad \text{and} \quad p = 1.$$

For $p = 1$ (and $\nu = 0$), the definition (1.1) coincides with that of the nth divided difference $\triangle^n a_0$. In this special case, Theorem 1 yields a (known) *nonlinear* transformation which is not recommended now–a–days (see, for details, Hartee [6] and Niethammer [10]).

The classical Laguerre polynomials $L_n^{(\alpha)}(x)$, of order α and degree n in x, are defined by (*cf.* Szegö [17, p. 101])

$$L_n^{(\alpha)}(x) := \sum_{k=0}^{n} \binom{n+\alpha}{n-k} \frac{(-x)^k}{k!} \tag{1.3}$$

or, equivalently, by

$$L_n^{(\alpha)}(x) = \binom{n+\alpha}{n} {}_1F_1(-n; \alpha+1; x), \tag{1.4}$$

where, as usual, ${}_lF_m$ denotes a generalized hypergeometric function with l numerator and m denominator parameters, defined by (*cf.*, *e.g.*, Erdélyi *et al.* [2, Vol. I, Chapter 4])

$${}_lF_m(\lambda_1, \ldots, \lambda_l; \mu_1, \ldots, \mu_m; z) := \sum_{n=0}^{\infty} \frac{(\lambda_1)_n \cdots (\lambda_l)_n}{(\mu_1)_n \cdots (\mu_m)_n} \frac{z^n}{n!} \tag{1.5}$$

$$\begin{aligned}&\left(\lambda_j \in \mathbb{C}\ (j = 1, \ \ldots, l);\ \mu_j \in \mathbb{C}\backslash\mathbb{Z}_0^-\ (j = 1, \ \ldots, m);\right.\\ &\mathbb{Z}_0^- := \{0, -1, -2, \ldots\};\ l,\ m \in \mathbb{N}_0;\ l < m+1\\ &\text{and } z \in \mathbb{C};\ l = m+1 \text{ and } |z| < 1;\ l = m+1,\\ &\left.|z| = 1, \text{ and } \Re\left(\textstyle\sum_{j=1}^{m} \mu_j - \sum_{j=1}^{l} \lambda_j\right) > 0\right)\end{aligned}$$

with, of course, the Pochhammer symbol (or the *shifted* factorial) $(\lambda)_n$ given, in terms of the Gamma functions, by

$$(\lambda)_n := \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} = \begin{cases} 1 & (n=0) \\ \lambda(\lambda+1)\cdots(\lambda+n-1) & (n\in\mathbb{N}). \end{cases}$$

Indeed these polynomials are orthogonal over the interval $(0,\infty)$ with respect to the weight function $x^\alpha e^{-x}$; in fact, we have (*cf.*, *e.g.*, Szegö [17, p. 100])

$$\int_0^\infty x^\alpha e^{-x} L_m^{(\alpha)}(x)\, L_n^{(\alpha)}(x)\, dx = \Gamma(\alpha+1)\binom{n+\alpha}{n}\delta_{m,n} \tag{1.6}$$
$$(\Re(\alpha) > -1;\ m,n\in\mathbb{N}_0),$$

where $\delta_{m,n}$ denotes the Kronecker delta.

Making use of the classical Laguerre polynomials $L_n^{(\alpha)}(x)$, Gabutti and Lyness [3] gave an interesting generalization of (the Euler–Knopp) Theorem 1. Indeed, by means of several illustrative examples of slowly converging series, they investigated the optimal choice of the acceleration parameter p so that the resulting new series has the fastest convergence (see, for details, [3, p. 265 *et seq.*]). We recall here their *main* result contained in

Theorem 2 (Gabutti and Lyness [3]) *Let $g(t)$ be any function for which the following integral representation of the sequence $\left\{b_n^{(\alpha,\nu)}(\lambda,\mu)\right\}_{n=0}^\infty$ exists:*

$$b_n^{(\alpha,\nu)}(\lambda,\mu) := \binom{n+\nu}{n}^{-1}\int_0^\infty e^{-\lambda t}\, t^\alpha\, L_n^{(\alpha)}(t)\, g(\mu t)\, dt \tag{1.7}$$
$$(\Re(\lambda) > 0),$$

where $L_n^{(\alpha)}(x)$ denotes the Laguerre polynomials defined by (1.3)*.*

Then

$$\sum_{n=0}^\infty a_n\, b_n^{(\alpha,\nu)}(\lambda,\mu) = \sum_{n=0}^\infty A_n^{(\nu)}(p)\,(1-p)^{-n}\, b_n^{(\alpha,\nu)}\left(\lambda',\mu'\right), \tag{1.8}$$

where

$$\lambda' := p + (1-p)\lambda \quad \text{and} \quad \mu' := (1-p)\mu, \tag{1.9}$$

and the sequence $\left\{A_n^{(\nu)}(p)\right\}_{n=0}^\infty$ is defined by (1.1)*, it being assumed that α, ν, λ, μ, a_n, and p are so constrained that both sums in* (1.8) *exist and converge.*

For $\nu = 0$, (the Gabutti–Lyness) Theorem 2 obviously provides a new (possibly wider) family of series to which the Euler–Knopp transformation (Theorem 1) may be applied. Moreover, for other admissible values of ν, it provides an apparently new class of transformations. In this two–part series of lectures, we aim at presenting some recent developments in connection with the aforementioned series transformations, especially in the derivation of numerous families of generating functions (involving the Stirling numbers of the second kind) for a remarkably wide variety of special functions and polynomials in one, two, and more variables. We also consider relevant connections of the results presented here with those given in earlier works on the subject.

2. A Unified Presentation of the Laguerre and Hermite Polynomials

In the remarkably vast and widely scattered literature on special functions and polynomials, there are numerous generalizations of the classical Laguerre polynomials $L_n^{(\alpha)}(x)$ defined by (1.3) and of the classical Hermite polynomials $H_n(x)$ defined by (*cf.* Szegö [17, p. 106])

$$\begin{aligned} H_n(x) &:= \sum_{k=0}^{[n/2]} (-1)^k \binom{n}{2k} \frac{(2k)!}{k!} (2x)^{n-2k} \\ &= (2x)^n \, {}_2F_0\left(-\frac{1}{2}n, -\frac{1}{2}n+\frac{1}{2}; -; -x^{-2}\right), \end{aligned} \tag{2.1}$$

which satisfy the orthogonality property (*cf.*, *e.g.*, Szegö [17, p. 105]):

$$\int_{-\infty}^{\infty} e^{-x^2} H_m(x) H_n(x)\, dx = 2^n n! \sqrt{\pi}\, \delta_{m,n} \qquad (m, n \in \mathbb{N}_0), \tag{2.2}$$

where, as before in (1.6), $\delta_{m,n}$ is the Kronecker delta. Indeed the relationship of these polynomials with the classical Laguerre polynomials $L_n^{(\alpha)}(x)$ is given by

$$H_{2n+\varepsilon}(x) = (-1)^n 2^{2n+\varepsilon} n! x^{\varepsilon} L_n^{\left(\varepsilon-\frac{1}{2}\right)}\left(x^2\right) \qquad (\varepsilon = 0 \quad \text{or} \quad 1). \tag{2.3}$$

One of the aforementioned generalizations of the classical Laguerre and Hermite polynomials is provided by (for example) the generalized Hermite polynomials considered by Gould and Hopper [4]. In their attempt to present a unified investigation of many of these known extensions and generalizations of the classical Laguerre and Hermite polynomials, Srivastava and Singhal [16] introduced a sequence $\left\{G_n^{(\alpha)}(x, r, q, \kappa)\right\}_{n=0}^{\infty}$ of polynomials (of degree n in x^r *and* in α) defined by the following Rodrigues–type formula:

$$G_n^{(\alpha)}(x, r, q, \kappa) := \frac{x^{-\alpha-\kappa n}}{n!} \exp(qx^r) \left(x^{\kappa+1} \frac{d}{dx}\right)^n \{x^{\alpha} \exp(-qx^r)\}, \tag{2.4}$$

where the parameters α, q, r, and κ are unrestricted, in general (with, of course, $\kappa \neq 0$). The Srivastava–Singhal polynomials $G_m^{(\alpha)}(x, r, q, \kappa)$ are generated by [16, p. 78, Equation (3.2)]

$$\begin{aligned}(1-\kappa t)^{-\alpha/\kappa} \quad & \exp\left(qx^r\left[1-(1-\kappa t)^{-r/\kappa}\right]\right) \\ &= \sum_{n=0}^{\infty} G_n^{(\alpha)}(x, r, q, \kappa)\, t^n\end{aligned} \tag{2.5}$$

and, more generally, by [16, p. 78, Equation (3.1)]

$$\begin{aligned}\sum_{n=0}^{\infty} \quad & \frac{(\lambda_1)_n \cdots (\lambda_l)_n}{(\mu_1)_n \cdots (\mu_m)_n}\, G_n^{(\alpha)}(x, r, q, \kappa)\, t^n \\ &= \exp(qx^r) \sum_{n=0}^{\infty} \frac{(-qx^r)^n}{n!} \\ &\quad \cdot {}_{l+1}F_m\left(\lambda_1, \ \ldots, \ \lambda_l, \frac{\alpha+nr}{\kappa}; \mu_1, \ \ldots, \ \mu_m; \kappa t\right),\end{aligned} \tag{2.6}$$

which would immediately reduce to the generating function (2.5) upon setting

$$l = m \quad \text{and} \quad \lambda_j = \mu_j \quad (j = 1, \ \ldots, \ l \, (\text{or } m)).$$

We recall here the following relationships of the Srivastava–Singhal polynomials $G_n^{(\alpha)}(x, r, q, \kappa)$ with some other known polynomials which are seemingly relevant to our present investigation (*cf.* [16, p. 76]; see also a more recent work by Hubbell and Srivastava [7]):

$$\begin{aligned}G_n^{(\alpha)}(x, r, q, -1) \quad &= G_n^{(\alpha-n+1)}(x, r, q, 1) \\ &= \frac{(-x)^n}{n!} H_n^r(x, \alpha, q),\end{aligned} \tag{2.7}$$

where $H_n^r(x, \alpha, q)$ denotes the aforementioned Gould–Hopper generalization of the classical Hermite polynomials;

$$G_n^{(0)}(x, 2, 1, -1) = G_n^{(1-n)}(x, 2, 1, 1) = \frac{(-x)^n}{n!} H_n(x); \tag{2.8}$$

$$G_n^{(\alpha+1)}(x, 1, 1, \kappa) = \kappa^n \; Y_n^{\alpha}(x; \kappa) \qquad (\kappa \in \mathbb{N}), \tag{2.9}$$

where $Y_n^{\alpha}(x; \kappa)$ $(\alpha > -1; \kappa \in \mathbb{N})$ denotes the Konhauser *biorthogonal* polynomials (*cf.* [8]; see also [12]);

$$G_n^{(\alpha+n)}(x, 1, 1, -1) = G_n^{(\alpha+1)}(x, 1, 1, 1) = L_n^{(\alpha)}(x). \tag{2.10}$$

For the Srivastava–Singhal polynomials defined by (2.4), it is known also that (*cf.* [16, p. 79, Equation (3.6)])

$$\sum_{n=0}^{\infty}\binom{m+n}{n} G_{m+n}^{(\alpha)}(x,r,q,\kappa)\,t^n = (1-\kappa t)^{-m-\alpha/\kappa}\exp\left(qx^r\left[1-(1-\kappa t)^{-r/\kappa}\right]\right) \cdot G_m^{(\alpha)}\left(x(1-\kappa t)^{-1/\kappa},r,q,\kappa\right) \tag{2.11}$$

$$\left(m\in\mathbb{N}_0;\ \kappa\neq 0;\ |t|<|\kappa|^{-1}\right).$$

Upon replacing x in (2.11) by $x^{1/r}$ $(r\neq 0)$, if we set

$$t=-\frac{p}{1-\kappa p} \qquad \left(|p|<|\kappa|^{-1}\right),$$

we easily find from (2.11) that

$$G_m^{(\alpha)}\left(x^{1/r},r,q,\kappa\right)=(1-\kappa p)^{-m-\alpha/\kappa}\exp\left(qx\left[1-(1-\kappa p)^{-r/\kappa}\right]\right) \cdot\sum_{n=0}^{\infty}\tbinom{m+n}{n}G_{m+n}^{(\alpha)}\left(\tfrac{x^{1/r}}{(1-\kappa p)^{1/\kappa}},r,q,\kappa\right)\left(-\tfrac{p}{1-\kappa p}\right)^n \tag{2.12}$$

$$\left(m\in\mathbb{N}_0;\ \kappa,r\neq 0;\ |p|<|\kappa|^{-1}\right).$$

By applying the generating function (2.12), we now state and prove a generalization of (the Gabutti–Lyness) Theorem 2, which is given by (see, for details, Srivastava [13])

Theorem 3 *Let $g(t)$ be any function for which the following integral representation of the sequence $\left\{c_n^{(\alpha,\beta)}(\lambda,\ \mu;\ \nu)\right\}_{n=0}^{\infty}$ exists:*

$$c_n^{(\alpha,\beta)}(\lambda,\mu;\nu):=\binom{n+\nu}{n}^{-1}\int_0^{\infty}e^{-\lambda t}\,t^{\beta}\,G_n^{(\alpha)}\left(t^{1/r},r,q,\kappa\right)g(\mu t)\,dt$$

$$(\Re(\lambda)>0;\ \kappa,\ r\neq 0), \tag{2.13}$$

where $G_n^{(\alpha)}(x,r,q,\kappa)$ denotes the Srivastava–Singhal polynomials defined by (2.4).

Then

$$\sum_{n=0}^{\infty}a_n\,c_n^{(\alpha,\beta)}(\lambda,\mu;\nu)=\sum_{n=0}^{\infty}A_n^{(\nu)}(p)\,(1-\kappa p)^{-n-[\alpha-(\beta+1)r]/\kappa} \cdot c_n^{(\alpha,\beta)}\left(\lambda',\mu';\nu\right), \tag{2.14}$$

where

$$\lambda':=q+(\lambda-q)(1-\kappa p)^{r/\kappa} \qquad \textit{and} \qquad \mu':=\mu(1-\kappa p)^{r/\kappa}, \tag{2.15}$$

and the sequence $\left\{A_n^{(\nu)}(p)\right\}_{n=0}^{\infty}$ *is defined by* (1.1)*, it being assumed that* α*,* β*,* λ*,* μ*,* q*,* r*,* κ*,* a_n*, and* p *are so constrained that both sums in* (2.14) *exist and converge.*

Proof. Denote, for convenience, the infinite series on the left–hand side of (2.14) by $\Im$. Then, by appealing to the definition (2.13) as well as the generating function (2.12), we obtain

$$\begin{aligned}
\Im &:= \sum_{n=0}^{\infty} a_n\, c_n^{(\alpha,\beta)}(\lambda,\mu;\nu) \\
&= \sum_{n=0}^{\infty} a_n \binom{n+\nu}{n}^{-1} \int_0^{\infty} e^{-\lambda t}\, t^{\beta} g(\mu t)\,(1-\kappa p)^{-n-\alpha/\kappa} \exp\left(qt\left[1-(1-\kappa p)^{-r/\kappa}\right]\right) \\
&\qquad \cdot \sum_{j=0}^{\infty} \binom{n+j}{j} G_{n+j}^{(\alpha)}\left(\frac{t^{1/r}}{(1-\kappa p)^{1/\kappa}}, r, q, \kappa\right)\left(-\frac{p}{1-\kappa p}\right)^{j} dt \\
&= \sum_{n,j=0}^{\infty} a_n \binom{n+\nu}{n}^{-1}\binom{n+j}{j}(1-\kappa p)^{-n-j-\alpha/\kappa}(-p)^{j} \\
&\qquad \cdot \int_0^{\infty} \exp\left(-t\left\{\lambda - q\left[1-(1-\kappa p)^{-r/\kappa}\right]\right\}\right) t^{\beta} g(\mu t) \\
&\qquad\qquad G_{n+j}^{(\alpha)}\left(\frac{t^{1/r}}{(1-\kappa p)^{1/\kappa}}, r, q, \kappa\right) dt \\
&= \sum_{j=0}^{\infty}\sum_{n=0}^{j} a_n \binom{n+\nu}{n}^{-1}\binom{j}{j-n}(1-\kappa p)^{-j-\alpha/\kappa}(-p)^{j-n} \\
&\qquad \cdot \int_0^{\infty} \exp\left(-t\left\{\lambda - q\left[1-(1-\kappa p)^{-r/\kappa}\right]\right\}\right) t^{\beta} g(\mu t) \\
&\qquad\qquad \cdot G_{j}^{(\alpha)}\left(\tfrac{t^{1/r}}{(1-\kappa p)^{1/\kappa}}, r, q, \kappa\right) dt, \qquad (2.16)
\end{aligned}$$

provided that the inversions of the order of summation and integration can be justified by absolute convergence of the integral and series involved.

If we now set

$$t = (1-\kappa p)^{r/\kappa}\, u \quad \text{and} \quad dt = (1-\kappa p)^{r/\kappa}\, du,$$

we find from (2.16) that

$$\begin{aligned}
\Im &= \sum_{j=0}^{\infty}\sum_{n=0}^{j} a_n \binom{n+\nu}{n}^{-1}\binom{j}{j-n}(-p)^{j-n}\ \ (1-\kappa p)^{-j-[\alpha-(\beta+1)r]/\kappa} \\
&\qquad \cdot \int_0^{\infty} e^{-\lambda' u}\, u^{\beta}\, G_j^{(\alpha)}\left(u^{1/r}, r, q, \kappa\right) g\left(\mu' u\right) du, \qquad (2.17)
\end{aligned}$$

where $\lambda^{'}$ and $\mu^{'}$ are defined by (2.15).

Finally, we interpret the integral in (2.17) by means of the definition (2.13). We thus obtain

$$\begin{aligned}\Im &= \sum_{j=0}^{\infty}\sum_{n=0}^{j} a_n \binom{n+\nu}{n}^{-1} \binom{j}{j-n} (-p)^{j-n} \\ &\qquad \cdot (1-\kappa p)^{-j-[\alpha-(\beta+1)r]/\kappa} \binom{j+\nu}{j} c_j^{(\alpha,\beta)}\left(\lambda^{'},\mu^{'};\nu\right) \\ &= \sum_{j=0}^{\infty} (1-\kappa p)^{-j-[\alpha-(\beta+1)r]/\kappa}\, c_j^{(\alpha,\beta)}\left(\lambda^{'},\mu^{'};\nu\right) \\ &\qquad \cdot \sum_{n=0}^{j} \binom{j+\nu}{j-n} (-p)^{j-n}\, a_n,\end{aligned}$$

which, in view of the definition (1.1), is precisely the infinite series on the right–hand side of the assertion (2.14) of Theorem 3.

Since [15, p. 381, Equation 7.6(19); see also Equation (2.9) above]

$$Y_n^{\alpha}(x;s) = s^{-n}\, G_n^{(\alpha+1)}(x,1,1,s) \qquad (s \in \mathbb{N}), \tag{2.18}$$

where $Y_n^{\alpha}(x;s)$ denotes one class of *biorthogonal* polynomials introduced by Konhauser (*cf.* [8]; see also [12]) for $\alpha > -1$ and $s \in \mathbb{N}$, by setting

$$q = r = 1 \quad \text{and} \quad \kappa = s \quad (s \in \mathbb{N})$$

and making some simple notational changes such as $\alpha \longmapsto \alpha+1$, we can specialize Theorem 3 in order to deduce

Corollary 1 *Let $g(t)$ be any function for which the following integral representation of the sequence $\left\{d_n^{(\alpha,\beta)}(\lambda,\mu;\nu)\right\}_{n=0}^{\infty}$ exists:*

$$\begin{aligned} d_n^{(\alpha,\beta)}(\lambda,\mu;\nu) &:= \binom{n+\nu}{n}^{-1} \int_0^{\infty} e^{-\lambda t}\, t^{\beta}\, Y_n^{\alpha}(t;s)\, g(\mu t)\, dt \\ &\quad (\Re(\lambda) > 0;\ s \in \mathbb{N}) \end{aligned} \tag{2.19}$$

in terms of the Konhauser biorthogonal polynomials $Y_n^{\alpha}(x;s)$ given by (2.18) *and* (2.4)*.*

Then

$$\begin{aligned}\sum_{n=0}^{\infty} a_n\, d_n^{(\alpha,\beta)}(\lambda,\mu;\nu) &= \sum_{n=0}^{\infty} A_n^{(\nu)}(p)\,(1-\kappa p)^{-n+(\beta-\alpha)/s} \\ &\qquad \cdot d_n^{(\alpha,\beta)}\left(\lambda^{'},\mu^{'};\nu\right),\end{aligned} \tag{2.20}$$

where

$$\lambda' := 1 + (\lambda - 1)(1-p)^{1/s} \quad \text{and} \quad \mu' := \mu(1-p)^{1/s}, \tag{2.21}$$

and the sequence $\left\{A_n^{(\nu)}(p)\right\}_{n=0}^{\infty}$ *is defined by* (1.1), *it being assumed that* α, β, λ, μ, a_n, *and* p *are so constrained that both sums in* (2.20) *exist and converge.*

Recalling that [*cf.* Equations (2.10) and (2.18)]

$$Y_n^{\alpha}(x;1) = L_n^{(\alpha)}(x) \qquad (n \in \mathbb{N}_0), \tag{2.22}$$

it is easily seen from the definitions (1.4) and (2.11) that

$$b_n^{(\alpha,\nu)}(\lambda,\mu) = d_n^{(\alpha,\alpha)}(\lambda,\mu;\nu)\Big|_{s=1}. \tag{2.23}$$

Thus the *main* result (Theorem 2) of Gabutti and Lyness [3] is a *further* special case of Corollary 1 above when

$$s = 1 \quad \text{and} \quad \alpha = \beta. \tag{2.24}$$

Numerous other corollaries and consequences of Theorem 3, in addition to Corollary 1 above, can indeed be deduced by suitably specializing the various parameters involved in Theorem 3. For instance, we can easily derive a corollary associated with the (Gould–Hopper) generalized Hermite polynomials $H_n^r(x,\alpha,q)$ by using the relationship (2.7).

3. Applications of Theorem 3 Involving Generating Functions

By appropriately choosing the function $g(t)$ in our definition (2.13) followed by some suitable variable as well as parameter changes or (alternatively) by appealing *directly* to the generating function (2.12) (with $p \longmapsto pz$), it is not difficult to show that

$$\begin{aligned}\sum_{n=0}^{\infty} a_n\, G_n^{(\alpha)}\left(x^{1/r}, r, q, \kappa\right) z^n \\ &= (1-\kappa pz)^{-\alpha/\kappa} \exp\left(qx\left[1-(1-\kappa pz)^{-r/\kappa}\right]\right) \\ &\quad \cdot \sum_{n=0}^{\infty} A_n^{(0)}(p)\, G_n^{(\alpha)}\left(\frac{x^{1/r}}{(1-\kappa pz)^{1/\kappa}}, r, q, \kappa\right)\left(\frac{z}{1-\kappa pz}\right)^n \\ &\qquad \left(|z| < |\kappa p|^{-1};\ \kappa, r, p \neq 0\right),\end{aligned} \tag{3.1}$$

where $A_n^{(0)}(p)$ is defined by (1.1) with, of course, $\nu = 0$.