A note on the ground state solutions for the nonlinear Schrödinger-Maxwell equations

A. Azzollini & A. Pomponio *

Abstract

In this paper we study the nonlinear Schrödinger-Maxwell equations

$$\begin{cases} -\Delta u + V(x)u + \phi u = |u|^{p-1}u & \text{in } \mathbb{R}^3, \\ -\Delta \phi = u^2 & \text{in } \mathbb{R}^3. \end{cases}$$

If V is a positive constant, we prove the existence of a ground state solution (u, ϕ) for $2 < p < 5$. The non-constant potential case is treated for $3 < p < 5$, and V possibly unbounded below.

1 Introduction

In this note, we present some results contained in [1]. We consider the following system

$$\begin{cases} -\Delta u + V(x)u + \phi u = f'(u) & \text{in } \mathbb{R}^3, \\ -\Delta \phi = u^2 & \text{in } \mathbb{R}^3, \end{cases} \tag{$\mathcal{SM}$}$$

where $V : \mathbb{R}^3 \to \mathbb{R}$ and $f \in C^1(\mathbb{R}, \mathbb{R})$. Such a system, also known as the nonlinear Schrödinger-Poisson, arises in an interesting physical context. In fact, according to a classical model, the interaction of a charge particle with an electro-magnetic field can be described by coupling the nonlinear Schrödinger's and the Maxwell's equations (we refer to [2] for more details on the physical aspects). In particular, if we are looking for electrostatic-type solutions, we just have to solve ($\mathcal{SM}$). We refer to [1] for the bibliography.

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The aim of our paper is to look for the existence of *ground state solutions* to the problem ($\mathcal{SM}$), namely couples (u, ϕ) which solve ($\mathcal{SM}$) and minimize the action functional associated to ($\mathcal{SM}$) among all possible solutions. Up to our knowledge, the literature does not contain any result in this direction.

We are interested in considering pure power type nonlinearities so that the problem we will deal with becomes

$$\begin{cases} -\Delta u + V(x)u + \phi u = |u|^{p-1}u & \text{in } \mathbb{R}^3, \\ -\Delta\phi = u^2 & \text{in } \mathbb{R}^3, \end{cases} \tag{1}$$

where $2 < p < 5$. The solutions $(u, \phi) \in H^1(\mathbb{R}^3) \times \mathcal{D}^{1,2}(\mathbb{R}^3)$ of (1) are the critical points of the action functional $\mathcal{E}\colon H^1(\mathbb{R}^3) \times \mathcal{D}^{1,2}(\mathbb{R}^3) \to \mathbb{R}$, defined as

$$\mathcal{E}(u, \phi) := \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2 + V(x)u^2 - \frac{1}{4}\int_{\mathbb{R}^3} |\nabla\phi|^2 + \frac{1}{2}\int_{\mathbb{R}^3} \phi u^2 - \frac{1}{p+1}\int_{\mathbb{R}^3} |u|^{p+1}.$$

We are interested in finding a ground state solution of (1), that is a solution (u_0, ϕ_0) of (1) such that $\mathcal{E}(u_0, \phi_0) \leqslant \mathcal{E}(u, \phi)$, for any solution (u, ϕ) of (1).

The action functional $\mathcal{E}$ exhibits a strong indefiniteness, namely it is unbounded both from below and from above on infinite dimensional subspaces. This indefiniteness can be removed using the reduction method described in [3], by which we are led to study a one variable functional that does not present such a strongly indefinite nature.

The main difficulty related with the problem of finding the critical points of the new functional consists in the lack of compactness of the Sobolev spaces embeddings in the unbounded domain $\mathbb{R}^3$. Usually, at least when V is radially symmetric, such a difficulty is overcome by restricting the functional to the natural constraint of the radial functions where compact embeddings hold. In particular, in [4] a radial solution having minimal energy among all the radial solutions has been found. However we are not able to say if that solution actually is a ground state for our equation. This is the reason why we will use an alternative method, based on a concentration-compactness argument on suitable measures, to recover compactness. Such an approach, very standard in studying the compactness in problems involving the Schrödinger equation, seems to be quite new for the nonlinear Schrödinger-Maxwell equations and presents several difficulties due to the coupling.

We analyze two different situations. First we assume that V is a positive constant and we look for a minimizer of the reduced functional restricted to a suitable manifold $\mathcal{M}$ introduced by Ruiz in [9]. Such a manifold has two interesting features: it is a natural constraint for the reduced functional and it contains, in a sense that we will explain later (see Remark 3.1), every solution of the problem (1). The main result we get is the following

Theorem 1.1. *If V is a positive constant, then the problem* (1) *has a ground state solution for any $p \in]2,5[$.*

Remark 1.2. *By using the strong maximum principle and quite standard arguments, it is easy to see that such a ground state solution does not change sign, so we can assume it positive.*

Afterwards we study (1) assuming the following hypotheses on V :

(V1) $V\colon \mathbb{R}^3 \to \mathbb{R}$ is a measurable function;

(V2) $V_\infty := \lim_{|y|\to\infty} V(y) \geqslant V(x)$, for almost every $x \in \mathbb{R}^3$, and the inequality is strict in a non-zero measure domain;

(V3) there exists $\bar{C} > 0$ such that, for any $u \in H^1(\mathbb{R}^3)$,

$$\int_{\mathbb{R}^3} |\nabla u|^2 + V(x)u^2 \geqslant \bar{C}\|u\|^2.$$

Remark 1.3. *These hypotheses on V, which have been introduced to study singular nonlinear Schrödinger equations in [5], are satisfied by a large class of potentials including those most meaningful by a physical point of view. An example of admissible potentials is $V\colon \mathbb{R}^3 \to \mathbb{R}$ defined as $V(x) = V_1(x) - \lambda V_2(x)$, where V_1 is a potential bounded below by a positive constant and satisfying (***V2***), λ is a sufficiently small positive constant and V_2 is a positive function such that*

$$\exists \alpha_1 > 0, \alpha_2 \geqslant 0\colon \int_{\mathbb{R}^3} V_2(x)u^2 \leqslant \int_{\mathbb{R}^3} \alpha_1|\nabla u|^2 + \alpha_2 u^2, \text{ for any } u \in H^1(\mathbb{R}^3),$$

and

$$\lim_{|x|\to+\infty} V_2(x) = 0.$$

Because of technical difficulties related with the presence of the potential, we are not allowed to use the same device as in the previous case. In particular the use of the Ruiz' constraint appears quite involved, and minimizing the functional on the Nehari manifold turns out to be a more natural approach. However this causes that only the case $3 < p < 5$ can be considered.

Another difficulty consists in the fact that we are not allowed to repeat the same concentration and compactness argument on positive measures as in the constant potential case. The reason is that, since V may have some singularities, we have no way to affirm that the integral

$$\int_\Omega |\nabla u|^2 + V(x)|u|^2$$

is nonnegative for any $u \in H^1(\mathbb{R}^3)$ and $\Omega \subset \mathbb{R}^3$, and consequently the measures could be not positive. We get the following

Theorem 1.4. *If V satisfies (**V1-3**) then the problem* (1) *has a ground state solution for any $p \in]3,5[$.*

Theorems 1.1 and 1.4 will be proved, respectively, in Section 3 and 4.

It is remarkable that, up to our knowledge, this latter theorem is the first existence result obtained for (1) when V is non-radial, and the nonlinearity is superlinear. Actually, in [10], existence and nonexistence results have been proved when the nonlinearity is asymptotically linear. However, the device used in [10] seems that does not work for nonlinearities such as $|u|^{p-1}u$, with $1 < p < 5$.

2 Some preliminary results

We first recall some well-known facts. For every $u \in L^{12/5}(\mathbb{R}^3)$, there exists a unique $\phi_u \in \mathcal{D}^{1,2}(\mathbb{R}^3)$ solution of

$$-\Delta\phi = u^2, \qquad \text{in } \mathbb{R}^3.$$

It can be proved that $(u, \phi) \in H^1(\mathbb{R}^3) \times \mathcal{D}^{1,2}(\mathbb{R}^3)$ is a solution of (1) if and only if $u \in H^1(\mathbb{R}^3)$ is a critical point of the functional $I\colon H^1(\mathbb{R}^3) \to \mathbb{R}$ defined as

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2 + V(x)u^2 + \frac{1}{4}\int_{\mathbb{R}^3} \phi_u u^2 - \frac{1}{p+1}\int_{\mathbb{R}^3} |u|^{p+1}, \tag{2}$$

and $\phi = \phi_u$.
The functions ϕ_u possess the following properties (see [4] and [9])

Lemma 2.1. *For any $u \in H^1(\mathbb{R}^3)$, we have:*

i) $\|\phi_u\|_{\mathcal{D}^{1,2}(\mathbb{R}^3)} \leqslant C\|u\|^2$, where C does not depend from u. As a consequence there exists $C' > 0$ such that

$$\int_{\mathbb{R}^3} \phi_u u^2 \leqslant C'\|u\|^4;$$

ii) $\phi_u \geqslant 0$;

iii) *for any* $t > 0$: $\phi_{tu} = t^2\phi_u$;

iv) *for any* $\theta > 0$: $\phi_{u_\theta}(x) = \theta^2\phi_u(\theta x)$, *where* $u_\theta(x) = \theta^2 u(\theta x)$;

v) *for any* $\Omega \subset \mathbb{R}^3$ *measurable,*

$$\int_\Omega \phi_u u^2 = \int_\Omega \int_{\mathbb{R}^3} \frac{u^2(x)u^2(y)}{|x-y|} dx\, dy.$$

3 The constant potential case

In this section we will assume that V is a positive constant. Without lost of generality, we suppose $V \equiv 1$. It can be proved (see [9]) that if $(u, \phi) \in H^1(\mathbb{R}^3) \times \mathcal{D}^{1,2}(\mathbb{R}^3)$ is a solution of (1), then it satisfies the following Pohozaev type identity

$$\int_{\mathbb{R}^3} \frac{1}{2}|\nabla u|^2 + \frac{3}{2}u^2 + \frac{5}{4}\phi u^2 - \frac{3}{p+1}|u|^{p+1} = 0. \tag{3}$$

As in [9], we introduce the following manifold

$$\mathcal{M} := \left\{ u \in H^1(\mathbb{R}^3) \setminus \{0\} \;\middle|\; G(u) = 0 \right\},$$

where

$$G(u) := \int_{\mathbb{R}^3} \frac{3}{2}|\nabla u|^2 + \frac{1}{2}u^2 + \frac{3}{4}\phi_u u^2 - \frac{2p-1}{p+1}|u|^{p+1}.$$

Remark 3.1. *Observe that if* $u \in H^1(\mathbb{R}^3)$ *is a nontrivial critical point of* I*, then* $u \in \mathcal{M}$*, since* $G(u) = 0$ *can be obtained by a linear combination of* $\langle I'(u), u\rangle = 0$ *and* (3)*, with* $\phi = \phi_u$*. As a consequence if* $(u, \phi) \in H^1(\mathbb{R}^3) \times \mathcal{D}^{1,2}(\mathbb{R}^3)$ *is a solution of* (1)*, then* $u \in \mathcal{M}$*.*

The next lemma, whose proof is contained in [9], describes some properties of the manifold $\mathcal{M}$:

Lemma 3.2. *1. For any* $u \in H^1(\mathbb{R}^3)$*,* $u \neq 0$*, there exists a unique number* $\bar\theta > 0$ *such that* $u_{\bar\theta} \in \mathcal{M}$ *(where* $u_{\bar\theta}$ *is defined in Lemma 2.1). Moreover*

$$I(u_{\bar\theta}) = \max_{\theta \geqslant 0} I(u_\theta);$$

2. there exists a positive constant C*, such that for all* $u \in \mathcal{M}$*,* $\|u\|_{p+1} \geqslant C$*;*

3. *$\mathcal{M}$ is a natural constraint of I, namely every critical point of $I|_{\mathcal{M}}$ is a critical point for I.*

By 3 of Lemma 3.2 we are allowed to look for critical points of I restricted to $\mathcal{M}$.
With an abuse of notations, we denote by $\theta : H^1(\mathbb{R}^3) \setminus \{0\} \to \mathbb{R}_+$ also the map such that for any $u \in H^1(\mathbb{R}^3)$, $u \neq 0$:

$$I\left(u_{\theta(u)}\right) = \max_{\theta \geqslant 0} I(u_\theta).$$

By 1 of Lemma 3.2, it is well defined.
Set

$$c_1 = \inf_{g \in \Gamma} \max_{\theta \in [0,1]} I(g(\theta));$$
$$c_2 = \inf_{u \neq 0} \max_{\theta \geqslant 0} I(u_\theta);$$
$$c_3 = \inf_{u \in \mathcal{M}} I(u);$$

where

$$\Gamma = \left\{ g \in C\big([0,1], H^1(\mathbb{R}^3)\big) \mid g(0) = 0,\ I(g(1)) \leqslant 0,\ g(1) \neq 0 \right\}. \tag{4}$$

Lemma 3.3. *The following equalities hold*

$$c := c_1 = c_2 = c_3.$$

Remark 3.4. *By point 3 of Lemma 3.2 and Remark 3.1, we argue that if $u \in \mathcal{M}$ is such that $I(u) = c$, then (u, ϕ_u) is a ground state solution of* (1).

3.1 Proof of Theorem 1.1

Let $(u_n)_n \subset \mathcal{M}$ such that

$$\lim_n I(u_n) = c. \tag{5}$$

We define the functional $J\colon H^1(\mathbb{R}^3) \to \mathbb{R}$ as:

$$J(u) = \int_{\mathbb{R}^3} \frac{p-2}{2p-1} |\nabla u|^2 + \frac{p-1}{2p-1} u^2 + \frac{p-2}{2(2p-1)} \phi_u u^2.$$

Observe that for any $u \in \mathcal{M}$, by ii of Lemma 2.1 we have $I(u) = J(u) \geqslant 0$. By (5), we deduce that $(u_n)_n$ is bounded in $H^1(\mathbb{R}^3)$, so there exists $\bar{u} \in H^1(\mathbb{R}^3)$ such that, up to a subsequence,

$$\begin{aligned} u_n &\rightharpoonup \bar{u} \quad \text{weakly in } H^1(\mathbb{R}^3), \\ u_n &\to \bar{u} \quad \text{in } L^s(B), \text{with } B \subset \mathbb{R}^3, \text{bounded, and } 1 \leqslant s < 6. \end{aligned} \tag{6}$$

To prove Theorem 1.1, we need some compactness on the sequence $(u_n)_n$. To this end, we use a concentration-compactness argument on the positive measures so defined: for every $u_n \in H^1(\mathbb{R}^3)$,

$$\nu_n(\Omega) = \int_\Omega \frac{p-2}{2p-1}|\nabla u_n|^2 + \frac{p-1}{2p-1}u_n^2 + \frac{p-2}{2(2p-1)}\phi_{u_n}u_n^2. \tag{7}$$

By (5) we have

$$\nu_n(\mathbb{R}^3) = J(u_n) \to c$$

and then, by P.L. Lions [6], there are three possibilities:

vanishing : for all $r > 0$

$$\lim_n \sup_{\xi\in\mathbb{R}^3} \int_{B_r(\xi)} d\nu_n = 0;$$

dichotomy : there exist a constant $\tilde{c} \in (0, c)$, two sequences $(\xi_n)_n$ and $(r_n)_n$, with $r_n \to +\infty$ and two nonnegative measures ν_n^1 and ν_n^2 such that

$$\begin{aligned} 0 \leqslant \nu_n^1 + \nu_n^2 \leqslant \nu_n, \qquad & \nu_n^1(\mathbb{R}^3) \to \tilde{c}, \;\; \nu_n^2(\mathbb{R}^3) \to c - \tilde{c}, \\ \mathrm{supp}(\nu_n^1) \subset B_{r_n}(\xi_n), \qquad & \mathrm{supp}(\nu_n^2) \subset \mathbb{R}^3 \setminus B_{2r_n}(\xi_n); \end{aligned}$$

compactness : there exists a sequence $(\xi_n)_n$ in $\mathbb{R}^3$ with the following property: for any $\delta > 0$, there exists $r = r(\delta) > 0$ such that

$$\int_{B_r(\xi_n)} d\nu_n \geqslant c - \delta.$$

In [1], we proved the following

Lemma 3.5. *Compactness holds for the sequence of measures $(\nu_n)_n$, defined in* (7).

Now we are able to yield the following

Proof of Theorem 1.1 Let $(u_n)_n$ be a sequence in $\mathcal{M}$ such that (5) holds. We define the measures $(\nu_n)_n$ as in (7); by Lemma 3.5 there exists a sequence $(\xi_n)_n$ in $\mathbb{R}^N$ with the following property: for any $\delta > 0$, there exists $r = r(\delta) > 0$ such that

$$\int_{B_r^c(\xi_n)} \frac{p-2}{2p-1}|\nabla u_n|^2 + \frac{p-1}{2p-1}u_n^2 + \frac{p-2}{2(2p-1)}\phi_{u_n}u_n^2 < \delta. \tag{8}$$

We define the new sequence of functions $v_n := u_n(\cdot - \xi_n) \in H^1(\mathbb{R}^3)$. It is easy to see that $\phi_{v_n} = \phi_{u_n}(\cdot - \xi_n)$, and hence $v_n \in \mathcal{M}$. Moreover, by (8), we have that for any $\delta > 0$, there exists $r = r(\delta) > 0$ such that

$$\|v_n\|_{H^1(B_r^c)} < \delta \text{ uniformly for } n \geqslant 1. \tag{9}$$

Since, by (6), $(v_n)_n$ is bounded in $H^1(\mathbb{R}^3)$, certainly there exist a subsequence (likewise labelled) and $\bar{v} \in H^1(\mathbb{R}^3)$ such that

$$v_n \rightharpoonup \bar{v} \quad \text{weakly in } H^1(\mathbb{R}^3), \tag{10}$$

$$v_n \to \bar{v} \quad \text{in } L^s(B), \text{with } B \subset \mathbb{R}^3, \text{bounded, and } 1 \leqslant s < 6. \tag{11}$$

By (9), (10) and (11), we have that, taken $s \in [2, 6[$, for any $\delta > 0$ there exists $r > 0$ such that, for any $n \geqslant 1$ large enough

$$\begin{aligned}\|v_n - \bar{v}\|_{L^s(\mathbb{R}^3)} &\leqslant \|v_n - \bar{v}\|_{L^s(B_r)} + \|v_n - \bar{v}\|_{L^s(B_r^c)} \\ &\leqslant \delta + C\left(\|v_n\|_{H^1(B_r^c)} + \|\bar{v}\|_{H^1(B_r^c)}\right) \leqslant (1 + 2C)\delta,\end{aligned}$$

where $C > 0$ is the constant of the embedding $H^1(B_r^c) \hookrightarrow L^s(B_r^c)$. We deduce that

$$v_n \to \bar{v} \text{ in } L^s(\mathbb{R}^3), \text{ for any } s \in [2, 6[. \tag{12}$$

Since ϕ is continuous from $L^{12/5}(\mathbb{R}^3)$ to $\mathcal{D}^{1,2}(\mathbb{R}^3)$, from (12) we deduce that

$$\begin{aligned}&\phi_{v_n} \to \phi_{\bar{v}} \text{ in } \mathcal{D}^{1,2}(\mathbb{R}^3), \qquad \text{as } n \to \infty, \\ &\int_{\mathbb{R}^3} \phi_{v_n} v_n^2 \to \int_{\mathbb{R}^3} \phi_{\bar{v}} \bar{v}^2, \qquad \text{as } n \to \infty.\end{aligned} \tag{13}$$

Since $(v_n)_n$ is in $\mathcal{M}$, by 2 of Lemma 3.2 $(\|v_n\|_{p+1})_n$ is bounded below by a positive constant. As a consequence, (12) implies that $\bar{v} \neq 0$. Proceeding as in [9, Theorem 3.2, Step 4], by (12) and (13) we can show that $v_n \to \bar{v}$ in $H^1(\mathbb{R}^3)$ so that $\bar{v} \in \mathcal{M}$ and $I(\bar{v}) = c$. By Remark 3.4, we have that $(\bar{v}, \phi_{\bar{v}})$ is a ground state solution of (1). $\square$

4 The non-constant potential case

In this section we suppose that the potential V satisfies (**V1-3**) and that $p \in]3, 5[$.

In order to get our result, we will use a very standard device: we will look for a minimizer of the functional (2) restricted to the Nehari manifold

$$\mathcal{N} = \left\{u \in H^1(\mathbb{R}^3) \setminus \{0\} \mid \widetilde{G}(u) = 0\right\},$$

where

$$\widetilde{G}(u) := \int_{\mathbb{R}^3} |\nabla u|^2 + V(x)u^2 + \phi_u u^2 - |u|^{p+1}.$$

The following lemma describes some properties of the Nehari manifold $\mathcal{N}$:

Lemma 4.1. *1. For any $u \neq 0$ there exists a unique number $\bar{t} > 0$ such that $\bar{t}u \in \mathcal{N}$ and*

$$I(\bar{t}u) = \max_{t \geqslant 0} I(tu);$$

2. there exists a positive constant C, such that for all $u \in \mathcal{N}$, $\|u\|_{p+1} \geqslant C$;

3. $\mathcal{N}$ is a C^1 manifold.

The Nehari manifold $\mathcal{N}$ is a natural constraint for the functional I, therefore we are allowed to look for critical points of I restricted to $\mathcal{N}$.

In view of this, we assume the following definition

$$c_V := \inf_{u \in \mathcal{N}} I(u),$$

so that our goal is to find $\bar{u} \in \mathcal{N}$ such that $I(\bar{u}) = c_V$, from which we would deduce that $(\bar{u}, \phi_{\bar{u}})$ is a ground state solution of (1).

First we recall some preliminary lemmas which can be obtained by using the same arguments as in [8].

As a consequence of the Lemma 4.1, we are allowed to define the map $t : H^1(\mathbb{R}^3) \setminus \{0\} \to \mathbb{R}_+$ such that for any $u \in H^1(\mathbb{R}^3)$, $u \neq 0$:

$$I\left(t(u)u\right) = \max_{t \geqslant 0} I(tu).$$

Now define

$$I_\infty(u) := \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2 + V_\infty u^2 + \frac{1}{4}\int_{\mathbb{R}^3} \phi_u u^2 - \frac{1}{p+1}\int_{\mathbb{R}^3} |u|^{p+1},$$

$$c_\infty := c_{V_\infty}.$$

As in [8], we have

Lemma 4.2. *If V satisfies (**V1-3**), we get $c_V < c_\infty$.*

Proof By Theorem 1.1, there exists $(w, \phi_w) \in H^1(\mathbb{R}^3) \times \mathcal{D}^{1,2}(\mathbb{R}^3)$ a ground state solution of the problem

$$\begin{cases} -\Delta u + V_\infty u + \phi u = |u|^{p-1}u & \text{in } \mathbb{R}^3, \\ -\Delta \phi = u^2 & \text{in } \mathbb{R}^3. \end{cases}$$

Let $t(w) > 0$ be such that $t(w)w \in \mathcal{N}$. By **(V2)**, we have

$$\begin{aligned} c_\infty &= I_\infty(w) \geqslant I_\infty\big(t(w)w\big) \\ &= I\big(t(w)w\big) + \int_{\mathbb{R}^N} \big(V_\infty - V(x)\big)|t(w)w|^2 > c_V, \end{aligned}$$

and then we conclude. □

4.1 Proof of Theorem 1.4

Let $(u_n)_n \subset \mathcal{N}$ such that

$$\lim_n I(u_n) = c_V. \tag{14}$$

We define the functional $J\colon H^1(\mathbb{R}^3) \to \mathbb{R}$ as:

$$J(u) = \left(\frac{1}{2} - \frac{1}{p+1}\right) \int_{\mathbb{R}^3} |\nabla u|^2 + V(x)u^2 + \left(\frac{1}{4} - \frac{1}{p+1}\right) \int_{\mathbb{R}^3} \phi_u u^2.$$

Observe that for any $u \in \mathcal{N}$, we have $I(u) = J(u)$.
By **(V3)** and (14), we deduce that $(u_n)_n$ is bounded in $H^1(\mathbb{R}^3)$, so there exists $\bar{u} \in H^1(\mathbb{R}^3)$ such that, up to a subsequence,

$$u_n \rightharpoonup \bar{u} \quad \text{weakly in } H^1(\mathbb{R}^3), \tag{15}$$

$$u_n \to \bar{u} \quad \text{in } L^s(B), \text{with } B \subset \mathbb{R}^3, \text{bounded, and } 1 \leqslant s < 6. \tag{16}$$

To prove Theorem 1.4, we need some compactness on the sequence $(u_n)_n$.
We denote by ν_n the measure

$$\nu_n(\Omega) = \left(\frac{1}{2} - \frac{1}{p+1}\right) \int_{\Omega} |\nabla u_n|^2 + V(x)u_n^2 + \left(\frac{1}{4} - \frac{1}{p+1}\right) \int_{\Omega} \phi_{u_n} u_n^2.$$

Observe that, since there is no lower boundedness condition on the potential V, the measures ν_n may be not positive, and then we are not allowed to use the Lions' concentration arguments [6, 7] on them. However, in the following theorem (for the proof see [1]) we are able to show that the functions u_k concentrate in the $H^1(\mathbb{R}^3)-$norms.

Theorem 4.3. *For any $\delta > 0$ there exists $\tilde{R} > 0$ such that for any $n \geqslant \tilde{R}$*

$$\int_{|x|>\tilde{R}} (|\nabla u_n|^2 + |u_n|^2) < \delta.$$

Proof of Theorem 1.4 By Theorem 4.3, for any $\delta > 0$ there exists $r > 0$ such that

$$\|u_n\|_{H^1(B_r^c)} < \delta, \quad \text{uniformly for } n \geqslant 1. \tag{17}$$

Hence, arguing as in the constant potential case, we deduce that

$$u_n \to \bar{u} \text{ in } L^s(\mathbb{R}^3), \text{ for any } s \in [2, 6[. \tag{18}$$

Moreover

$$\begin{aligned} \phi_{u_n} \to \phi_{\bar{u}} \ \text{ in } \mathcal{D}^{1,2}(\mathbb{R}^3), \qquad & \text{as } n \to \infty, \\ \int_{\mathbb{R}^3} \phi_{u_n} u_n^2 \to \int_{\mathbb{R}^3} \phi_{\bar{u}} \bar{u}^2, \qquad & \text{as } n \to \infty, \end{aligned} \tag{19}$$

and for any $\psi \in C_0^\infty(\mathbb{R}^3)$

$$\int_{\mathbb{R}^3} \phi_{u_n} u_n \psi \to \int_{\mathbb{R}^3} \phi_{\bar{u}} \bar{u} \psi. \tag{20}$$

By (14), we can suppose (see [11]) that $(u_n)_n$ is a Palais-Smale sequence for $I|_{\mathcal{N}}$ and, as a consequence, it is easy to see that $(u_n)_n$ is a Palais-Smale sequence for I. By (15), (18) and (20), we conclude that $I'(\bar{u}) = 0$.
Since $(u_n)_n$ is in $\mathcal{N}$, by 3 of Lemma 4.1 $(\|u_n\|_{p+1})_n$ is bounded below by a positive constant. As a consequence, (18) implies that $\bar{u} \neq 0$ and so $\bar{u} \in \mathcal{N}$.
Finally, by (14), (15), (18) and (19) and by (**V2-3**) we get

$$c_V \leqslant I(\bar{u}) \leqslant \liminf I(u_n) = c_V,$$

so we can conclude that $(\bar{u}, \phi_{\bar{u}})$ is a ground state solution of (1). □

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A. Azzollini
Dipartimento di Matematica
Università degli Studi di Bari
Via Orabona, 4
70125 Bari
Italy
azzollini@dm.uniba.it

A. Pomponio
Dipartimento di Matematica
Politecnico di Bari
Via Orabona, 4
70125 Bari
Italy
a.pomponio@poliba.it