Linear independence of translates implies linear independence of affine Parseval frames on LCA groups

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Abstract

Motivated by Bownik and Speegle's result on linear independence of wavelet Parseval frames, we consider affine systems (analogous to wavelet systems) defined on a second countable, locally compact abelian group G, where the translations are replaced by the action of a countable, discrete subgroup Γ of G acting as a group of unitary operators on $L^2(G)$. The dilation operation in the wavelet setting is replaced by integer powers of a unitary operator δ onto $L^2(G)$. We show that, under some compatibility conditions between δ and the action of the group Γ, the linear independence of the translates of any function in $L^2(G)$ by elements of Γ implies the linear independence of affine Parseval frames in $L^2(G)$.

Keywords: LCA group, Parseval frame, range function, multiplicity function.
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1. Introduction

The study of doubly invariant subspaces of measurable vector functions defined in the unit circle and taking values in a separable Hilbert space, as developed by Helson in [13], has been retrieved in the context of shift invariant subspaces of $L^2(\mathbb{R}^n)$ by de Boor, DeVore, Ron [10], and Bownik [4], leading to what is commonly known as Helson's theorem.

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A shift invariant subspace of $L^2(\mathbb{R}^n)$ is any closed subspace $V \subset L^2(\mathbb{R}^n)$ which is closed under integer shifts, i.e. such that $f \in V$ implies $T_k f = f(\cdot - k) \in V$, for all $k \in \mathbb{Z}^n$. The Fourier transform is defined as $\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x)\, e^{-ix\cdot\xi}\, dx$.

Theorem (Helson)**.** *A closed subspaces $V \subset L^2(\mathbb{R}^n)$ is shift invariant if and only if*

$$V = \{f \in L^2(\mathbb{R}^n),\, (\hat{f}(\xi - 2\pi k))_{k\in\mathbb{Z}^n} \in J(\xi),\, \textit{for a.e. } \xi \in \mathbb{T}^n\},$$

where J is a measurable range function

$$J : \mathbb{T}^n \to \{\textit{closed subspaces of } \ell^2(\mathbb{Z}^n)\}.$$

The correspondence between V and J is one-to-one, under the convention that the range functions are identified if they are equal a.e.. Furthermore, if

$$V = \overline{span}\{T_k\varphi,\ k \in \mathbb{Z}^n, \varphi \in \mathscr{A}\},$$

for an at most countable set $\mathscr{A} \subset L^2(\mathbb{R}^n)$, then,

$$J(\xi) = \overline{span}\{(\hat{\varphi}(\xi - 2\pi k))_{k\in\mathbb{Z}^n},\ \varphi \in \mathscr{A}\}, \quad a.e.\, \xi \in \mathbb{T}^n.$$

This result has led to significant progress in the study of linear independence of wavelet systems (see the work by Bownik and Speegle [6]). Given $\psi \in L^2(\mathbb{R}^n)$, a wavelet system in $\mathbb{R}^n$ is

$$|\det A|^{j/2}\, \psi(A^j x - k), \quad x \in \mathbb{R}^n,\ k \in \mathbb{Z}^n,\ j \in \mathbb{Z},$$

where A is an $n \times n$ integer-valued, non-singular, expansive matrix.

Helson's theorem has been generalized to locally compact abelian groups, by various authors: Kamyabi Gol and Raisi Tousi [16],[17], Cabrelli and Paternostro [9], Bownik and Ross [5]. In all these works, albeit with several distinctions, a subgroup Γ of a LCA group G acts as a group of translations on $L^2(G)$. When finishing this paper we became aware also of the work by Barbieri, Hernández, and Paternostro [3], on characterization of invariant spaces in terms of range functions and the generalized Zak transform, as well as of Iverson's work [15] in the same direction.

One of the aims of this work is to extend Helson's theorem in case G is a LCA, second countable, Hausdorff group and Γ is a closed countable subgroup of G, with compact dual group of characters $\widehat{\Gamma}$, which acts as a group of unitary operators on the space $L^2(G)$, meaning that there is a unitary representation

$$\pi : \Gamma \to \mathcal{U}(L^2(G)).$$

We do not require Γ to be co-compact, but we assume that the measure μ on $\widehat{\Gamma}$, arising from the spectral theorem applied to the representation π, is absolutely continuous with respect to the Haar measure.

We obtain a characterization of π-invariant subspaces (i.e. closed subspaces $V \subset L^2(G)$ such that $\pi(\gamma)V \subset V$, for all $\gamma \in \Gamma$) in terms of the range function

$$J : \widehat{\Gamma} \to \{\text{closed subspaces of } \ell^2(\Gamma)\},$$

(see Theorem 4.6).

Also, motivated by the above mentioned paper by Bownik and Speegle, we consider affine systems of functions defined on G, (analogous to wavelet systems)

$$\{\delta^j\, \pi(\gamma)\, \psi, \quad \gamma \in \Gamma,\ j \in \mathbb{Z}\},$$

where $\psi \in L^2(G)$, the translations are replaced by the action of Γ as a group of unitary operators on $L^2(G)$, and the dilation operation of the wavelet setting is replaced by integer powers of a unitary operator δ onto $L^2(G)$, verifying a compatibility condition with the representation π described below.

We suppose that there is a one to one endomorphism $\alpha : \Gamma \to \Gamma$ such that the subgroup $\alpha(\Gamma)$ has finite index in Γ, $\bigcap_{n\geq 1} \alpha^n(\Gamma) = \{0\}$, and the following relation holds

$$\delta^{-1}\pi(\gamma)\delta = \pi(\alpha(\gamma)), \quad \text{for all } \gamma \in \Gamma.$$

Affine systems include wavelet systems for a particular choice of the group and the representation.

This approach has been used by Baggett and his collaborators in [2], in the context of the GMRA. By the spectral theorem of Stone and von Neumann, in conjunction with the characterization of spectral measure, any subrepresentation of π, arising from an invariant subspace, is realized as a direct integral. We obtain, see Lemma 5.5, that the multiplicity functions associated with the subrepresentations of π on an invariant subspace V and its "dilation" $\delta(V)$ verify the same relation obtained by Bownik and Rzeszotnik in the case $G = \mathbb{R}^n$,[7, Corollary 2.5], for the corresponding dimension functions. It is worth to recall that the dimension function of an invariant subspace V is defined as $\dim_V(\xi) = \dim J(\xi)$, where J is the range function provided by Helson's theorem and that in case of translations the dimension function is indeed equivalent to the multiplicity function.

This fact allows us to prove, by extending the technique used in [6], that every affine Parseval frame is linearly independent if it is assumed that the translations of each function in $L^2(G)$ are linearly independent. The hypothesis on linear independence is certainly verified in the case of $G = \mathbb{R}^n$, while it is not, for example, in the case of finite groups (see the work of Rosenblatt [20] for a discussion on this topic).

We say that a sequence $(e_n)_{n\in\mathbb{Z}}$ in a Hilbert space $\mathcal{H}$ is *linearly independent* if every finite subsequence of $(e_n)_{n\in\mathbb{Z}}$ is linearly independent, i.e.

$$c_n \in \mathbb{C},\ n \in F \text{ finite}, \quad \sum_{n\in F} c_n\, e_n = 0 \Rightarrow\ c_n = 0, \quad \text{for all } n \in F.$$

We decided to label the following hypothesis, since we shall assume it several times:

(A) For any $0 \neq f \in L^2(G)$, the sequence of translates $\{T_\gamma f,\ \gamma \in \Gamma\}$ is linearly independent.

As it will be shown in Section 6, our hypotheses on Γ and μ guarantee that linear independence of translates implies the linear independence of the sequence $\{\pi(\gamma)f,\ \gamma \in \Gamma\}$, for any $0 \neq f \in L^2(G)$.

Before stating the main results, we need the definition of a (Parseval) frame. Frames in a separable Hilbert space provide redundant but stable expansions for elements of the space itself. Frames play key roles in many settings, such as sampling theory, wavelet analysis, and time-frequency (Gabor) analysis. We say that a sequence $(e_n)_{n\in\mathbb{Z}}$ in a Hilbert space $\mathcal{H}$ is a frame if there exist constants $A, B > 0$ such that

$$A\|x\| \leq \sum_{n\in\mathbb{Z}} | < x, e_n > |^2 \leq B\|x\|, \quad \text{for all } x \in \mathcal{H}.$$

If $A = B$, we say that $(e_n)_{n\in\mathbb{Z}}$ is a tight frame, if $A = B = 1$, a Parseval frame.

The main results of the paper are

Theorem 1. *Assume hypothesis* **(A)**. *Suppose $\psi \in L^2(G)$ and its space of negative dilates $V_0 = \overline{span}\{\delta^j\, \pi(\gamma)\, \psi,\ j < 0, \gamma \in \Gamma\}$ is π- invariant.*

Then the affine system $\{\delta^j\, \pi(\gamma)\, \psi,\ j \in \mathbb{Z}, \gamma \in \Gamma\}$ is linearly independent.

Theorem 2. *Assume hypothesis* **(A)**. *Suppose* $\psi \in L^2(G)$. *If the affine system* $\{\delta^j \pi(\gamma)\,\psi, j \in \mathbb{Z}, \gamma \in \Gamma\}$ *is a Parseval frame, then it is linearly independent.*

We are mainly interested in affine systems on $L^2(G)$, so we decided to state and prove all results in that context, but, as the reader can easily check, still being valid hypothesis **(A)** on translates in $L^2(G)$, all results remain valid if we replace $L^2(G)$ by any separable Hilbert space $\mathcal{H}$, and the representation π by any unitary representation

$$\pi' : \Gamma \to \mathcal{U}(\mathcal{H}),$$

verifying the analogous compatibility condition to (1) together with a unitary operator δ' onto $\mathcal{H}$.

We tried to separate those results that do not need neither all the machinery of representation theory of LCA groups, nor the characterization of π-invariant spaces by Helson from those who do. So after the main hypotheses in Section 2, we state a first result on linear independence in Section 3. In Section 4 we extend Helson's theorem to π-invariant spaces, and in Section 5 we prove the main properties of the multiplicity function. The proof of Theorem 1 and Theorem 2 are given in Section 6.

2. Hypotheses and Notations

In this section we collect all the hypotheses and notations needed in this paper. We assume that G is a locally compact abelian, second countable, Hausdorff group (LCA) (so that the Hilbert space $L^2(G)$ is separable) and $\Gamma \subset G$ is a closed countable subgroup of G with compact dual group of characters $\widehat{\Gamma}$. We do not require Γ to be co-compact. Note that $\widehat{\Gamma}$ is compact and metrizable, hence separable and second countable.

We suppose that there is a one to one endomorphism $\alpha : \Gamma \to \Gamma$ such that the subgroup $\alpha(\Gamma)$ has finite index in Γ, i.e. the quotient group

$$\Gamma/\alpha(\Gamma)$$

has a finite number of elements, say $N > 1$.

We consider the dual endomorphism onto $\widehat{\Gamma}$, $\alpha^* : \widehat{\Gamma} \to \widehat{\Gamma}$, defined, in any character $\chi \in \widehat{\Gamma}$, by $\alpha^*(\chi) = \chi \circ \alpha$. We assume that $\bigcup_{n\geq 1} \ker \alpha^{*n}$ is dense in $\widehat{\Gamma}$, which is equivalent to require $\bigcap_{n\geq 1} \alpha^n(\Gamma) = \{0\}$. Note that $|\ker \alpha^*| = N$ and that α^* is ergodic with respect to the normalized Haar measure λ on $\widehat{\Gamma}$.

We assume

$$\pi : \Gamma \to \mathcal{U}(L^2(G)),$$

is a unitary representation of Γ on $L^2(G)$, and $\delta : L^2(G) \to L^2(G)$ a unitary operator verifying the following relation

$$\delta^{-1}\pi(\gamma)\delta = \pi(\alpha(\gamma)), \quad \text{for all } \gamma \in \Gamma. \tag{1}$$

It follows that

$$\pi(\gamma)\delta^j = \delta\pi(\alpha(\gamma))\delta^{j-1} \cdots = \delta^j\pi(\alpha^j(\gamma)), \quad j \geq 1, \quad \text{for all } \gamma \in \Gamma. \tag{2}$$

For any given σ-finite measure μ on $\widehat{\Gamma}$, by $L^2(\widehat{\Gamma}, \ell^2(\Gamma), \mu)$ we mean the Hilbert space of (equivalence class of) vector functions F defined on $\widehat{\Gamma}$, attaining values in $\ell^2(\Gamma)$, which are measurable and square integrable with respect to the measure μ, i.e. such that

$$\|F\|_2 = \left(\int_{\widehat{\Gamma}} \|F(\chi)\|_{\ell^2(\Gamma)}\, d\mu(\chi)\right)^{1/2} < +\infty.$$

The scalar product in $L^2(\widehat{\Gamma}, \ell^2(\Gamma), \mu)$ is given by

$$(F, G) = \int_{\widehat{\Gamma}} (F(\chi), G(\chi))\, d\mu(\chi),$$

where the inner product inside the integral is the one in $\ell^2(\Gamma)$.

We recall some consequences of the spectral theorem that we shall need in the paper.

By the spectral theorem of Stone and von Neumann, [11], and the characterization of spectral measure, see [14], since $L^2(G)$ is a separable Hilbert space and

$$\pi : \Gamma \to \mathcal{U}(L^2(G)),$$

is a unitary representation, there exists a finite measure μ on Borel subsets of $\widehat{\Gamma}$ (we normalize it so that $\mu(\widehat{\Gamma}) = 1$) and measurable subsets

$$\ldots \sigma_i \subset \cdots \subset \sigma_2 \subset \sigma_1 \subset \widehat{\Gamma},$$

there exists a unitary map

$$T : L^2(G) \to \bigoplus_i L^2(\sigma_i, \mu) \hookrightarrow L^2(\widehat{\Gamma}, \ell^2(\Gamma), \mu), \tag{3}$$

such that

$$[T(\pi(\gamma)f)]_i(\chi) = (\gamma,\chi)[T(f)]_i(\chi), \quad \text{for all } \gamma\in\Gamma, f\in L^2(G),\ \mu\text{-a.e. } \chi\in\widehat{\Gamma}, \tag{4}$$

where by $L^2(\sigma_i,\mu)$ we mean (scalar) functions defined in $\widehat{\Gamma}$ with support in σ_i.

Note that, since Γ is countable, $T(f)(\chi)$ is identified with the sequence $((Tf)_i(\chi))_i \in \ell^2(\Gamma)$, and if $\chi\in\sigma_j\setminus\sigma_{j+1}$ all entries following position j are zero.

Furthermore (4) implies the following identity for $F\in L^2(\widehat{\Gamma},\ell^2(\Gamma),\mu)$, $f\in L^2(G)$, and the inner product in $\ell^2(\Gamma)$,

$$(F(\chi), T(\pi(\gamma)f)(\chi)) = \overline{(\gamma,\chi)}(F(\chi), Tf(\chi)), \quad \text{for all } \gamma\in\Gamma,\ \mu\text{-a.e. } \chi\in\widehat{\Gamma}. \tag{5}$$

As reported in the work by Aldroubi et al., [1], if there exists some $f\in L^2(G)$ such that the collection $\{\pi(\gamma)f,\ \gamma\in\Gamma\}$ is a frame for $L^2(G)$, then T maps $L^2(G)$ unitarily onto $L^2(\sigma_1,\mu)$.

We assume, since in general this is not the case, that μ is absolute continuous with respect to the Haar measure λ on $\widehat{\Gamma}$.

A closed subspace $V\subset L^2(G)$ is said π-invariant if $\pi(\gamma)V\subset V$, for all $\gamma\in\Gamma$. We use the following notation for a fixed $\psi\in L^2(G)$,

$$Y = \{\delta^j\,\pi(\gamma)\,\psi,\ j\in\mathbb{Z},\ \gamma\in\Gamma\},$$

$$V_k = \overline{\operatorname{span}}\{\delta^j\,\pi(\gamma)\,\psi,\ j<k,\ \gamma\in\Gamma\} = \delta^k(V_0), \quad k\in\mathbb{Z}.$$

The indicator function of any set A is denoted by I_A. All Hilbert spaces in this paper are separable.

3. Extension of Bownik and Speegle result

Results in this section extend some work by Bownik and Weber [8], and Bownik and Speegle [6] to abstract context. We include the proofs for sake of completeness.

Definition 3.1. The frame operator for a frame $(e_j)_{j\in J}$ in the Hilbert space $\mathcal{H}$ is

$$S:\mathcal{H}\to\mathcal{H}, \quad S(x) = \sum_{j\in J} < x, e_j > e_j.$$

It is a bounded, positive, invertible operator. The frame is a tight frame if and only if $S = AI$, where I is the identity operator. The frame is a Parseval frame iff $S = I$ i.e.

$$\sum_{j \in J} < x, e_j > e_j = x, \quad \text{for all } x \in \mathcal{H}.$$

Theorem 3.2. *If Y is a Parseval frame then for all $k \in \mathbb{N}$ the set*

$$V_k = \overline{span}\,\{\delta^j \pi(\gamma)\,\psi, j < k, \gamma \in \Gamma\}$$

is π-invariant.

Proof. Fix $k \geq 0$. Since Y is a Parseval frame, for all $f \in L^2(G)$, we have

$$\begin{aligned} f &= \sum_{j \geq k} \sum_{\gamma \in \Gamma} < f, \delta^j \pi(\gamma)\psi > \delta^j \pi(\gamma)\psi \\ &\quad + \sum_{j < k} \sum_{\gamma \in \Gamma} < f, \delta^j \pi(\gamma)\psi > \delta^j \pi(\gamma)\psi \\ &= B_1 f + B_2 f, \end{aligned}$$

where, by the frame property, the linear operators B_i are bounded.

Now, if $\eta \in \Gamma$, since k is positive, by (2),

$$\begin{aligned} \pi(\eta) B_1 f &= \sum_{j \geq k} \sum_{\gamma \in \Gamma} < f, \delta^j\, \pi(\gamma)\,\psi > \pi(\eta)\delta^j \pi(\gamma)\,\psi \\ &= \sum_{j \geq k} \sum_{\gamma \in \Gamma} < f, \delta^j \pi(\gamma)\,\psi > \delta^j \pi(\alpha^j(\eta))\pi(\gamma)\,\psi. \end{aligned}$$

If we set $\nu = \alpha^j(\eta)\gamma$ (on the other hand any ν can be written obviously as $\alpha^j(\eta)[\alpha^j(\eta)]^{-1}\nu$), and we use (2), by unitariness of π the latter is equal to

$$\begin{aligned} & \sum_{j \geq k} \sum_{\nu \in \Gamma} < f, \delta^j \pi([\alpha^j(\eta)]^{-1})\pi(\nu)\,\psi > \delta^j \pi(\nu)\,\psi \\ =\; & \sum_{j \geq k} \sum_{\nu \in \Gamma} < f, \pi(\eta)^* \delta^j \pi(\nu)\,\psi > \delta^j \pi(\nu)\,\psi \\ =\; & \sum_{j \geq k} \sum_{\nu \in \Gamma} < \pi(\eta) f, \delta^j \pi(\nu)\,\psi > \delta^j \pi(\nu)\,\psi \\ =\; & B_1(\pi(\eta) f). \end{aligned}$$

It follows that $B_2(L^2(G)\ )$ is π-invariant since

$$\pi(\eta)B_2 f = \pi(\eta)(f - B_1 f) = \pi(\eta)f - B_1(\pi(\eta)f) = B_2(\pi(\eta)f),$$

and so $\overline{B_2(L^2(G)\ )}$ is π-invariant as well.

Next we show that $\overline{B_2(L^2(G)\ )} = V_k$, from which we obtain that V_k is π-invariant. Indeed obviously $B_2(L^2(G)\ ) \subset V_k$ and so we get $\overline{B_2(L^2(G)\ )} \subset V_k$. Conversely, if $f \in B_2(L^2(G)\ )^{\perp}$, then

$$0 =< f, B_2 f >= \sum_{j<k} \sum_{\gamma\in\Gamma} | < f, \delta^j \pi(\gamma)\psi > |^2,$$

so $f \in V_k^{\perp}$ and everything is proved. $\square$

Lemma 3.3. *Assume $V \subset L^2(G)$ is a π-invariant closed subspace, and let P_V be the orthogonal projection onto V. Then, for any $\gamma \in \Gamma$, we have*

$$P_V \pi(\gamma) = \pi(\gamma) P_V.$$

Proof. Since V is π-invariant, then $V^{\perp}$ is π-invariant, too. Indeed, if $v \in V$ and $w \in V^{\perp}$ then

$$< \pi(\gamma)w, v >=< w, \pi(\gamma)^* v >=< w, \underbrace{\pi(\gamma^{-1})v}_{\in V} >= 0.$$

Let $u \in L^2(G)$ and write $u = v + w$, where $v \in V$ and $w \in V^{\perp}$. For any $\gamma \in \Gamma$ we have

$$\pi(\gamma)P_V u = \pi(\gamma)v = P_V(\pi(\gamma)v + \pi(\gamma)w) = P_V(\pi(\gamma)(v + w)) = P_V \pi(\gamma)u.$$

$\square$

Theorem 3.4. *Assume $V_0 = \overline{span}\{\delta^j \pi(\gamma)\psi, j < 0, \gamma \in \Gamma\}$ is π-invariant, $V_0 \neq V_1 = \delta(V_0)$. Assume that for any $0 \neq f \in L^2(G)$ the collection $\{\pi(\gamma)f,\ \gamma \in \Gamma\}$ is linearly independent.*

Then the affine system Y is linearly independent.

Proof. Suppose there exist a finite number of non-zero constants $c_{j,\gamma} \in \mathbb{C}$ such that

$$\sum_{j\in\mathbb{Z}} \sum_{\gamma\in\Gamma} c_{j,\gamma}\, \delta^j \pi(\gamma)\psi = 0. \tag{6}$$

By applying either δ or its inverse as many times as we need, we can suppose that the biggest j in the sum (6) such that $c_{j,\gamma} \neq 0$, for some $\gamma \in \Gamma$, is $j = 0$.

So (6) leads to

$$\sum_{\gamma\in\Gamma} c_{0,\gamma}\,\pi(\gamma)\psi = -\sum_{j<0}\sum_{\gamma\in\Gamma} c_{j,\gamma}\,\delta^j\pi(\gamma)\psi \in V_0.$$

If P_{V_0} is the orthogonal projection onto V_0, then, by Lemma 3.3

$$\begin{aligned} 0 &= (I - P_{V_0})[\sum_{\gamma\in\Gamma} c_{0,\gamma}\,\pi(\gamma)\psi] \\ &= \sum_{\gamma\in\Gamma} c_{0,\gamma}\,(I - P_{V_0})\pi(\gamma)\psi \\ &= \sum_{\gamma\in\Gamma} c_{0,\gamma}\,\pi(\gamma)(I - P_{V_0})\psi. \end{aligned} \tag{7}$$

Note that $\psi \notin V_0$, otherwise, since V_0 is π-invariant, we get $\pi(\gamma)\psi \in V_0$ for any $\gamma \in \Gamma$, and the contradiction

$$V_1 = \overline{\mathrm{span}}\,\{\delta^j\pi(\gamma)\psi, j < 1, \gamma \in \Gamma\} = \overline{\mathrm{span}}\,\{\delta^j\pi(\gamma)\psi, j \leq 0, \gamma \in \Gamma\} = V_0.$$

Therefore $(I - P_{V_0})\psi \neq 0$, and (7) leads to a contradiction of our hypothesis on linear independence. $\square$

The above theorem obviously holds if we only assume that for any $0 \neq f \in V_0^{\perp} \subset L^2(G)$ the collection

$$\{\pi(\gamma)f, \gamma \in \Gamma\}$$

is linearly independent. A closer look to its proof shows that it generalizes to more than one function, say $0 \neq \psi_1, \ldots, \psi_n \in L^2(G)$, assuming that the set

$$\{\pi(\gamma)(I - P_{V_0})\psi_i, \gamma \in \Gamma,\, i = 1, \ldots, n\}$$

is linearly independent and no ψ_i belongs to V_0.

This last remark is used in the following example, taken from [6], to show how the use of more general groups leads to results that cannot be reached just using the group $\mathbb{Z}$.

Example. Given $\varepsilon > 0$, let us define the function $\psi = \psi_0 + \varepsilon\psi_1$ where

$$\hat{\psi}_0 = \mathbf{1}_{[-1/4,-1/8]\cup[1/8,1/4]}, \quad \hat{\psi}_1 = \mathbf{1}_{[-1/2,-1/4]\cup[1/4,3/4]}.$$

We note that the system $\{D_{2^j}T_k\psi\}$ is a frame for sufficiently small $\varepsilon > 0$, even if this does not matter here.

The space of negative dilates is

$$\{f \in L^2(\mathbb{R})\ ,\ \mathrm{supp}\hat{f} \subset [-1/4, 3/8],\ \hat{f}(\xi - 1/2) = \hat{f}(\xi) \text{ for a.e. } \xi \in [1/4, 3/8]\}$$

which is $2\mathbb{Z}$-shift invariant but not shift invariant.

Thus Bownik and Speegle's theorem does not apply, while a direct calculation shows that $\{D_{2^j}T_k\psi\}$ is linearly independent.

On the other hand, we note that

$$\{D_{2^j}T_k\psi,\ j,k \in \mathbb{Z}\} = \{D_{2^j}T_{2k}\phi,\ j,k \in \mathbb{Z},\ \phi = \psi, T_1\psi\},$$

the space of negative dilates being obviously the same $2\mathbb{Z}$-shift invariant space. Furthermore both ψ and $T_1\psi$ do not belong to V_0.

If we prove that the set $\{T_{2k}(I - P_{V_0})\psi, T_{2k}(I - P_{V_0})T_1\psi,\ k \in \mathbb{Z}\}$ is linearly independent, we can apply the (generalization of the) above theorem with $\Gamma = 2\mathbb{Z}$, $\pi(2k) = T_{2k}$ to conclude that $\{D_{2^j}T_k\psi, j, k \in \mathbb{Z}\}$ is linearly independent.

Now an easy calculation shows that $(I - P_{V_0})\psi$ and $(I - P_{V_0})T_1\psi$ have Fourier transform, respectively equal to

$$\varepsilon\mathbf{1}_{[-1/2,-1/4]\cup[3/8,3/4]} + (\frac{1-\varepsilon}{2})(\mathbf{1}_{[-1/4,-1/8]} - \mathbf{1}_{[1/4,3/8]})$$

and

$$\varepsilon e^{-2\pi i\xi}\mathbf{1}_{[-1/2,-1/4]\cup[3/8,3/4]} + (\frac{1+\varepsilon}{2})e^{-2\pi i\xi}\mathbf{1}_{[-1/4,-1/8]\cup[1/4,3/8]}.$$

Hence, since the intervals $[-1/4, -1/8]$ and $[1/4, 3/8]$ have disjoint intersection with $[-1/2, -1/4] \cup [3/8, 3/4]$, the linear independence follows.

4. Invariant spaces and range functions

The purpose of this section is to provide a version of Helson's theorem, [13, Theorem 8], adapted to π-invariant spaces. In the case of translations, a proof can be found in the work by Bownik [4] for uniform lattices in $\mathbb{R}^n$, Cabrelli and Paternostro [9] for uniform lattices in LCA group, and Bownik and Ross [5] for not necessarily discrete subgroups.

Definition 4.1. Assume μ is a σ-finite measure on $\widehat{\Gamma}$. A range function is any map

$$J : \widehat{\Gamma} \to \{\text{closed subspaces of } \ell^2(\Gamma)\}.$$

J is said measurable if, denoted by $P(\chi)$ the orthogonal projection onto $J(\chi)$, for all $a, b \in \ell^2(\Gamma)$, the map

$$\chi \in \widehat{\Gamma} \mapsto (P(\chi)a, b) \in \mathbb{C},$$

is μ-measurable.

Range functions are identified if they are a.e. equal with respect to the measure μ on $\widehat{\Gamma}$.

Definition 4.2. Let J be a range function, let μ be a σ-finite measure on $\widehat{\Gamma}$. We define

$$M_J = \{F \in L^2(\widehat{\Gamma}, \ell^2(\Gamma), \mu),\ F(\chi) \in J(\chi),\ \mu\text{-a.e. } \chi \in \widehat{\Gamma}\}. \tag{8}$$

Remark 4.3. [13, p. 57][14, p.6, ex.2]

It is useful to recall that, for every sequence $F_n \in L^2(\widehat{\Gamma}, \ell^2(\Gamma), \mu)$, $n \in \mathbb{N}$, converging to $F \in L^2(\widehat{\Gamma}, \ell^2(\Gamma), \mu)$ in norm, there exists a subsequence F_{n_k}, $k \in \mathbb{N}$, converging to $F(\chi)$, pointwise a.e..

It follows that M_J is a closed subspace of $L^2(\widehat{\Gamma}, \ell^2(\Gamma), \mu)$.

The next lemma, proved in [13, p.58] for $\widehat{\Gamma} = \mathbb{T}$, extends, mutatis mutandis, to the general setting.

Lemma 4.4. *Let J be a measurable range function. Let M_J be the space defined by* (8)*. Let*

$$\mathscr{P} : L^2(\widehat{\Gamma}, \ell^2(\Gamma), \mu) \to L^2(\widehat{\Gamma}, \ell^2(\Gamma), \mu)$$

be the orthogonal projection onto M_J and, for any $\chi \in \widehat{\Gamma}$, let us denote by $P(\chi) : \ell^2(\Gamma) \to \ell^2(\Gamma)$ the orthogonal projection onto $J(\chi)$. Then, for any $F \in L^2(\widehat{\Gamma}, \ell^2(\Gamma), \mu)$, we have

$$(\mathscr{P}F)(\chi) = P(\chi)F(\chi), \quad \mu - a.e.\ \chi \in \widehat{\Gamma}. \tag{9}$$

Consider the unitary map defined in (3), arising from the spectral theorem applied to π, $T : L^2(G) \to \bigoplus_i L^2(\sigma_i, \mu) \hookrightarrow L^2(\widehat{\Gamma}, \ell^2(\Gamma), \mu)$.
The multiplicity function $m : \widehat{\Gamma} \to \{0, 1, \dots, +\infty\}$ is defined as $m(\chi) = \sharp\{\sigma_j,\ \chi \in \sigma_j\} = \sum_j I_{\sigma_j}(\chi)$.

As proved at the end of the theorem on characterization of spectral measures [14, p.17], the range of T is an M_J for a certain measurable range function J

$$T : L^2(G) \to M_J. \tag{10}$$

T commutes between the spectral measure Π on $\widehat{\Gamma}$ (whose domain is the σ-algebra of Borel sets and whose values are self-adjoint projections in $L^2(G)$) and the standard spectral measure (given by multiplication of characteristic functions of Borel sets), i.e. $T \circ \Pi(E)(f) = I_E\, T(f)$, for all $f \in L^2(G)$ and any Borel set $E \subset \widehat{\Gamma}$.

The next theorem, which generalizes Helson's theorem, says that the same T maps unitarily any π invariant subspace V onto a certain M_{J_V}.

We need a preliminary lemma.

Lemma 4.5. *Let $g : \widehat{\Gamma} \to \mathbb{C}$ be in $L^1(\widehat{\Gamma}, \mu)$, such that, for all $\gamma \in \Gamma$,*

$$\int_{\widehat{\Gamma}} (\gamma, \chi)\, g(\chi)\, d\mu(\chi) = 0,$$

where μ is a finite measure on Borel sets of $\widehat{\Gamma}$ which is absolutely continuous with respect to the Haar measure λ on $\widehat{\Gamma}$. Then for μ-almost all $\chi \in \widehat{\Gamma}$, we have $g(\chi) = 0$.

Proof. Let $h \in L^1(\widehat{\Gamma}, \lambda)$, be given by the Radon-Nikodým theorem, such that $\mu(E) = \int_E h(\chi)\, d\lambda(\chi)$, and

$$\int_{\widehat{\Gamma}} (\gamma, \chi)\, g(\chi) h(\chi)\, d\lambda(\chi) = 0, \quad \text{for all}\, \gamma \in \Gamma.$$

By Pontryagin duality theorem and Fourier uniqueness theorem, we get, for λ almost all $\chi \in \widehat{\Gamma}$, $g(\chi)h(\chi) = 0$.

Let us denote by $A \subset \widehat{\Gamma}$ the set where $g(\chi)h(\chi) \neq 0$, and by $B \subset \widehat{\Gamma}$ the set where $h(\chi) = 0$. Then $\lambda(A) = 0$ and so $\mu(A) = 0$ by absolute continuity. Also $\mu(B) = \int_B h(\chi)\, d\lambda(\chi) = 0$. But

$$g(\chi) \neq 0 \Longrightarrow \chi \in A \cup B,$$

so the result follows. $\square$

Theorem 4.6. *Let $V \subset L^2(G)$ be a π-invariant closed subspace, where $\pi : \Gamma \to \mathcal{U}(L^2(G))$ is a unitary representation. Then*

$$V = \{f \in L^2(G)\ ,\ Tf(\chi) \in J_V(\chi),\ \mu\text{-a.e.}\, \chi \in \widehat{\Gamma}\}, \tag{11}$$

where T is the unitary map in (3) *and J_V is a μ-measurable range function. The correspondence between V and J_V is one-to-one. Moreover*

$$V = \overline{span}\{\pi(\gamma)\varphi,\ \gamma \in \Gamma, \varphi \in \mathscr{A}\}, \tag{12}$$

for an at most countable set $\mathscr{A} \subset L^2(G)$, and for any such $\mathscr{A}$ verifying (12) *we have,*

$$J_V(\chi) = \overline{span}\{T\varphi(\chi),\ \varphi \in \mathscr{A}\}, \quad \mu\text{-a.e.}\, \chi \in \widehat{\Gamma}.$$

Proof. We omit the incessant reference to the measure μ, hence it is assumed in this proof that a.e. means μ-a.e..

In order to prove (11), we need to show that

$$T(V) = \{F \in L^2(\widehat{\Gamma}, \ell^2(\Gamma), \mu),\ F = Tf,\ f \in V\} = M_{J_V},$$

for a suitable measurable range function J_V. Indeed, if the latter is true, $f \in V$ implies $Tf \in T(V) = M_{J_V}$, which means, by definition, that $Tf(\chi) \in J_V(\chi)$, a.e.. Conversely, if $Tf(\chi) \in J_V(\chi)$, a.e., then $Tf \in M_{J_V} = T(V)$ and so $Tf = Tg$ for some $g \in V$, yielding $f = g$, since T is one to one.

Once we prove $T(V) = M_{J_V}$, the uniqueness of J_V comes from Lemma 4.4. Indeed, assume $T(V) = M_{J_V} = M_K$ for two measurable range functions. Let $\mathscr{P}$ be the orthogonal projection onto $T(V)$ and $P(\chi)$, $Q(\chi)$ be the orthogonal projections onto $J_V(\chi)$ and $K(\chi)$ respectively. Then Lemma 4.4 implies that for a.e. $\chi \in \widehat{\Gamma}$, and for all $F \in L^2(\widehat{\Gamma}, \ell^2(\Gamma), \mu)$

$$P(\chi)F(\chi) = (\mathscr{P}F)(\chi) = Q(\chi)F(\chi).$$

In particular, for any $a \in \ell^2(\Gamma)$,

$$P(\chi)a = P(\chi)P(\chi)a = Q(\chi)P(\chi)a, \quad \text{and} \quad Q(\chi)a = P(\chi)Q(\chi)a,$$

hence the range of $P(\chi)$ equals the range of $Q(\chi)$ that means a.e. $J_V(\chi) = K(\chi)$, i.e. $J_V = K$.

Now we prove (12).

Let $(e_n)_n$ be an orthonormal basis for $\ell^2(\Gamma)$. Since Γ is countable, let us denote by γ_k, $k \in \mathbb{Z}$, the elements of Γ. Let us consider the following elements in $L^2(\widehat{\Gamma}, \ell^2(\Gamma), \mu)$,

$$F_{k,n}(\chi) = (\gamma_k, \chi)\, e_n \in \ell^2(\Gamma).$$

We prove that $\overline{\text{span}}\{F_{k,n}, k, n \in \mathbb{Z}\} = L^2(\widehat{\Gamma}, \ell^2(\Gamma), \mu)$.

Indeed, if $F \in L^2(\widehat{\Gamma}, \ell^2(\Gamma), \mu)$ is such that $0 = (F_{k,n}, F)$ for all $k, n \in \mathbb{Z}$, then

$$0 = (F_{k,n}, F) = \int_{\widehat{\Gamma}} (F_{k,n}(\chi), F(\chi))\, d\mu(\chi) = \int_{\widehat{\Gamma}} (\gamma_k, \chi)(e_n, F(\chi))\, d\mu(\chi).$$

It follows that the L^1 function (n fixed)

$$\chi \in \widehat{\Gamma} \mapsto (e_n, F(\chi)) \in \mathbb{C},$$

verifies the hypotheses of Lemma 4.5, hence $(e_n, F(\chi)) = 0$ a.e. $\chi \in \widehat{\Gamma}$, for all $n \in \mathbb{Z}$. Since $(e_n)_n$ is an orthonormal basis of $\ell^2(\Gamma)$ we have $F(\chi) = 0$ a.e. $\chi \in \widehat{\Gamma}$, and so $F \equiv 0$, as desired.

Let $\mathcal{P}(F_{k,n})$ be the projection onto $T(V)$, then $\mathcal{P}(F_{k,n}) = T\varphi_{k,n}$, for some $\varphi_{k,n} \in V$.

By above, the set $\{\mathcal{P}(F_{k,n}) = T\varphi_{k,n},\ k, n \in \mathbb{Z}\}$ spans the range of $\mathcal{P}$, i.e. $T(V)$.

We claim that

$$V = \overline{\text{span}}\{\pi(\gamma)\varphi_{k,n},\ \gamma \in \Gamma, k, n \in \mathbb{Z}\}.$$

Indeed, since V is π-invariant, it is obvious that $\pi(\gamma)\varphi_{k,n} \in V$, so that

$$\overline{\text{span}}\{\pi(\gamma)\varphi_{k,n},\ \gamma \in \Gamma, k, n \in \mathbb{Z}\} \subset V.$$

On the other hand, if $f \in V$ such that $0 = (f, \pi(\gamma)\varphi_{k,n})$, for all $\gamma \in \Gamma, k, n \in \mathbb{Z}$, since T is a unitary map, by (5)

$$\begin{aligned} 0 &= (Tf, T(\pi(\gamma)\varphi_{k,n})) = \int_{\widehat{\Gamma}} (Tf(\chi), T(\pi(\gamma)\varphi_{k,n})(\chi))\, d\mu(\chi) \\ &= \int_{\widehat{\Gamma}} \overline{(\gamma, \chi)}(Tf(\chi), \mathcal{P}(F_{k,n})(\chi))\, d\mu(\chi). \end{aligned}$$

Again, by Lemma 4.5 (k, n fixed), $(Tf(\chi), \mathcal{P}(F_{k,n})(\chi)) = 0$ a.e. $\chi \in \widehat{\Gamma}$, for all $k, n \in \mathbb{Z}$. Hence

$$(Tf, \mathcal{P}(F_{k,n})) = \int_{\widehat{\Gamma}} (Tf(\chi), \mathcal{P}(F_{k,n})(\chi))\, d\mu(\chi) = 0.$$

It follows, since $\{\mathcal{P}(F_{k,n}) = T\varphi_{k,n},\ k,n \in \mathbb{Z}\}$ spans $\mathcal{P}(L^2(\widehat{\Gamma}, \ell^2(\Gamma), \mu) = T(V)$, that $f \equiv 0$, and so the claim is proved. Hence (12) is proved with $\mathscr{A}$ being the collection of all $\varphi_{k,n}$.

Next let $\mathscr{A}$ be an at most countable set verifying (12). Let us define the range function J_V as

$$J_V(\chi) = \overline{\operatorname{span}}\{T\varphi(\chi),\ \varphi \in \mathscr{A}\} \subset \ell^2(\Gamma), \quad \text{a.e. } \chi \in \widehat{\Gamma}.$$

We show that $T(V) = M_{J_V}$.

If $F \in T(V)$, let $f \in V$ such that $Tf = F$. Taken a sequence $f_n \in \operatorname{span}\{\pi(\gamma)\varphi,\ \gamma \in \Gamma, \varphi \in \mathscr{A}\}$ such that $f_n \to f$ in norm, it follows that $Tf_n \to Tf = F$.

Now, for any $\gamma \in \Gamma$, $\varphi \in \mathscr{A}$, and a.e. $\chi \in \widehat{\Gamma}$, (4) implies

$$T(\pi(\gamma)\varphi)(\chi) = (\gamma, \chi) T\varphi(\chi) \in J_V(\chi),$$

since $(\gamma, \chi) \in \mathbb{C}$. So a.e. also $Tf_n(\chi) \in J_V(\chi)$.

As we pointed in Remark 4.3, there exists a subsequence such that $Tf_{n_k} \to F$ a.e.. Since $J_V(\chi)$ is closed, this implies that $F(\chi) \in J_V(\chi)$, a.e. and so $F \in M_{J_V}$ as required.

Conversely, assume that $T(V) \subsetneqq M_{J_V}$. Then, there exists a non zero $F \in M_{J_V}$, such that $F \in T(V)^\perp$. This yields, for all $\gamma \in \Gamma$, and $\varphi \in \mathscr{A}$,

$$\begin{aligned} 0 &= \int_{\widehat{\Gamma}} (T(\pi(\gamma)\varphi)(\chi), F(\chi))\, d\mu(\chi) \\ &= \int_{\widehat{\Gamma}} (\gamma, \chi)(T\varphi(\chi), F(\chi))\, d\mu(\chi), \end{aligned}$$

and the same reasoning above yields $(T\varphi(\chi), F(\chi)) = 0$ for all $\varphi \in \mathscr{A}$, and a.e. $\chi \in \widehat{\Gamma}$. In particular, since $F(\chi) \in J_V(\chi)$, we get $(F(\chi), F(\chi)) = 0$ a.e., which implies the contradiction $\|F\|^2 = 0$.

Finally, it remains to show that J_V is measurable. Let $\mathscr{P}$ be the orthogonal projection onto $T(V) = M_{J_V}$, and $P(\chi)$ the orthogonal projection onto $J_V(\chi)$. By Lemma 4.4, if we take the constant function $F(\chi) = a \in \ell^2(\Gamma)$, for a fixed a, by (9) and every $b \in \ell^2(\Gamma)$,

$$(P(\chi)a, b) = (P(\chi)F(\chi), b) = ((\mathscr{P}F)(\chi), b).$$

The function $\chi \mapsto ((\mathscr{P}F)(\chi), b)$ is measurable since $\mathscr{P}F$ is, so for all $a, b \in \ell^2(\Gamma)$ the function $\chi \mapsto (P(\chi)a, b)$ is measurable and everything is proved. $\square$

Corollary 4.7. *Let $V \subset L^2(G)$ be a closed π-invariant subspace, $\varphi \in V$, and $f \in L^2(G)$. If T denotes the unitary map in* (3)*, suppose that for μ-almost all $\chi \in \widehat{\Gamma}$ there exists a constant $c(\chi) \in \mathbb{C}$ such that, for all i,*

$$[Tf]_i(\chi) = c(\chi)[T\varphi]_i(\chi). \tag{13}$$

Then $f \in V$.

Proof. Since V is π-invariant, for all $\gamma \in \Gamma$ we have $\pi(\gamma)\varphi \in V$, and

$$S := \overline{\text{span}}\{\pi(\gamma)\varphi,\ \gamma \in \Gamma\} \subset V.$$

Hence it suffices to prove that $f \in S$.

S is π-invariant so, by Theorem 4.6, we can write it in terms of its range function

$$S = \{g \in L^2(G)\ ,\ Tg(\chi) \in J_S(\chi),\ \mu\text{-}a.e.\ \chi \in \widehat{\Gamma}\}, \tag{14}$$

where, since S is generated by φ,

$$J_S(\chi) = \overline{\text{span}}\{T\varphi(\chi)\} = \{\lambda\, T\varphi(\chi),\ \lambda \in \mathbb{C}\}.$$

But (13) implies that, for μ-a.e. χ, $Tf(\chi) \in J_S(\chi)$ and so $f \in S$ by (14). □

Definition 4.8. Let $V \subset L^2(G)$ be a π-invariant closed space and let J_V be a range function associated with V as in (11) of Theorem 4.6. If $\mathscr{A}$ is a countable set verifying (12), we define, for μ-almost all $\chi \in \widehat{\Gamma}$,

$$\dim_V(\chi) = \dim J_V(\chi) = \dim \overline{\text{span}}\{T\psi(\chi),\, \psi \in \mathscr{A}\},$$

where the latter means dimension as a vector subspace in $\ell^2(\Gamma)$. If $\mathscr{A}$ is finite, we say that V is finitely generated. If $V = L^2(G)$ we simply write $\dim J(\chi)$.

An elementary argument about vector spaces is at the basis of the following proposition.

Proposition 4.9. *Let $V \subset L^2(G)$ be a π-invariant closed space and let J_V be a range function associated with V as in* (11) *of Theorem 4.6. Let $\mathscr{A}$ be a countable set verifying* (12)*. If V is finitely generated, then, for μ-almost all $\chi \in \widehat{\Gamma}$,* $\dim J_V(\chi) \leq \sharp\mathscr{A} < +\infty$.

5. The multiplicity function

This section is devoted to the multiplicity function and some basic formulas for it.

As we have already seen, any time we are given a unitary representation

$$\pi : \Gamma \to \mathcal{U}(L^2(G)),$$

we have a Borel measure μ on $\widehat{\Gamma}$, a unitary operator T defined as in (3), and an associated multiplicity function m.

Assume now $V \subset L^2(G)$ is invariant under π. It determines, in an obvious way, a unitary representation

$$\widetilde{\pi} : \Gamma \to \mathcal{U}(V), \quad \widetilde{\pi}(\gamma)f = \pi(\gamma)f, \quad f \in V,$$

called the subrepresentation of π on V.

As above, we get a Borel measure $\widetilde{\mu}$ on $\widehat{\Gamma}$, and measurable subsets

$$\dots \widetilde{\sigma}_i \subset \dots \subset \widetilde{\sigma}_2 \subset \widetilde{\sigma}_1 \subset \widehat{\Gamma},$$

a unitary map

$$\widetilde{T} : V \to \bigoplus_i L^2(\widetilde{\sigma}_i, \widetilde{\mu}) \hookrightarrow L^2(\widehat{\Gamma}, \ell^2(\Gamma), \widetilde{\mu}),$$

such that

$$[\widetilde{T}(\widetilde{\pi}(\gamma)f)]_i(\chi) = (\gamma, \chi)[\widetilde{T}(f)]_i(\chi), \quad \text{for all } \gamma \in \Gamma, f \in V, \ \widetilde{\mu}\,\text{a.e. } \chi \in \widehat{\Gamma}. \quad (15)$$

The multiplicity function $\widetilde{m} : \widehat{\Gamma} \to \{0, 1, \dots, +\infty\}$ is defined as

$$\widetilde{m}(\chi) = \sharp\{\sigma_j, \ \chi \in \sigma_j\} = \sum_j I_{\widetilde{\sigma}_j}(\chi).$$

Furthermore, we denote by $\widetilde{J}$ the range function such that $\widetilde{T} : V \to M_{\widetilde{J}}$, see (10).
We shall always affix the *tilde* to objects related to subrepresentations of π.

Remark 5.1. Note that, since $\widetilde{\pi}$ is the subrepresentation of π on V, the measure $\widetilde{\mu}$ is absolutely continuous with respect to μ (and hence absolute continuous with respect to the Haar measure on $\widehat{\Gamma}$).

This is a consequence of a general theorem on type I representation (see Theorem 3.26 in Führ's monograph [12]), but we can also prove it directly, and we choose to do so, since this allows to introduce some elements of spectral theory that we will need later.

We recall that, by the spectral theorem, see [14], representations π and $\widetilde{\pi}$, are linked respectively to the spectral measures Π and $\widetilde{\Pi}$, in the following way:

$$\pi(\gamma) = \int_{\widehat{\Gamma}} (\gamma, \chi)\, d\Pi(\chi), \quad \widetilde{\pi}(\gamma) = \int_{\widehat{\Gamma}} (\gamma, \chi)\, d\widetilde{\Pi}(\chi). \tag{16}$$

The meaning of (16) is that for any $f, g \in L^2(G)$, and $f', g' \in V$, we have

$$(\pi(\gamma)f, g) = \int_{\widehat{\Gamma}} (\gamma, \chi)\, dm_{f,g}(\chi), \quad (\widetilde{\pi}(\gamma)f', g') = \int_{\widehat{\Gamma}} (\gamma, \chi)\, d\widetilde{m}_{f',g'}(\chi), \tag{17}$$

where measures $m_{f,g}$ and $\widetilde{m}_{f',g'}$ are defined by

$$m_{f,g}(E) = (\Pi(E)f, g), \quad \widetilde{m}_{f',g'}(E) = (\widetilde{\Pi}(E)f', g'), \tag{18}$$

for all Borel sets $E \subset \widehat{\Gamma}$.

An application of Zorn's lemma yields that there exist $f \in L^2(G)$ and $g \in V$ such that, for any Borelian $E \subset \widehat{\Gamma}$,

$$\mu(E) = m_{f,f}(E) = (\Pi(E)f, f), \quad \widetilde{\mu}(E) = \widetilde{m}_{g,g}(E) = (\widetilde{\Pi}(E)g, g).$$

Moreover, for any other $h \in L^2(G)$ we have $m_{h,h} \ll \mu$, and, since $\widetilde{\pi}$ is the subrepresentation of π on V, we have also, by uniqueness of Fourier-Stieltjies transform,

$$\widetilde{\mu}(E) = (\widetilde{\Pi}(E)g, g) = (\Pi(E)g, g), \quad \text{since } g \in V.$$

Finally, if $\mu(E) = 0$ then $m_{g,g}(E) = 0$ and, by above, $\widetilde{\mu}(E) = 0$.

An additional consequence of Stone's theorem is the following

Lemma 5.2. *Suppose that π is a unitary representation of the abelian group Γ acting on a Hilbert space $\mathcal{H}$, and let ν and τ_i be, respectively, the Borel measure and the Borel measurable sets as in Stone's theorem.*

Suppose that τ'_j is another collection of, not necessarily nested, Borel subsets of $\widehat{\Gamma}$, and T' is a unitary operator,

$$T' : \mathcal{H} \to \bigoplus_j L^2(\tau'_j, \nu)$$

satisfying

$$[T'(\pi(\gamma)f)]_j(\chi) = (\gamma,\chi)[T'(f)]_j(\chi), \quad \textit{for all } \gamma \in \Gamma, f \in \mathcal{H},\ \nu \textit{ a.e. } \chi \in \widehat{\Gamma}.$$

Then, for $\nu-$*almost all* $\chi \in \widehat{\Gamma}$,

$$\sum_i I_{\tau_i}(\chi) = \sum_j I_{\tau'_j}(\chi).$$

Remark 5.3. We can recover, by the lemma above, that the multiplicity function of the representation π on $L^2(G)$ satisfies

$$m(\chi) = \dim J(\chi), \quad \mu\text{-a.e } \chi \in \widehat{\Gamma},$$

if J is the range function associated with the operator T as in (10).

Indeed it suffices to take $T' = T$, $\tau_i = \sigma_i$, and τ'_j being the set where $\dim J(\chi) = j$.

The same argument applies for a subrepresentation on a π-invariant subspace as well.

Remark 5.4. A Borel cross-section for the quotient map $q : \widehat{\Gamma} \to \widehat{\Gamma}/\ker\alpha^*$ is a Borel measurable right inverse for q, i.e. a map $\widetilde{s} : \widehat{\Gamma}/\ker\alpha^* \to \widehat{\Gamma}$ such that $q \circ \widetilde{s} = \mathrm{Id}_{\widehat{\Gamma}/\ker\alpha^*}$.

Since $\ker\alpha^*$ is closed, and $\widehat{\Gamma}$ is compact and metrizable (hence separable), a Borel cross-section for q exists by Mackey's result in [19, Lemma 1.1].

It follows that there exists a measurable map $s : \widehat{\Gamma} \to \widehat{\Gamma}$, such that $\alpha^*(s(\chi)) = \chi$.

Indeed, for any $\chi \in \widehat{\Gamma}$ let us denote by $[\chi]$ the corresponding equivalence class in $\widehat{\Gamma}/\ker\alpha^*$. Then $[\widetilde{s}([\chi])] = q(\widetilde{s}([\chi])) = [\chi]$ implies $\widetilde{s}([\chi])^{-1}\chi \in \ker\alpha^*$. It follows that there exists a unique element $\eta = \widetilde{s}([\chi])^{-1}\chi \in \ker\alpha^*$ such that $\chi = \widetilde{s}([\chi])\eta$.

Now let us define $s : \widehat{\Gamma} \to \widehat{\Gamma}$, as $s(\chi) = \widetilde{s}([\xi])$ if $\alpha^*(\xi) = \chi$. The map s is well defined since $\alpha^*(\xi) = \alpha^*(\xi')$ implies $[\xi] = [\xi']$. Finally we get, if $\eta \in \ker\alpha^*$ is the unique element such that $\xi = \widetilde{s}([\xi])\eta = s(\chi)\eta$,

$$\alpha^*(s(\chi)) = \alpha^*(\widetilde{s}([\xi])) = \alpha^*(\widetilde{s}([\xi])\eta) = \alpha^*(\xi) = \chi.$$

In the sequel we shall need the following formula, contained implicitly in [2], which generalizes the analogous formula obtained by Bownik and Rzeszotnik, in Corollary 2.5 of [7], for shift invariant spaces in $L^2(\mathbb{R}^n)$.

Lemma 5.5. *Assume we are given two closed subspaces $V, W \subset L^2(G)$, such that $W = \delta(V)$, where $\delta : L^2(G) \to L^2(G)$ is a unitary map verifying* (1)*. Suppose V is π-invariant. Let us denote by $\widetilde{\mu}$, $\widetilde{T}$, and $\widetilde{m}$ the usual objects given by Stone's theorem related to the subrepresentation of π on V and by $\mu^\sharp$, $T^\sharp$, and $m^\sharp$ the corresponding objects for W. Then we have, for $\mu^\sharp$-almost all $\chi \in \widehat{\Gamma}$,*

$$m^\sharp(\chi) = \sum_{\alpha^*(\xi)=\chi} \widetilde{m}(\xi). \tag{19}$$

Proof. Let $s : \widehat{\Gamma} \to \widehat{\Gamma}$ be the map linked to the Borel cross-section for the quotient map $q : \widehat{\Gamma} \to \widehat{\Gamma}/\ker \alpha^*$, as in Remark 5.4.

For any index i and any $\eta \in \ker \alpha^*$(recall that $|\ker \alpha^*| = N$), let us define a collection of Borel measurable sets by

$$\tau_{i,\eta} = \{\chi \in \widehat{\Gamma},\ s(\chi)\,\eta \in \widetilde{\sigma}_i\},$$

and the map

$$T' : \delta(V) \to \bigoplus_{i,\eta} L^2(\tau_{i,\eta}, \mu^\sharp),$$

by

$$[T'(f)]_{i,\eta}(\chi) = [\widetilde{T}(\delta^{-1}(f))]_i(s(\chi)\eta), \quad \chi \in \tau_{i,\eta}.$$

We have by (1), and (15), for $f \in \delta(V)$, $\chi \in \tau_{i,\eta}$,

$$\begin{aligned}
[T'(\pi(\gamma)f)]_{i,\eta}(\chi) &= [\widetilde{T}(\delta^{-1}(\pi(\gamma)f))]_i\,(s(\chi)\,\eta) \\
&= [\widetilde{T}(\pi(\alpha(\gamma))\delta^{-1}(f))]_i\,(s(\chi)\,\eta) \\
&= (\alpha(\gamma), s(\chi)\,\eta)\,[\widetilde{T}(\delta^{-1}(f))]_i\,(s(\chi)\,\eta) \\
&= (\gamma, \alpha^*(s(\chi)\,\eta))\,[T'(f)]_{i,\eta}(\chi) \\
&= (\gamma, \chi)\,[T'(f)]_{i,\eta}(\chi).
\end{aligned}$$

Hence, by Lemma 5.2, the multiplicity function $m^\sharp$ is given, $\mu^\sharp$-a.e. $\chi \in \widehat{\Gamma}$, by

$$\begin{aligned}
m^\sharp(\chi) &= \sum_{i,\eta} I_{\tau_{i,\eta}}(\chi) = \sum_{\eta}\sum_{i} I_{\widetilde{\sigma}_i}(s(\chi)\,\eta) \\
&= \sum_{\eta} \widetilde{m}(s(\chi)\,\eta) = \sum_{\alpha^*(\xi)=\chi} \widetilde{m}(\xi).
\end{aligned}$$

□

Any time we have a π-invariant subspace V, we are given two range functions: J_V by Theorem 4.6, and $\widetilde{J}$. It is worth to compare the respective dimensions $\dim J_V(\chi)$ and $\dim \widetilde{J}(\chi)$.

In the following proposition we use the fact that $\widetilde{\mu}$ is absolutely continuous with respect to μ to show that a relation always exists.

Proposition 5.6. *Let $V \subset L^2(G)$ be a π-invariant subspace, and J_V be the range function associated with V, provided by Theorem 4.6. Consider the subrepresentation of π on V, $\widetilde{\pi} : \Gamma \to \mathcal{U}(V)$, the range function $\widetilde{J}$ associated with the unitary map $\widetilde{T}$, and the multiplicity function $\widetilde{m}$.*

Then for $\widetilde{\mu}$ almost all $\chi \in \widehat{\Gamma}$, $\widetilde{m}(\chi) = \dim \widetilde{J}(\chi) \leq \dim J_V(\chi)$.

Proof. By Helson's theorem, Theorem 4.6, we get $T(V) = M_{J_V}$. Consider the unitary map $\widetilde{T} : V \to M_{\widetilde{J}}$ as in (10). The composition $T_{|V} \circ \widetilde{T}^{-1} : M_{\widetilde{J}} \to M_{J_V}$ is a unitary map. We call it $T \circ \widetilde{T}^{-1}$ for short.

Consider $n \in \mathbb{N}$, and the set $\widetilde{\tau}_n$ where

$$\widetilde{m}(\chi) = \dim \widetilde{J}(\chi) = n.$$

Let $F_1, \dots, F_n \in M_{\widetilde{J}}$, be pointwise orthonormal $\widetilde{\mu}$ a.e. on $\widetilde{\tau}_n$ and vanishing out of it (see [14] p.12 problem 4). We aim to show that pointwise orthonormality of $F_1(\chi), \dots, F_n(\chi) \in \widetilde{J}(\chi)$ implies pointwise orthonormality of a same number of elements in $J_V(\chi)$. By (3) we have

$$\begin{aligned} & \int_{\widehat{\Gamma}} (\gamma, \chi)\, (T \circ \widetilde{T}^{-1}(F_j)(\chi), T \circ \widetilde{T}^{-1}(F_h)(\chi))\, d\mu(\chi) \qquad (20) \\ = & \int_{\widehat{\Gamma}} (T(\pi(\gamma)\widetilde{T}^{-1}F_j)(\chi), T(\widetilde{T}^{-1}F_h)(\chi))\, d\mu(\chi). \end{aligned}$$

Since T is a unitary operator, by recalling the definition of inner product in the vector space $L^2(\widehat{\Gamma}, \ell^2(\Gamma), \mu)$, the latter is equal to

$$(\pi(\gamma)\, \widetilde{T}^{-1}F_j, \widetilde{T}^{-1}F_h) = (\widetilde{\pi}(\gamma)\, \widetilde{T}^{-1}F_j, \widetilde{T}^{-1}F_h) = \int_{\widehat{\Gamma}} (\gamma, \chi)\, d\widetilde{\mu}_{\widetilde{T}^{-1}F_j, \widetilde{T}^{-1}F_h}(\chi),$$

see (17). The measure $\widetilde{\mu}_{\widetilde{T}^{-1}F_j, \widetilde{T}^{-1}F_h}$ is defined in any Borel set $E \subset \widehat{\Gamma}$, in terms of the spectral measure $\widetilde{\Pi}$ associated with $\widetilde{\mu}$, see (18), by

$$\begin{aligned} \widetilde{\mu}_{\widetilde{T}^{-1}F_j, \widetilde{T}^{-1}F_h}(E) &= (\widetilde{\Pi}(E)\, \widetilde{T}^{-1}F_j, \widetilde{T}^{-1}F_h) \\ &= (\widetilde{T}^{-1}(I_E\, F_j), \widetilde{T}^{-1}(F_h)) \\ &= (I_E\, F_j, F_h), \end{aligned}$$

the last two lines justified by the commuting properties of $\widetilde{T}^{-1}$ between $\widetilde{\Pi}$ and the standard spectral measure, and the unitariness of the operator $\widetilde{T}^{-1}$.

But, for our choice of $F_1, \dots, F_n \in M_{\widetilde{J}}$,

$$\begin{aligned}(I_E\, F_j, F_h) &= \int_E (F_j(\chi), F_h(\chi))\, d\widetilde{\mu}(\chi) = \int_{E\cap\widetilde{\tau}_n} (F_j(\chi), F_h(\chi))\, d\widetilde{\mu}(\chi) \\ &= \delta_{j,h}\, \widetilde{\mu}(E\cap\tau_n).\end{aligned}$$

Hence the measure $\widetilde{\mu}_{\widetilde{T}^{-1}F_j, \widetilde{T}^{-1}F_h}$ is identically zero whenever $j \neq h$, while for $j = h$ is nothing else that $\widetilde{\mu}$ restricted to $\widetilde{\tau}_n$.

It follows that (20) is identically zero whenever $j \neq h$, while for $j = h$, if we call w the nonnegative measurable function on $\widehat{\Gamma}$, provided by the Radon-Nikodym Theorem, such that $\widetilde{\mu}(E) = \int_E w(\chi)\, d\mu(\chi)$, for any Borel set E, (20) is equal to

$$\int_{\widehat{\Gamma}} (\gamma, \chi)\, \|T \circ \widetilde{T}^{-1}(F_j)(\chi)\|^2\, d\mu(\chi) = \int_{\widehat{\Gamma}} (\gamma, \chi)\; I_{\widetilde{\tau}_n}(\chi)\; w(\chi)\; d\mu(\chi).$$

By Lemma 4.5, we get for μ (and hence $\widetilde{\mu}$) almost all $\chi \in \widetilde{\tau}_n$,

$$(T \circ \widetilde{T}^{-1}(F_j)(\chi), T \circ \widetilde{T}^{-1}(F_h)(\chi)) = \delta_{j,h}\, w(\chi).$$

The set where $w(\chi) = 0$ has zero measure with respect to $\widetilde{\mu}$, so we get for $\widetilde{\mu}$ almost all $\chi \in \widetilde{\tau}_n$, $0 \not\equiv T \circ \widetilde{T}^{-1}(F_j)(\chi) \in J_V(\chi)$, and $\dim J_V(\chi) \geq n$.

The same argument shows that in the set where $\dim \widetilde{J}(\chi) = +\infty$ then also $\dim J_V(\chi) = +\infty$, (see [14, Theorem 2, p.8]), hence we get, $\widetilde{\mu}$ a.e., $\dim J_V(\chi) \geq \dim \widetilde{J}(\chi)$, as desired. □

6. Main results

We first briefly discuss the linear independence of translates of one function $\{T_\gamma f,\, \gamma \in \Gamma\}$, compared to linear independence of $\{\pi(\gamma) f,\, \gamma \in \Gamma\}$,(recall that $T_\gamma f = f(\cdot - \gamma)$).

In [18, Corollary 4.3.14] Kutyniok has proved, among other things, that, for any countable set $\Gamma \subset G$, the set of (left) translates $\{T_\gamma f,\, \gamma \in \Gamma\}$ is linearly independent for any $0 \neq f \in L^2(G)$, if and only if, for any finite subset $\Lambda \subset \Gamma$, and for any $(c_\gamma)_{\gamma\in\Lambda} \subset \mathbb{C}$, $(c_\gamma)_{\gamma\in\Lambda} \neq 0$, we have

$$\sum_{\gamma\in\Lambda} c_\gamma\, (\gamma, \chi) \neq 0, \quad a.e.\ \chi \in \widehat{G},$$

where a.e. means with respect the Haar measure on $\widehat{G}$.

It turns out that the above equivalence still holds if the countable set $\Gamma \subset G$ and the left translation are replaced, respectively, by a countable, closed subgroup and a unitary representation with corresponding measure μ absolutely continuous with respect the Haar measure, as shown in the following lemma.

Lemma 6.1. *Let G be a locally compact abelian group. Let $\Gamma \subset G$ be a countable closed subgroup. Let λ be the Haar measure on $\widehat{\Gamma}$.*

Then the following conditions are equivalent.

(i) The set of (left) translates $\{T_\gamma f,\, \gamma \in \Gamma\}$ is linearly independent for any $0 \neq f \in L^2(G)$;

(ii) For any finite set $\Lambda \subset \Gamma$, and for any $(c_\gamma)_{\gamma\in\Lambda} \subset \mathbb{C}$, $(c_\gamma)_{\gamma\in\Lambda} \neq 0$, we have
$$\sum_{\gamma\in\Lambda} c_\gamma\,(\gamma,\chi) \neq 0, \quad \lambda - a.e.\ \chi \in \widehat{\Gamma};$$

(iii) For any unitary representation $\pi : \Gamma \to \mathcal{U}(L^2(G))$, with corresponding measure $\mu \ll \lambda$, and for any $0 \neq f \in L^2(G)$, the set $\{\pi(\gamma)f,\, \gamma \in \Gamma\}$ is linearly independent.

Proof. (i) implies (ii) is part of the statement of Corollary 4.3.14 in [18] together with the observation that the hypotheses on Γ guarantee that any character of Γ extends to a character of G, [21]. Obviously (iii) implies (i), so we need to show (ii) implies (iii). To this purpose we recall the unitary operator T defined in (3) and associated with π. Let $0 \neq f \in L^2(G)$, and $F \subset \Gamma$ be a finite subset. If $(c_\gamma)_{\gamma\in F} \subset \mathbb{C}$, $(c_\gamma)_{\gamma\in F} \neq 0$, is such that $\sum_{\gamma\in F} c_\gamma \pi(\gamma) f \equiv 0$, we get that $0 \equiv \sum_{\gamma\in F} c_\gamma\, T(\pi(\gamma)f)$ implies

$$\left(\sum_{\gamma\in F} c_\gamma\,(\gamma,\chi)\right)[Tf]_i(\chi) = \sum_{\gamma\in F} c_\gamma\,[T(\pi(\gamma)f)]_i(\chi) = 0, \quad \mu - \text{a.e.}\ \chi \in \widehat{\Gamma},$$

for any index i arising in the definition of T. By (ii) and absolute continuity of the measure, the sum $\left(\sum_{\gamma\in F} c_\gamma\,(\gamma,\chi)\right)$ is μ-a.e. different from zero, yielding $[Tf]_i(\chi) = 0$ for all i and μ-almost all $\chi \in \widehat{\Gamma}$. This implies $f \equiv 0$, since T is unitary, against the assumption $f \neq 0$. $\square$

We now return to questions about linear independence of the affine system Y and the role played by the endomorphism α defined in Section 2.

A little technical lemma anticipates one of the main results, which explores the behavior of the space V_0 of negative translates, as in [6, Theorem 3.4].

Lemma 6.2. *Under the hypotheses* (1) *on* α *and* $\alpha(\Gamma)$, *there exists a finite set* $\nu_1, \dots, \nu_N \in \Gamma$ *with the following property: for any* $\gamma \in \Gamma$, *and for any* $j \in \mathbb{N}$, *there exists* $\eta \in \Gamma$ *such that* $\gamma = \alpha^j(\eta)\nu_i$ *for some* $i = 1, \dots, N$.

Proof. By a recursive argument it is sufficient to consider $j = 1$.

Assume

$$|\Gamma/\alpha(\Gamma)| = N,$$

and let $\nu_1, \nu_2, \dots, \nu_N \in \Gamma$ be a complete set of coset representatives. If $\gamma \in \Gamma$, let $i = 1, \dots, N$ such that $[\gamma] = [\nu_i]$. Then $\gamma {\nu_i}^{-1} \in \alpha(\Gamma)$ and so $\gamma = \alpha(\eta)\nu_i$ for some $\eta \in \Gamma$. ☐

As in the previous section we denote by T, μ and m the usual objects linked to the representation π.

The following remark is crucial in the proof of what follows.

We observe that the compatibility condition (2) implies that, for any $M \in \mathbb{N}$, the representation $\pi \circ \alpha^M : \Gamma \to \mathcal{U}(L^2(G))$, is unitarily equivalent to π, being $\delta^M : L^2(G) \to L^2(G)$ the intertwining operator. It follows that the corresponding measures μ and, say μ_M, are equivalent and the multiplicity functions agree up to a set of measure 0 (with respect to either measure). In particular, μ_M is absolutely continuous with respect to the Haar measure on $\widehat{\Gamma}$.

The compatibility condition (1) is satisfied too, since π does,

$$\delta^{-1}\pi(\alpha^M(\gamma))\delta = \pi(\alpha(\alpha^M(\gamma))) = \pi \circ \alpha^M(\alpha(\gamma)).$$

We can conclude that all results of this paper so far hold for the representation $\pi \circ \alpha^M$ as well.

Theorem 6.3. *Assume hypothesis* **(A)**.

If the system $Y = \{\delta^j \pi(\gamma)\psi,\ j \in \mathbb{Z}, \gamma \in \Gamma\}$ *is linearly dependent, then* $V_0 = \overline{span}\{\delta^j \pi(\gamma)\psi,\ j < 0, \gamma \in \Gamma\}$ *is* $\pi \circ \alpha^M$*-invariant for some* $M \geq 0$.

Moreover, if we consider the unitary representation

$$\pi_M := \pi \circ \alpha^M : \Gamma \to \mathcal{U}(L^2(G)),$$

and the subrepresentation $\widetilde{\pi_M}$ on V_0, with corresponding measure $\widetilde{\mu_M}$ and multiplicity function $\widetilde{m_M}$, we have

$$\widetilde{m_M}(\chi) < +\infty, \quad \widetilde{\mu_M} - a.e.\, \chi \in \widehat{\Gamma}.$$

Proof. The first part follows the proof of Theorem 3.4 in [6].

If the system Y is linearly dependent, there exists a finite set $F \subset \Gamma$ and a finite non zero sequence $c_{j,\gamma} \in \mathbb{C}$, $j \in \mathbb{Z}$, $\gamma \in F$, such that

$$0 = \sum_{j \in \mathbb{Z}} \sum_{\gamma \in F} c_{j,\gamma}\, \delta^j\, \pi(\gamma)\psi. \tag{21}$$

After several applications of δ, we can assume that the smallest j in the sum (21) is 0. Call the largest M. Hence

$$0 = \sum_{j=0}^{M} \sum_{\gamma \in F} c_{j,\gamma}\, \delta^j\, \pi(\gamma)\psi.$$

We define

$$f := \sum_{\gamma \in F} c_{0,\gamma}\, \pi(\gamma)\psi = -\sum_{j=1}^{M} \sum_{\gamma \in F} c_{j,\gamma}\, \delta^j\, \pi(\gamma)\psi. \tag{22}$$

For any $h, k \in \mathbb{Z}$, we consider the subspaces in $L^2(G)$

$$V_{h,k} = \overline{\mathrm{span}}\{\delta^j\, \pi(\gamma)\psi,\ h \le j \le k,\ \gamma \in \Gamma\},$$

and we note first that $f \in V_{1,M}$, and $\delta(V_{h,k}) = V_{h+1,k+1}$; secondly, each $V_{h,k}$ is π-invariant whenever $h \ge 0$. Indeed, by (2),

$$\pi(\eta)\delta^j\, \pi(\gamma)\psi = \delta^j \pi(\alpha^j(\eta))\, \pi(\gamma)\psi = \delta^j \pi(\alpha^j(\eta)\gamma)\psi \in V_{h,k}.$$

By (22) and (4), we get, for all i and μ-a.e. $\chi \in \sigma_i$,

$$[Tf]_i(\chi) = \sum_{\gamma \in \Gamma} c_{0,\gamma}\, [T(\pi(\gamma)\psi)]_i(\chi) = \left(\sum_{\gamma \in \Gamma} c_{0,\gamma}\, (\gamma, \chi) \right) [T(\psi)]_i(\chi).$$

By (ii) of Lemma 6.1, the hypothesis (**A**) on linear independence of translates implies that

$$\sum_{\gamma \in \Gamma} c_{0,\gamma}\, (\gamma, \chi) \neq 0, \quad \lambda\text{-a.e.}\, \chi \in \widehat{\Gamma},$$

(λ is the Haar measure) hence, by absolute continuity, μ-a.e.

$$[T(\psi)]_i(\chi) = \frac{1}{\sum_{\gamma\in\Gamma} c_{0,\gamma}(\gamma,\chi)}\,[Tf]_i(\chi),$$

and we obtain, by Corollary 4.7, that $\psi \in V_{1,M}$.

Therefore, since $V_{1,M}$ is π-invariant, $\{\pi(\gamma)\psi,\, \gamma \in \Gamma\} \subset V_{1,M}$, yielding $V_{0,M} \subset V_{1,M}$, and so $V_{1,M} = V_{0,M}$. By several application of δ^k, we get also $V_{k+1,M+k} = V_{k,M+k}$. The argument goes on as in the proof of [6], we include it for completeness. By induction it is proved that

$$V_{r,M} = V_{1,M}, \quad \text{for all}\, r \leq 0.$$

Indeed, by above the statement is true for $r = 0$. Suppose it is true for $r+1 \leq 0$, and consider $r \leq 0$, then obviously $V_{r+1,M+r} \subset V_{r+1,M}$ and we have

$$V_{r,M+r} = V_{r+1,M+r} \subset V_{r+1,M} = V_{1,M},$$

the latter equality being the induction hypothesis.

So the inclusion

$$\begin{aligned}\{\delta^j\pi(\gamma)\psi,\, r \leq j \leq M,\, \gamma \in \Gamma\} \quad \subset \quad & \{\delta^j\pi(\gamma)\psi,\, r \leq j \leq M+r,\, \gamma \in \Gamma\} \cup \\ & \{\delta^j\pi(\gamma)\psi,\, r+1 \leq j \leq M,\, \gamma \in \Gamma\}\end{aligned}$$

implies, since $r \leq 0$,

$$\begin{aligned}V_{r,M} \quad &= \quad \overline{\text{span}}\{\delta^j\pi(\gamma)\psi,\, r \leq j \leq M,\, \gamma \in \Gamma\} \\ &\subset \quad V_{1,M} \cup V_{r+1,M} = V_{1,M} \cup V_{1,M} = V_{1,M} \subset V_{r,M},\end{aligned}$$

as needed.

Hence we obtain

$$V_{M+1} = \overline{\text{span}}\{\delta^j\pi(\gamma)\psi,\, j \leq M,\, \gamma \in \Gamma\} = \bigcup_{r\leq 0} V_{r,M} = V_{1,M},$$

and $V_0 = \delta^{-(M+1)}(V_{M+1}) = V_{-M,-1}$.

Now we prove that V_0 is $\pi \circ \alpha^M$-invariant. Indeed, for $-M \leq j \leq -1$, and $\eta, \nu \in \Gamma$, by equality (2) we have

$$\begin{aligned}\pi \circ \alpha^M(\eta)(\delta^j\pi(\nu)\psi) \quad &= \quad \delta^{-M}\pi(\eta)\delta^M(\delta^j\pi(\nu)\psi) = \delta^{-M}\pi(\eta)\delta^{M+j}\pi(\nu)(\psi) \\ &= \quad \delta^{-M}\delta^{M+j}\pi(\alpha^{M+j}(\eta))\pi(\nu)(\psi) = \delta^j\pi(\alpha^{M+j}(\eta)\nu)(\psi),\end{aligned}$$

and the latter is again in $V_{-M,-1} = V_0$. Furthermore, by Lemma 6.2, any element of Γ is of the form $\alpha^{M+j}(\eta)\nu$, for ν varying in a finite set and $\eta \in \Gamma$, hence by the above equality we get also that

$$V_0 = \overline{\operatorname{span}}\{\pi \circ \alpha^M(\eta)(\delta^j \pi(\nu_i)\psi),\ -M \le j \le -1,\ i = 1, \dots, N,\ \eta \in \Gamma\}.$$

Finally let us consider the unitary representation $\pi \circ \alpha^M : \Gamma \to \mathcal{U}(L^2(G))$, the associate unitary map T_M as defined in (3) together with the measure μ_M. By Theorem 4.6, if $J_{V_0,M}$ denotes a μ_M measurable range function corresponding to V_0 we have, μ_M-a.e. $\chi \in \widehat{\Gamma}$,

$$J_{V_0,M}(\chi) = \overline{\operatorname{span}}\{T_M(\delta^j \pi(\nu_i)\psi)(\chi),\ -M \le j \le -1,\ i = 1, \dots, N\}.$$

By Proposition 4.9, we have $\dim J_{V_0,M}(\chi) < +\infty$, and so, by Proposition 5.6, we get $\widetilde{m_M}(\chi) < +\infty$, $\widetilde{\mu_M}$-a.e., as required. $\square$

Theorem 6.4. *Let π be a unitary representation of Γ on $L^2(G)$ verifying* (1)*, and suppose that hypothesis* **(A)** *holds true. Assume $V \subset L^2(G)$ is π-invariant and $V = \delta V$.*

Let μ and $\widetilde{\mu}$ denote the obvious measures and assume that μ is absolutely continuous with respect the Haar measure on $\widehat{\Gamma}$. Let $\widetilde{m}$ be the multiplicity function associated with the subrepresentation of π on V. Then we have, for $\widetilde{\mu}$-almost all $\chi \in \widehat{\Gamma}$, $\widetilde{m}(\chi) = +\infty$.

Proof. Since $V = \delta V$, the multiplicity function, $m^\sharp$, associated with the subrepresentation of π on δV coincides with $\widetilde{m}$. Hence, by (19) we get $\widetilde{\mu}$-almost all $\chi \in \widehat{\Gamma}$,

$$\widetilde{m}(\chi) = m^\sharp(\chi) = \sum_{\alpha^*(\xi)=\chi} \widetilde{m}(\xi). \tag{23}$$

The proof follows now the same standard ergodic argument as in [6, Lemma 3.5].

Let $E = \{\chi \in \widehat{\Gamma},\ \widetilde{m}(\chi) \ge 1\}$, then $\alpha^*(E) \subset E$, i.e. $E \subset (\alpha^*)^{-1}(E)$. Since α^* is measure-preserving, in the sense that $\lambda((\alpha^*)^{-1}(E)) = \lambda(E)$, λ being the Haar measure on $\widehat{\Gamma}$, we have that $E = (\alpha^*)^{-1}(E)$ modulo null-sets. Since α^* is ergodic we must have either $\lambda(E) = 0$ or $\lambda(E) = 1$, and since $V \neq \{0\}$, it follows that $\lambda(E) = 1$.

So $\widetilde{m}(\chi) \ge 1$ for a.e. $\chi \in \widehat{\Gamma}$. From (23) above and the fact that all elements ξ verifying $\alpha^*(\xi) = \chi$ yield the same coset in $\widehat{\Gamma}/\ker \alpha^*$, it follows first that

$\widetilde{m}(\chi) \geq N > 1$ for a.e. $\chi \in \widehat{\Gamma}$, $(N = |\ker \alpha^*|)$ and secondly $\widetilde{m}(\chi) = +\infty$ for a.e. $\chi \in \widehat{\Gamma}$. The hypothesis $\mu \ll \lambda$ and the fact that $\widetilde{\mu} \ll \mu$, see Remark 5.1, complete the proof. $\square$

Proof of Theorem 1 By hypothesis V_0 is π-invariant.
If $V_0 \neq V_1 = \delta(V_0)$ then Theorem 3.4 yields the linear independence of Y.
If $V_0 = V_1$, then the restriction of δ to V_0, say δ_{V_0}, is a unitary map onto V_0.

If we assume that Y is linear dependent, by Theorem 6.3 there exists an $M \geq 0$ such that V_0 is $\pi_M := \pi \circ \alpha^M$-invariant, and the multiplicity function $\widetilde{m_M}$ verifies $\widetilde{m_M}(\chi) < +\infty$, for $\widetilde{\mu_M}$-almost all $\chi \in \widehat{\Gamma}$.

But the π_M-invariance implies that the subrepresentations $\widetilde{\pi_M}$ and $\widetilde{\pi}$ on V_0 are equivalent, $\delta_{V_0}^M$ being the intertwining operator, and so the corresponding measures $\widetilde{\mu_M}$ and $\widetilde{\mu}$ are equivalent. Furthermore the multiplicity functions agree up to a set of measure 0 (with respect to either measure), so $\widetilde{m}(\chi) < +\infty$, for $\widetilde{\mu}$-almost all $\chi \in \widehat{\Gamma}$, leading to a contradiction of Theorem 6.4. $\square$

Proof of Theorem 2
It follows by Theorem 3.2 and Theorem 1. $\square$

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