Self-Similar Pyramidal Structures and Signal Reconstruction

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ABSTRACT

Pyramidal structures are defined which are locally a combination of low and highpass filtering. The structures are analogous to but different from wavelet packet structures. In particular, new frequency decompositions are obtained; and these decompositions can be parametrized to establish a correspondence with a large class of Cantor sets. Further correspondences are then established to relate such frequency decompositions with more general self-similarities. The role of the filters in defining these pyramidal structures gives rise to signal reconstruction algorithms, and these, in turn, are used in the analysis of speech data.

Keywords: Pyramidal structures, self-similarities, frequency decomposition, speech analysis

1. INTRODUCTION

We shall define pyramidal structures in the form of dyadic trees, see Figure 1. The nodes at any level will be function spaces on the real line; and the nodes at level m will be subspaces of the nodes at level $m-1$.

There are many examples of such trees, and, in the realm of wavelet theory and signal processing, a standard example of such a pyramidal structure is defined by the Walsh functions and the Haar multiresolution analysis. This particular example is generalized by the theory of wavelet packets, due to Coifman, Meyer, and Wickerhauser.[1] Wavelet packets provide a particular type of frequency decomposition for a given pair of quadrature mirror filters (QMFs).

Our pyramidal structures are also associated with filter pairs, but they determine a different frequency decomposition than that of wavelet packets. Since we shall use Paley-Wiener spaces as nodes, we refer to our dyadic trees as Paley-Wiener pyramidal structures. These are defined precisely and concretely in Section 2, and we shall see that a large class of them can be characterized in terms of the class of Cantor sets having constant ratio of dissection.[2]

In Section 3 we generalize and simplify the approach of Section 2 by using set theoretic methods from the analysis of self-similar processes. In this setting, the correspondence between Cantor sets and pyramidal stuctures from Section 2 can be generalized to include other self-similarities. Further, we shall see that these more general self-similar pyramidal structures defined in terms of specific filter pairs provide perfect reconstruction dyadic trees in the same sense as wavelet packets, the major difference being the difference in frequency decomposition. It is natural to investigate the effectiveness of self-similar pyramidal structures with regard to this signal reconstruction. We have chosen to implement our algorithm on speech data, and our results are contained in Section 4. This is part of an ongoing study including a comparison with Fourier and wavelet packet results.

Notation. We shall use standard notation from mathematical analysis,[3] but we do mention the following. $\mathbb{R}$ is the real line, and $\widehat{\mathbb{R}}$ is also the real line, but considered as the frequency axis. The Fourier transform of a function f on $\mathbb{R}$ is the function $\widehat{f}$ defined on $\widehat{\mathbb{R}}$ as

$$\forall \gamma \in \widehat{\mathbb{R}}, \quad \widehat{f}(\gamma) = \int f(t) e^{-2\pi i t \gamma}\, dt,$$

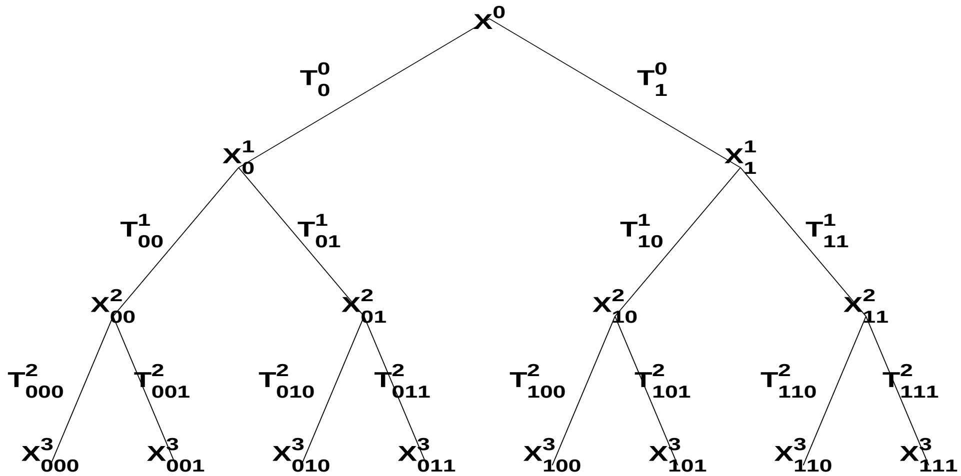

Figure 1. Pyramidal Structure Nodes and Mappings

where integration is over $\mathbb{R}$. The support of a function F is designated by supp F, the characteristic function of a set S is 1_S, the measure of S is $|S|$, and a disjoint union is denoted by $\dot{\bigcup}$.

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2. CONSTRUCTION OF PYRAMIDAL STRUCTURES

A *pyramidal structure* is defined as a double sequence of spaces X_n^m, accompanied by mappings

$$T_{\epsilon_1\ldots\epsilon_m}^{m-1} : X_{\epsilon_1\ldots\epsilon_{m-1}}^{m-1} \longrightarrow X_{\epsilon_1\ldots\epsilon_m}^m$$

which depend on specific geometric operations and digital or analogue filters. The subscripts and superscripts have the following properties: $m = 1, 2, \ldots$, $n = 0, 1, \ldots., 2^m - 1$ and $n = \sum_{j=1}^m \epsilon_j 2^{j-1}$ for $\epsilon_j = 0, 1$. Thus X_n^m is the same as $X_{\epsilon_1\ldots\epsilon_m}^m$.

It is convenient to begin the process at level 0 with a subspace X^0 and mappings

$$T_0^0 : X^0 \longrightarrow X_0^1, \text{and} T_1^0 : X^0 \longrightarrow X_1^1.$$

It is also natural to think of $\{X_n^m\}$ as nodes of a binary tree, where m denotes a particular level and where $X_{\epsilon_1\ldots\epsilon_{m-1}0}^m$ and $X_{\epsilon_1\ldots\epsilon_{m-1}1}^m$ are the nodes at level m coming from $X_{\epsilon_1\ldots\epsilon_{m-1}}^{m-1}$, see Figure 1.

By our designation of the nodes at level m from the nodes at level $m-1$, we see that for any fixed m the ordering of the spaces X_n^m from left to right is bit reversal ordering. Thus, at level 1, we have the ordering 0, 1; at level 2, we have the ordering $0, 2, 1, 3$; at level 3, we have the ordering $0, 4, 2, 6, 1, 5, 3, 7$; etc.

We shall deal with *Paley-Wiener pyramidal structures* by taking $X^0 = \widehat{PW}_\Omega$, where PW_Ω is the Paley-Wiener space of Ω-bandlimited, finite energy signals, i.e.,

$$PW_\Omega = \{f : \mathbb{R} \longrightarrow \mathbf{C} : \mathbf{f} \in \mathbf{L^2}(\mathbb{R}) \quad \text{and} \quad \operatorname{supp} \widehat{\mathbf{f}} \subseteq [-\Omega, \Omega]\}.$$

In order to define the mappings T_n^m we shall begin by considering analogue filters $H_0,\ H_1 \in \widehat{PW}_\Omega \bigcap L^\infty(\widehat{\mathbb{R}})$. Generally, we shall take $H_0,\ H_1$ to be lowpass and highpass filters, respectively. These functions will also be considered as mappings

$$H_0, H_1 : X^0 \longrightarrow X^0$$

defined by $H_\epsilon(F)(\gamma) = H_\epsilon(\gamma)\,F(\gamma)$, for $\epsilon = 0, 1$ and $F \in X^0$.

Then, for an arbitrary $c > 1$ and $\omega \in \widehat{\mathbb{R}}$ we consider the dilation and translation operators

$$D_c(F)(\gamma) = F(c\gamma) \text{ and } \tau_\omega(F)(\gamma) = F(\gamma - \omega);$$

and we use these operators to define the operators

$$D_0 F = D_c F \quad \text{and} \quad D_1 F = D_{\frac{2c}{c-1}}\left(\tau_{\frac{c+1}{c-1}\Omega} F + \tau_{-\frac{c+1}{c-1}\Omega} F\right) \tag{1}$$

as well as

$$D_{\epsilon_1 \ldots \epsilon_m} F = (D_{\epsilon_1} \ldots D_{\epsilon_{m-1}} H_{\epsilon_m})(F), \tag{2}$$

where $F \in X^0$ and $D_{\epsilon_1} \ldots D_{\epsilon_{m-1}} H_{\epsilon_m}$ denote the m-fold composition operator consisting of the D_{ϵ_j}s and H_{ϵ_m}. The second operator in (1) has the same effect as the dissection procedure used in the construction of perfect symmetric sets.[2]

To fix ideas, let $0 < \alpha = \frac{1}{c} < 1$ and let H_0 and H_1 be the ideal lowpass and highpass filters defined as

$$H_0 = 1_{[-\alpha\Omega, \alpha\Omega]} \quad \text{and} \quad H_1 = (1 - H_0)\, 1_{[-\Omega,\Omega]}. \tag{3}$$

Note that

$$H_0 = D_0 1_{[-\Omega,\Omega]} \quad \text{and} \quad H_1 = D_1 1_{[-\Omega,\Omega]}$$

as elements of X^0. Thus, in this case of ideal filters, the operator $D_{\epsilon_1 \ldots \epsilon_m}$ defined by (2) asserts that

$$\forall F \in X^0, \quad D_{\epsilon_1 \ldots \epsilon_m} F = (D_{\epsilon_1} D_{\epsilon_2} \ldots D_{\epsilon_m})(F).$$

Example 2.1.

a. It is not necessarily true that $\operatorname{supp} D_\epsilon F \subseteq \operatorname{supp} F$ for $F \in X^0$, cf., *Proposition 2.2*. For example, if $F = 1_{[\beta,\Omega]}$, $\beta > 0$, then $\operatorname{supp} D_0 F = [\frac{\beta}{c}, \frac{\Omega}{c}]$.

b. Further, it is not necessarily true that $\operatorname{supp} D_0 D_1 F \subseteq \operatorname{supp} D_1 F$ even in the case $\operatorname{supp} F = [-\Omega, \Omega]$. For example, if $F = 1_{[-\Omega,\Omega]}$, then $\operatorname{supp} D_1 F = [-\Omega, -\frac{\Omega}{c}] \bigcup [\frac{\Omega}{c}, \Omega]$, whereas

$$\operatorname{supp} D_0 D_1 F = [-\frac{\Omega}{c}, -\frac{\Omega}{c^2}] \bigcup [\frac{\Omega}{c^2}, \frac{\Omega}{c}].$$

c. On the other hand, if $\operatorname{supp} F = [-\Omega, \Omega]$, then $\operatorname{supp} D_\epsilon F \subseteq \operatorname{supp} F$ and

$$\operatorname{supp} D_{\epsilon_1} D_{\epsilon_2} F \subseteq \operatorname{supp} D_{\epsilon_1} F.$$

In particular, dealing with the positive axis, we have

$$[\frac{\Omega}{c^2}, \frac{\Omega}{c}] \subseteq [0, \frac{\Omega}{c}]$$

for the case $\operatorname{supp} D_0 D_1 F \subseteq \operatorname{supp} D_0 F$,

$$[\frac{c^2+1}{2c^2}\Omega, \frac{c^2+2c-1}{2c^2}\Omega] \subseteq [\frac{\Omega}{c}, \Omega]$$

for the case $\operatorname{supp} D_1 D_0 F \subseteq \operatorname{supp} D_1 F$,

$$[0, \frac{\Omega}{c^2}] \subseteq [0, \frac{\Omega}{c}]$$

for the case supp $D_0D_0F \subseteq$ supp D_0F, and

$$[\frac{\Omega}{c}, \frac{c^2+1}{2c^2}\Omega]\bigcup[\frac{c^2+2c-1}{2c^2}\Omega, \Omega] \subseteq [\frac{\Omega}{c}, \Omega]$$

for the case supp $D_1D_1F \subseteq$ supp D_1F.

d. More generally, let supp $F = [a,b] \subseteq [-\Omega,\Omega]$. Then supp $D_0F = [\frac{a}{c}, \frac{b}{c}] \subseteq [\frac{-\Omega}{c}, \frac{\Omega}{c}] \subseteq [-\Omega,\Omega]$ and

$$\text{supp } D_1F = [\frac{c-1}{2c}a + \frac{c+1}{2c}\Omega, \frac{c-1}{2c}b + \frac{c+1}{2c}\Omega]\bigcup[\frac{c-1}{2c}a - \frac{c+1}{2c}\Omega, \frac{c-1}{2c}b - \frac{c+1}{2c}\Omega] = I_+ \bigcup I_-,$$

a disjoint union with the property that $I_+ \subseteq [\frac{\Omega}{2c}, \Omega]$ and $I_- \subseteq [-\Omega, -\frac{\Omega}{2c}]$. Geometrically, D_0F will shrink by a factor of $\frac{1}{c}$ the support of F; and D_1F gives rise to two copies of F shrunk by a factor $\frac{c-1}{2c}$ and sent in opposite directions away from 0.

e. Let $B(\gamma,r)$ be the interval centered at γ with radius $r>0$. If $[a,b] \subseteq [-\Omega,\Omega]$ and $B(\gamma,r) \subseteq [-\Omega,\Omega]$, then

$$D_0 1_{B(\gamma,r)} = 1_{B(\frac{\gamma}{c}, \frac{r}{c})}$$

and

$$D_1 1_{B(\gamma,r)} = 1_{B(\frac{c-1}{2c}\gamma + \frac{c+1}{2c}\Omega, \frac{c-1}{2c}r)} + 1_{B(\frac{c-1}{2c}\gamma - \frac{c+1}{2c}\Omega, \frac{c-1}{2c}r)}.$$

To simplify notation, we write

$$E^m_{\epsilon_1,\ldots,\epsilon_m} = \text{supp } D_{\epsilon_1\ldots\epsilon_m}1_{[-\Omega,\Omega]}.$$

Proposition 2.2. *If* $F \in X^0$ *has the property that* supp $F = [-\Omega,\Omega]$, *then* supp $D_{\epsilon_1\ldots\epsilon_m}F \subseteq$ supp $D_{\epsilon_1\ldots\epsilon_{m-1}}F$, *i.e.*,

$$E^m_{\epsilon_1,\ldots,\epsilon_m} \subseteq E^{m-1}_{\epsilon_1,\ldots,\epsilon_{m-1}}. \tag{4}$$

The following calculation is a *formal* proof of *Proposition 2.2*, cf., *Theorem 3.6*, where we deal rigorously with this situation. It is formal since the intuitive change of variables can not be justified.

Proof. Without loss of generality, let $F = 1_{[-\Omega,\Omega]}$. We shall prove that if A and B are closed sets for which $A \subseteq B$, then

$$\text{supp } D_{\epsilon_1\ldots\epsilon_{m-1}}1_A \subseteq \text{supp } D_{\epsilon_1\ldots\epsilon_{m-1}}1_B. \tag{5}$$

The inclusion (5) is sufficient to prove the result since supp $D_{\epsilon_m}F = A \subseteq B =$ supp $F = [-\Omega,\Omega]$.

To prove (5), let $D_{\epsilon_1\ldots\epsilon_{m-1}}1_B = 0$ on an open set $U \subseteq \widehat{\mathbb{R}}$ and let $\phi \geqslant 0$ have support contained in U. We obtain the desired conclusion by the definition of support and the following calculation:

$$0 \leqslant \int D_{\epsilon_1\ldots\epsilon_{m-1}}1_A(\gamma)\phi(\gamma)d\gamma = \int 1_A(\gamma)D^{-1}_{\epsilon_{m-1}}\ldots D^{-1}_{\epsilon_1}\phi(\gamma)d\gamma$$
$$\leqslant \int 1_B(\gamma)D^{-1}_{\epsilon_{m-1}}\ldots D^{-1}_{\epsilon_1}\phi(\gamma)d\gamma = \int D_{\epsilon_1\ldots\epsilon_{m-1}}1_B(\gamma)\phi(\gamma)d\gamma = 0. \quad \square$$

We can now complete our definition of the Paley-Wiener pyramidal structure.

Definition 2.3.

a. The *Paley-Wiener pyramidal structure* corresponding to the ideal filters in (3) is the double sequence of spaces

$$X^m_n = D_{\epsilon_1\ldots\epsilon_m}X^0, \; X^0 = \widehat{PW}_\Omega$$

and the family of internodal and interlevel mappings

$$T^{m-1}_{\epsilon_1\ldots\epsilon_m} : X^{m-1}_{\epsilon_1\ldots\epsilon_{m-1}} \longrightarrow X^m_{\epsilon_1\ldots\epsilon_m}$$

defined as

$$T^{m-1}_{\epsilon_1\ldots\epsilon_m} F = D_{\epsilon_1\ldots\epsilon_{m-1}\epsilon_m} F$$

see *Figure 2*.

b. Note that

$$X^m_{\epsilon_1\ldots\epsilon_m} = X^0 1_E, \quad \text{where} \quad E = E^m_{\epsilon_1,\ldots,\epsilon_m}$$

and

$$T^{m-1}_{\epsilon_1\ldots\epsilon_m} F = 1_E F.$$

Example 2.4.

We can establish a bijective correspondence between perfect symmetric Cantor sets[2] determined by $\xi \in (0, 1/2)$ and the Paley-Wiener pyramidal structures of *Definition 2.3*, where the ideal filters in (3) are defined by $\alpha = 1 - 2\xi$. We proceed as follows.

At level 0 consider the interval $[-\Omega, \Omega]$, and compute $D_1 1_{[-\Omega,\Omega]}$. This produces two disjoint intervals $C_{1,1} = [-\Omega, -\alpha\Omega]$ and $C_{1,2} = [\alpha\Omega, \Omega]$ at level 1, each of length $(1-\alpha)\Omega = 2\Omega\xi$; and we have thrown away the middle interval of length $2\Omega(1-2\xi)$. Let $C_1 = C_{1,1} \bigcup C_{1,2}$. Next, we compute $D_{11} 1_{[-\Omega,\Omega]}$. This produces 4 disjoint intervals $C_{2,1}$, $C_{2,2}$, $C_{2,3}$, $C_{2,4}$, where $C_{2,1} \bigcup C_{2,2} \subseteq C_{1,1}$ and $C_{2,3} \bigcup C_{2,4} \subseteq C_{1,2}$. Let $C_2 = C_{2,1} \bigcup \cdots \bigcup C_{2,4}$.

We proceed in this way and compute $D_{11\ldots1} 1_{[-\Omega,\Omega]}$, an m-fold dilation/translation operator.

We obtain disjoint sets $\{C_{m,n} : n = 1, 2, \ldots 2^m\}$ with the property that each $|C_{m,n}| = 2\Omega\xi^m$; and if we set $C_m = C_{m,1} \bigcup \cdots \bigcup C_{m,2^m}$, then $C_\xi = \bigcap C_m$ is the perfect symmetric Cantor set determined by ξ.

Thus, the extreme right branch of $\{X^m_n\}$, defined by the m-fold dilations and translations $D_{11\ldots1}$ originating at X^0, corresponds to the Cantor set C_ξ in the sense that

$$C_\xi = \bigcap_m \text{supp}\, D_{11\ldots1} 1_{[-\Omega,\Omega]}.$$

The structure of this process is self-replicating if we consider right branches emanating from any node, and the allocation of subintervals defined by these dilations and translations is essentially of bit reversal type.

The procedure of *Example 2.4* is generalized and rigorized in *Section 3*.

3. SELF-SIMILAR PYRAMIDAL STRUCTURES

We shall now reformulate the constructive approach of *Section 2* in terms of elementary but abstract set theoretical techniques which give rise to a more general setting than *Section 2*.

To begin, let $A \subseteq \mathbb{R}^d$ and let

$$g_i^0 : \mathbb{R}^d \longrightarrow \mathbb{R}^d, \ i \in I_0, \quad \text{and} \quad g_i^1 : \mathbb{R}^d \longrightarrow \mathbb{R}^{d,} \ i \in I_1,$$

be *bijections* such that

$$A = \left(\dot{\bigcup_{i\in I_0}} g_i^0(A) \right) \dot{\bigcup} \left(\dot{\bigcup_{i\in I_1}} g_i^1(A) \right),$$

or, equivalently,

$$1_A = \sum_{i\in I_0} 1_{g_i^0(A)} + \sum_{i\in I_1} 1_{g_i^1(A)}.$$

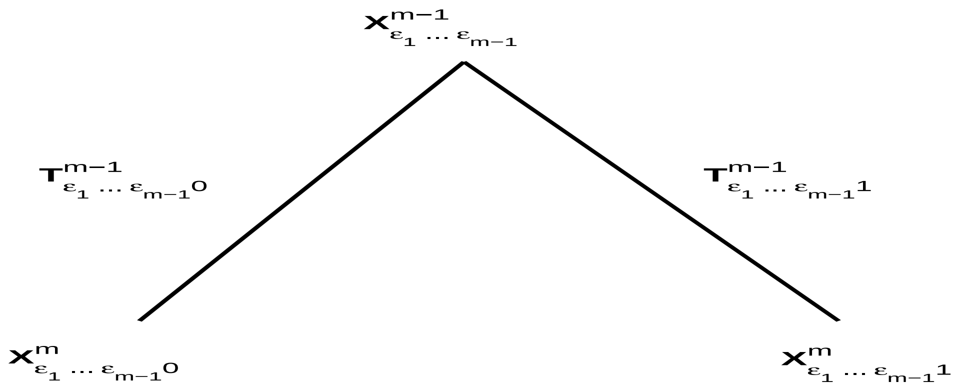

Figure 2. Pyramidal Structure Nodes and Mappings from Level $m-1$

Next, we define filters $H_0, H_1 : \mathbb{R}^d \longrightarrow \mathbb{R}$ as

$$H_0(\gamma) = 1_{\bigcup\limits_{i \in I_0} g_i^0(A)} \quad \text{and} \quad H_1(\gamma) = 1_{\bigcup\limits_{i \in I_1} g_i^1(A)}; \tag{6}$$

and for real-valued functions F on $\mathbb{R}^d$, we define

$$D_0 F(x) = \sum_{i \in I_0} F(g_i^{0^{-1}}(x)) \quad \text{and} \quad D_1 F(x) = \sum_{i \in I_1} F(g_i^{1^{-1}}(x)). \tag{7}$$

¿From the definitions of D_0 and D_1 we have

$$D_{\epsilon_1} \dots D_{\epsilon_m} F = \sum_{i_1 \in I_{\epsilon_1}} \cdots \sum_{i_m \in I_{\epsilon_m}} F g_{i_m}^{\epsilon_m -1} \dots g_{i_1}^{\epsilon_1 -1}$$

and, as in *Section 2*, $D_{\epsilon_1} \dots D_{\epsilon_m}$ will be denoted $D_{\epsilon_1 \dots \epsilon_m}$.

Lemma 3.1. $H_0 = D_0 1_A$ *and* $H_1 = D_1 1_A$.

Proof.

$$H_0(\gamma) = 1 \iff \gamma \in \bigcup_{i \in I_0} g_i^0(A) \iff g_i^{0^{-1}}(\gamma) \in A \text{ for some } i \in I_0$$

$$\iff \sum_{i_1 \in 0} 1_A(g_i^{0^{-1}}(\gamma)) = 1 \iff D_0 1_A(\gamma) = 1.$$

This proves the result since the range of both functions is $\{0,1\}$. The proof is similar for H_1. □

Lemma 3.2. *Let* $\epsilon_1 \dots \epsilon_m \in \{0,1\}$, $\epsilon'_1 \dots \epsilon'_m \in \{0,1\}$, $i_j \in I_{\epsilon_j}$, $i'_j \in I_{\epsilon'_j}$. *If*

$$\forall X, Y \subseteq A, \quad g_{i_1}^{\epsilon_1} \dots g_{i_m}^{\epsilon_m}(X) \bigcap g_{i'_1}^{\epsilon'_1} \dots g_{i'_m}^{\epsilon'_m}(Y) \neq \emptyset,$$

then $\epsilon_j = \epsilon'_j$ *and* $i_j = i'_j$ *for all* $j = 1, 2, \dots m$.

Proof. Since the images of different $g_i^\epsilon(A)$s are disjoint, the result is true for m=1. Assume now that the result is true for m and that

$$g_{i_1}^{\epsilon_1} \dots g_{i_{m+1}}^{\epsilon_{m+1}}(X) \bigcap g_{i'_1}^{\epsilon'_1} \dots g_{i'_{m+1}}^{\epsilon'_{m+1}}(Y) \neq \emptyset.$$

Then

$$g_{i_1}^{\epsilon_1} \left(g_{i_2}^{\epsilon_2} \dots g_{i_{m+1}}^{\epsilon_{m+1}}(X) \right) \bigcap g_{i'_1}^{\epsilon'_1} \left(g_{i'_2}^{\epsilon'_2} \dots g_{i'_{m+1}}^{\epsilon'_{m+1}}(Y) \right) \neq \emptyset.$$

Since the result is true for $m = 1$ it follows that $\epsilon_1 = \epsilon_1'$ and $i_1 = i_1'$. Then , since the g_i^ϵs are one-to-one,

$$g_{i_2}^{\epsilon_2} \dots g_{i_{m+1}}^{\epsilon_{m+1}}(X) \bigcap g_{i_2'}^{\epsilon_2'} \dots g_{i_{m+1}'}^{\epsilon_{m+1}'}(Y) \neq \emptyset,$$

and the result follows by induction. □

Lemma 3.3.

$$\text{supp } D_{\epsilon_1 \dots \epsilon_m} H_0 = \bigcup_{i_1 \in I_{\epsilon_1}} \cdots \bigcup_{i_m \in I_{\epsilon_m}} \bigcup_{i \in I_0} g_{i_1}^{\epsilon_1} \dots g_{i_m}^{\epsilon_m} g_i^0(A) \tag{8}$$

and

$$\text{supp } D_{\epsilon_1 \dots \epsilon_m} H_1 = \bigcup_{i_1 \in I_{\epsilon_1}} \cdots \bigcup_{i_m \in I_{\epsilon_m}} \bigcup_{i \in I_1} g_{i_1}^{\epsilon_1} \dots g_{i_m}^{\epsilon_m} g_i^1(A). \tag{9}$$

Proof. Using the fact that g_i^0 and g_i^1 are bijections, we have

$$D_{\epsilon_1 \dots \epsilon_m} H_0(\gamma) = 1 \longleftrightarrow g_{i_m}^{\epsilon_m -1} \dots g_{i_1}^{\epsilon_1 -1}(\gamma) \in \bigcup_{i \in I_0} g_i^0(A) \quad \text{for some} \quad \epsilon_j, i_j \Longleftrightarrow$$
$$\gamma \in \bigcup_{i_1 \in I_{\epsilon_1}} \cdots \bigcup_{i_m \in I_{\epsilon_m}} \bigcup_{i \in I_0} g_{i_1}^{\epsilon_1} \dots g_{i_m}^{\epsilon_m} g_i^0(A).$$

This proves the claim since the range of $D_{\epsilon_1 \dots \epsilon_m} H_0$ is $\{0, 1\}$. The proof is similar for H_1. □

¿From *Lemmas 3.2* and *3.3* we obtain the following result.

Lemma 3.4.

$$\text{supp } D_{\epsilon_1 \dots \epsilon_m} \ H_0 \bigcap \text{ supp } D_{\epsilon_1 \dots \epsilon_m} H_1 = \emptyset.$$

Lemma 3.5.

$$\text{supp } D_{\epsilon_1 \dots \epsilon_m} H_0 \ \dot{\bigcup} \text{ supp } D_{\epsilon_1 \dots \epsilon_m} H_1 = \text{supp } D_{\epsilon_1 \dots \epsilon_{m-1}} H_{\epsilon_m}.$$

Proof. The sets on the left side are disjoint by *Lemma 3.4*. Also, we can use *Lemma 3.3* to evaluate the left side as follows:

$$\left(\bigcup_{i_1 \in I_{\epsilon_1}} \cdots \bigcup_{i_m \in I_{\epsilon_m}} \bigcup_{i \in I_0} g_{i_1}^{\epsilon_1} \dots g_{i_m}^{\epsilon_m} g_i^0(A) \right) \bigcup \left(\bigcup_{i_1 \in I_{\epsilon_1}} \cdots \bigcup_{i_m \in I_{\epsilon_m}} \bigcup_{i \in I_1} g_{i_1}^{\epsilon_1} \dots g_{i_m}^{\epsilon_m} g_i^1(A) \right) =$$
$$\bigcup_{i_1 \in I_{\epsilon_1}} \cdots \bigcup_{i_m \in I_{\epsilon_m}} \bigcup_{i \in I_1} g_{i_1}^{\epsilon_1} \dots g_{i_m}^{\epsilon_m} \left(\dot{\bigcup_{i \in I_0}} g_i^0(A) \right) \dot{\bigcup} \left(\dot{\bigcup_{i \in I_1}} g_i^1(A) \right) =$$
$$\bigcup_{i_1 \in I_{\epsilon_1}} \cdots \bigcup_{i_m \in I_{\epsilon_m}} g_{i_1}^{\epsilon_1} \dots g_{i_m}^{\epsilon_m}(A) = \text{supp } D_{\epsilon_1 \dots \epsilon_{m-1}} H_{\epsilon_m}. \quad \square$$

These results allow us to make the following decomposition.

Theorem 3.6. *Let X^0 be a space of real-valued functions supported by a closed set $A \subseteq \mathbb{R}^d$, and define the function spaces X_n^m and mappings $T_{\epsilon_1 \dots \epsilon_m}^{m-1}$ as in Section 2, but in terms of the filters H_0, H_1 and mappings D_0, D_1 defined in (6) and (7). Then*

$$X_{\epsilon_1 \dots \epsilon_m}^m = \{F \in X : \text{supp } F = \bigcup_{i_1 \in I_{\epsilon_1}} \dots \bigcup_{i_m \in I_{\epsilon_m}} g_{i_1}^{\epsilon_1} \dots g_{i_m}^{\epsilon_m}(A)\},$$

$$X^m_{\epsilon_1\ldots\epsilon_m} = X^{m+1}_{\epsilon_1\ldots\epsilon_m 0} \oplus X^{m+1}_{\epsilon_1\ldots\epsilon_m 1},$$

and

$$X^0 = \underset{\epsilon_i=0,1}{\oplus} X^m_{\epsilon_1\ldots\epsilon_m} = \underset{n=0,1,\ldots,2^m-1}{\oplus} X^m_n.$$

EXAMPLE 3.7.

a. We shall now revisit the Paley-Wiener pyramidal structure of *Section 2* in terms of bijections g_i^0, $i \in I_0$, and g_i^1, $i \in I_1$. Let $c > 1$ and define

$$I_0 = \{1\}, I_1 = \{1,2\}, g_1^0(x) = \frac{x}{c}, g_1^1(x) = \frac{c-1}{2c}x + \frac{c+1}{2c}\Omega, g_2^1(x) = \frac{c-1}{2c}x - \frac{c+1}{2c}\Omega. \tag{10}$$

Then

$$g_1^{0^{-1}}(x) = cx, g_1^{1^{-1}}(x) = \frac{2c}{c-1}x - \frac{c+1}{c-1}\Omega, g_2^{1^{-1}}(x) = \frac{2c}{c-1}x + \frac{c+1}{c-1}\Omega,$$

thus giving H_0, H_1, D_0, D_1, and $E^m_{\epsilon_1,\ldots,\epsilon_m}$ as defined in *Section 2* for $X^0 = \widehat{PW}_\Omega$.

If $B(u,r)$ denotes the interval centered at u with radius r (as in *Example 2.1e*), we have

$$g_1^0\left(B(u,r)\right) = B(u/c, r/c),$$

$$g_1^1\left(B(u,r)\right) = B\left(\frac{c-1}{2c}u + \frac{c+1}{2c}\Omega, \frac{c-1}{2c}r\right),$$

$$g_1^1\left(B(u,r)\right) = B\left(\frac{c-1}{2c}u + \frac{c+1}{2c}\Omega, \frac{c-1}{2c}r\right);$$

and, thus, $length(g_1^0(B(u,r))) = r/c$ and $length(g_1^1(B(u,r))) = \frac{c-1}{2c} = lenght(g_2^1(B(u,r)))$.

b. Because of part a we see that the nodes $X^m_{\epsilon_1\ldots\epsilon_m}$ of the Paley-Wiener pyramidal structure are the functions in $\widehat{PW}_\Omega$ supported by $E^m_{\epsilon_1,\ldots,\epsilon_m}$, where $E^m_{\epsilon_1,\ldots,\epsilon_m}$ is a disjoint union of 2^k intervals of length $2\Omega(\frac{1}{c})^{m-k}(\frac{c-1}{2c})^k$ centered at $\{g_{i_1}^{\epsilon_1}\ldots g_{i_m}^{\epsilon_m}(0)\}$, and k is the number of ϵ_js $= 1$. Thus, if $\alpha = 1/c$ and $k = m$, then

$$2\Omega(\frac{1}{c})^{m-k}(\frac{c-1}{2c})^k = 2\Omega(\frac{1-\alpha}{2})^m = 2\Omega\xi^m,$$

and $C_m = E^m_{1,1,\ldots,1}$, where ξ and C_m were defined in *Example 2.4*.

c. In fractal terminology, $(\frac{c-1}{2c}, \frac{c-1}{2c})$ is a *contracting ratio list* and (g_1^1, g_2^1) is an *iterated system of similarities* that realizes the ratio list.[4] The Cantor sets C_ξ is the limit of its approximants C_m in a well-defined norm, and it is the unique set C with the property that $C = g_1^1(C) \bigcup g_2^1(C)$.

d. There are analogous constructions for spaces X^0 of functions supported in an equilateral triangle or in a square. In these cases, the subspaces $X^m_{1,\ldots,1}$ of X^0 are supported in the m-th approximation of the Sierpinski gasket or Sierpinski carpet, respectively.

4. SIGNAL RECONSTRUCTION IN SELF-SIMILAR PYRAMIDAL STRUCTURES

The Paley-Wiener and self-similar pyramidal structures of *Sections 2* and *3*, respectively, are determined by filter pairs. As such, there are signal processing (for frequency decomposition) and signal reconstruction methods associated with these structures. We shall compare these methods with wavelet packet signal reconstruction.[1]

We shall analyze the speech signals depicted in *Figure 3*.

First, we take fixed values of $c > 0$ and compute with the filters defined in (3). In *Figure 4*, the 3 columns of graphs correspond to the values $c = 2,3,4$, respectively, and the graphs depict the processed signal at level $m = 3$. Thus, for each value of c, there are $2^3 = 8$ graphs, and each of the graphs is the inverse Fourier transform of the product of $F = \widehat{s}$ and the characteristic function of E^3_n. For example, the graph corresponding to $c = 2$, $m = 3$, and $n = 3$ is evaluated as

$$s_{1,1,0} = \text{ifft}[(\text{fft}\, s)\mathbf{1}_{\mathbf{E^3_{1,1,0}}}].$$

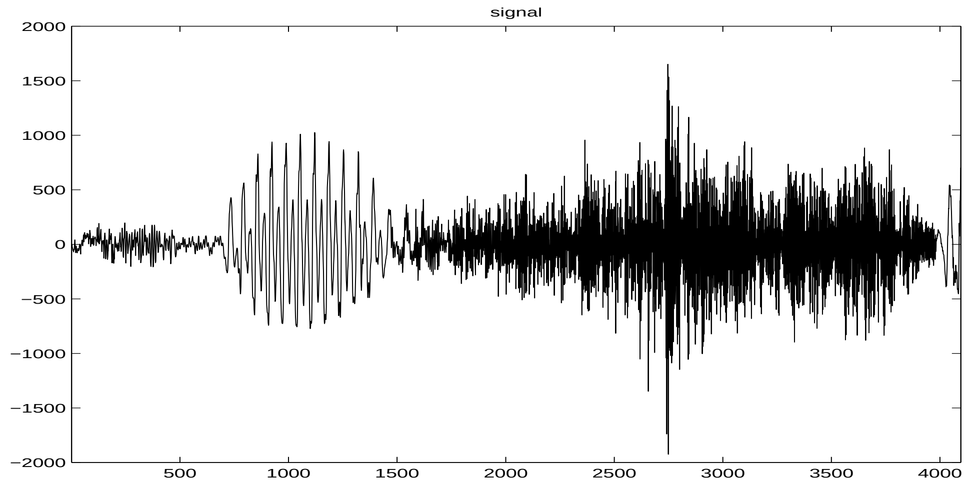

Figure 3. Speech Signal

With this simple approach we have

$$s = \sum s_{\epsilon_1,\epsilon_2,\epsilon_3}.$$

Figure 5 compares the effect in the time domain of the frequency decompositions of self-similar and wavelet packet structures. The first column of graphs in *Figure 5* is the same as the first column of *Figure 4*, viz., the Paley-Wiener structure for $c = 2$ and $m = 3$. The other two columns are the wavelet packet processing for the wavelets db5 and coif4, respectively. The differences between the first and second two columns are apparent, and analysis is underway to exploit such differences in the area of phoneme discrimination.

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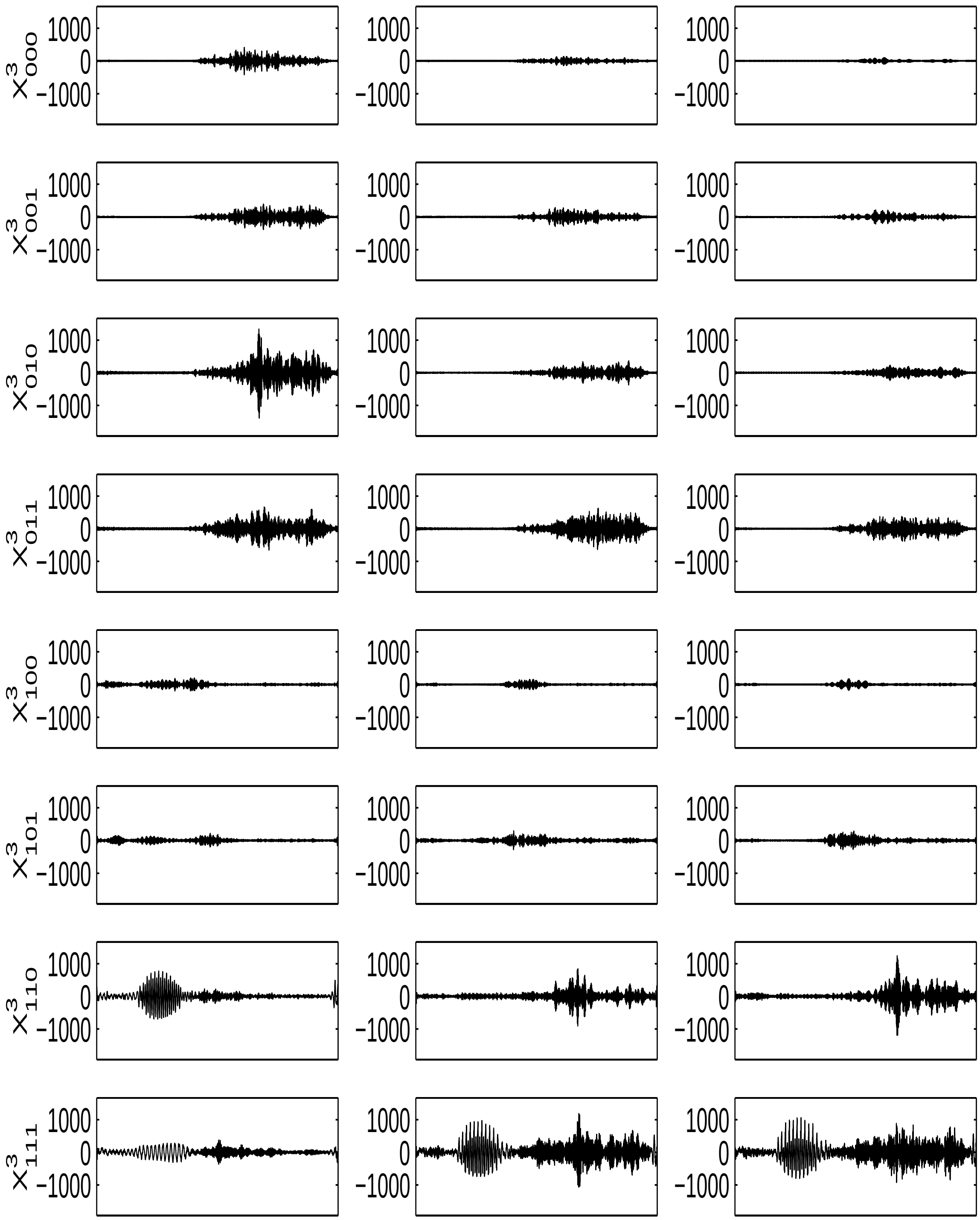

Figure 4. Transforms at level 3 for c=2,3,4

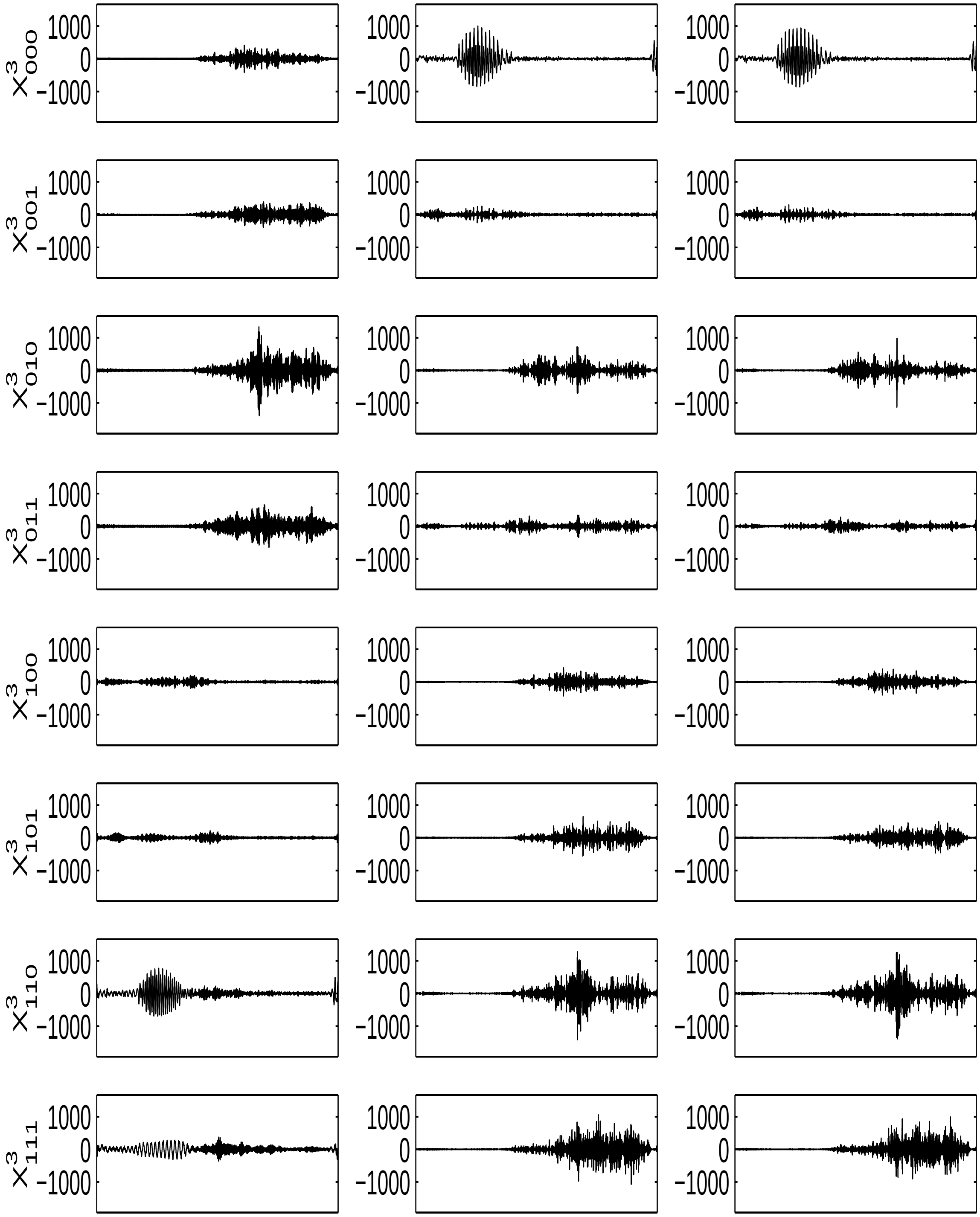

Figure 5. Nodes level 3 for c=2 and db5 and coif4