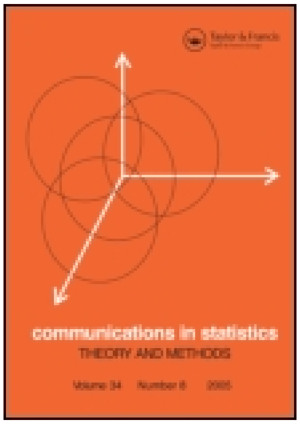

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On Some Applications of a Symbolic Representation of Non Centered Lévy Processes

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By using a symbolic technique known in the literature as the classical umbral calculus, we characterize two classes of polynomials related to Lévy processes: the Kailath-Segall and the time-space harmonic polynomials. We provide the Kailath-Segall formula in terms of cumulants and we recover simple closed-forms for several families of polynomials with respect to not centered Lévy processes, such as the Hermite polynomials with Brownian motion, Poisson-Charlier polynomials with Poisson processes, actuarial polynomials with Gamma processes, first kind Meixner polynomials with Pascal processes, and Bernoulli, Euler, and Krawtchuk polynomials with suitable random walks.

Keywords Cumulant; Kailath-Segall polynomial; Lévy process; Time-space harmonic polynomial; Umbral calculus.

1. Introduction

The umbral calculus is a symbolic method, known in the literature since the XIX century, consisting in a set of mathematical tricks, dealing with number sequences, whose subscripts were treated as they were powers. No formal setting for this theory was given until 1964, when Gian-Carlo Rota disclosed the "umbral magic art" of lowering and raising exponents, bringing to the light the underlying linear functional (Rota, 1964). From 1964 on, the umbral calculus was deeply developed. In particular, in 1994, Rota and Taylor (Rota and Taylor, 1994) provided a simple presentation of the umbral calculus in a framework very similar to the theory of random variables and, in 2001, Di Nardo and Senato (Di Nardo and Senato, 2001) gave a complete formalization of the matter.

Here, we refer to the classical umbral calculus as a syntax consisting in an alphabet $\mathcal{A} = \{\alpha, \beta, \gamma, \dots\}$ of symbols, called *umbrae*, and a suitable linear functional E, called *evaluation*, which resembles the expectation operator in probability theory. Therefore, umbrae look like the framework of random variables,

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with no reference to any probability space. The key point of the theory is the idea of associating a unital number sequence $1, a_1, a_2, \ldots$ to a sequence $1, \alpha, \alpha^2, \ldots$ of powers of α by means of the evaluation functional.

In this framework, the notion of summation of umbrae can be extended to the case of a non integer number of addends, thus leading us to a symbolic version of the infinite divisibility property and therefore of Lévy processes (Sato, 1999).

In 1997, together with Wallstrom (Rota and Wallstrom, 1997), Rota conceived a combinatorial definition of stochastic integration in the setting of random measures. The starting point is the Kailath-Segall formula (Kailath and Segall, 1976) interpreted in combinatorial terms and applied to derive recursion relations for some classes of orthogonal polynomials. The Kailath-Segall formula links the variations $\{X_t^{(n)}\}_{t\geq 0}$ of a Lévy process

$$X_t^{(1)} = X_t, \quad X_t^{(2)} = [X, X]_t, \quad X_t^{(n)} = \sum_{s\geq t} (\Delta X_s)^n, \quad n \geq 3, \tag{1.1}$$

to its iterated stochastic integrals

$$P_t^{(0)} = 1, \quad P_t^{(1)} = X_t, \quad P_t^{(n)} = \int_0^t P_{s-}^{(n-1)} \mathrm{d}X_s, \quad n \geq 2 \tag{1.2}$$

by using suitable polynomials, named the *Kailath-Segall polynomials*. In this article, we give an umbral expression of this class of polynomials highlighting the role played by their cumulants. We show that the Kailath-Segall formula is a suitable generalization of the well-known formulae giving elementary symmetric polynomials in terms of power sum symmetric polynomials.

Cumulants play the same role in the umbral expression of time-space harmonic polynomials with respect to not necessarily centered Lévy processes. A family of polynomials $\{P(x, t)\}_{t\geq 0}$ is said to be time-space harmonic with respect to a Lévy process $\{X_t\}_{t\geq 0}$ if $E[P(X_t, t) \mid \mathscr{F}_v] = P(X_v, v)$, for all $v \leq t$, where $\mathscr{F}_v = \sigma(X_\tau : \tau \leq v)$ is the natural filtration associated with $\{X_t\}_{t\geq 0}$. A Lévy process is not necessarily a martingale. Therefore, to find polynomials such that it is a martingale the stochastic process obtained by replacing the indeterminate x with the Lévy process $\{X_t\}_{t\geq 0}$, becomes fundamental, especially for applications in mathematical finance Cuchiero et al. (2008). In Solé and Utzet (2008), to get a characterization of time-space harmonic polynomials, the authors used the Teugels martingale and refer to centered Lévy processes for which the martingale property holds. In this article, we focus our attention on non-centered Lévy processes, which do not share the martingale property and we show how the classical umbral calculus allows us to get more general results without taking advantage of the martingale property. Moreover, the umbral expression of these polynomials relies on a very simple closed-form of the corresponding coefficients which can be easily implemented in any symbolic software; see Di Nardo and Oliva (2009) as an example.

This article is structured as follows. Section 2 is provided for readers unaware of the classical umbral calculus. We have chosen to recall terminology, notation, and the basic definitions strictly necessary to deal with the object of this article. We skip any proof. The reader interested in is referred to Di Nardo and Senato (2001, 2006). Section 3 gives the umbral expression of Lévy processess and analyses the classes of Kailath-Segall and time-space harmonic polynomials. In Sec. 4, we give umbral expressions of many classical families of polynomials as time-space harmonic with respect to suitable Lévy processes.

2. Background on the Classical Umbral Calculus

The classical umbral calculus is a syntax consisting of the following data:

(i) a set $\mathcal{A} = \{\alpha, \beta, \gamma, \ldots\}$ of objects, called *umbrae*;
(ii) an *evaluation* linear functional $E : \mathbb{R}[x][\mathcal{A}] \longrightarrow \mathbb{R}[x]$, where $\mathbb{R}$ is the field of real numbers, such that $E[1] = 1$ and the *uncorrelation property* holds

$$E[x^n \alpha^i \beta^j \gamma^k \cdots] = x^n E[\alpha^i] E[\beta^j] E[\gamma^k] \cdots$$

for all $\alpha, \beta, \gamma, \cdots \in \mathcal{A}$ and for all non negative integers $n, i, j, k, , \ldots$
(iii) the *augmentation* umbra $\epsilon \in \mathcal{A}$, with $E[\epsilon^n] = \delta_{0,n}$, for all non negative integers n, where $\delta_{0,n}$ is the Kronecker symbol, that is, $\delta_{0,n}$ is equal to 1 if $n = 0$ and 0 otherwise;
(iv) the *unity umbra* $u \in \mathcal{A}$, with $E[u^n] = 1$, for all non negative integers n.

A sequence $a_0 = 1, a_1, a_2, \ldots \in \mathbb{R}[x]$ is *umbrally represented* by an umbra α if $E[\alpha^n] = a_n$, for all $n \geq 0$. The element a_n is the n-th *moment* of the umbra α. An umbra is said to be *scalar* (respectively, *polynomial*) if its moments are in $\mathbb{R}$ (respectively, in $\mathbb{R}[x]$). A polynomial in $\mathbb{R}[\mathcal{A}]$ is an *umbral polynomial*. The *generating function* of an umbra $\alpha \in \mathcal{A}$ is the formal power series $f(\alpha, z) = 1 + \sum_{n \geq 1} a_n z^n / n!$, for which we do not take into account any question of convergence (Stanley, 1997).

Special umbrae are the *singleton umbra* $\chi \in \mathcal{A}$, with $f(\chi, z) = 1 + z$; the *boolean unity* $\bar{u} \in \mathcal{A}$, with $f(\bar{u}, z) = 1/(1 - z)$; the *Bell umbra* $\beta \in \mathcal{A}$, with $f(\beta, z) = \exp(e^z - 1)$ and moments the Bell numbers; the *Bernoulli umbra* ι, with $f(\iota, z) = z/(e^z - 1)$ and moments the Bernoulli numbers; the *Euler umbra* η, with $f(\eta, z) = 2e^z/(1 + e^{2z})$ and moments the Euler numbers.

The alphabet $\mathcal{A}$ can be extended with new symbols arising from operations among umbrae. These new umbrae are called *auxiliary umbrae* and the resulting umbral calculus is said to be *saturated* (Rota and Taylor, 1994). Some useful auxiliary umbrae are recalled in the following.

Disjoint sum and difference. Given $\alpha, \gamma \in \mathcal{A}$, their *disjoint sum* $\alpha \dot{+} \gamma$ (respectively, *disjoint difference* $\alpha \dot{-} \gamma$) is such that $f(\alpha \dot{+} \gamma, z) = f(\alpha, z) + f(\gamma, z) - 1$ (respectively, $f(\alpha \dot{-} \gamma, z) = f(\alpha, z) - f(\gamma, z) + 1$).

Dot-product. First, let us observe that there are infinitely many and distinct umbrae representing the same sequence of moments. More precisely, the umbrae α and γ are said to be *similar* if $E[\alpha^n] = E[\gamma^n]$ for all non negative integers n, in symbols $\alpha \equiv \gamma$. Now let us consider n uncorrelated umbrae $\alpha', \alpha'', \ldots, \alpha'''$ similar to α and their summation: the resulting umbra $\alpha' + \alpha'' + \cdots + \alpha'''$ is denoted by the symbol $n.\alpha$. The umbra $n.\alpha$ is called the *dot-product* of the integer n and the umbra α. Its generating function is $f(n.\alpha, z) = (f(\alpha, z))^n$ and the moments are $E[(n.\alpha)^i] = \sum_{j=1}^{i} (n)_j B_{i,j}(a_1, \ldots, a_{i-j+1})$, where $(n)_j$ is the lower factorial and $B_{i,j}$ are the partial exponential Bell polynomials (Di Nardo and Senato, 2006). The integer n can be replaced by any $t \in \mathbb{R}$ so that

$$E[(t.\alpha)^i] = \sum_{j=1}^{i} (t)_j B_{i,j}(a_1, \ldots, a_{i-j+1}). \tag{2.1}$$

In particular, we have

$$t.(\alpha + \gamma) \equiv t.\alpha + t.\gamma. \tag{2.2}$$

Set $t = -1$. The umbra $-1.\alpha$ is called the inverse of α. We have $-1.\alpha + \alpha \equiv \epsilon$. In (2.1), we can replace t by any umbra $\gamma \in \mathcal{A}$, for more details see Di Nardo and Senato (2006). If $(\gamma)_j = \gamma(\gamma - 1)\cdots(\gamma - j + 1)$ denotes the lower factorial polynomial, then we have $E[(\gamma.\alpha)^i] = \sum_{j=1}^{i} E[(\gamma)_j]B_{i,j}(a_1, \ldots, a_{i-j+1})$. The umbra $\gamma.\alpha$ is called the *dot-product* of the umbrae α and γ. Special dot-product umbrae are $\chi.\alpha$ and $\beta.\alpha$. The umbra $\chi.\alpha$ is denoted by the symbol κ_α and called the *α-cumulant umbra* (Di Nardo et al., 2008), since $f(\kappa_\alpha, z) = 1 + \log(f(\alpha, z))$. The umbra $\beta.\alpha$ is called the *α-partition umbra*. In particular, we have $\alpha \equiv \beta.\kappa_\alpha$ and $\beta.\chi \equiv \chi.\beta \equiv u$. Later on, we will often use the following properties for the cumulant umbra and the partition umbra:

$$\chi.(\alpha + \gamma) \equiv \chi.\alpha \dot{+} \chi.\gamma, \quad \beta.(\alpha \dot{+} \gamma) \equiv \beta.\alpha + \beta.\gamma. \tag{2.3}$$

We also recall the distributive property of the summation with respect to the dot-product, that is $(\alpha + \gamma).\vartheta \equiv \alpha.\vartheta + \gamma.\vartheta$.

Composition umbra. The *composition umbra* of α and γ is denoted by the symbol $\gamma.\beta.\alpha$, where β is the Bell umbra. Its generating function is the composition of $f(\alpha, z)$ and $f(\gamma, z)$, that is $f(\gamma.\beta.\alpha, z) = f(\gamma, f(\alpha, z) - 1)$. The moments are (Di Nardo and Senato, 2006):

$$E[(\gamma.\beta.\alpha)^i] = \sum_{j=1}^{i} E[\gamma^j]B_{i,j}(a_1, \ldots, a_{i-j+1}). \tag{2.4}$$

As example of composition umbra, the *compositional inverse umbra* $\alpha^{<-1>}$ of an umbra α is such that $\alpha^{<-1>}.\beta.\alpha \equiv \chi \equiv \alpha.\beta.\alpha^{<-1>}$. In particular, we have $f(\alpha^{<-1>}, z) = f^{<-1>}(\alpha, z)$, where $f^{<-1>}$ denotes the compositional inverse of $f(\alpha, z)$ (Stanley, 1997).

3. Lévy Processes

Now we focus our attention on the family of auxiliary umbrae $\{t.\alpha\}_{t\geq 0}$. If the moments of α are all finite, this family is the symbolic counterpart of a stochastic process $\{X_t\}_{t\geq 0}$ such that $E[X_t^k] = E[(t.\alpha)^k]$, given in (2.1), for all non negative integers k. This stochastic process is a Lévy process.

Theorem 3.1. *Let $\{X_t\}_{t\geq 0}$ be a Lévy process and let α be the umbra such that $f(\alpha, z) = E[e^{zX_1}]$. The Lévy process $\{X_t\}_{t\geq 0}$ is umbrally represented by the family of auxiliary umbrae $\{t.\alpha\}_{t\geq 0}$.*

Proof. Recall that a Lévy process $\{X_t\}_{t\geq 0}$ is a stochastic process which starts at 0, with independent and stationary increments. If we denote by $\phi(z, t)$ the moment generating function of the increment $X_{t+s} - X_s$ and by $\phi(z)$ the moment generating function of X_1, then $\phi(z, t) = (\phi(z))^t$, due to the infinite divisibility property (Sato, 1999). The result follows by observing that we also have $f(t.\alpha, z) = [f(\alpha, z)]^t$. □

A fundamental result of the classical umbral calculus is that any umbra is a partition umbra. In particular, if κ_α is the α-cumulant umbra, then $\alpha \equiv \beta.\kappa_\alpha$ (Di Nardo and Senato, 2006). Referring to Lévy processes, this means that $f(t.\alpha, z) = f(t.\beta.\kappa_\alpha, z) = \exp\{t[f(\kappa_\alpha, z) - 1]\}$ which is very similar to the Lévy-Khintchine formula (Schoutens, 2000), provided that we specify the expression of $f(\kappa_\alpha, z)$. Indeed, if we denote by $E[e^{zX_t}] = (\phi(z))^t$ the moment generating function of a Lévy process $\{X_t\}_{t\geq 0}$, then the Lévy-Khintchine formula is

$$\phi(z) = \exp\left\{z\, m + \frac{1}{2}s^2z^2 + \int_{\mathbb{R}} \left(e^{zx} - 1 - zx\mathbf{1}_{\{|x|\leq 1\}}\right) \mathrm{d}(v(x))\right\}. \tag{3.1}$$

The tern (m, s^2, v) is called *Lévy triplet* and v is the *Lévy measure*. If v is a measure admitting all moments and if we set $c_0 = m + \int_{\{|x|\geq 1\}} x\, \mathrm{d}(v(x))$, then the Lévy-Khintchine formula (3.1) becomes

$$\phi(z) = \exp\left\{c_0 z + \frac{1}{2}s^2z^2\right\} \exp\left\{\int_{\mathbb{R}} (e^{zx} - 1 - zx)\, \mathrm{d}(v(x))\right\}. \tag{3.2}$$

The following theorem gives an umbral version of a Lévy process, according to the Lévy-Khintchine formula (3.2).

Theorem 3.2. *A Lévy process $\{X_t\}_{t\geq 0}$ is umbrally represented by the family $\{t.\beta.[c_0\chi\dot{+}s\delta\dot{+}\gamma]\}_{t\geq 0}$, where γ is the umbra associated to the Lévy measure, that is $f(\gamma, z) = 1 + \int_{\mathbb{R}} (e^{zx} - 1 - zx)\, \mathrm{d}(v(x))$, and δ is an umbra with $f(\delta, z) = 1 + z^2/2$.*

Proof. We have $f(t.\beta.[c_0\chi \dot{+} s\delta \dot{+} \gamma], z) = \exp\{t[f(c_0\chi \dot{+} s\delta \dot{+} \gamma, z) - 1]\}$. Since $f(c_0\chi \dot{+} s\delta \dot{+} \gamma, z) = f(c_0\chi, z) + f(s\delta, z) + f(\gamma, z) - 2$, where $f(c_0\chi, z) = 1 + c_0 z$, we have $f(c_0\chi \dot{+} s\delta \dot{+} \gamma, z) - 1 = \log\phi(z)$, with $\phi(z)$ given in (3.2). □

Remark 3.1. As introduced in Di Nardo and Oliva (2009), the *Gaussian umbra* is the umbra $m + \beta.(s\delta)$, where $m \in \mathbb{R}$, $s > 0$ and δ is the umbra given in Theorem 3.2. Recalling that $m \equiv \beta.\chi.m \equiv \beta.(m\chi)$, we have $m + \beta.(s\delta) \equiv \beta.(m\chi\dot{+}s\delta)$, due to the latter of (2.3). Thanks to Theorem 3.2 and the latter of (2.3), a Lévy process $\{X_t\}_{t\geq 0}$ is umbrally represented by the family $t.\beta.[c_0\chi\dot{+}s\delta\dot{+}\gamma] \equiv t.\beta.[c_0\chi\dot{+}s\delta] + t.\beta.\gamma$. By recalling that the auxiliary umbra $t.\beta.\alpha$ is the umbral counterpart of a compound Poisson process $S_N = Y_1 + \cdots + Y_N$, with $\{Y_i\}$ independent and identically distributed random variables and N a Poisson random variable of parameter t, then a Lévy process is the summation of two compound Poisson processes: in the first, the random variables Y_i are Gaussian with c_0 mean and variance s^2, in the second the random variables Y_i correspond to the umbra γ associated to the Lévy measure.

A centered Lévy process is such that $E[X_t] = 0$ for all $t \geq 0$. This is equivalent to choose $c_0 = 0$ in equivalence (3.2).

Corollary 3.1. *A centered Lévy process is umbrally represented by $\{t.\beta.(s\delta\dot{+}\gamma)\}_{t\geq 0}$.*

The Lévy process corresponding to (3.2) is a martingale if and only if $c_0 = 0$, see Theorem 5.2.1 in Applebaum (2004). This means that the singleton umbra χ

plays a central role in the martingale property of a Lévy process. Indeed, if $c_0 = 0$, no contribution is given by the singleton umbra which indeed does not admit a probabilistic counterpart.

3.1. *The Kailath-Segall Formula*

Let $\{X_t\}_{t\geq 0}$ be a centered Lévy process with moments of all orders and let $\{X_t^{(n)}\}_{t\geq 0}$ be the variations (1.1) of the process. The iterated stochastic integrals (1.2) are related to the variations $\{X_t^{(n)}\}_{t\geq 0}$ by the Kailath-Segall formula (Kailath and Segall, 1976)

$$P_t^{(n)} = \frac{1}{n}\left(P_t^{(n-1)}X_t^{(1)} - P_t^{(n-2)}X_t^{(2)} + \cdots + (-1)^{n+1}P_t^{(0)}X_t^{(n)}\right). \tag{3.3}$$

Then, $P_t^{(n)} = P_n\left(X_t^{(1)}, \ldots, X_t^{(n)}\right)$ is a polynomial in $X_t^{(1)}, X_t^{(2)}, \ldots, X_t^{(n)}$, called the n-th *Kailath-Segall polynomial*. Let us introduce the family of umbrae $\{\Upsilon_t\}_{t\geq 0}$ such that $E[\Upsilon_t^n] = n!E\left[P_t^{(n)}\right]$ and $\{\sigma_t\}_{t\geq 0}$ such that $E[\sigma_t^n] = E[X_t^{(n)}]$, for all non negative integers n. The following theorem states the umbral Kailath-Segall formula and its inversion.

Theorem 3.3. *We have* $\Upsilon_t \equiv \beta.[(\chi.\chi)\sigma_t]$ *and* $(\chi.\chi)\sigma_t \equiv \chi.\Upsilon_t$.

Proof. Assume $\psi_t \equiv (\chi.\chi)\sigma_t$ where $E[(\chi.\chi)^n] = (-1)^{n-1}(n-1)!$ (Di Nardo and Senato, 2006). The recurrence relation (3.3) is equivalent to $E[\Upsilon_t^n] = E[\psi_t(\Upsilon_t + \psi_t)^{n-1}]$, for all $n \geq 1$. Indeed, by definition of umbrae ψ_t and Υ_t, we have

$$\begin{aligned} E[\Upsilon_t^n] &= n!\frac{1}{n}\left\{\frac{E\left[\Upsilon_t^{n-1}\right]E\left[\psi_t\right]}{(n-1)!} + \frac{E\left[\Upsilon_t^{n-2}\right]E\left[\psi_t^2\right]}{(n-2)!} + \cdots + \frac{E\left[\psi_t^n\right]}{(n-1)!}\right\} \\ &= \sum_{j=0}^{n-1}\binom{n-1}{j}E\left[\Upsilon_t^{n-1-j}\right]E\left[\psi_t^{j+1}\right] = E\left[\psi_t(\Upsilon_t + \psi_t)^{n-1}\right]. \end{aligned}$$

By using the first equivalence of Theorem 3.1 in Di Nardo and Oliva (2009), we have $\psi_t \equiv (\chi.\chi)\sigma_t \equiv \chi.\Upsilon_t$ (inversion of the Kailath-Segall formula). The second equivalence follows by observing that $\psi_t \equiv \chi.\Upsilon_t \Leftrightarrow \beta.\psi_t \equiv \beta.\chi.\Upsilon_t$ and $\beta.\chi \equiv u$. $\square$

By recalling that the moments of $\beta.\alpha$ are the (exponential) complete Bell polynomials (Comtet, 1974) in the moments of α, see Di Nardo and Senato (2001) formula (29), then the *Kailath-Segall polynomials* are the complete Bell exponential polynomials in $\{(-1)^{n-1}(n-1)!E[X_t^{(n)}]\}$. From the inversion of the Kailath-Segall formula and equivalence (2.4), the following corollary follows.

Corollary 3.2. *If* $c_i = i!E\left[P_t^{(i)}\right]$ *for* $i = 1, \ldots, n$, *then*

$$E[X_t^{(n)}] = \sum_{j=1}^{n}\frac{(-1)^{n-j}}{(n-1)_{n-j}}B_{n,j}(c_1, c_2, \ldots, c_{n-j+1}).$$

The inversion of the Kailath-Segall formula in Theorem 3.3 is a generalization of formula (3.2) in Di Nardo et al. (2008) which gives the elementary symmetric polynomials in terms of power sum symmetric polynomials. That is, if we replace the jumps $\{\Delta X_s\}$ in $X_t^{(n)}$ with suitable indeterminates $\{x_s\}$, then the Kailath-Segall polynomials reduce to the polynomials given in Avram and Taqqu (1986).

3.2. *Umbral Time-space Harmonic Polynomials*

Let us recall the definition of conditional evaluation given in Di Nardo and Oliva (2011). Denote by $\mathscr{X}$ the set $\mathscr{X} = \{\alpha\}$.

Definition 3.1. The linear operator $E(\cdot \mid \alpha) : \mathbb{R}[x][\mathscr{A}] \longrightarrow \mathbb{R}[\mathscr{X}]$ such that

(i) $E(1 \mid \alpha) = 1$;
(ii) $E(x^m \alpha^n \gamma^i \delta^j \cdots \mid \alpha) = x^m \alpha^n E[\gamma^i] E[\delta^j] \cdots$ for uncorrelated umbrae $\alpha, \gamma, \delta, \ldots$ and for non negative integers $m, n, i, j, \ldots$

is called *conditional evaluation* with respect to α.

In other words, Definition 3.1 says that the conditional evaluation with respect to α handles the umbra α as it was an indeterminate.

Definition 3.2. Let $\{P(x, t)\} \in \mathbb{R}[x]$ be a family of polynomials indexed by $t \geq 0$. $P(x, t)$ is said to be a time-space harmonic polynomial with respect to the family of auxiliary umbrae $\{q(t)\}_{t \geq 0}$ if and only if $E[P(q(t), t) \mid q(s)] = P(q(s), s)$ for all $0 \leq s \leq t$.

Theorem 3.4. *The family of polynomials* $\{Q_k(x, t)\}_{t \geq 0} \in \mathbb{R}[x]$, *where* $Q_k(x, t) = E[(x - t.\alpha)^k]$ *for all non negative integers* k, *is time-space harmonic with respect to* $\{t.\alpha\}_{t \geq 0}$.

The proof of Theorem 3.4 is in Di Nardo and Oliva (2011).

Remark 3.2. Every linear combination of $\{Q_k(x, t)\}_{t \geq 0}$ is a time-space harmonic polynomial with respect to $\{t.\alpha\}_{t \geq 0}$.

Theorem 3.4 guarantees that the polynomial

$$Q_k(x, t) = E[(x - t.\beta.[c_0\chi \dot{+} s\delta \dot{+} \gamma])^k], \tag{3.4}$$

of degree k in the variable x and depending on the parameter t, is time-space harmonic with respect to the family of auxiliary umbrae $\{t.\beta.[c_0\chi \dot{+} s\delta \dot{+} \gamma]\}_{t \geq 0}$, that is with respect to a Lévy process, thanks to Theorem 3.2. The following theorem generalizes Corollary 2(a) in Solé and Utzet (2008).

Proposition 3.1. *We have* $Q_k(x, t) = Y_k(x - tc_0, -t(s^2 + m_2), -tm_3, \ldots, -tm_k)$, *for all non negative integers* $k \geq 1$, *where* Y_k *are the complete Bell polynomials and* $m_i = E[\gamma^i]$, *for all* $i = 2, \ldots, k$.

Proof. As proved in Di Nardo and Senato (2006), if $a_k = E[\alpha^k]$ and $b_i = E[\kappa_\alpha^i]$, for $i = 1, \ldots, k$ then $a_k = Y_k(b_1, b_2, \ldots, b_k)$, where Y_k are the complete Bell polynomials.

By definition of cumulant umbra and by virtue of (3.4), we have

$$E\{[\chi.(x - t.\beta.[c_0\chi\dot{+}s\delta\dot{+}\gamma])]^k\} = \begin{cases} x + E\{\chi.(-t).\beta.[c_0\chi\dot{+}s\delta\dot{+}\gamma]\}, & \text{if } k = 1 \\ E\{(\chi.(-t).\beta.[c_0\chi\dot{+}s\delta\dot{+}\gamma])^k\}, & \text{if } k > 1. \end{cases}$$

Therefore, since $E\{\chi.(-t).\beta.[c_0\chi\dot{+}s\delta\dot{+}\gamma]\} = -tc_0$, $E\{(\chi.(-t).\beta.[c_0\chi\dot{+}s\delta\dot{+}\gamma])^2\} = -t(s^2 + m_2)$ and $E\{(\chi.(-t).\beta.[c_0\chi\dot{+}s\delta\dot{+}\gamma])^k\} = -tm_k$, for $k \geq 3$, the result follows. Indeed we have proved that the cumulant umbra of $x - t.\beta.[c_0\chi\dot{+}s\delta\dot{+}\gamma]$ has the first k moments given by $x - tc_0, -t(s^2 + m_2), \ldots, -tm_k$. □

We observe that the polynomial umbra $x - t.\beta.[c_0\chi\dot{+}s\delta\dot{+}\gamma]$ is an Appell umbra with respect to the indeterminate x (Di Nardo et al., 2010). Therefore the moments $\{Q_k(x,t)\}_{k\in\mathbb{N}}$ in (3.4) are Appell polynomials such that $\partial Q_k(x,t)/\partial x = kQ_{k-1}(x,t)$. With respect to t, the polynomial umbra $x - t.\beta.[c_0\chi\dot{+}s\delta\dot{+}\gamma]$ is a Sheffer umbra (Di Nardo et al., 2010), so that the Sheffer identity holds $Q_k(x, t+v) = \sum_{j=0}^{k}\binom{k}{j}P_j(v)Q_{k-j}(x,t)$, where $Q_{k-j}(x,t)$ are given in (3.4) and $P_j(v) = Q_j(0,v)$, for all non negative integers j.

4. Examples

4.1. *Sum of Two Independent Lévy Processes*

Let us consider two independent Lévy processes $W = \{W_t\}_{t\geq 0}$ and $Z = \{Z_t\}_{t\geq 0}$, umbrally represented by $\{t.\alpha\}_{t\geq 0}$ and $\{t.\gamma\}_{t\geq 0}$, respectively. Due to the distributive property (2.2), the process $X = W + Z$ is umbrally represented by $t.(\alpha + \gamma) \equiv t.\alpha + t.\gamma$. If we replace $\mathbb{R}[x]$ with $\mathbb{R}[x, w, z]$ (Di Nardo et al., 2008), and denote by $\{Q_k(x,t)\}_{k\in N}$, $\{Q'_k(x,t)\}_{k\in N}$ and $\{Q''_k(x,t)\}_{k\in N}$ the time-space harmonic polynomials with respect to $\{X_t\}_{t\geq 0}$, $\{W_t\}_{t\geq 0}$ and $\{Z_t\}_{t\geq 0}$, respectively, we have $Q_k(x,t) = \sum_{j=0}^{k}\binom{k}{j}Q'_j(w,t)Q''_{k-j}(z,t)$, if $x = w + z$.

4.2. *Brownian Motion*

The Brownian motion $\{B_t\}_{t\geq 0}$ is a Lévy process whose increments are Gaussian random variables with zero mean, variance s^2 and zero Lévy measure. Hence, thanks to Theorem 3.2, the symbolic counterpart of $\{B_t\}_{t\geq 0}$ is given by the family of umbrae $\{t.\beta.(s\delta)\}_{t\geq 0}$. The standard Brownian motion is recovered by setting $s = 1$.

From Theorem 3.4, for all non negative integers k, the polynomials $Q_k(x,t) = E[(x - t.\beta.(s\delta))^k]$ are time-space harmonic with respect to the Brownian motion $\{B_t\}_{t\geq 0}$.

Proposition 4.1. *For all non negative integers $k \geq 1$, we have $Q_k(x,t) = H_k^{(s^2 t)}(x)$.*

Proof. Recall that the generalized Hermite polynomials $\{H_k^{(s^2)}(x)\}_{t\geq 0}$ have generating function

$$\sum_{k\geq 0} H_k^{(s^2)}(x)\frac{z^k}{k!} = \exp\left\{xz - \frac{s^2z^2}{2}\right\}.$$

In Di Nardo and Oliva (2011), we have proved that $H_k^{(s^2)}(x) = E\{[x - 1.\beta.(s\delta)]^k\}$. In particular, we have $H_k^{(s^2 t)}(x) = E[(x - 1.\beta.(\sqrt{t}s\delta))^k]$. The result follows by observing that $-1.\beta.(\sqrt{t}s\delta) \equiv -t.\beta.s\delta$. □

4.3. *Poisson Process*

The Poisson process $\{N_t\}_{t\geq 0}$ is a pure jump Lévy process, whose increments follow a Poisson distribution with parameter $\lambda > 0$. The moment generating function is $(\phi(z))^t = (\exp\{\lambda(e^t - 1)\})^t$, so the Poisson process of intensity parameter λ is umbrally represented by the family of umbrae $\{t.\lambda.\beta\}_{t\geq 0}$. Thanks to Theorem 3.4, the polynomials $Q_k(x, \lambda t) = E[(x - t.\lambda.\beta)^k]$ are time-space harmonic with respect to the Poisson process $\{N_t\}_{t\geq 0}$.

Proposition 4.2 states that also the Poisson-Charlier polynomials $\{\widetilde{C}_k(x, \lambda t)\}$ are time-space harmonic with respect to the Poisson process $\{N_t\}_{t\geq 0}$.

Proposition 4.2. *We have* $\widetilde{C}_k(x, \lambda t) = \sum_{j=1}^k s(k, j)Q_j(x, \lambda t)$, *where* $s(k, j)$ *are the Stirling numbers of the first kind.*

Proof. Recall that the Poisson-Charlier polynomials $\widetilde{C}_k(x, \lambda t)$ have generating function

$$\sum_{k\geq 0} \widetilde{C}_k(x, \lambda t)\frac{z^k}{k!} = e^{-\lambda t z}(1 + z)^x,$$

so $\widetilde{C}_k(x, \lambda t) = E[(x.\chi - t.\lambda.u)^k]$. Since $x.\chi - t.\lambda.u \equiv (x - t.\lambda.\beta).\chi$ and by recalling that $E[(\alpha.\chi)^k] = E[(\alpha)_k]$; see Di Nardo and Senato (2006), we have

$$\widetilde{C}_k(x, \lambda t) = E[(x - t.\lambda.\beta)_k] = \sum_{j=0}^{k} s(k, j)E[(x - t.\lambda.\beta)^k].$$

□

4.4. *Gamma Process*

The Gamma process $\{G_t(\lambda, b)\}_{t\geq 0}$ with scale parameter $\lambda > 0$ and shape parameter $b > 0$ is a Lévy process with stationary, independent and Gamma-distributed increments. If we set $b = 1$, the moment generating function of the Gamma process is $(\phi(z))^t = [(1 - z)^{-\lambda}]^t$. Thus, the umbral representation of the Gamma process $\{G_t(\lambda, 1)\}_{t\geq 0}$ is given by the family of umbrae $\{(\lambda t).\bar{u}\}_{t\geq 0}$, where $\bar{u}$ is the *boolean unity*.

There are two families of polynomials time-space harmonic with respect to Gamma processes, according to the value of the scale parameter λ: the *Laguerre polynomials* $\{\mathscr{L}_k^{t-k}(x)\}$ and the *actuarial polynomials* $\{g_k(x, \lambda t)\}$.

As regards the former, we have

$$(-1)^k k!\mathscr{L}_k^{t-k}(x) = E[(x + t.(-\chi))^k], \quad k = 0, 1, 2, \ldots \tag{4.1}$$

since the Laguerre polynomials $\{\mathscr{L}_k^{t-k}(x)\}$ have generating function $\sum_{k\geq 0}(-1)^k \mathfrak{L}_k^{t-k}(x)z^k = (1 - z)^t e^{zx}$.

Theorem 4.1. *The Laguerre polynomials* $\{\mathscr{L}_k^{t-k}(x)\}_{t\geq 0}$ *are time-space harmonic with respect to the Gamma process* $\{G_t(1,1)\}_{t\geq 0}$.

Proof. Theorem 3.4 implies that the polynomials $Q_k(x,t) = E[(x - t.\bar{u})^k]$ are time-space harmonic with respect to the Gamma process $\{G_t(1,1)\}_{t\geq 0}$. Moreover, we have $-1.\bar{u} \equiv -\chi$, so $-t.\bar{u} \equiv t.(-\chi)$ and $x - t.\bar{u} \equiv x + t.(-\chi)$. Then, thanks to (4.1), we have $Q_k(x,t) = k!(-1)^k \mathscr{L}_k^{t-k}(x)$. □

For the latter, Roman (1984) defines the class of actuarial polynomials as the sequence of polynomials with generating function

$$\sum_{k\geq 0} g_k(x, \lambda t)\frac{z^k}{k!} = \exp\{\lambda t z + x(1 - e^z)\}. \tag{4.2}$$

To get the symbolic expression of $g_k(x, \lambda t)$ we use the umbral Lévy-Sheffer systems. Recall that a Lévy-Sheffer system (Di Nardo and Oliva, 2011) is a sequence of polynomials $\{R_k(x,t)\}$ such that

$$\sum_{k\geq 0} R_k(x,t)\frac{z^k}{k!} = (f(z))^t \exp\{x u(z)\}, \tag{4.3}$$

where $f(z)$ and $u(z)$ are analytic in the neighborhood of $z = 0$, $u(0) = 0$, $f(0) = 1$, $u'(0) \neq 0$ and $1/f(\tau(z))$ is an infinitely divisible moment generating function, with $\tau(z)$ such that $\tau(u(z)) = z$. If $f(z) = f(\alpha, z)$ and $u(z) = f(\gamma, z) - 1$, then $R_k(x,t) = E[(t.\alpha + x.\beta.\gamma)^k]$, for all non negative integers k. By comparing (4.3) with (4.2), we obtain $\alpha \equiv (\lambda t).u$ and $\gamma \equiv (\chi.(-\chi))^{<-1>}$, where $(\chi.(-\chi))^{<-1>}$ is the compositional inverse of the umbra $\chi.(-\chi)$. This leads to the umbral version of the actuarial polynomials, that is, for all non negative integers k,

$$g_k(x, \lambda t) = E\left\{\left[\lambda t + x.\beta.(\chi.(-\chi))^{<-1>}\right]^k\right\}. \tag{4.4}$$

Theorem 4.2. *The actuarial polynomials* $\{g_k(x, \lambda t)\}_{t\geq 0}$ *are time-space harmonic with respect to the Gamma process* $\{G_t(\lambda, 1)\}_{t\geq 0}$.

Proof. By virtue of Theorem 3.4, $Q_k(x,t) = E[(x - (\lambda t).\bar{u})^k]$ are time-space harmonic polynomials for all $k \geq 0$ with respect to the Gamma process $\{G_t(\lambda, 1)\}_{t\geq 0}$. On the other hand, $\lambda t + x.\beta.(\chi.(-\chi))^{<-1>} \equiv (x + (\lambda t).(-\chi)).\beta.(\chi.(-\chi))^{<-1>}$. Then, by virtue of (4.4) and (2.4), we have

$$g_k(x, \lambda t) = \sum_{j=1}^{k} E[(x + (\lambda t).(-\chi))^j] B_{k,j}(m_1, \ldots, m_{k-j+1}),$$

where $m_i = E[(\chi.(-\chi))^{<-1>})^i]$. Observe that $\bar{u} \equiv -1.(-\chi)$, thus

$$g_k(x, \lambda t) = \sum_{j=1}^{k} E[(x - (\lambda t).\bar{u})^j] B_{k,j}(m_1, \ldots, m_{k-j+1}) = \sum_{j=1}^{k} Q_k(x,t) B_{k,j}(m_1, \ldots, m_{k-j+1}).$$

The result follows from Remark 3.2. □

4.5. *Pascal Process*

Let $\{Pa(t,p)\}_{t\geq 0}$ be a Pascal process, that is, a Lévy process whose increments have Pascal distribution with mean td, where $d = p/q$ and $p+q=1$. As the moment generating function of the Pascal process is $(\phi(z))^t = [q/(1-pe^z)]^t$, with some calculations we obtain that a Pascal process is umbrally represented by the family of umbrae $\{t.\bar{u}.d.\beta\}_{t\geq 0}$, where $\bar{u}$ is the boolean unity. By virtue of Theorem 3.4, the time-space harmonic polynomials with respect to the Pascal process are the $Q_k(x,t) = E[(x - t.\bar{u}.d.\beta)^k]$ are for all non negative integers k.

Consider the family of Meixner polynomials of the first kind $\{M_k(x,t,p)\}$ (Schoutens, 2000) such that

$$\sum_{k\geq 0}(-1)^k (t)_k M_k(x,t,p)\frac{z^k}{k!} = \left(1+\frac{z}{p}\right)^x (1+z)^{-x-t}. \tag{4.5}$$

From (4.5), the symbolic version of the Meixner polynomials of the first kind is

$$(-1)^k (t)_k M_k(x,t,p) = E\left\{\left[x.\left(-1.\chi + \frac{\chi}{p}\right) - t.\chi\right]^k\right\}. \tag{4.6}$$

Theorem 4.3. *The Meixner polynomials of the first kind are time-space harmonic with respect to the Pascal process* $\{Pa(p,t)\}_{t\geq 0}$.

Proof. The Meixner polynomials of the first kind is a Lévy-Sheffer system, so they are represented by the polynomial umbra $x.\beta.(\chi.(-1.\chi+\chi/p)) + t.(-1.\chi)$, with $E[(-1.\chi+\chi/p)] \neq 0$. This hypothesis guarantees that the compositional inverse umbra exists, so

$$x.\left(-1.\chi+\frac{\chi}{p}\right) - t.\chi \equiv (x + t.(-1.\chi).\beta.\bar{u}.d.\beta).\beta.\left(\chi.\left(-1.\chi+\frac{\chi}{p}\right)\right).$$

Thus, by (2.4) and (4.6), the Meixner polynomials of first kind can be written in the following way

$$\begin{aligned}(-1)^k (t)_k M_k(x,t,p) &= \sum_{j=1}^{k} E[(x - t.\bar{u}.d.\beta)^j] B_{k,j}(m_1,\ldots,m_{k-j-1})\\ &= \sum_{j=1}^{k} Q_k(x,t) B_{k,j}(m_1,\ldots,m_{k-j-1}),\end{aligned}$$

where $m_i = E[\{\chi.(-1.\chi+\chi/p)\}^i]$. The result follows thanks to Remark 3.2. □

4.6. *Random Walks*

The results in the literature involving the polynomials we are going to introduce refer to an integer parameter n. In order to highlight their time-space harmonic property, we consider the discrete version of a Lévy process, that is a random walk $S_n = X_1 + X_2 + \cdots + X_n$, with $\{X_i\}$ independent and identically distributed random variables. For the symbolic representation of Lévy processes we have dealt with, a random walk is umbrally represented by $n.\alpha$. The generality of the symbolic approach shows that if the parameter n is replaced by t, that is if the random walk is replaced by a Lévy process, more general classes of polynomials can be recovered for which many of the properties here introduced still hold.

Bernoulli polynomials. The Bernoulli polynomials $\{B_k(x,n)\}$ is defined by the generating function (Roman, 1984)

$$\sum_{k\geq 0} B_k(x,n)\frac{z^k}{k!} = \left(\frac{z}{e^z-1}\right)^n e^{zx}.$$

Therefore we have $B_k(x,n) = E[(x+n.\iota)^k]$ for all non negative integers k.

Theorem 4.4. *The Bernoulli polynomials $\{B_k(x,n)\}_{n\geq 0}$ are time-space harmonic with respect to the random walk $\{n.(-1.\iota)\}_{n\geq 0}$.*

Proof. Let us consider the random walk $S_n = X_1 + X_2 + \cdots + X_n$ such that $X_1, X_2, \ldots, X_n$ are n independent and identically distributed random variables with uniform distribution on the interval $[0,1]$. Since X_i is umbrally represented by the umbra $-1.\iota$, the random walk S_n is umbrally represented by the family of auxiliary umbrae $\{n.(-1.\iota)\}_{n\geq 0}$. Theorem 3.4 ensures that the polynomials $Q_k(x,t) = E[(x - n.(-1.\iota))^k]$ are time-space harmonic with respect to S_n, for all $k \geq 0$. On the other hand $n.(-1.\iota) \equiv -n.\iota$, hence $E[(x-n.(-1.\iota))^k] = E[(x+n.\iota)^k]$, that is $B_k(x,n) = Q_k(x,n)$. □

Euler polynomials. The Euler polynomials $\{\mathscr{E}_k(x,n)\}$ is defined by the generating function (Roman, 1984)

$$\sum_{k\geq 0} \mathscr{E}_k(x,n)\frac{z^k}{k!} = \left(\frac{2}{e^z+1}\right)^n e^{zx}.$$

Therefore we have $\mathscr{E}_k(x,n) = E\left[\left(x + n.\left[\frac{1}{2}(-1.u+\eta)\right]\right)^k\right]$ for all non negative integers k.

Theorem 4.5. *The Euler polynomials $\{\mathscr{E}_k(x,n)\}$ are time-space harmonic with respect to the random walk $\{n.\left[\frac{1}{2}(-1.\eta+u)\right]\}_{n\geq 0}$.*

Proof. Let us consider the random walk $S_n = X_1 + X_2 + \cdots + X_n$ such that $X_1, X_2, \ldots, X_n$ are n independent and identically distributed Bernoulli random variables with parameter $1/2$. The result follows by using arguments similar to the proof of Theorem 4.4, as X_i is umbrally represented by the umbra $\frac{1}{2}(-1.\eta+u)$. □

Krawtchouk polynomials. The Krawtchouk polynomials $\{\mathscr{K}_k(x,p,n)\}$ are defined by the generating function (Roman, 1984)

$$\sum_{k\geq 0}\binom{n}{k}\mathscr{K}_k(x,p,n)z^k = \left(1 - \frac{1-p}{p}z\right)^x (1+z)^{n-x}. \tag{4.7}$$

The umbra with generating function (4.7) is $(n-x).\chi + x.(-\chi/d) \equiv n.\chi + x.(-1.\chi - \chi/d)$, with $d = p/q$ and $p+q=1$. Then, for all non negative integers k we have

$$\frac{n!}{(n-k)!}\mathcal{K}_k(x,p,n) = E\left\{\left[n.\chi + x.\left(-1.\chi - \frac{\chi}{d}\right)\right]^k\right\}. \tag{4.8}$$

Theorem 4.6. *The Krawtchouk polynomials are time-space harmonic with respect to the random walk* $\{n.(-1.\mu)\}_{n\geq 0}$, *where* $-1.\mu$ *is the umbral counterpart of a Bernoulli random variable with parameter* p.

Proof. For $i = 1, \ldots, n$, let X_i be a random variable with Bernoulli distribution of parameter p. Let $\mu \equiv -1.\chi.p.\beta$ be the umbra such that $f(\mu, z) = 1/(pe^z + (1-p))$, so the random walk $S_n = X_1 + X_2 + \cdots + X_n$ is umbrally represented by the family of auxiliary umbrae $\{n.(-1.\mu)\}_{n\geq 0.}$ From Theorem 3.4, the polynomials $Q_k(x,n) = E[(x - n.(-1.\mu))^k] = E[(x + n.\mu)^k]$ are time-space harmonic with respect to S_n for all non negative integers k. From (4.8), we have

$$n.\chi + x.\left(-1.\chi - \frac{\chi}{d}\right) \equiv \left(x + n.\left(\chi.\left(-1.\chi - \frac{\chi}{d}\right)\right)^{<-1>}\right).\beta.\left(\chi.\left(-1.\chi - \frac{\chi}{d}\right)\right).$$

By applying (2.4), the Krawtchouk polynomials are such that

$$\frac{n!}{(n-k)!}\mathcal{K}_k(x,p,n) = \sum_{j=1}^{k} E\left[\left\{x + n.\left[\chi.\left(-1.\chi - \frac{\chi}{d}\right)\right]^{<-1>}\right\}^j\right] B_{k,j}(m_1, \ldots, m_{k-j+1}), \tag{4.9}$$

where $m_i = E[(\chi.(-1.\chi - \chi/d)^i]$. Via generating functions, it is straightforward to prove that $-1.(\chi.(-1.\chi - \chi/d))^{<-1>} \equiv \mu$, therefore $E[\{x + n.(\chi.(-1.\chi - \chi/d))^{<-1>}\}^j] = E[(x + n.\mu)^j] = Q_j(x,n)$. By replacing this last equality in (4.9), we have

$$\frac{n!}{(n-k)!}\mathcal{K}_k(x,p,n) = \sum_{j=1}^{k} Q_j(x,n) B_{k,j}(m_1, \ldots, m_{k-j+1}),$$

and the result follows thanks to Remark 3.2. □

Pseudo-Narumi polynomials. The family of pseudo-Narumi polynomials $\{N_k(x, an)\}$, $a \in \mathbb{N}$, is the sequence of coefficients of the following power series (Roman, 1984)

$$\sum_{k\geq 0} N_k(x, an) z^k = \left(\frac{\log(1+z)}{z}\right)^{an} (1+z)^x. \tag{4.10}$$

From (4.10), the pseudo-Narumi polynomials result to be the moments of the umbra $x.\chi + (an).u_P^{<-1>}$, where $u_P^{<-1>}$ is the primitive umbra of the compositional inverse $u^{<-1>}$. We recall that, given an umbra $\alpha \in \mathcal{A}$, the *α-primitive umbra* α_P is such that $f(\alpha_P, z) = (f(\alpha, z) - 1)/z$. For all non negative integers k, we have

$$k! N_k(x, an) = E\{[(an).u_P^{<-1>} + x.\chi]^k\}. \tag{4.11}$$

Theorem 4.7. *The pseudo-Narumi polynomials are time-space harmonic with respect to the random walk* $\{(an).(-1.\iota)\}_{n\geq 0}$.

Proof. Consider the random walk $S_n = X_1 + X_2 + \cdots + X_n$, where, for $i = 1, \ldots, n$, X_i is a sum of $a \in \mathbb{N}$ random variables with uniform distribution on the interval $[0, 1]$. Therefore, for $i = 1, \ldots, n$, X_i is umbrally represented by $a.(-1.\iota)$ and S_n is umbrally represented by $\{n.a.(-1.\iota)\}_{n\geq 0}$. By applying Theorem 3.4, it is straightforward to prove that the polynomials $Q_k(x, n) = E[(x - (an).(-1.\iota))^k]$ are time-space harmonic with respect to S_n. On the other hand, $x.\chi + (an).u_P^{<-1>} \equiv (x + (an).u_P^{<-1>}.\beta).\beta.u^{<-1>}$, and then, from (4.11)

$$k!N_k(x, an) = \sum_{j=1}^{k} E[(x + (an).u_P^{<-1>}.\beta)^j] B_{k,j}(m_1, \ldots, m_{k-j+1}),$$

where $m_i = E[(u^{<-1>})^i]$. To prove the result, it is sufficient to show that $E[(x + (an).u_P^{<-1>}.\beta)^j]$ fits with the j-th time-space harmonic polynomial $Q_j(x, n)$. Via generating functions, we have $u_P^{<-1>}.\beta \equiv \iota$, which gives

$$k!N_k(x, an) = \sum_{j=1}^{k} Q_j(x, n) B_{k,j}(m_1, \ldots, m_{k-j+1}),$$

and the result follows thanks to Remark 3.2. □

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