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MIXED FINITE ELEMENTS FOR THE ELASTOPLASTIC ANALYSIS OF 2D CONTINUA

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Abstract. *The paper aims to formulate assumed stress finite elements for the analysis of elastoplastic structures. The interpolations of the displacement and stress fields, typical of the elastic version of the mixed elements, is enriched with the FEM representation of the plastic strain field. The formulation of the elastoplastic problem of the element is then established, consistently, with respect to its variational basis based on the weak enforcement of the compatibility condition. Its correlation with the Haar–Karman principle leads to a minimization problem of a quadratic functional subjected to a linearized form of the plastic admissibility constraints.*

1 INTRODUCTION

The numerical modelling of elastoplastic materials can be applied in several engineering fields. The elastoplastic theory, initially proposed for the analysis of metal structures [1], is now also successfully used in the analysis of concrete and masonry structures [2, 3], and in the modelling of geomaterials [4]. The latter applications bring to light the degree of maturity of the elastoplastic approach with respect to the basic theoretical framework [5], or with respect to the computational tools [6, 7].

Despite this *state of the art*, the FEM technology mainly used in the analysis of elastoplastic problems is based on the standard compatible formulation with the solution of the elastoplastic problem independently at each Gauss point of the element. Alternative FEM approaches hardly ever have been considered in the FEM literature and the number of the explored proposals is not so large. For example, in [9] and [10], the problem of the limit analysis of structures is approached with the use of a mixed triangular element based on the evaluation of the stress in each node. In [11], the mixed approach is used to formulate the elastoplastic problem which is then solved taking into account all the Gauss points of the element together. In [12] the adoption of generalized variables is advocated from the first inception of the variational framework of the element.

The present work considers a class of assumed stress finite elements which, in the elastic field, has been studied deeply [13, 14, 15]. The basic FEM ingredients of this kind of element, i.e. the displacement and the stress field interpolations, are enriched with the description of the plastic strain field. The reason for this choice is twofold: to maintain the formulation of the compatibility condition at the element level, avoiding resorting to a solution at the level of the generic Gauss point, and to restate the problem as the minimization of a quadratic function constrained by linear inequalities [17], i.e. the Haar–Karman principle [18] subjected to linear constraints is recovered.

The paper is organized as follows. The next section describes the adopted FEM model with the details of the equations involved. In section 3 the elastoplastic problem of the element is formulated as minimum condition of the Haar–Karman functional. Section 4 presents a four node element for the analysis of plane stress problems characterized by a Mises yielding function. The last section contains some numerical results and some final remarks.

2 THE FEM MODEL

This section presents the features of the FEM model adopted. The model can be considered an extension to the analysis of elastic, perfectly plastic structures of the assumed stress approach mainly due to Pian and coworkers [13, 14, 15]. The finite element equations of the problem will be presented in the following. Now let us introduce the fields interpolated by the proposed model:

$$\boldsymbol{u} = \boldsymbol{N}\boldsymbol{d} \qquad \boldsymbol{\sigma} = \boldsymbol{S}\boldsymbol{\beta} \qquad \boldsymbol{\varepsilon}^p = \boldsymbol{R}\boldsymbol{\gamma} \tag{1}$$

The first field, the displacement, is usually interpolated in a standard manner, i.e imposing the continuity among the elements. The second field, the stress, is interpolated without any kind of continuity across the elements. It can be reasonably assumed coincident with the same interpolation already tested in the linear field where the stress interpolation is usually devised in order to obtain high performance capabilities as shown, for example, in [15] and [16].

The third field, the plastic strain, is interpolated in order to characterize, in a concise manner, the plastic response of the material at the element level. The usual scheme for testing the plasticity condition of the material, i.e. on the basis of the Gauss points of the element, is avoided and the compatibility condition is maintained at the element level, consistently with the variational formulation.

This choice, as will be shown in the next section, has an important influence mainly on the formulation and the solution of the elastoplastic problem of the element. The assumed plastic strain field will determine the shape of admissible domain in the space of the stress parameters $\boldsymbol{\beta}$, a domain which will be defined by a set of linear conditions. Further details on the selection of the plastic strain fields will be discussed in section 4 where the formulation of a specific plane, four node element will be presented.

In order to make the context of the work proposed clearer, the discussion can be conveniently referred to a typical path–following analysis in which each step of the analysis can be split into two distinct phases.

I. Solution of the nonlinear compatibility equations. In this phase the unknowns are stresses and plastic strains and the displacement parameters are kept fixed. This phase is performed at element level.

II. Solution of the equilibrium conditions. The unknowns are now the displacements, the non linear response of the material is known and the problem is posed at structural level involving all the kinematic degrees of freedom.

In the following, attention will be focused on phase I, i.e. the formulation of the elasto-plastic problem for a generic assumed stress element. However in order to recall some ingredients of the assumed stress formulation, the equilibrium problem of the element will also be quickly sketched.

2.1 The element equations

In the context of a step–by–step analysis in which the equilibrium path of the structure is determined through the evaluation of a series of successive equilibrium points, the nonlinear compatibility equation involved in phase I of the analysis can be expressed, for each element, as follows

$$\int_{\Omega_e} \delta\boldsymbol{\sigma}_{\mathrm{n+1}}^T \boldsymbol{\mathcal{C}}^{-1} \boldsymbol{\sigma}_{\mathrm{n+1}} = \int_{\Omega_e} \delta\boldsymbol{\sigma}_{\mathrm{n+1}}^T \left(\mathcal{D}\boldsymbol{u}_{\mathrm{n+1}} - \boldsymbol{\varepsilon}_{\mathrm{n+1}}^p \right) \tag{2}$$

where Ω_e is the domain of the element, $\mathcal{D}$ is the continuous operator linking displacements and strains, $\boldsymbol{\mathcal{C}}$ is the constitutive matrix of the mechanical model and the suffix $\mathrm{n}+1$ means the values of the fields relative to the new step to be evaluated in the path–following analysis. According to the fact that the stress and plastic strain require no kind of continuity among the elements and that the displacement is a datum in this phase, the FEM counter–part of the previous equation can be formulated at element level as

$$\boldsymbol{\beta}_{\mathrm{n}+1} = \boldsymbol{H}^{-1}\boldsymbol{Q}\boldsymbol{d}_{n+1} - \boldsymbol{H}^{-1}\hat{\boldsymbol{Q}}\boldsymbol{\gamma}_{n+1} \tag{3}$$

obtained by introducing the FEM interpolations (1). The following discrete operators have been defined

$$\boldsymbol{H} = \int_{\Omega_e} \boldsymbol{S}^T\boldsymbol{\mathcal{C}}^{-1}\boldsymbol{S} \qquad \boldsymbol{Q} = \int_{\Omega_e} \boldsymbol{S}^T\boldsymbol{B} \qquad \hat{\boldsymbol{Q}} = \int_{\Omega_e} \boldsymbol{S}^T\boldsymbol{R} \tag{4}$$

$\boldsymbol{B} = \mathcal{D}\boldsymbol{N}$ is the discrete compatibility operator. For the latter operator $\hat{\boldsymbol{Q}}$ the actual integration bounds coincide with the domain of definition of the plastic fields. It is worth of noting that they could be defined in the element domain along discontinuity lines too.

Phase II of the nonlinear strategy implies the solution of the equilibrium condition which, as usual, is formulated as an instance of the principle of virtual work relative to the whole domain Ω of the problem

$$\int_{\Omega} (\mathcal{D}\delta\boldsymbol{u}_{\mathrm{n}+1})^T \boldsymbol{\sigma}_{\mathrm{n}+1} - \delta\boldsymbol{u}^T_{\mathrm{n}+1}\boldsymbol{W}'_{\mathrm{ext}} = 0 \tag{5}$$

$\boldsymbol{W}_{\mathrm{ext}}$ is the work of the applied loads and the prime indicates its Frechet derivative with respect to the displacement field. The discrete form of the latter equation is as follows

$$\mathcal{A}_e(\boldsymbol{Q}^T\boldsymbol{\beta}_{\mathrm{n}+1} - \lambda_{\mathrm{n}+1}\boldsymbol{f}) = 0 \tag{6}$$

Where $\boldsymbol{f}$ is the FEM representation of the applied loads, the bulk loads $\boldsymbol{b}$ and the boundary traction $\boldsymbol{t}$:

$$\boldsymbol{f} = \int_{\Omega_e} \boldsymbol{N}^T\boldsymbol{b} + \int_{\partial\Omega_e} \boldsymbol{N}^T\boldsymbol{t}$$

$\lambda_{\mathrm{n}+1}$ is the multiplier of the applied loads at step $\mathrm{n}+1$, loads linearly proportional to a unique multiplier are assumed. The symbol $\mathcal{A}_e$ denotes the assembling operation ruled by the continuity of the FEM displacement field.

REMARK 1 - The elastic version of the equation (3) is simply $\boldsymbol{\beta}_{\mathrm{n}+1} = \boldsymbol{H}^{-1}\boldsymbol{Q}\boldsymbol{d}_{n+1}$, a relation which can be used to eliminate the stress parameters and to formulate the FEM model in terms of the following form of the equilibrium conditions

$$\mathcal{A}_e(\boldsymbol{Q}^T\boldsymbol{H}^{-1}\boldsymbol{Q}\boldsymbol{d}_{n+1} - \lambda_{\mathrm{n}+1}\boldsymbol{f}) = 0$$

REMARK 2 - The core of the nonlinear problem, i.e. phase I of the path–following analysis, can be restated in terms of the evaluation, for each element of the structure, of the plastic multipliers $\boldsymbol{\gamma}_{\rm n+1}$ involved in equation (3).

REMARK 3 - In the analysis of plastic structures by means of standard finite elements, the problem stated in phase I is resolved in each Gauss point of the element. In the FEM language it is said that the nonlinear constitutive equations are resolved in the strong form. In the present context, the nonlinear constitutive equations are resolved in the weak form. The FEM interpolations adopted here make it possible to formulate the plastic problem in the weak form, coherently with the assumed stress approach which is based on a weak statement of the compatibility equations. In the work [11] the assumed stress approach is also proposed, but the use of the element Gauss points is adopted too.

3 THE HAAR–KARMAN PRINCIPLE

Behind the solution of the elastoplastic problem of phase I there is the issue, deeply studied in the computational literature, see [6] for a review, of the integration of the plastic strain defined only incrementally through the equation

$$\dot{\boldsymbol{\varepsilon}}^p = \dot{\gamma}\boldsymbol{r} \tag{7}$$

where $\dot{\boldsymbol{\varepsilon}}^p$ is the increment of plastic strain, $\dot{\gamma}$ is the plastic multiplier and $\boldsymbol{r}$ is the plastic flow vector. The solution procedure usually adopted in FEM implementations is based on numerical algorithms [19], such as the well known implicit Euler algorithm. In our context, aiming at the formulation of the plastic problem at the element level, the holonomic approach based on the extremal path theory of Ponter and Martin [20] can be effectively exploited. In particular, for a perfectly plastic behaviour, the extremal path theory coincides with the Haar–Karman principle [21] which, as will be shown, suits our FEM formulation well.

Assuming a perfectly plastic behaviour, the elastoplastic stress solution must satisfy the Haar–Karman principle which can be formulated as follows

$$\frac{1}{2}\int_{\Omega_e} \boldsymbol{\sigma}^T_{\rm n+1}\boldsymbol{\mathcal{C}}^{-1}\boldsymbol{\sigma}_{\rm n+1} - \int_{\partial\Omega_u} (\boldsymbol{n}\boldsymbol{\sigma}_{\rm n+1})^T\bar{\boldsymbol{u}} = \min. \tag{8}$$

under the condition

$$f[\boldsymbol{\sigma}_{\rm n+1}] \leq 0 \tag{9}$$

where $\partial\Omega_u$ is the constrained part of the boundary, $\boldsymbol{n}$ is its outward normal and $f[\boldsymbol{\sigma}]$ is the yielding function. Equation (8) coincides with the minimum of the complementary energy and it can be rewritten with respect to an elastic stress solution too. This allows the term on the boundary to be eliminated from (8) and an equivalent formulation of the principle of Haar–Karman to be obtained

$$\frac{1}{2}\int_{\Omega_e} \left(\boldsymbol{\sigma}_{\rm n+1} - \boldsymbol{\sigma}^*_{\rm n+1}\right)^T \boldsymbol{\mathcal{C}}^{-1} \left(\boldsymbol{\sigma}_{\rm n+1} - \boldsymbol{\sigma}^*_{\rm n+1}\right) = \min. \tag{10}$$

where $\boldsymbol{\sigma}^*_{\mathrm{n}+1}$ is the elastic solution at step $\mathrm{n}+1$, i.e. assuming a null increment of the plastic strain field from step n to step $\mathrm{n}+1$.

The principle requires that the solution must also belong to the admissible domain defined in the stress space by the chosen yielding condition. In our case we also represent the plastic strain field via a FEM interpolation, then the admissibility condition can be restated in the following way:

$$\int_{\Omega_e} \boldsymbol{r}_i^T \left(\boldsymbol{\sigma}_{n+1} - \boldsymbol{\sigma}_y[\boldsymbol{r}_i]\right) \leq 0 \qquad i = 1 \dots n_p \tag{11}$$

where $\boldsymbol{r}_i$ is the ith plastic mode assumed in the interpolation of $\boldsymbol{\varepsilon}^p$, i.e. it is the ith column of the matrix $\boldsymbol{R}$, see equation (1), and $\boldsymbol{\sigma}_y[\boldsymbol{r}_i]$ is the stress field associated to $\boldsymbol{r}_i$ through the yielding criterion. In other words $\boldsymbol{\sigma}_y[\boldsymbol{r}_i]$ is evaluated by imposing the conditions:

$$\boldsymbol{r}_i = \frac{\partial f}{\partial \boldsymbol{\sigma}} \qquad f[\boldsymbol{\sigma}_y[\boldsymbol{r}_i]] = 0 \tag{12}$$

The first condition states the normality rule for the plastic strain, as it is always true for stable materials. The second condition requires that $\boldsymbol{\sigma}_y[\boldsymbol{r}_i]$ be on the yielding surface.

3.1 FEM representation of the minimum condition

On the basis of the assumed stress interpolation, the minimum condition (10) can be restated as follows

$$\frac{1}{2}\left(\boldsymbol{\beta}_{\mathrm{n}+1} - \boldsymbol{\beta}^*_{\mathrm{n}+1}\right)^T \boldsymbol{H} \left(\boldsymbol{\beta}_{\mathrm{n}+1} - \boldsymbol{\beta}^*_{\mathrm{n}+1}\right) = \mathrm{min.} \tag{13}$$

$\boldsymbol{\beta}^*_{\mathrm{n}+1}$ is the discrete form of the elastic predictor and it is expressed by

$$\boldsymbol{\beta}^*_{\mathrm{n}+1} = \boldsymbol{H}^{-1}\boldsymbol{Q}\boldsymbol{d}_{n+1} - \boldsymbol{H}^{-1}\hat{\boldsymbol{Q}}\boldsymbol{\gamma}_n \tag{14}$$

The latter equation and the equation (3) relative to $\boldsymbol{\beta}_{\mathrm{n}+1}$ allow the following rewriting of the minimum condition:

$$\frac{1}{2}\left(\boldsymbol{\gamma}_{\mathrm{n}+1} - \boldsymbol{\gamma}_{\mathrm{n}}\right)^T \hat{\boldsymbol{Q}}^T \boldsymbol{H}^{-1}\hat{\boldsymbol{Q}} \left(\boldsymbol{\gamma}_{\mathrm{n}+1} - \boldsymbol{\gamma}_{\mathrm{n}}\right) = \mathrm{min.} \tag{15}$$

or equivalently

$$\frac{1}{2}\Delta\boldsymbol{\gamma}^T \hat{\boldsymbol{Q}}^T \boldsymbol{H}^{-1}\hat{\boldsymbol{Q}}\Delta\boldsymbol{\gamma} = \mathrm{min.} \tag{16}$$

in which the increment of plastic strain is introduced:

$$\boldsymbol{\gamma}_{\mathrm{n}+1} = \boldsymbol{\gamma}_{\mathrm{n}} + \Delta\boldsymbol{\gamma} \tag{17}$$

3.2 FEM representation of the constraints

The set of linear constraints given in equation (11) assume the following FEM representation:

$$\hat{\boldsymbol{Q}}^T \boldsymbol{\beta}_{\mathrm{n}+1} \leq \mathrm{diag}(\hat{\boldsymbol{Q}}_y) \tag{18}$$

where the operator diag() gives as a vector the terms of the main diagonal of the matrix $\hat{\boldsymbol{Q}}_y$. The new operator $\hat{\boldsymbol{Q}}_y$ introduced is given by the following expression:

$$\hat{\boldsymbol{Q}}_y = \int_{\Omega_e} \boldsymbol{S}_y^T \boldsymbol{R} \tag{19}$$

where the columns of the matrix $\boldsymbol{S}_y$ are constituted by the stress fields $\boldsymbol{\sigma}_y[\boldsymbol{r}_i]$ defined in (12).

Also in this case the relations (3) and (17) can be exploited in order to reformulate (18):

$$\hat{\boldsymbol{Q}}^T \boldsymbol{H}^{-1} \hat{\boldsymbol{Q}} \Delta\boldsymbol{\gamma} \geq \hat{\boldsymbol{Q}}^T \boldsymbol{H}^{-1} \boldsymbol{Q} \boldsymbol{d}_{\mathrm{n}+1} - \hat{\boldsymbol{Q}}^T \boldsymbol{H}^{-1} \hat{\boldsymbol{Q}} \boldsymbol{\gamma}_{\mathrm{n}} - \mathrm{diag}(\hat{\boldsymbol{Q}}_y) \tag{20}$$

4 A FOUR NODE PLANE STRESS ELEMENT FOR MISES–LIKE MATERIALS

The selection of the plastic strain fields is strictly linked to the following factors: the kind of nonlinear behaviour to be modeled, i.e. the yield criterion of the basic material; the kind of element; the use the element is designed for, i.e. fine or coarse meshes. These factors strongly determine the more suitable fields to be adopted.

In order to perform an experimentation on some possible FEM interpolations of the plastic strain field avoiding the burden of particular constitutive prescriptions, the standard Mises criterion in plane stress condition has been chosen. The discussion will be focused on the four node element proposed by Pian and Sumihara which constitutes one of the best performers in the family of the assumed stress elements proposed for the elastic field. The basic interpolation fields of this element are as follows:

$$\boldsymbol{u} = \begin{bmatrix} u[\xi,\eta] \\ v[\xi,\eta] \end{bmatrix} = \boldsymbol{N}\boldsymbol{d} = \begin{bmatrix} N_1[\xi,\eta] & 0 & \dots & N_4[\xi,\eta] & 0 \\ 0 & N_1[\xi,\eta] & \dots & 0 & N_4[\xi,\eta] \end{bmatrix} \begin{bmatrix} u_1 \\ v_1 \\ \dots \\ u_4 \\ v_4 \end{bmatrix} \tag{21}$$

with

$$N_1[\xi,\eta] = \tfrac{1}{4}(1-\xi)(1-\eta)\eta \qquad N_2[\xi,\eta] = \tfrac{1}{4}(1+\xi)(1-\eta)\eta$$
$$N_3[\xi,\eta] = \tfrac{1}{4}(1+\xi)(1+\eta)\eta \qquad N_4[\xi,\eta] = \tfrac{1}{4}(1-\xi)(1+\eta)\eta$$

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{bmatrix} = \boldsymbol{S}\boldsymbol{\beta} = \begin{bmatrix} 1 & 0 & 0 & a_1^2\,\eta & a_3^2\,\xi \\ 0 & 1 & 0 & b_1^2\,\eta & b_3^2\,\xi \\ 0 & 0 & 1 & a_1\,b_1\;\eta & a_3\,b_3\;\xi \end{bmatrix} \begin{bmatrix} \beta_1 \\ \dots \\ \beta_5 \end{bmatrix} \tag{22}$$

where

$$\begin{bmatrix} a_0 & b_0 \\ a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & 1 & -1 \\ -1 & -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ x_4 & y_4 \end{bmatrix}$$

are the usual coefficients adopted to describe the geometry of the element.

4.1 Continuous interpolation

A simple continuous interpolation of the plastic strain field of the element can be built on the basis of the same fields adopted in the interpolation of the stress. In particular the $\boldsymbol{R}$ can be assumed as:

$$\boldsymbol{R} = \begin{bmatrix} \boldsymbol{S} & -\boldsymbol{S} \end{bmatrix} \tag{23}$$

which represents ten plastic fields, five for positive strains and the others for the negative case.

The subsequent step is the evaluation of the fields contained in the matrix $\boldsymbol{S}_y$ on the basis of the conditions (12). The latter and the Mises surface for the plane stress case,

$$f[\boldsymbol{\sigma}] = \sqrt{\frac{1}{2}\boldsymbol{\sigma}^T \boldsymbol{P} \boldsymbol{\sigma}} - \sigma_y \tag{24}$$

with

$$\boldsymbol{P} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 6 \end{bmatrix},$$

allow us to obtain the following expression for the generic stress field associated to $\boldsymbol{r}_i$

$$\boldsymbol{\sigma}_y[\boldsymbol{r}_i] = \frac{\sigma_y}{\left(\frac{1}{2}\boldsymbol{r}_i^T \boldsymbol{P}^{-1} \boldsymbol{r}_i\right)^{1/2}} \boldsymbol{P}^{-1} \boldsymbol{r}_i \tag{25}$$

and then the matrix $\boldsymbol{S}_y$.

In the evaluation of $\mathrm{diag}(\hat{\boldsymbol{Q}}_y)$ either equation (19) can be used or it can be directly calculated through the following formula

$$\mathrm{diag}(\hat{\boldsymbol{Q}}_y)_i = \sqrt{2}\sigma_y \int_{\Omega_e} \sqrt{\boldsymbol{r}_i^T \boldsymbol{P}^{-1} \boldsymbol{r}_i} \qquad i = 1 \ldots n_p \tag{26}$$

4.2 Interpolation along discontinuity lines

Another way to formulate the plastic fields inside the element can be based on the adoption of discontinuity lines along which the plastic strain is concentrated. Let us consider the discontinuity lines reported in Figure 1. Each of the assumed plastic fields will be defined only on one of these lines, then the integrals involved in the calculation of

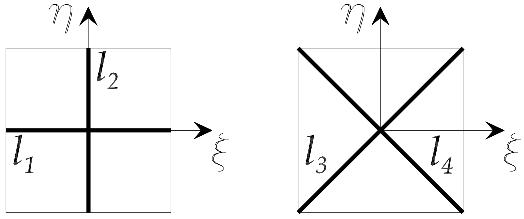

Figure 1: Plastic strain lines

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$\hat{\boldsymbol{Q}}$ and diag($\hat{\boldsymbol{Q}}_y$) must be rewritten in the correct form. Let $g[\xi, \eta]$ be the generic argument of such integrals, they take the following form

$$\int_{-1}^{1} g[\xi, \eta = 0]\frac{l_1}{2}\mathrm{d}\xi \quad \int_{-1}^{1} g[\xi = 0, \eta]\frac{l_2}{2}\mathrm{d}\eta \quad \int_{-1}^{1} g[\xi, \eta = \xi]\frac{l_3}{2}\sqrt{2}\mathrm{d}\xi \quad \int_{-1}^{1} g[\xi, \eta = -\xi]\frac{l_4}{2}\sqrt{2}\mathrm{d}\xi$$

along the segments l_1, l_2, l_3 and l_4 respectively, and defined by the following expressions

$$l_1 = 2\sqrt{a_1^2 + b_1^2} \qquad l_2 = 2\sqrt{a_3^2 + b_3^2} \qquad l_3 = 2\sqrt{(a_1 + a_3)^2 + (b_1 + b_3)^2} \qquad l_4 = 2\sqrt{(a_1 - a_3)^2 + (b_1 - b_3)^2}$$

The adopted plastic strain fields $\boldsymbol{r}_i$ are as follows

$$\begin{array}{llllllll}
\text{along line } l_1: & \pm & \begin{array}{c}1\\0\\0\end{array} & \pm & \begin{array}{c}0\\1\\0\end{array} & \pm & \begin{array}{c}0\\0\\1\end{array} \\
\\
\text{along line } l_2: & \pm & \begin{array}{c}1\\0\\0\end{array} & \pm & \begin{array}{c}0\\1\\0\end{array} & \pm & \begin{array}{c}0\\0\\1\end{array} \\
\\
\text{along line } l_3: & \pm & \begin{array}{r}\frac{1}{2}\\\frac{1}{2}\\-\frac{1}{2}\end{array} & \pm & \begin{array}{r}\frac{1}{2}\\\frac{1}{2}\\\frac{1}{2}\end{array} & \pm & \begin{array}{r}1\\-1\\0\end{array} \\
\\
\text{along line } l_4: & \pm & \begin{array}{r}\frac{1}{2}\\\frac{1}{2}\\\frac{1}{2}\end{array} & \pm & \begin{array}{r}\frac{1}{2}\\\frac{1}{2}\\-\frac{1}{2}\end{array} & \pm & \begin{array}{r}-1\\1\\0\end{array}
\end{array} \qquad (27)$$

As shown in the following numerical results, this kind of choice leads to results nearly coincident with the continuous interpolation. The main differences regard the description of the plastic strain distribution. Moreover a sensible simplification of the operators $\hat{\boldsymbol{Q}}$ and $\hat{\boldsymbol{Q}}_y$ is obtained.

5 NUMERICAL RESULTS

The problems analyzed are depicted in Figure 2 where all the information, geometry, constraints, constitutive prescriptions and a sample mesh, are shown. The analyses were performed by a path–following analysis driven by a Riks–like strategy [22]. Table 1

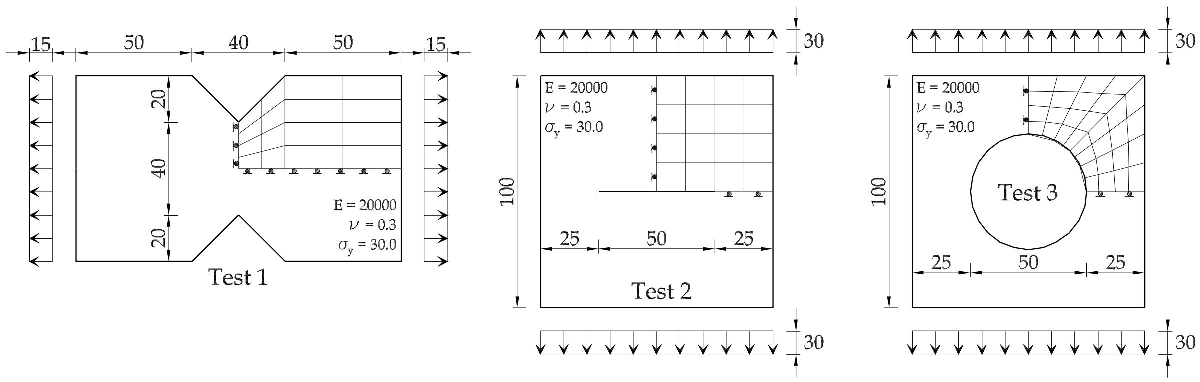

Figure 2: Test problems analyzed

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reports the values of the load multiplier at collapse obtained refining the mesh and on the basis of different representations of the plastic strain field. The column labeled $\lambda_c^\star$ is relative to the approach based on the continuous interpolation, the other column, with the label $\lambda_c^\dagger$, is obtained through the interpolation along discontinuity lines. The reference values are theoretical, when available, or they are obtained by means of other FEM codes. With respect to the load multiplier at collapse, the two approaches considered are nearly

Test 1			Test 2			Test 3		
nodes	$\lambda_c^\star$	$\lambda_c^\dagger$	nodes	$\lambda_c^\star$	$\lambda_c^\dagger$	nodes	$\lambda_c^\star$	$\lambda_c^\dagger$
35	1.4434	1.4434	25	0.8660	0.8660	45	0.6095	0.5793
81	1.2990	1.2990	81	0.7217	0.7217	153	0.5067	0.4906
289	1.2269	1.2269	289	0.6495	0.6495	561	0.4605	0.4532
1089	1.1908	1.1908	1089	0.6134	0.6134	2145	0.4468	0.4266
4225	1.1727	1.1727	4225	0.5954	0.5954	-	-	-
ADINA	1.1747		ADINA	0.5163		Other	0.4101	
Hill (∞)	1.1555		Theory	0.498–0.522		Theory	-	

Table 1: Values of the load multiplier at collapse

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coincident and differ only in the 3rd test considered. But if we observe the distribution of the elements in the plastic phase at collapse, see Figure 3, we can note some differences.

In particular the solution obtained with an approach based on a continuous interpolation of the plastic strain is far from being a realistic solution. The 2nd test, see Table 1, shows

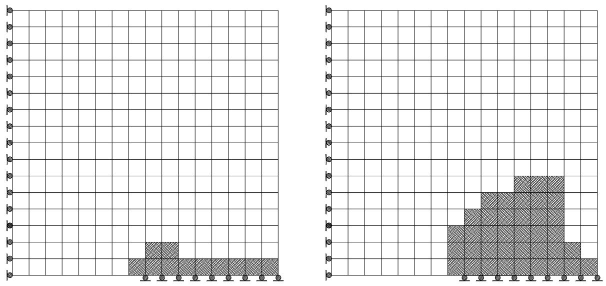

Figure 3: Test 2: elements in plastic phase at collapse (mesh with 289 nodes)

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a slower convergence of the collapse multiplier toward the reference value. This can be related to a non optimal choice of the plastic strain interpolation which in this case should be capable of representing high gradients in the neighborhood of the tip of the slit.

6 CONCLUSIONS

The formulation of mixed elements for the analysis of elastoplastic structures has been examined. The elastoplastic response of the element is evaluated by posing the problem as a minimization of a functional subjected to a set of linear constraints. This is obtained on the basis of the Haar–Karman principle and the interpolation of the plastic strain field of the element. Two different kinds of interpolations have been tested in the context of Mises materials, one continuous and the other defined along assigned discontinuity lines. The latter approach shows a better behaviour in the description of the plastic strain field, however further experimentations should be performed in order to device an optimal FEM description of the plastic strain and to extend the approach to the modelling of structures with different constitutive behaviours.

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