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Research Paper

Approximation of the Hilbert transform on the half–line

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ABSTRACT

The paper concerns the weighted Hilbert transform of locally continuous functions on the semiaxis. By using a filtered de la Vallée Poussin type approximation polynomial recently introduced by the authors, it is proposed a new "truncated" product quadrature rule (VP- rule). Several error estimates are given for different smoothness degrees of the integrand ensuring the uniform convergence in Zygmund and Sobolev spaces. Moreover, new estimates are proved for the weighted Hilbert transform and for its approximation (L-rule) by means of the truncated Lagrange interpolation at the same Laguerre zeros. The theoretical results are validated by the numerical experiments that show a better performance of the VP-rule versus the L-rule.

1. Introduction

Hilbert transforms arise in several mathematical problems of applied sciences, such as signal processing, image analysis, optics, electrodynamics, fluid mechanics, etc. (see e.g. [18, Vol I, II] and the references therein). They can also be the main part of singular integral equations [36,27,17,32], as well as their derivatives can appear in some kind of hypersingular integral equations (for instance [1,16,3,29,12,9]), where in turn both singular and hypersingular integral equations are tools for modeling several physical problems [30,20,19].

Here we consider the weighted Hilbert transform of f on the positive semiaxis,

$$\mathcal{H}(f,t) = \int_0^\infty \frac{f(x)}{x-t} w_\alpha(x)dx = \lim_{\epsilon\to 0}\left\{\int_0^{t-\epsilon} + \int_{t+\epsilon}^\infty\right\} \frac{f(x)}{x-t} w_\alpha(x)dx, \; t>0, \tag{1}$$

being $w_\alpha(x) = e^{-x}x^\alpha$ the Laguerre weight of parameter $\alpha \geq 0$.

The case we treat here involves function f which are locally continuous on $(0,\infty)$, with a possible algebraic singularity in 0, and/or exponentially growing as $x \to \infty$.

In view of the relevance of the topic, there is a wide literature dealing with numerical methods to approximate Hilbert transforms, especially in the case of bounded intervals (see e.g. [13,28,2,5,4,15] and the references therein), but also over unbounded regions (see e.g. [6,7,11,14,31] and the references therein).

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As far as approximation methods based on orthogonal polynomials are concerned, we recall the product integration rule based on truncated Lagrange polynomials interpolating f at Laguerre zeros (L-rule) [7]. Here we introduce a product integration rule, obtained approximating f by discrete de la Vallée Poussin polynomials $V_n^m(w_\alpha, f)$, introduced and studied in [35]. The new rule, shortly denoted by VP-rule, is a truncated product rule again, of not interpolatory type, which uses the same number of samples of the L-rule. The truncation of the rule not only reduces the number of function evaluations, but it is crucial to "fight" or delay, possible overflows arising when f exponentially grows.

One of the advantages of the VP rule is due to the presence of the additional parameter m $(1 \le m \le n-1)$, which can be suitably modulated in order to improve the quadrature error of the L-rule. This improvement is especially appreciable in case of density functions f presenting isolated "pathologies" (peaks, cusps, etc.). Indeed, in such cases, the Lagrange polynomial is affected by the Gibbs phenomenon, and shows oscillations also "far" from the pathological point. Also in $[-1,1]$ a similar behavior is observed for filtered quadrature rules [33], obtained by de la Vallée Poussin approximation w.r.t. Jacobi polynomials (see e.g. [39,34,40]).

The paper is organized as follows. In Section 2 we recall some basic results about the approximation spaces and the polynomial tools we will employ. Section 3 contains a study on the mapping properties of the Hilbert transform in such spaces of functions, while in Section 4 it is introduced the VP rule, with some computational details for its implementation. Error estimates for both VP and L-rules are provided in Section 5, while Section 6 contains some experimental results. In Appendix are given the proofs of some technical lemmas, moved here to improve the readability of paper.

2. Notation and preliminary results

Throughout the paper, C denotes a positive constant that may have different meaning in different formulas. In particular, we write $C \ne C(a,b,\dots)$ in order to say that C is independent of the parameters $a,b,\dots$, and $C = C(a,b,\dots)$ to say that C depends on $a,b,\dots$. Moreover, if $A,B>0$ are quantities depending on some parameters, we write $A \sim B$, if there exists a constant $0 < C \ne C(A,B)$ such that $C^{-1} \le A/B \le C$.

If $D \subset [0,+\infty)$ then $C^0(D)$ denotes the space of all continuous functions on D and $\|f\|_D = \sup_{x\in D}|f(x)|$.

For any $\gamma \ge 0$ we set $u_\gamma(x) = x^\gamma e^{-x/2}$ and denote by C_{u_γ} the following functional space

$$C_{u_\gamma} = \begin{cases} \{f \in C^0((0,\infty)) : \lim_{x\to+\infty}(fu_\gamma)(x) = 0 = \lim_{x\to 0^+}(fu_\gamma)(x)\}, & \text{if } \gamma > 0, \\ \left\{ f \in C^0([0,\infty)) : \lim_{x\to+\infty}(fu_\gamma)(x) = 0 \right\}, & \text{if } \gamma = 0, \end{cases}$$

equipped with the norm

$$\|f\|_{C_{u_\gamma}} = \|fu_\gamma\| := \sup_{x\ge 0}\left|(fu_\gamma)(x)\right|.$$

Note that functions in C_{u_γ} can grow exponentially as $x \to +\infty$, and have an algebraic singularity at the origin.

The previous limit conditions are crucial for polynomial approximation of $f \in C_{u_\gamma}$. More precisely, for all $n \in \mathbb{N}$, let $\mathbb{P}_n$ denote the space of the algebraic polynomials of degree at most n, and let

$$E_n(f)_{u_\gamma} := \inf_{P_n \in \mathbb{P}_n} \|(f - P_n)u_\gamma\|, \qquad f \in C_{u_\gamma}$$

be the error of best approximation of f in $\mathbb{P}_n$. It is known that [21,22,25,26]

$$\lim_{n\to\infty} E_n(f)_{u_\gamma} = 0, \qquad \forall f \in C_{u_\gamma},$$

and the rate of convergence depends on the smoothness of $f \in C_{u_\gamma}$. This can be measured by the following main part φ–modulus of smoothness [8,21]

$$\Omega_\varphi^r(f,\tau)_{u_\gamma} = \sup_{0<h\le\tau} \|u_\gamma \Delta_{h\varphi}^r f\|_{I_{rh}}, \qquad t > 0, \qquad r \in \mathbb{N},$$

where $I_{rh} = \left[4r^2h^2, \frac{C}{h^2}\right]$ with C a fixed positive constant,

$$\Delta_{h\varphi}^r f(x) = \sum_{k=0}^r (-1)^k \binom{r}{k} f(x + h\varphi(x)(r-k)),$$

and throughout the paper we set $\varphi(x) = \sqrt{x}$.

We recall that the following Jackson and Stechkin type inequalities hold [8]

$$E_n(f)_{u_\gamma} \le C \int_0^{\frac{1}{\sqrt{n}}} \frac{\Omega_\varphi^r(f,\tau)_{u_\gamma}}{\tau} d\tau, \qquad C \ne C(n,f), \tag{2}$$

$$\Omega_\varphi^r(f,\tau)_{u_\gamma} \le C\tau^r \sum_{k=0}^{\lfloor \tau^{-2} \rfloor} (k+1)^{\frac{r}{2}-1} E_k(f)_{u_\gamma}.$$

Consequently, for all $s > 0$, we get

$$E_n(f)_{u_\gamma} = \mathcal{O}(n^{-s/2}) \Longleftrightarrow \Omega_\varphi^r(f,\tau)_{u_\gamma} = \mathcal{O}(\tau^s), \quad r > s.$$

Based on the previous moduli of smoothness, the Zygmund-type spaces are defined as follows

$$Z_s(u_\gamma) = \left\{ f \in C_{u_\gamma} \ : \ \sup_{\tau>0} \frac{\Omega_\varphi^r(f,\tau)_{u_\gamma}}{\tau^s} < \infty \right\} \quad \forall s > 0,$$

equipped with the norm

$$\|f\|_{Z_s(u_\gamma)} = \|fu_\gamma\| + \sup_{\tau>0} \frac{\Omega_\varphi^r(f,\tau)_{u_\gamma}}{\tau^s}.$$

Equivalently, we can also define

$$Z_s(u_\gamma) = \left\{ f \in C_{u_\gamma} \ : \ \sup_{k>0}(k+1)^{\frac{r}{2}} E_k(f)_{u_\gamma} < \infty \right\} \quad \forall s > 0,$$

and

$$\|f\|_{Z_s(u_\gamma)} = \|fu_\gamma\| + \sup_{k>0}(k+1)^{\frac{s}{2}} E_k(f)_{u_\gamma}.$$

Moreover, for all integers $r \in \mathbb{N}$, we recall the Sobolev–type space

$$W_r(u_\gamma) = \{ f \in C_{u_\gamma} \ : \ \|f^{(r)}\varphi^r u_\gamma\| < \infty \} \quad r \in \mathbb{N},$$

equipped with the norm

$$\|f\|_{W_r(u_\gamma)} = \|fu_\gamma\| + \|f^{(r)}\varphi^r u_\gamma\|.$$

Finally, we point out that for all $s > 0$ it is

$$E_n(f)_{u_\gamma} \le C \frac{\|f\|_{Z_s(u_\gamma)}}{n^{s/2}}, \qquad \forall f \in Z_s(u_\gamma), \quad C \ne C(n,f), \tag{3}$$

and for all $r \in \mathbb{N}$

$$E_n(f)_{u_\gamma} \le C \frac{\|f\|_{W_r(u_\gamma)}}{n^{r/2}}, \qquad \forall f \in W_r(u_\gamma), \quad C \ne C(n,f). \tag{4}$$

2.1. A truncated Lagrange interpolation process on $(0,\infty)$

Let $w_\alpha(x) = e^{-x}x^\alpha$ be the Laguerre weight of parameter $\alpha > -1$ and let $\{p_n(w_\alpha)\}_n$ be the corresponding sequence of orthonormal Laguerre polynomials with positive leading coefficients. For all $n \in \mathbb{N}$ we denote by $x_k := x_{n,k}(w_\alpha)$, $k = 1, \dots, n$, the zeros of $p_n(w_\alpha)$ in increasing order, being (see [38])

$$\frac{C}{n} < x_1 < x_2 < \dots < x_n < 4n + 2\alpha - Cn^{\frac{1}{3}}.$$

Throughout the paper, for any fixed $0 < \rho < 1$, the integer $j := j(n)$ denotes the index of the first zero of $p_n(w_\alpha)$ not less than $4n\rho$, i.e.

$$j = \min_{k=1,2,\dots,n} \left\{ k : x_k \ge 4n\rho \right\}. \tag{5}$$

Based on the first j zeros of $p_n(w_\alpha)$, the truncated Lagrange polynomial $L_{n+1}^*(w_\alpha, f)$ is defined as [23] (see also [24])

$$L_{n+1}^*(w_\alpha, f, x) := \sum_{k=1}^{j} f(x_k)\ell_{n+1,k}(x),$$

where $\ell_{n+1,k}(x)$ are the fundamental Lagrange polynomials corresponding to the zeros of $p_n(w_\alpha, x)(4n - x)$, represented as (see e.g. [22, (4.1.4)])

$$\ell_{n+1,k}(x) = \lambda_{n,k}(w_\alpha) \sum_{i=0}^{n-1} p_i(w_\alpha, x_k)p_i(w_\alpha, x)\frac{4n - x}{4n - x_k}, \qquad k = 1, 2, \dots, j, \tag{6}$$

being $\lambda_{n,k}(w_\alpha) = \left[\sum_{r=0}^{n} p_r(w_\alpha, x_k)\right]^{-1}$ the Laguerre–Christoffel numbers. Note that we have

$$L^*_{n+1}(w_\alpha, f, x_k) = f(x_k), \quad k = 1, \dots, j. \tag{7}$$

Moreover, setting

$$\mathcal{P}^*_n = \left\{ P \in \mathbb{P}_n : P(x_k) = P(4n) = 0, \quad k > j \right\} \subset \mathbb{P}_n, \tag{8}$$

$L^*_{n+1}(w_\alpha)$ projects C_{u_γ} onto $\mathcal{P}^*_n$, i.e. for all functions $f \in C_{u_\gamma}$ we have $L^*_{n+1}(w_\alpha, f) \in \mathcal{P}^*_n$, and

$$L^*_{n+1}(w_\alpha, f) = f, \qquad \forall f \in \mathcal{P}^*_n. \tag{9}$$

We recall the following result concerning the norm of the map $L^*_{n+1}(w_\alpha) : C_{u_\gamma} \to C_{u_\gamma}$ and the approximation error provided by the truncated Lagrange polynomial [23].

Theorem 2.1. *[23] Let $w_\alpha(x) = e^{-x}x^\alpha$, $\alpha > -1$ and $u_\gamma(x) = e^{-\frac{x}{2}}x^\gamma$, $\gamma \ge 0$, be such that*

$$\max\left(0, \frac{\alpha}{2} + \frac{1}{4}\right) \le \gamma \le \frac{\alpha}{2} + \frac{5}{4}.$$

For all $n \in \mathbb{N}$ and $f \in C_{u_\gamma}$, we have

$$\|L^*_{n+1}(w_\alpha, f)u_\gamma\| \le C\|f u_\gamma\| \log n, \qquad C \ne C(n, f),$$

and

$$\|(f - L^*_{n+1}(w_\alpha, f))u_\gamma\| \le C\left[E_N(f)_{u_\gamma} \log n + e^{-An}\|f u_\gamma\|\right], \quad C \ne C(n, f),$$

where $N = \left\lfloor n\frac{\rho}{(1+\rho)} \right\rfloor$ with $\rho \in]0,1[$ defining j in (5), and $A > 0$ independent of n, f.

In [7] the truncated Lagrange polynomial $L^*_{n+1}(w_\alpha, f)$ has been employed to derive a product integration rule for the Hilbert transform $\mathcal{H}(f, t)$. More precisely, approximating f in (1) by $L^*_{n+1}(w_\alpha, f)$, i.e.

$$\mathcal{H}(f, t) = \mathcal{H}_n(f, t) + \mathcal{R}_n(f, t), \qquad t > 0, \tag{10}$$

the quadrature rule $\mathcal{H}_n(f, t) = \mathcal{H}(L^*_{n+1}(w_\alpha, f), t)$, briefly denoted by L-rule, takes the form

$$\mathcal{H}_n(f, t) = \sum_{k=1}^{j} f(x_k)\mathcal{B}_k(t), \qquad \mathcal{B}_k(t) = \int_0^\infty \frac{\ell_{n+1,k}(x)}{x - t} w_\alpha(x)dx,$$

with j defined in (5). In view of (6), the coefficients of the product rule take the form

$$\mathcal{B}_k(t) = \lambda_{n,k}(w_\alpha) \sum_{i=0}^{n-1} \frac{p_i(w_\alpha, x_k)}{4n - x_k} \widetilde{M}_i(t), \quad \widetilde{M}_i(t) = \mathcal{H}(p_i(w_\alpha, \cdot)(4n - \cdot), t), \tag{11}$$

where the functions $\{\widetilde{M}_i(t)\}_i$ are the so-called modified moments.

By (9), the quadrature error

$$\mathcal{R}_n(f, t) = \mathcal{H}(f, t) - \mathcal{H}_n(f, t) = \mathcal{H}((f - L^*_{n+1}(w_\alpha, f)), t),$$

satisfies

$$\mathcal{R}_n(f, t) = 0, \quad \forall f \in \mathcal{P}^*_n,$$

with $\mathcal{P}^*_n$ defined in (8). Moreover, from a more general result [7, Thm. 4.1], the following error estimate is easily deduced

$$\sup_{t \le 4\rho n - 1} |\mathcal{R}_n(f, t)| \le C\frac{\|f\|_{W_r(u_\alpha)}}{n^{r/2}} \log^2 n, \qquad \forall f \in W_r(u_\alpha), \ r \in \mathbb{N}, \ \alpha \ge \frac{1}{2}.$$

Hence, the convergence is ensured for any $f \in W_r(u_\alpha)$, $r \ge 1$, under the assumption $\alpha \ge \frac{1}{2}$. In Section 5 we will able to prove other estimates of the error $\mathcal{R}_n(f, t)$, for functions f in Zygmund type spaces.

2.2. The filtered VP approximation

For all pairs of integers $0 < m < n$ and any Laguerre weight $w_\alpha(x) = x^\alpha e^{-x}$, $\alpha > -1$, the filtered VP approximation polynomial is defined as [35]

$$V_n^m(w_\alpha, f, x) = \sum_{k=1}^{j} f(x_k)\Phi_{n,k}^m(x),$$

where $j = j(n)$ is defined in (5) and $\Phi_{n,k}^m(x)$ are the following fundamental VP polynomials

$$\Phi_{n,k}^m(x) = \lambda_{n,k}(w_\alpha) \sum_{i=0}^{n+m-1} \mu_{n,i}^m p_i(w_\alpha, x_k) p_i(w_\alpha, x),$$

being

$$\mu_{n,i}^m := \begin{cases} 1 & \text{if} \quad i = 0, \dots, n-m, \\ \dfrac{n+m-i}{2m} & \text{if} \quad n-m+1 \le i \le n+m-1. \end{cases}$$

Note that for any choice of the positive integer $m < n$, $V_n^m(w_\alpha, f)$ is based on the same nodes of $L_{n+1}^*(w_\alpha, f)$, but the interpolation property (7) does not extend to $V_n^m(w_\alpha, f)$, in general. However, by using the same data but taking $V_n^m(w_\alpha, f)$ instead of $L_{n+1}^*(w_\alpha, f)$ we succeed in cutting off the typical $\log n$ factor appearing in Theorem 2.1, as shown by the following theorem.

Theorem 2.2. *[35] Let $w_\alpha(x) = e^{-x}x^\alpha$, $\alpha > -1$ and $u_\gamma(x) = e^{-\frac{x}{2}}x^\gamma$, $\gamma \ge 0$ be such that*

$$\max\left(0, \frac{\alpha}{2} - \frac{1}{4}\right) < \gamma < \min\left(\frac{\alpha}{2} + \frac{7}{6}, \alpha + 1\right),$$

being also possible the case that $\gamma = \frac{\alpha}{2} + \frac{7}{6} < \alpha + 1$. Moreover, let $0 < m < n$ be such that $m \sim n$, i.e. assume that $m < n \le Cm$ holds with $C \ne C(n,m)$. Then, for any $f \in C_{u_\gamma}$, we have

$$\|V_n^m(w_\alpha, f)u_\gamma\| \le C\|fu_\gamma\|, \qquad C \ne C(n,m,f),$$

and

$$\|(f - V_n^m(w_\alpha, f))u_\gamma\| \le C\left[E_q(f)_{u_\gamma} + e^{-An}\|fu_\gamma\|\right], \qquad C \ne C(n,m,f),$$

where $q = \min(n-m, N)$, $N = \left\lfloor n^{\frac{\rho}{(1+\rho)}} \right\rfloor$ with $\rho \in]0,1[$ defining j in (5), and $A > 0$ is independent of n, m, f.

As regards the choice of the parameter m, we remark that by taking

$$m = \lfloor \theta n \rfloor, \qquad \text{with } 0 < \theta < 1 \tag{12}$$

we certainly have $m \sim n$ and in addition $(n-m) \sim N \sim n$ holds too.

Consequently, assuming that w_α and u_γ satisfy the hypotheses of Theorem 2.2, if (12) holds then we have

$$\lim_{n\to\infty} \|(f - V_n^m(w_\alpha, f))u_\gamma\| = 0, \qquad \forall f \in C_{u_\gamma},$$

and the convergence order is comparable with that one of the error of best polynomial approximation $E_n(f)_{u_\gamma}$. We point out that a similar result cannot be achieved by using the truncated Lagrange interpolation polynomial $L_{n+1}^*(w_\alpha, f)$.

In the sequel, we are interested in the case $\gamma = \alpha$. Indeed, recalling (2), (3) and (4), by the previous theorem we get the following

Corollary 2.1. *Let $n \in \mathbb{N}$ and let m be given by (12). Moreover, assume that $0 < \alpha \le \frac{7}{3}$ holds. Then we have*

$$\|(f - V_n^m(w_\alpha, f))u_\alpha\| \le C \int_0^{\frac{1}{\sqrt{n}}} \frac{\Omega_\varphi^r(f,\tau)_{u_\alpha}}{\tau} d\tau, \qquad \forall f \in C_{u_\alpha},$$

$$\|(f - V_n^m(w_\alpha, f))u_\alpha\| \le C \frac{\|f\|_{Z_s(u_\alpha)}}{n^{s/2}}, \qquad \forall f \in Z_s(u_\alpha), \quad s > 0,$$

$$\|(f - V_n^m(w_\alpha, f))u_\alpha\| \le C \frac{\|f\|_{W_r(u_\alpha)}}{n^{r/2}}, \qquad \forall f \in W_r(u_\alpha), \quad r \in \mathbb{N},$$

with in all cases it is $C \ne C(n,f)$.

3. Mapping properties of $\mathcal{H}(f)$

The following theorem provides a pointwise estimate of the Hilbert transform of functions in C_{u_α}. It is based on the following weight function

$$v(t)=\begin{cases}1, & \text{if} \quad t\geq 1 \text{ and } \alpha\geq 0,\\ \log^{-1}\left(\frac{e}{t}\right), & \text{if} \quad 0<t<1 \text{ and } \alpha=0,\\ t^{\alpha}, & \text{if} \quad 0<t<1 \text{ and } \alpha>0,\end{cases} \tag{13}$$

that satisfies

$$0<v(t)\leq 1, \qquad \forall t>0. \tag{14}$$

Theorem 3.1. *Assume $\alpha\geq 0$ and let $v(t)$ be the weight function defined by* (13)*. Then, for all $f\in C_{u_\alpha}$ and $t>0$, we have*

$$v(t)|\mathcal{H}(f,t)|\leq C\left(|f(t)u_\alpha(t)|+\|fu_\alpha\|+\int_0^1\frac{\Omega_\varphi(f,y)_{u_\alpha}}{y}dy\right), \tag{15}$$

where $C\neq C(f,t)$.

In particular, this theorem ensures that $\mathcal{H}(f,t)$ exists for all $t>0$ if $f\in C_{u_\alpha}$ satisfies the following Dini–type condition

$$\int_0^1\frac{\Omega_\varphi(f,y)_{u_\alpha}}{y}dy<\infty.$$

Proof of Theorem 3.1. Let us consider the following decomposition

$$\begin{aligned}v(t)|\mathcal{H}(f,t)|\leq v(t)\left|\int_{|x-t|>1}\frac{f(x)}{x-t}w_\alpha(x)dx\right|+v(t)\left|\int_{|x-t|\leq 1}\frac{f(x)-f(t)}{x-t}w_\alpha(x)dx\right|\\ +|f(t)|v(t)\left|\int_{|x-t|\leq 1}\frac{w_\alpha(x)}{x-t}dx\right|=:I_1(t)+I_2(t)+I_3(t).\end{aligned} \tag{16}$$

As regards $I_1(t)$, using (14) and taking into account that $|x-t|^{-1}\leq 1$, we have

$$I_1(t)\leq\int_{|x-t|>1}\frac{|f(x)|}{|x-t|}w_\alpha(x)dx\leq\int_{|x-t|>1}|f(x)|x^\alpha e^{-x}dx\leq C\|fu_\alpha\|, \tag{17}$$

where throughout the proof we always mean that $C\neq C(f,t)$. In order to estimate the other addenda we apply the following lemmas whose technical proof can be found in the Appendix.

Lemma 3.2. *For all $\alpha\geq 0$ and any $t>0$ we have*

$$\int_{|x-t|<1}\frac{w_\alpha(x)}{(x-t)}dx\leq C\begin{cases}w_\alpha(t), & \text{if } t\geq 1 \text{ and } \alpha\geq 0,\\ \log\left(\frac{e}{t}\right), & \text{if } 0<t<1 \text{ and } \alpha=0,\\ 1, & \text{if } 0<t<1 \text{ and } \alpha>0,\end{cases}$$

where $C\neq C(t)$.

Lemma 3.3. *For any $f\in C_{u_\alpha}$ with $\alpha\geq 0$, and any $0<t<1$ we have*

$$v(t)\left|\int_{|x-t|<1}\frac{f(x)-f(t)}{x-t}w_\alpha(x)dx\right|\leq C\left(\int_0^1\frac{\Omega_\varphi(f,y)_{u_\alpha}}{y}dy+\|fu_\alpha\|+|f(t)u_\alpha(t)|\right),$$

where $0<C\neq C(f,t)$.

Lemma 3.4. *For any $f\in C_{u_\alpha}$ with $\alpha\geq 0$, and any $t\geq 1$ we have*

$$\left|\int_{|x-t|<1}\frac{f(x)-f(t)}{x-t}w_\alpha(x)dx\right|\leq C\left(\int_0^1\frac{\Omega_\varphi(f,y)_{u_\alpha}}{y}dy+\|fu_\alpha\|+|f(t)u_\alpha(t)|\right),$$

where $0<C\neq C(f,t)$.

By using Lemma 3.2 we get

$$v(t) \int_{|x-t|<1} \frac{w_\alpha(x)}{(x-t)} dx \le C \begin{cases} t^\alpha e^{-\frac{t}{2}}, & \text{if } t \ge 1 \text{ and } \alpha \ge 0, \\ e^{-\frac{t}{2}}, & \text{if } 0 < t < 1 \text{ and } \alpha = 0, \\ t^\alpha e^{-\frac{t}{2}}, & \text{if } 0 < t < 1 \text{ and } \alpha > 0, \end{cases}$$

and consequently

$$I_3(t) \le C|f(t)|u_\alpha(t), \qquad \forall t > 0. \tag{18}$$

Finally, by applying Lemmas 3.3 and 3.4, we obtain

$$I_2(t) \le C\left(\int_0^1 \frac{\Omega_\varphi(f,y)_{u_\alpha}}{y} dy + \|f u_\alpha\| + |f(t)u_\alpha(t)| \right), \qquad \forall t > 0,$$

and Theorem 3.1 follows. □

4. The product VP rule

For $f \in C_{u_\alpha}$ by $f \sim V_n^m(w_\alpha, f)$ we have

$$\mathcal{H}(f,t) = \mathcal{H}_n^m(f,t) + R_n^m(f,t), \tag{19}$$

where $R_n^m(f,t)$ denotes the quadrature error and $\mathcal{H}_n^m(f,t)$, briefly denoted by VP-rule, takes the form

$$\mathcal{H}_n^m(f,t) = \mathcal{H}(V_n^m(w_\alpha, f), t) = \sum_{k=1}^{j} f(x_k)\mathcal{A}_k(t), \tag{20}$$

with coefficients $\mathcal{A}_k(t)$ given by

$$\mathcal{A}_k(t) = \lambda_{n,k}(w_\alpha) \sum_{i=0}^{n+m-1} \mu_{n,i}^m p_i(w_\alpha, x_k) M_i(t), \quad M_i(t) = \mathcal{H}(p_i(w_\alpha, \cdot), t). \tag{21}$$

4.1. Computational details

As it is known, the main computational effort of product rules is due to the functions $\{M_i(t)\}_{i=0}^{n+m-1}$, in the general case computed via recurrence relations. Based on three-term recurrence relation for orthogonal polynomials, it is not hard to prove that they satisfy the following recurrence relation

$$\begin{cases} M_0(t) = \frac{1}{\sqrt{\Gamma(\alpha+1)}} \mathcal{H}(\mathbf{1}, t), \\ M_1(t) = \frac{1}{a_1} \left(M_0(t)(t-b_0) + \sqrt{\Gamma(\alpha+1)} \right), \\ M_{i+1}(t) = \frac{1}{a_{i+1}} \left(M_i(t)(t-b_i) - a_i M_{i-1}(t) \right), \quad i = 1, 2, \dots \end{cases}$$

where

$$a_i = \sqrt{i(i+\alpha)}, \quad b_i = 2i + \alpha + 1, \quad i = 0, 1, \dots,$$

are the coefficients of the three-term recurrence relation for the orthonormal Laguerre sequence w.r.t. the weight w_α (see e.g. [38]) and $\Gamma(z)$ is the Euler's Gamma function. About the computation of M_0 we recall the expression of the Hilbert transform of the weight w_α [37, p. 325, n. 16]

$$\mathcal{H}(\mathbf{1}, t) = \begin{cases} -e^{-t}\mathrm{Ei}(t), & \alpha = 0, \\ -\pi t^\alpha e^{-t} \cot((1+\alpha)\pi) + 2^\alpha \Gamma(\alpha) {}_1F_1(1, 1-\alpha, -t), & \alpha \ne 0, 1, 2, \dots, \end{cases} \tag{22}$$

where Ei(t) is the Exponential Integral function and ${}_1F_1(a,b,x)$ is the Confluent Hypergeometric function.

We conclude this section providing details about the coefficients $\mathcal{B}_k$ of the (11), whose construction requires the modified moments $\{\widetilde{M}_i(t)\}_{i=0}^{n-1}$. It is no hard to prove that, once $\{M_i(t)\}_{i=0}^{n-1}$ are given, the modified moments $\{\widetilde{M}_i(t)\}_{i=0}^{n-1}$ satisfy the following recurrence relation

$$\begin{cases} \widetilde{M}_0(t) = (4n-t)M_0(t) - \sqrt{\Gamma(\alpha+1)}, \\ \widetilde{M}_{i+1}(t) = (4n-t)M_{i+1}(t), \quad i = 0, 1, 2, \dots \end{cases}$$

5. Error estimates

In this section we study the convergence of the VP-rule, providing error estimates depending on the smoothness of the function f, when it belongs to different subspaces of C_{u_α}. In addition, in view of Theorem 3.1, we are able to prove analogous error estimates for the L-rule (10) based on the same nodes, allowing to us to make a comparison between both VP-rule and L-rule.

First of all, by Theorem 3.1 we derive the following general result

Theorem 5.1. *Assume $\alpha \geq 0$ and let $v(t)$ be defined in (13). Then for all $n, m \in \mathbb{N}$ with $m < n$, for any $f \in C_{u_\alpha}$, and all $t > 0$, the errors of the VP-rule (19) and L-rule, satisfy*

$$v(t)|R_n^m(f,t)| \leq C|(f(t) - V_n^m(w_\alpha, f, t))u_\alpha(t)| \\ + C\log n\|(f - V_n^m(w_\alpha, f))u_\alpha\| + C\int_0^{\frac{1}{\sqrt{n}}} \frac{\Omega_\varphi^r(f,y)_{u_\alpha}}{y}dy, \quad C \neq C(n,m,f,t), \tag{23}$$

$$v(t)|R_n(f,t)| \leq C|(f(t) - L_{n+1}^*(w_\alpha, f, t))u_\alpha(t)| \\ + C\log n\|(f - L_{n+1}^*(w_\alpha, f))u_\alpha\| + C\int_0^{\frac{1}{\sqrt{n}}} \frac{\Omega_\varphi^r(f,y)_{u_\alpha}}{y}dy, \quad C \neq C(n,f,t). \tag{24}$$

Proof of Theorem 5.1. We prove only (23), since the proof of (24) similarly follows, by replacing $V_n^m(w_\alpha, f)$ with $L_{n+1}^*(w_\alpha, f)$. Start from $R_n^m(f,t) = \mathcal{H}(f - V_n^m(w_\alpha, f), t)$ and by applying Theorem 3.1, we get

$$\begin{aligned} v(t)|R_n^m(f,t)| &= v(t)\left|\mathcal{H}\left(f - V_n^m(w_\alpha, f); t\right)\right| \\ &\leq C\left(|(f(t) - V_n^m(w_\alpha, f, t))u_\alpha(t)| + \|(f - V_n^m(w_\alpha, f))u_\alpha\|\right) \\ &\quad + C\int_0^1 \frac{\Omega_\varphi(f - V_n^m(w_\alpha, f), y)_{u_\alpha}}{y}dy, \end{aligned}$$

where here and in the following we mean $C \neq C(n,m,f,t)$.

Now, consider

$$\int_0^1 \frac{\Omega_\varphi(f - V_n^m(w_\alpha, f), y)_{u_\alpha}}{y}dy = \left\{\int_0^{\frac{1}{\sqrt{n}}} + \int_{\frac{1}{\sqrt{n}}}^1\right\} \frac{\Omega_\varphi(f - V_n^m(w_\alpha, f), y)_{u_\alpha}}{y}dy.$$

In view of [10, Lemma 6.1], we get

$$\int_0^{\frac{1}{\sqrt{n}}} \frac{\Omega_\varphi(f - V_n^m(w_\alpha, f), y)_{u_\alpha}}{y}dy \leq C\left(\|(f - V_n^m(w_\alpha, f))u_\alpha\| + \int_0^{\frac{1}{\sqrt{n}}} \frac{\Omega_\varphi^r(f,y)_{u_\alpha}}{y}dy\right).$$

On the other hand

$$\begin{aligned} \int_{\frac{1}{\sqrt{n}}}^1 \frac{\Omega_\varphi(f - V_n^m(w_\alpha, f), y)_{u_\alpha}}{y}dy &\leq C\|(f - V_n^m(w_\alpha, f))u_\alpha\| \int_{\frac{1}{\sqrt{n}}}^1 \frac{dy}{y} \\ &\leq C\|(f - V_n^m(w_\alpha, f))u_\alpha\| \log n. \end{aligned}$$

Hence, we get

$$\int_0^1 \frac{\Omega_\varphi(f - V_n^m(w_\alpha, f), y)_{u_\alpha}}{y}dy \leq C\left(\|(f - V_n^m(w_\alpha, f))u_\alpha\| \log n + \int_0^{\frac{1}{\sqrt{n}}} \frac{\Omega_\varphi^r(f,y)_{u_\alpha}}{y}dy\right),$$

and the Theorem follows. □

From (23) and Corollary 2.1, we deduce the following

Corollary 5.1. *Let $n \in \mathbb{N}$ and let m be given by (12). Moreover, assume that $0 < \alpha \le \frac{7}{3}$ holds and let $v(t)$ be given by (13). Then for all $f \in C_{u_\alpha}$ and any $t > 0$ the quadrature error by the VP-rule (19) satisfies the following estimate*

$$v(t)|R_n^m(f,t)| \le C|(f(t) - V_n^m(w_\alpha, f, t))u_\alpha(t)| + C\log n \int_0^{\frac{1}{\sqrt{n}}} \frac{\Omega_\varphi^r(f,\tau)_{u_\gamma}}{\tau} d\tau, \tag{25}$$

with $C \neq C(n, m, f, t)$. Moreover, for all $t > 0$ we have

$$v(t)|R_n^m(f,t)| \le C\log n \frac{\|f\|_{Z_s(u_\alpha)}}{n^{s/2}}, \qquad \forall f \in Z_s(u_\alpha), \quad s > 0, \tag{26}$$

$$v(t)|R_n^m(f,t)| \le C\log n \frac{\|f\|_{W_r(u_\alpha)}}{n^{r/2}}, \qquad \forall f \in W_r(u_\alpha), \quad r \in \mathbb{N}, \tag{27}$$

with $C \neq C(n, m, f, t)$ in both cases.

5.1. Error estimates for the L–rule

From (24), recalling Theorem 2.1, we get

Theorem 5.2. *Let $n \in \mathbb{N}$ and assume that $\frac{1}{2} \le \alpha \le \frac{5}{2}$ holds and let $v(t)$ be given by (13). Then for all $f \in C_{u_\alpha}$ and any $t > 0$ the quadrature error by the L-rule (10) satisfies the following estimate*

$$v(t)|R_n(f,t)| \le C|(f(t) - L_{n+1}^*(w_\alpha, f, t))u_\alpha(t)| + C\log^2 n \int_0^{\frac{1}{\sqrt{n}}} \frac{\Omega_\varphi^r(f,\tau)_{u_\gamma}}{\tau} d\tau, \tag{28}$$

with $C \neq C(n, m, f, t)$. Moreover, for all $t > 0$ we have

$$v(t)|R_n(f,t)| \le C\log^2 n \frac{\|f\|_{Z_s(u_\alpha)}}{n^{s/2}}, \qquad \forall f \in Z_s(u_\alpha), \quad s > 0, \tag{29}$$

$$v(t)|R_n(f,t)| \le C\log^2 n \frac{\|f\|_{W_r(u_\alpha)}}{n^{r/2}}, \qquad \forall f \in W_r(u_\alpha), \quad r \in \mathbb{N}, \tag{30}$$

with $C \neq C(n, f, t)$ in both cases.

Comparing Theorem 5.2 with Corollary 2.1 we observe that the bounds for α are different. In particular, for $0 < \alpha < \frac{1}{2}$ the convergence in Zygmund or Sobolev spaces is ensured for the VP rule but not for the L-rule. Moreover, when α satisfies the hypotheses of both Corollary 2.1 and Theorem 5.2, the error estimates of $R_n(f,t)$ present an additional $\log n$ factor. From the computational point of view, such an extra factor is not relevant, due to the slow divergence of $\log n$. However, even if the maximum errors of both the formulas are almost the same, we will show in the section "Numerical tests", a better pointwise performance of the VP-rule respect to the L-rule. This improvement is a consequence of the better ability in pointwise approximation of the filtered VP polynomial compared to the truncated Lagrange polynomial [35].

6. Numerical tests

In this section we propose some tests to evaluate the performance of the VP rule, compared to the L rule, highlighting differences between the global and the pointwise approximation. In each test, for fixed values of $t > 0$, we report in the Tables the errors $|R_n^m(f,t)|$ attained by the VP rule (19) for n, $m = \lfloor n\theta \rfloor$, setting in the first two columns the values of n and θ. In this regard, we point out that θ has been selected as that giving the minimal absolute error $|R_n^m(f,t)|$ reported in the fourth column. The third column contains the number j of functions evaluations in both VP-rule and L-rule (10). The errors $|R_n(f,t)|$ of the L-rule are reported in the fifth column, for some more significant tests we have provided also the graphics of the pointwise absolute errors, in order to evidence the different local behavior of the errors. In the tests 1-4, we have chosen $\frac{1}{2} \le \alpha < \frac{7}{3}$, range for which both VP and L rules converge.

Example 1.

$$\mathcal{H}(f,t) = \int_0^\infty \frac{|x-1|^{6.3}}{x-t} \sqrt{x} e^{-x} dx.$$

According to estimates (26) and (29), the expected errors are $|R_n^m(f,t)| = \mathcal{O}\left(\frac{\log n}{(\sqrt{n})^{6.3}}\right)$, and $|R_n(f,t)| = \mathcal{O}\left(\frac{\log^2 n}{(\sqrt{n})^{6.3}}\right)$. Since for $s = 6.3$ the norm $\|f\|_{Z_s(u_{1/2})} \le 6 \times 10^4$, the numerical estimate are adequate to the theoretical ones, and the VP rule provides more accurate results in the global approximation (see Fig. 1). (See Table 1.)

Table 1
Example 1: results for $t = 1$ (left), $t = 5$ (right).

n	θ	j	$\|R_n^m(f,t)\|$	$\|R_n(f,t)\|$
50	0.5	34	**1.5505e-04**	1.6343e-04
150	0.3	60	**1.8587e-06**	6.1514e-05
250	0.8	78	**9.7262e-07**	4.8089e-04
350	0.4	92	**1.4425e-06**	5.1379e-04
450	0.7	104	**1.5632e-07**	5.0562e-04
550	0.4	115	**6.5981e-07**	4.8194e-04

n	θ	j	$\|R_n^m(f,t)\|$	$\|R_n(f,t)\|$
50	0.5	34	**8.6014e-06**	3.7835e-05
150	0.1	60	**9.0595e-08**	9.4806e-06
250	0.7	78	**1.3233e-07**	4.1154e-06
350	0.8	92	**1.4518e-07**	2.8412e-05
450	0.9	104	**1.1936e-07**	5.7654e-05
550	0.7	115	**4.8854e-08**	2.2491e-05

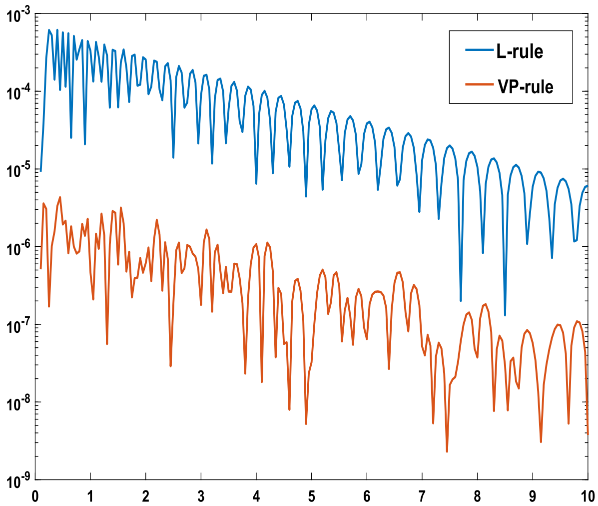

Fig. 1. Example 1: pointwise absolute errors for $n = 100, \theta = 0.5$.

Example 2.

$$\mathcal{H}(f,t) = \int_0^\infty \frac{1}{100 + 10(x-3)^2} \frac{e^{-x} x^{0.6}}{x-t} dx.$$

The function f is very smooth, eventhough presenting a peak for $x = 3$. As shown in Table 2, the VP rule provides better results, attaining two digits more than the L-rule, and catching also the machine precision for some n. The graphic of the local absolute errors confirms a better performance of the VP rule.

Example 3.

$$\mathcal{H}(f,t) = \int_0^\infty \frac{e^{-\frac{7}{8}x}}{100 + 100(x-5)^2} \frac{x^{0.7}}{x-t} dx.$$

.

Also in this case $f(x) = \frac{e^{\frac{x}{8}}}{100+100(x-5)^2}$ presents a peak in $x = 5$, exponentially growing, in addition. In Table 2 we observe that the errors if compared with the errors of the Example 2, slower decay to 0. In this case $\|f\|_{W_r(u)} < \infty$ for any r, with norms growing fast and fast. For instance, $\|f\|_{W_{10}(u)} \sim 10^6$, and $\|f\|_{W_{20}(u)} \sim 10^{21}$. (See Fig. 2.).

Table 2
Example 2: results for $t = 3$ (left), $t = 10$ (right).

n	θ	j	$\lvert R_n^m(f,t)\rvert$	$\lvert R_n(f,t)\rvert$
10	0.1	10	**4.1650e-06**	**4.1650e-06**
30	0.2	18	**6.4742e-07**	7.9230e-07
50	0.2	23	**3.0621e-08**	3.9514e-08
70	0.1	27	3.9253e-09	**2.6740e-09**
90	0.2	31	**7.1253e-11**	1.3562e-10
130	0.1	37	**7.8966e-12**	9.1980e-12
150	0.1	40	**8.4584e-13**	1.2355e-12
250	0.5	52	**3.6094e-14**	2.9953e-12

n	θ	j	$\lvert R_n^m(f,t)\rvert$	$\lvert R_n(f,t)\rvert$
10	0.8	10	**1.2733e-07**	2.8347e-07
30	0.2	18	**5.3418e-08**	5.4457e-08
50	0.5	23	**1.2786e-10**	1.9187e-10
70	0.5	27	**5.3111e-12**	8.0498e-11
110	0.3	34	**9.3038e-13**	1.0592e-12
130	0.2	37	**4.9112e-14**	2.4359e-13
250	0.2	51	**1.5375e-15**	3.4953e-13
350	0.5	60	**8.5457e-16**	1.7962e-13

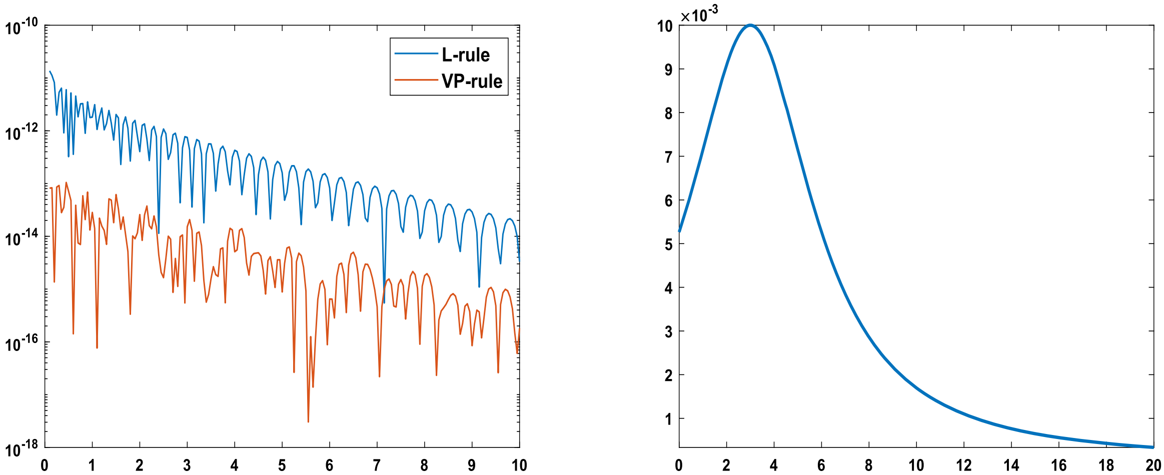

Fig. 2. Example 2: pointwise absolute errors for $n = 350$, $\theta = 0.2$ (left), function f (right).

Example 4.

$$\mathcal{H}(f,t) = \int_0^\infty \frac{f(x)}{x-t} e^{-x} x^{0.6} dx,$$

$$f(x) = \frac{1}{1+100(x-0.5)^2} + \frac{1}{1+1000\sqrt{x^2+0.5}}.$$

The expected error are $|R_n^m(f,t)| = \mathcal{O}\left(\frac{\log n}{n^r}\right)$, and $|R_n(f,t)| = \mathcal{O}\left(\frac{\log^2 n}{n^r}\right)$, for any $r \geq 1$ (see Table 3). In this case we observe that the seminorms of the function $f \in W_r(u_\alpha)$ increase faster as r increases. For instance, with $r = 8$, the seminorm $\|f^{(r)}\varphi^r u_\alpha\| \sim 10^{11}$ and for $r = 15$ it is $\|f^{(r)}\varphi^r u_\alpha\| \sim 10^{24}$. This fast growth justifies the slow convergence shown in Table 4. In Fig. 4 are reported the graphics of the approximations of $\mathcal{H}(f)$ by VP and L rules. The function $\mathcal{H}_{m,n}(f,t)$ oscillates less than the function $\mathcal{H}_n(f)$ also in subintervals not including $\bar{t} = 0.5$, which is a critical point for both the functions f and $\mathcal{H}(f)$. (See Fig. 3.)

Table 3
Example 3: results for $t = 1$ (left), $t = 0.01$ (right).

n	θ	j	$\lvert R_n^m(f,t)\rvert$	$\lvert R_n(f,t)\rvert$
50	0.9	23	**9.9142e-07**	2.8904e-05
250	0.2	52	**2.8436e-08**	1.2252e-06
350	0.2	62	**3.0644e-08**	5.1455e-08
450	0.5	70	**4.1101e-09**	7.3813e-08

n	θ	j	$\lvert R_n^m(f,t)\rvert$	$\lvert R_n(f,t)\rvert$
50	0.9	23	**7.6703e-07**	5.0043e-05
150	0.4	40	**9.5262e-08**	4.6366e-06
450	0.2	70	**2.1288e-09**	1.6132e-07
650	0.2	84	**7.0889e-10**	2.5365e-09
750	0.1	90	**6.7631e-11**	7.4402e-09

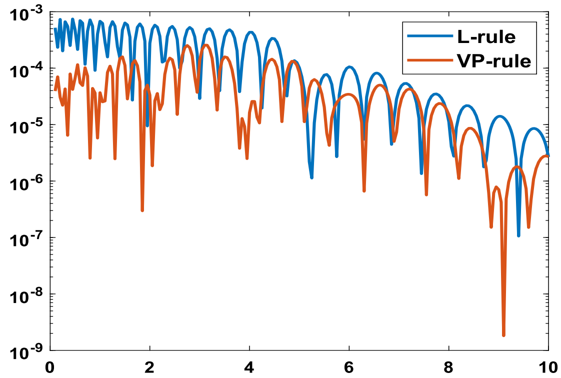

Fig. 3. Example 3: absolute errors $n = 200, \theta = 0.5$.

Table 4
Example 4: results for $t = 0.3$ (left), $t = 3$ (right).

n	θ	j	$\|R_n^m(f,t)\|$	$\|R_n(f,t)\|$
50	0.9	22	**9.3140e-01**	2.0118e-01
150	0.9	39	**6.5537e-02**	2.3136e-02
250	0.2	50	**3.3369e-04**	2.9756e-03
850	0.5	92	**5.3155e-04**	1.4570e-02
1050	0.5	103	**4.7642e-05**	3.3843e-03

n	θ	j	$\|R_n^m(f,t)\|$	$\|R_n(f,t)\|$
50	0.9	22	**2.6598e-03**	1.0016e-03
150	0.7	39	**1.3897e-03**	2.9817e-03
350	0.5	59	**3.5231e-04**	1.9588e-04
450	0.4	67	**1.1087e-06**	9.4390e-06
550	0.8	74	**4.7642e-07**	3.3843e-07

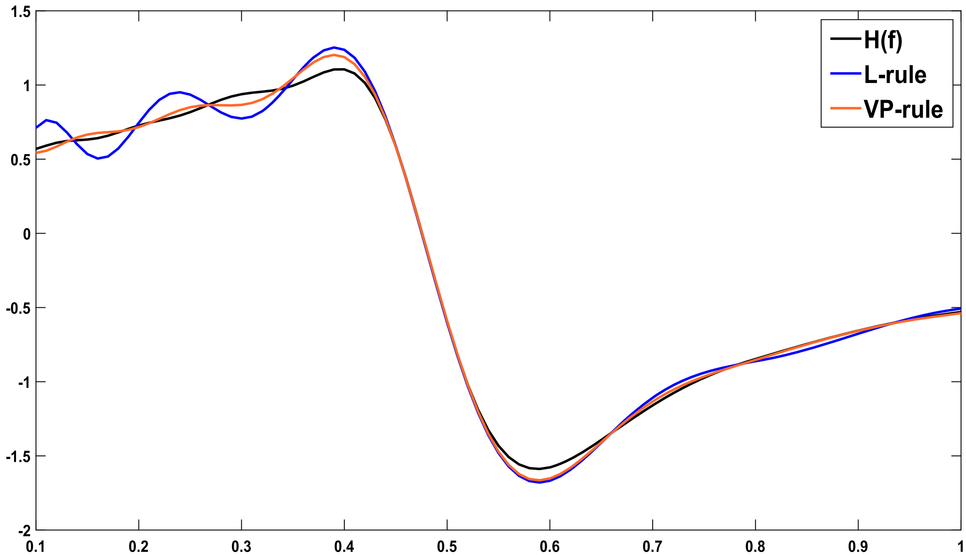

Fig. 4. Example 4: $n = 800, \theta = 0.6$.

Example 5.

$$\mathcal{H}(f,t) = \int_0^\infty \frac{\sin(x)}{x-t} e^{-x} dx.$$

In the end, we propose a test for which α is out of range for both the rules. However, $\alpha = 0$ represents a limit case for which we conjecture VP-rule converges. By Table 5 and Fig. 5 we observe that both the rules converge, the VP-rule faster than the L-rule.

Table 5
Example 5: results for $t = 1$ (left) and $t = 10$ (right).

n	θ	j	$\|R_n^m(f,t)\|$	$\|R_n(f,t)\|$
50	0.1	26	**3.7011e-08**	7.4960e-08
150	0.5	46	**8.6364e-13**	2.8181e-09
250	0.6	59	**3.6118e-12**	1.0138e-09
350	0.7	70	**1.1182e-12**	8.6256e-10
450	0.4	79	**5.4001e-13**	3.1962e-10

n	θ	j	$\|R_n^m(f,t)\|$	$\|R_n(f,t)\|$
50	0.1	26	**5.9148e-08**	1.1380e-07
150	0.2	46	**7.2381e-11**	1.7277e-09
350	0.3	70	**9.4333e-12**	7.4801e-10
450	0.6	79	**6.4215e-13**	1.4242e-10

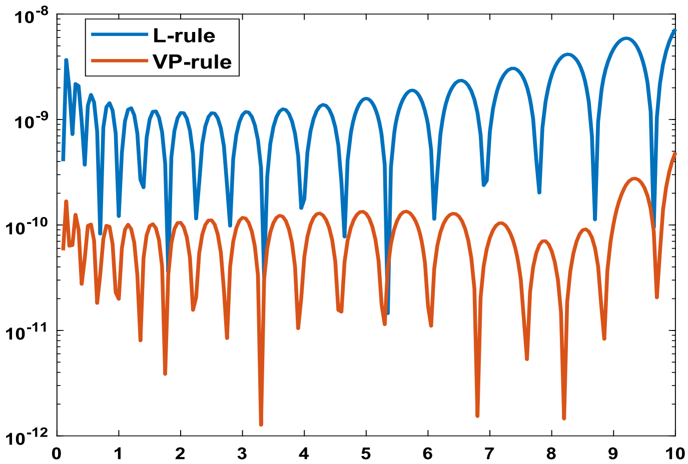

Fig. 5. Example 5: pointwise absolute errors $n = 100, n = 10$.

CRediT authorship contribution statement

Donatella Occorsio: Project administration, Methodology, Investigation, Funding acquisition, Formal analysis, Data curation, Conceptualization. **Woula Themistoclakis:** Methodology, Investigation, Formal analysis, Data curation, Conceptualization.

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Appendix A

In this section we prove Lemmas 3.2–3.4. In all proofs we will always mean that $C > 0$ as well as the constants in $\sim$ are independent of t, x and f.

Proof of Lemma 3.2. Let us distinguish the following cases.

Case $\alpha > 0$.

Since $e^{-x} = e^{-t}e^{t-x}$, we firstly note that $|t - x| < 1$ implies that $e^{-x} \sim e^{-t}$ holds.

For $0 < t < 1$, using $e^{-x} \sim e^{-t}$ and taking into account that $\int_0^{2t} \frac{dx}{x-t} = 0$, we have

$$\left| \int\limits_{|x-t|<1} \frac{w_\alpha(x)}{x-t} dx \right| = \left| \int\limits_0^{t+1} \frac{x^\alpha e^{-x}}{x-t} dx \right| \leq Ce^{-t} \left| \int\limits_0^{t+1} \frac{x^\alpha}{x-t} dx \right|$$

$$\leq Ce^{-t} \left(\left| \int\limits_0^{2t} \frac{x^\alpha - t^\alpha}{x-t} dx \right| + \left| \int\limits_{2t}^{t+1} \frac{x^\alpha}{x-t} dx \right| \right).$$

Hence, by applying the change of variable $x = ut$ in the first integral, and the inequality $(x-t) \geq \frac{x}{2}$ in the second integral, we get

$$\left| \int_{|x-t|<1} \frac{w_\alpha(x)}{x-t} dx \right| \leq C \left(\int_0^{2t} \frac{|x^\alpha - t^\alpha|}{|x-t|} dx + \int_{2t}^{t+1} \frac{x^\alpha}{x-t} dx \right)$$
$$\leq C \left(t^\alpha \int_0^2 |u-1|^{\alpha-1} du + \int_{2t}^{t+1} x^{\alpha-1} dx \right) \leq C,$$

having used $0 < t < 1$ to get the last inequality.

For $t \geq 1$, by using $e^{-x} \sim e^{-t}$ and $\int_{t-1}^{t+1} \frac{dx}{x-t} = 0$, we obtain

$$\left| \int_{|x-t|<1} \frac{w_\alpha(x)}{(x-t)} dx \right| = \left| \int_{t-1}^{t+1} \frac{x^\alpha e^{-x}}{(x-t)} dx \right| \leq Ce^{-t} \left| \int_{t-1}^{t+1} \frac{x^\alpha}{x-t} dx \right|$$
$$= Ce^{-t} \left| \int_{t-1}^{t+1} \frac{x^\alpha - t^\alpha}{x-t} dx \right| \leq Ce^{-t} \int_{t-1}^{t+1} \frac{|x^\alpha - t^\alpha|}{|x-t|} dx$$
$$= Ce^{-t} t^\alpha \int_{1-\frac{1}{t}}^{1+\frac{1}{t}} |u-1|^{\alpha-1} du \leq C\, w_\alpha(t).$$

<u>Case $\alpha = 0$</u>.

For $0 < t < 1$, by using $\int_0^{2t} \frac{dx}{x-t} = 0$, the Lagrange mean value theorem, and $e^{-x} \sim e^{-t}$, we have

$$\left| \int_{|x-t|<1} \frac{e^{-x}}{x-t} dx \right| \leq \left| \int_0^{2t} \frac{e^{-x} - e^{-t}}{x-t} dx \right| + \left| \int_{2t}^{t+1} \frac{e^{-x}}{x-t} dx \right|$$
$$\leq \int_0^{2t} dx + Ce^{-t} \int_{2t}^{t+1} \frac{dx}{x-t} dx = 2t + Ce^{-t} \log\left(\frac{1}{t}\right)$$
$$\leq C \left[1 + \log\left(\frac{e}{t}\right) \right] \leq C \log\left(\frac{e}{t}\right).$$

For $t \geq 1$, by using $\int_{t-1}^{t+1} \frac{dx}{x-t} = 0$ and the Lagrange mean value theorem, we get

$$\left| \int_{|x-t|<1} \frac{e^{-x}}{x-t} dx \right| = \left| \int_{t-1}^{t+1} \frac{e^{-x} - e^{-t}}{(x-t)} dx \right| \leq \int_{t-1}^{t+1} e^{-\xi} dx \leq C\, e^{-t}. \quad \square$$

Proof of Lemma 3.3. First of all, note that for all $0 < t < 1$ and $\alpha \geq 0$ we have

$$\left| \int_{|x-t|<1} \frac{f(x)-f(t)}{x-t} w_\alpha(x) dx \right| = \left| \int_0^{t+1} \frac{f(x)-f(t)}{x-t} w_\alpha(x) dx \right|$$
$$\leq \left| \int_0^{2t} \frac{f(x)-f(t)}{x-t} w_\alpha(x) dx \right| + \left| \int_{2t}^{t+1} \frac{f(x)-f(t)}{x-t} w_\alpha(x) dx \right|$$
$$=: J_1(t) + J_2(t). \tag{31}$$

Now let us distinguish the following cases.

<u>Case $\alpha > 0$</u>

As regards $J_1(t)$, taking into account that $e^{-x} \sim e^{-t} \sim 1$, we have

$$J_1(t) \leq Ce^{-t} \left| \int_0^{2t} \frac{f(x)-f(t)}{x-t} x^\alpha dx \right|$$

$$= Ce^{-t}\left|\int_0^t \frac{f(x)-f(t)}{x-t}x^\alpha dx + \int_t^{2t}\frac{f(x)-f(t)}{x-t}x^\alpha dx\right|.$$

Hence, we introduce the changes of variable $x = t - \frac{h}{2}\varphi(t)$ and $x = t + \frac{h}{2}\varphi(t)$ in the first and second integrals, respectively, and by adding and subtracting $f\left(t+\frac{h}{2}\varphi(t)\right)\left(t-\frac{h}{2}\varphi(t)\right)^\alpha$, we get

$$
\begin{aligned}
J_1(t) &= Ce^{-t}\left|\int_0^{2\varphi(t)} \frac{[f(t)-f\left(t-\frac{h}{2}\varphi(t)\right)]\left(t-\frac{h}{2}\varphi(t)\right)^\alpha + [f\left(t+\frac{h}{2}\varphi(t)\right)-f(t)]\left(t+\frac{h}{2}\varphi(t)\right)^\alpha}{h}dh\right| \\
&\le Ce^{-t}\int_0^{2\varphi(t)}\left(t-\frac{h}{2}\varphi(t)\right)^\alpha \frac{\left|f\left(t+\frac{h}{2}\varphi(t)\right)-f\left(t-\frac{h}{2}\varphi(t)\right)\right|}{h}dh \\
&\quad + Ce^{-t}\int_0^{2\varphi(t)}\left|f\left(t+\frac{h}{2}\varphi(t)\right)\right| \frac{\left|\left(t-\frac{h}{2}\varphi(t)\right)^\alpha-\left(t+\frac{h}{2}\varphi(t)\right)^\alpha\right|}{h}dh \\
&\quad + Ce^{-t}|f(t)|\int_0^{2\varphi(t)}\frac{\left|\left(t-\frac{h}{2}\varphi(t)\right)^\alpha-\left(t+\frac{h}{2}\varphi(t)\right)^\alpha\right|}{h}dh \\
&\le Ce^{-t}\int_0^{2\varphi(t)} t^\alpha\frac{\left|f\left(t+\frac{h}{2}\varphi(t)\right)-f\left(t-\frac{h}{2}\varphi(t)\right)\right|}{h}dh \\
&\quad + C\int_0^{2\varphi(t)} e^{-\left(t+\frac{h}{2}\varphi(t)\right)}\left|f\left(t+\frac{h}{2}\varphi(t)\right)\right| \frac{\left(t+\frac{h}{2}\varphi(t)\right)^\alpha-\left(t-\frac{h}{2}\varphi(t)\right)^\alpha}{h}dh \\
&\quad + Ce^{-t}|f(t)|\int_0^{2\varphi(t)}\frac{\left(t+\frac{h}{2}\varphi(t)\right)^\alpha-\left(t-\frac{h}{2}\varphi(t)\right)^\alpha}{h}dh \\
&= C\int_0^{2\varphi(t)}\frac{|\Delta_{h\varphi}f(t)|u_\alpha(t)}{h}dh \\
&\quad + C\int_0^{2\varphi(t)}\left|f\left(t+\frac{h}{2}\varphi(t)\right)\right|u_\alpha\left(t+\frac{h}{2}\varphi(t)\right)\frac{\left(t+\frac{h}{2}\varphi(t)\right)^\alpha-\left(t-\frac{h}{2}\varphi(t)\right)^\alpha}{\left(t+\frac{h}{2}\varphi(t)\right)^\alpha h}dh \\
&\quad + Ce^{-t}|f(t)|\int_0^{2\varphi(t)}\frac{\left(t+\frac{h}{2}\varphi(t)\right)^\alpha-\left(t-\frac{h}{2}\varphi(t)\right)^\alpha}{h}dh \\
&\le C\int_0^2\frac{\Omega_\varphi(f,h)_{u_\alpha}}{h}dh + \frac{\|fu_\alpha\|+|f(t)|u_\alpha(t)}{t^\alpha}\int_0^{2\varphi(t)}\frac{\left(t+\frac{h}{2}\varphi(t)\right)^\alpha-\left(t-\frac{h}{2}\varphi(t)\right)^\alpha}{h}dh.
\end{aligned}
\tag{32}
$$

On the other hand, recalling that $\Omega_\varphi(f,2y)_{u_\alpha} \le C\Omega_\varphi(f,y)_{u_\alpha}$ holds, we have

$$\int_0^2\frac{\Omega_\varphi(f,h)_{u_\alpha}}{h}dh = \int_0^1\frac{\Omega_\varphi(f,2y)_{u_\alpha}}{y}dy \le C\int_0^1\frac{\Omega_\varphi(f,y)_{u_\alpha}}{y}dy.$$

Moreover, by the Lagrange mean value theorem we note that

$$\frac{\left(t+\frac{h}{2}\varphi(t)\right)^\alpha-\left(t-\frac{h}{2}\varphi(t)\right)^\alpha}{h} = \alpha(t+\xi)^{\alpha-1}\varphi(t), \qquad |\xi| < \frac{h}{2}\varphi(t), \tag{33}$$

and consequently, by taking into account that

$$(t+\xi)^{\alpha-1} \le \begin{cases} (2t)^{\alpha-1}, & \text{if} \quad \alpha-1>0, \\ \left(t-\frac{h}{2}\varphi(t)\right)^{\alpha-1}, & \text{if} \quad \alpha-1<0, \end{cases} \tag{34}$$

we get

$$\int_0^{2\varphi(t)} \frac{\left(t+\frac{h}{2}\varphi(t)\right)^{\alpha}-\left(t-\frac{h}{2}\varphi(t)\right)^{\alpha}}{h}\,dh \le Ct^{\alpha}, \qquad \forall \alpha \ge 0. \tag{35}$$

Hence, by (32)–(35) we conclude

$$J_1(t) \le C\left[\int_0^1 \frac{\Omega_\varphi(f,y)_{u_\alpha}}{y}\,dy + \|fu_\alpha\| + |f(t)|u_\alpha(t)\right]. \tag{36}$$

As regards $J_2(t)$, by taking into account that $(x-t) > \frac{x}{2}$, we have

$$\begin{aligned} J_2(t) &= \left|\int_{2t}^{t+1} \frac{f(x)-f(t)}{x-t} w_\alpha(x)\,dx\right| \le \int_{2t}^{t+1} \frac{|f(x)|+|f(t)|}{x-t} w_\alpha(x)\,dx \\ &\le \|fu_\alpha\| \int_{2t}^{t+1} \frac{x^\alpha e^{-\frac{x}{2}}}{x^\alpha(x-t)}\,dx + |f(t)|e^{-t}\int_{2t}^{t+1} \frac{x^\alpha e^{-\frac{x}{2}}}{x-t}\,dx \\ &\le \left(\frac{\|fu_\alpha\|}{(2t)^\alpha} + |f(t)|e^{-t}\right)\int_{2t}^{t+1} \frac{x^\alpha}{x-t}\,dx \\ &\le \left(\frac{\|fu_\alpha\|}{(2t)^\alpha} + |f(t)|e^{-t}\right)\int_{2t}^{t+1} x^{\alpha-1}\,dx \le C\left(\frac{\|fu_\alpha\|}{(2t)^\alpha} + |f(t)|e^{-t}\right), \end{aligned}$$

having used $0 < t < 1$ in the last inequality.

Consequently, since $v(t) = t^\alpha$ in this case, we get

$$v(t)J_2(t) \le C\left(\|fu_\alpha\| + |f(t)|u_\alpha(t)\right). \tag{37}$$

Thus, for any $\alpha > 0$, by (31), (36) and (37), recalling also (14), we obtain

$$\begin{aligned} v(t)\left|\int_{|x-t|<1} \frac{f(x)-f(t)}{x-t} w_\alpha(x)\,dx\right| &\le J_1(t) + v(t)J_2(t) \\ \le C\left[\int_0^1 \frac{\Omega_\varphi(f,y)_{u_\alpha}}{y}\,dy + \|fu_\alpha\| + |f(t)|u_\alpha(t)\right]. \end{aligned} \tag{38}$$

<u>Case $\alpha = 0$</u> The proof is analogous to the previous case.

More precisely, as regards $J_1(t)$, applying the same changes of variable made in the case $\alpha > 0$, and taking into account that

$$1 \le e^{\frac{h}{2}\varphi(t)} \le e^t \le e, \qquad \forall h \in [0, 2\varphi(t)],$$

by simple computations, we get

$$\begin{aligned} J_1(t) &= \left|\int_0^t \frac{f(x)-f(t)}{x-t} e^{-x}\,dx + \int_t^{2t} \frac{f(x)-f(t)}{x-t} e^{-x}\,dx\right| \\ &\le C\int_0^1 \frac{\Omega_\varphi(f,h)_{u_0}}{y}\,dy + C\left(\|fu_0\| + f(t)u_0(t)\right)\int_0^{2\varphi(t)} \frac{e^{\frac{h}{2}\varphi(t)} - e^{-\frac{h}{2}\varphi(t)}}{h}\,dh. \end{aligned}$$

On the other hand, we note that

$$\int_0^{2\varphi(t)} \frac{e^{\frac{h}{2}\varphi(t)} - e^{-\frac{h}{2}\varphi(t)}}{h} dh = 2\int_0^{2\varphi(t)} \frac{\sinh(\frac{h}{2}\varphi(t))}{h} dh = 2\int_0^t \frac{\sinh(x)}{x} dx \le 2\mathrm{Shi}(1) = C,$$

being $\mathrm{Shi}(z) = \int_0^z \frac{\sinh(t)}{t} dt$ the Hyperbolic Sine Integral.

Hence, (36) holds for $\alpha = 0$ too.

Finally, as regards $J_2(t)$ we have

$$\begin{aligned} J_2(t) &= \left| \int_{2t}^{t+1} \frac{f(x) - f(t)}{x - t} e^{-x} dx \right| \le \int_{2t}^{t+1} \frac{|f(x)| + |f(t)|}{x - t} e^{-x} dx \\ &\le \|f u_0\| \int_{2t}^{t+1} \frac{dx}{x - t} dx + |f(t)| u_0(t) \int_{2t}^{t+1} \frac{dx}{x - t} dx \\ &\le C \left(\|f u_0\| + |f(t)| u_0(t) \right) \log\left(\frac{e}{t}\right), \end{aligned}$$

and consequently (37) holds for $\alpha = 0$ too, since in this case we have

$$v(t) J_2(t) = \log^{-1}\left(\frac{e}{t}\right) J_2(t) \le C \left(\|f u_0\| + |f(t)| u_0(t) \right).$$

In conclusion, (38) holds for $\alpha = 0$ too. □

Proof of Lemma 3.4. The proof can be achieved by similar arguments used in proving Lemma 3.3. The main changes regard the integration range that in the case $t \ge 1$ differs from the case $0 < t < 1$, being

$$\int_{|x-t|<1} \frac{f(x) - f(t)}{x - t} w_\alpha(x) dx = \int_{t-1}^{t+1} \frac{f(x) - f(t)}{x - t} w_\alpha(x) dx.$$

Case $\alpha > 0$

In this case we use $e^{-x} \sim e^{-t}$ and apply the same changes of variable of Lemma 3.3 getting

$$\begin{aligned} &\left| \int_{|x-t|<1} \frac{f(x) - f(t)}{x - t} w_\alpha(x) dx \right| \\ &\le C e^{-t} \left| \int_{t-1}^{t} \frac{f(x) - f(t)}{x - t} x^\alpha dx + \int_t^{t+1} \frac{f(x) - f(t)}{x - t} x^\alpha dx \right| \\ &= C e^{-t} \left| \int_0^{\frac{2}{\varphi(t)}} \frac{[f(t) - f\left(t - \frac{h}{2}\varphi(t)\right)]\left(t - \frac{h}{2}\varphi(t)\right)^\alpha}{h} dh + \int_0^{\frac{2}{\varphi(t)}} \frac{[f\left(t + \frac{h}{2}\varphi(t)\right) - f(t)]\left(t + \frac{h}{2}\varphi(t)\right)^\alpha}{h} dh \right| \\ &\le C e^{-t} \int_0^{\frac{2}{\varphi(t)}} \left(t - \frac{h}{2}\varphi(t)\right)^\alpha \frac{\left| f\left(t + \frac{h}{2}\varphi(t)\right) - f\left(t - \frac{h}{2}\varphi(t)\right) \right|}{h} dh \\ &\quad + C e^{-t} \int_0^{\frac{2}{\varphi(t)}} \left| f\left(t + \frac{h}{2}\varphi(t)\right) \right| \frac{\left| \left(t - \frac{h}{2}\varphi(t)\right)^\alpha - \left(t + \frac{h}{2}\varphi(t)\right)^\alpha \right|}{h} dh \\ &\quad + C e^{-t} |f(t)| \int_0^{\frac{2}{\varphi(t)}} \frac{\left| \left(t - \frac{h}{2}\varphi(t)\right)^\alpha - \left(t + \frac{h}{2}\varphi(t)\right)^\alpha \right|}{h} dh \\ &\le C \int_0^1 \frac{\Omega_\varphi(f, h)_{u_\alpha}}{h} dh + \frac{\|f u_\alpha\| + |f(t)| u_\alpha(t)}{t^\alpha} \int_0^{\frac{2}{\varphi(t)}} \frac{\left(t + \frac{h}{2}\varphi(t)\right)^\alpha - \left(t - \frac{h}{2}\varphi(t)\right)^\alpha}{h} dh. \end{aligned} \tag{39}$$

On the other hand, by using (33)–(34) and taking into account that now $t \ge 1$, if $\alpha - 1 \ge 0$ then we have

$$\int_0^{\frac{2}{\varphi(t)}} \frac{\left(t+\frac{h}{2}\varphi(t)\right)^{\alpha}-\left(t-\frac{h}{2}\varphi(t)\right)^{\alpha}}{h}dh \le Ct^{\alpha-1}\le Ct^{\alpha},$$

and for $\alpha - 1 < 0$ we get

$$\int_0^{\frac{2}{\varphi(t)}} \frac{\left(t+\frac{h}{2}\varphi(t)\right)^{\alpha}-\left(t-\frac{h}{2}\varphi(t)\right)^{\alpha}}{h}dh \le 2[t^{\alpha}-(t-1)^{\alpha}]\le Ct^{\alpha}.$$

Hence, similarly to the previous lemma, we obtain

$$\int_0^{\frac{2}{\varphi(t)}} \frac{\left(t+\frac{h}{2}\varphi(t)\right)^{\alpha}-\left(t-\frac{h}{2}\varphi(t)\right)^{\alpha}}{h}dh \le Ct^{\alpha}, \qquad \forall\alpha\ge 0, \tag{40}$$

which together with (39) yields the statement.

Case $\alpha = 0$

In this case by applying the usual changes of variables, and taking into account that $e^{\frac{h}{2}\varphi(t)} \sim 1$ for all $h\in[0,\frac{2}{\varphi(t)}]$, we have

$$\begin{aligned}
&\left|\int_{|x-t|<1} \frac{f(x)-f(t)}{x-t}w_0(x)dx\right| \\
&=\left|\int_{t-1}^{t} \frac{f(x)-f(t)}{x-t}e^{-x}dx+\int_{t}^{t+1} \frac{f(x)-f(t)}{x-t}e^{-x}dx\right| \\
&=\left|\int_0^{\frac{2}{\varphi(t)}} \frac{[f(t)-f\left(t-\frac{h}{2}\varphi(t)\right)]e^{-\left(t-\frac{h}{2}\varphi(t)\right)}}{h}dh+\int_0^{\frac{2}{\varphi(t)}} \frac{[f\left(t+\frac{h}{2}\varphi(t)\right)-f(t)]e^{-\left(t+\frac{h}{2}\varphi(t)\right)}}{h}dh\right| \\
&\le \int_0^{\frac{2}{\varphi(t)}} e^{-\left(t-\frac{h}{2}\varphi(t)\right)}\frac{\left|f\left(t+\frac{h}{2}\varphi(t)\right)-f\left(t-\frac{h}{2}\varphi(t)\right)\right|}{h}dh \\
&\quad+\int_0^{\frac{2}{\varphi(t)}} \left|f\left(t+\frac{h}{2}\varphi(t)\right)\right| \frac{\left|e^{-\left(t-\frac{h}{2}\varphi(t)\right)}-e^{-\left(t+\frac{h}{2}\varphi(t)\right)}\right|}{h}dh \\
&\quad+|f(t)|\int_0^{\frac{2}{\varphi(t)}} \frac{\left|e^{-\left(t-\frac{h}{2}\varphi(t)\right)}-e^{-\left(t+\frac{h}{2}\varphi(t)\right)}\right|}{h}dh \\
&\le C\int_0^1 \frac{\Omega_{\varphi}(f,h)_{u_0}}{h}dh + C\left(\|fu_0\|+|f(t)|u_0(t)\right)\int_0^{\frac{2}{\varphi(t)}} \frac{e^{\frac{h}{2}\varphi(t)}-e^{-\frac{h}{2}\varphi(t)}}{h}dh,
\end{aligned}$$

and the statement follows by taking into account that

$$\int_0^{\frac{2}{\varphi(t)}} \frac{e^{\frac{h}{2}\varphi(t)}-e^{-\frac{h}{2}\varphi(t)}}{h}dh = 2\int_0^{\frac{2}{\varphi(t)}} \frac{\sinh(\frac{h}{2}\varphi(t))}{h}dh = 2\int_0^1 \frac{\sinh(x)}{x}dx \le C. \qquad \square$$

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